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A common fixed point theorem for a commuting family of weak* continuous nonexpansive mappings

by

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Abstract. It is shown that if $\mathcal{S}$ is a commuting family of weak* continuous nonexpansive mappings acting on a weak* compact convex subset C of the dual Banach space E, then the set of common fixed points of $\mathcal{S}$ is a nonempty nonexpansive retract of C. This partially solves an open problem in metric fixed point theory in the case of commutative semigroups.

1. Introduction. A subset C of a Banach space E is said to have the *fixed point property* if every nonexpansive mapping $T : C \to C$ (that is, $\|Tx - Ty\| \leq \|x - y\|$ for $x, y \in C$) has a fixed point. A general problem, initiated by the works of F. Browder, D. Göhde and W. A. Kirk and studied by numerous authors for over 40 years, is to classify those E and C which have the fixed point property. For a fuller discussion of this topic we refer the reader to [3, 6].

In this paper we concentrate on weak* compact convex subsets of a dual Banach space E. In 1976, L. Karlovitz [5] proved that if C is a weak* compact convex subset of ℓ_1 (as the dual to c_0) then every nonexpansive mapping $T : C \to C$ has a fixed point. His result was extended by T. C. Lim [11] to the case of left reversible topological semigroups. On the other hand, C. Lennard gave an example of a weak* compact convex subset of ℓ_1 with the weak* topology induced by its predual c and an affine contractive mapping without fixed points (see [12, Example 3.2]). This shows that, apart from nonexpansiveness, some additional assumptions have to be made to obtain the fixed points.

Let S be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each $t \in S$, the mappings $s \mapsto t \cdot s$ and $s \mapsto s \cdot t$ from S into S are continuous. Consider the following fixed point property:

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(F_*) *Whenever* $\mathcal{S} = \{T_s : s \in S\}$ *is a representation of* S *as norm-nonexpansive mappings on a nonempty weak* compact convex set* C *of a dual Banach space* E *and the mapping* $(s, x) \mapsto T_s(x)$ *from* $S \times C$ *to* C *is jointly continuous, where* C *is equipped with the weak* topology of* E*, then there is a common fixed point for* $\mathcal{S}$ *in* C*.*

It is not difficult to show (see, e.g., [9, p. 528]) that property (F_*) implies that S is left amenable (in the sense that LUC(S), the space of bounded complex-valued left uniformly continuous functions on S, has a left invariant mean). Whether the converse is true is a long-standing open problem, posed by A. T.-M. Lau [8] (see also [9, Problem 2], [10, Question 1]).

It is well known that all commutative semigroups are left amenable. The aim of this paper is to give a partial answer to the above problem by showing that every commuting family $\mathcal{S}$ of weak* continuous nonexpansive mappings acting on a weak* compact convex subset C of a dual Banach space E has common fixed points. Moreover, we prove that the set $\operatorname{Fix}\mathcal{S}$ of fixed points is a nonexpansive retract of C.

Note that the structure of $\operatorname{Fix}\mathcal{S}$ (with $\mathcal{S}$ commutative) was examined by R. Bruck [1, 2] who proved that if every nonexpansive mapping $T : C \to C$ has a fixed point in every nonempty closed convex subset of C which is invariant under T, and C is convex and weakly compact or separable, then $\operatorname{Fix}\mathcal{S}$ is a nonexpansive retract of C. We are able to mix the elements of Bruck's method with some properties of w^*-continuous and nonexpansive mappings to get the desired result.

2. Preliminaries. Let E be the dual of a Banach space E_*. In this paper we focus on the weak* topology—the weakest locally convex topology on E satisfying the condition: for all $e \in E$, the functional $\hat{e}(x) = x(e)$ is continuous (in the strong topology). This definition opens up the possibility to consider the so-called weak* properties, for example, w^*-compactness (compactness in the w^*-topology), w^*-completeness, etc. In this topology, E becomes a locally convex Hausdorff space. We say that a dual Banach space E has the w^*-FPP if every nonexpansive self-mapping defined on a nonempty w^*-compact convex subset of E has a fixed point. It is known that $\ell_1 = c_0^*$ and some other Banach lattices have w^*-FPP, while $\ell_1 = c^*$ and the dual of $C(\Omega)$, where Ω is an infinite compact Hausdorff topological space, do not possess this property.

A nonvoid set $D \subset C$ is said to be a *nonexpansive retract* of C if there exists a nonexpansive retraction $R : C \to D$ (i.e., a nonexpansive mapping $R : C \to D$ such that $R_{|D} = I$). Since we deal a lot with w^*-continuous nonexpansive mappings, we abbreviate them to w^*-CN.

We conclude by recalling the following consequence of the Ishikawa theorem [4]: if C is a bounded convex subset of a Banach space X, $\gamma \in (0, 1)$, and $T : C \to C$ is nonexpansive, then the mapping $T_\gamma = (1-\gamma)I + \gamma T$ is *asymptotically regular*, i.e., $\lim_{n\to\infty} \|T_\gamma^{n+1}x - T_\gamma^n x\| = 0$ for every $x \in C$. We use this theorem in Lemma 3.5.

3. Fixed-point theorems. We begin with a structural result concerning a single w^*-continuous nonexpansive mapping $T : C \to C$.

THEOREM 3.1. *Let C be a nonempty weak* compact convex subset of a dual Banach space. Then for any w^*-CN self-mapping T of C, the set* $\operatorname{Fix} T$ *of fixed points of T is a (nonempty) nonexpansive retract of C.*

The proof will follow by constructing consecutively (and establishing properties of) three functions, each one defined in terms of the earlier ones, and the last one being the retraction from C to $\operatorname{Fix} T$.

Proof. Notice first that C is complete in the strong topology. Now, for $x \in C$ and a positive integer n, consider a mapping $T_x : C \to C$ defined by

$$T_x z = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)Tz, \qquad z \in C.$$

It is not difficult to see that T_x is a contraction:

$$\|T_x y - T_x z\| \le \left(1 - \frac{1}{n}\right)\|y - z\|.$$

It follows from the Banach Contraction Principle that there exists exactly one point $F_n x \in C$ such that $T_x F_n x = F_n x$. This defines a mapping $F_n : C \to C$ satisfying

$$F_n x = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)TF_n x \tag{1}$$

for $x \in C$. Thus

$$\|TF_n x - F_n x\| = \frac{1}{n}\|TF_n x - x\| \le \frac{1}{n}\operatorname{diam} C$$

and consequently

$$\lim_n \|TF_n x - F_n x\| = 0$$

since C is norm bounded as a weak* compact subset of a Banach space.

Notice that for $x \in \operatorname{Fix} T$ we have

$$T_x x = x$$

and consequently $F_n x = x$.

Furthermore, $F_n x$ is nonexpansive, which follows from

$$F_n x - F_n y = T_x F_n x - T_y F_n y = \frac{1}{n}(x - y) + \left(1 - \frac{1}{n}\right)(TF_n x - TF_n y) \tag{2}$$

and nonexpansiveness of T.

Notice that we can view C^C as the product space of copies of C, where each copy is endowed with the w^*-topology. Then, according to Tikhonov's theorem, C^C is compact in the product topology generated in this way ("w^*-product topology"). It follows that the sequence $(F_n)_{n\in\mathbb{N}}$ of elements in C^C has a convergent subnet $(F_{n_\alpha})_{\alpha\in\Lambda}$ and we can define

$$R = w^*\text{-}\lim_{\alpha} F_{n_\alpha},$$

where the above limit should be understood as taken in the aforementioned w^*-product topology. Now we can treat the application of R to some $x \in C$ as the projection of the mapping onto the xth coordinate and since such projections are continuous in the product topology, we obtain

$$Rx = w^*\text{-}\lim_{\alpha} F_{n_\alpha} x,$$

where this limit is an ordinary w^*-limit. With this approach, we are able to construct one subnet which guarantees convergence for all $x \in C$.

Notice that

$$TRx = w^*\text{-}\lim_{\alpha} TF_{n_\alpha} x$$

since T is weak* continuous. Now, it follows from the weak* lower semicontinuity of the norm that for any $x \in C$,

$$\|TRx - Rx\| = \Big\|w^*\text{-}\lim_{\alpha}(TF_{n_\alpha}x - F_{n_\alpha}x)\Big\| \le \liminf_{\alpha} \|TF_{n_\alpha}x - F_{n_\alpha}x\| = 0$$

and hence

$$TRx = Rx,$$

which means that $Rx \in \operatorname{Fix} T$. Furthermore, $Rx = x$ if $x \in \operatorname{Fix} T$.

We can now use (2) and the weak* lower semicontinuity of the norm to prove that R is nonexpansive:

$$\begin{aligned}\|Rx - Ry\| &= \Big\|w^*\text{-}\lim_{\alpha}(F_{n_\alpha}x - F_{n_\alpha}y)\Big\| \\ &\le \liminf_{\alpha} \Big\|\frac{1}{n_\alpha}(x-y) + \Big(1 - \frac{1}{n_\alpha}\Big)(Tx - Ty)\Big\| \\ &\le \limsup_{\alpha} \frac{1}{n_\alpha}\|x-y\| + \limsup_{\alpha}\Big(1 - \frac{1}{n_\alpha}\Big)\|Tx - Ty\| \\ &= \|Tx - Ty\| \le \|x - y\|.\end{aligned}$$

Thus we conclude that $\operatorname{Fix} T$ is indeed a nonexpansive retract of C. ∎

Remark 3.2. The w^*-continuity of T cannot be omitted in the assumptions of Theorem 3.1. Indeed, otherwise we would conclude that any dual Banach space has w^*-FPP. But it is known (see, e.g., [12, Example 3.2]) that ℓ_1 (as the dual to the Banach space c) fails the w^*-FPP, a contradiction.

The following example shows that we cannot relax the assumption of nonexpansiveness of T to continuity, even if we only postulate the existence of a (continuous) retraction.

EXAMPLE 3.3. Let $\ell_1 = c_0^*$ and define

$$T(x_1, x_2, x_3, \ldots) = ((x_1)^2, 0, x_2, x_3, \ldots)$$

on the unit ball B_{ℓ_1}. Notice that $T : B_{\ell_1} \to B_{\ell_1}$ is w^*-continuous and $\operatorname{Fix} T = \{(0, 0, \ldots), (1, 0, \ldots)\}$. But a disconnected set cannot be a retract of the ball.

Our next objective is to generalize Theorem 3.1 to a commuting family of w^*-continuous nonexpansive mappings. If $\mathcal{S} = \{T_s : s \in S\}$ is a family of mappings, we denote by

$$\operatorname{Fix} \mathcal{S} = \bigcap_{s \in S} \operatorname{Fix} T_s$$

the set of common fixed points of $\mathcal{S}$.

We first prove a lemma which resembles [1, Lemma 6].

LEMMA 3.4. *Let $\mathcal{S}$ be a family of commuting self-mappings of a set C and suppose that there exists a retraction R of C onto* $\operatorname{Fix} \mathcal{S}$. *If $\widetilde{T}$ commutes with every element of the family $\mathcal{S}$, then*

$$\operatorname{Fix} \mathcal{S} \cap \operatorname{Fix} \widetilde{T} = \operatorname{Fix}(\widetilde{T} R).$$

Proof. The inclusion from left to right follows from the simple observation that if $x \in \operatorname{Fix} \mathcal{S} \cap \operatorname{Fix} \widetilde{T}$, then $Rx = x$ and $\widetilde{T} x = x$.

For the other direction, assume $x \in \operatorname{Fix}(\widetilde{T} R)$, which means $\widetilde{T} R x = x$. Then, for every $T \in \mathcal{S}$, it follows from the commutativity and the fact that $Rx \in \operatorname{Fix} T$ that

$$T \widetilde{T} R x = \widetilde{T}(T R x) = \widetilde{T} R x.$$

Therefore $\widetilde{T} R x \in \operatorname{Fix} T$ for every $T \in \mathcal{S}$ and consequently

$$x = \widetilde{T} R x \in \operatorname{Fix} \mathcal{S}.$$

Since R is a retraction onto $\operatorname{Fix} \mathcal{S}$, we have $Rx = x$ and hence $\widetilde{T} x = x$. It follows that $x \in \operatorname{Fix} \mathcal{S} \cap \operatorname{Fix} \widetilde{T}$, which proves the desired inclusion. ∎

LEMMA 3.5. *Suppose that C is as in Theorem* 3.1 *and $\mathcal{S}_n = \{T_1, \ldots, T_n\}$ is a finite commuting family of w^*-CN self-mappings on C. Then* $\operatorname{Fix} \mathcal{S}_n$ *is a nonexpansive retract of C.*

Proof. We will show by induction on n that there exists a nonexpansive retraction R_n from C onto $\operatorname{Fix} \mathcal{S}_n$. The base case $n = 1$ follows directly from Theorem 3.1 since $\operatorname{Fix} \mathcal{S}_1 = \operatorname{Fix} T_1$.

Now assume that that there exists a nonexpansive retraction R_n of C onto $\operatorname{Fix} \mathcal{S}_n$. We need to show the existence of a nonexpansive retraction R_{n+1} of C onto $\operatorname{Fix} \mathcal{S}_{n+1}$.

Let

$$\widetilde{R}_n x = \frac{1}{2}x + \frac{1}{2}T_{n+1}R_n x, \qquad x \in C,$$

and consider the sequence $(\widetilde{R}_n^k)_{k\in\mathbb{N}}$ of successive iterations of $\widetilde{R}_n$. As in the proof of Theorem 3.1, we can view C^C as the product space, compact with respect to the w^*-topology on C. Hence the sequence $(\widetilde{R}_n^k)_{k\in\mathbb{N}}$ has a convergent subnet $(\widetilde{R}_n^{k_\alpha})_{\alpha\in\Lambda}$ and we can define

$$R_{n+1}x = w^*\text{-}\lim_{\alpha} \widetilde{R}_n^{k_\alpha} x$$

for every $x \in C$.

Since $T_{n+1}R_n$ is nonexpansive as a composition of such mappings, it is easy to see that also $\widetilde{R}_n$ is nonexpansive. The nonexpansiveness of R_{n+1} now follows from the weak* lower semicontinuity of the norm. It is also easy to see that $\operatorname{Fix} T_{n+1}R_n \subset \operatorname{Fix} R_{n+1}$ and, by using Lemma 3.4, we conclude that

$$\operatorname{Fix} \mathcal{S}_{n+1} \subset \operatorname{Fix} R_{n+1}.$$

But this does not yet prove that R_{n+1} is a mapping we are looking for, nor that $\operatorname{Fix} \mathcal{S}_{n+1}$ is nonempty. To complete the proof, we must show that R_{n+1} is a mapping onto $\operatorname{Fix} \mathcal{S}_{n+1}$.

Since C is convex closed and bounded, and $\widetilde{R}_n$ is a convex combination of a nonexpansive mapping and the identity, it follows from the Ishikawa theorem [4] that $\widetilde{R}_n$ is asymptotically regular, i.e.,

$$\lim_{k\to\infty} \|\widetilde{R}_n^{k+1}x - \widetilde{R}_n^k x\| = 0$$

for every $x \in C$.

Now, fix x and notice that $(\widetilde{R}_n^{k_\alpha}x)_{\alpha\in\Lambda}$ is an approximate fixed point net for the mapping $T_{n+1}R_n$. To see this, use the equation

$$\widetilde{R}_n^{k_\alpha+1}x = \frac{1}{2}(\widetilde{R}_n^{k_\alpha}x - T_{n+1}R_n\widetilde{R}_n^{k_\alpha}x) + T_{n+1}R_n\widetilde{R}_n^{k_\alpha}x$$

and the asymptotical regularity in the following calculations:

$$\begin{aligned}
&\limsup_{\alpha} \|T_{n+1}R_n\widetilde{R}_n^{k_\alpha}x - \widetilde{R}_n^{k_\alpha}x\| \\
&\quad\le \limsup_{\alpha} \|T_{n+1}R_n\widetilde{R}_n^{k_\alpha}x - \widetilde{R}_n^{k_\alpha+1}x\| + \lim_{\alpha} \|\widetilde{R}_n^{k_\alpha+1}x - \widetilde{R}_n^{k_\alpha}x\| \\
&\quad= \limsup_{\alpha} \|T_{n+1}R_n\widetilde{R}_n^{k_\alpha}x - \widetilde{R}_n^{k_\alpha+1}x\| = \frac{1}{2}\limsup_{\alpha} \|T_{n+1}R_n\widetilde{R}_n^{k_\alpha}x - \widetilde{R}_n^{k_\alpha}x\|.
\end{aligned}$$

Thus we conclude that

$$\lim_{\alpha} \|T_{n+1}R_n\widetilde{R}_n^{k_\alpha}x - \widetilde{R}_n^{k_\alpha}x\| = 0, \tag{3}$$

as desired.

Now, for brevity, denote $r_\alpha = \widetilde{R}_n^{k_\alpha} x$ and notice that for every $m \le n$,

$$T_m T_{n+1} R_n r_\alpha = T_{n+1} T_m R_n r_\alpha = T_{n+1} R_n r_\alpha.$$

That is, $T_{n+1} R_n r_\alpha \in \operatorname{Fix} T_m$, which is equivalent to the statement that $T_{n+1} R_n r_\alpha$ belongs to $\operatorname{Fix} \mathcal{S}_n$. It follows that

$$T_{n+1} R_n r_\alpha = R_n T_{n+1} R_n r_\alpha.$$

and using (3), we obtain

$$\begin{aligned} (4) \qquad & \limsup_\alpha \|R_n r_\alpha - r_\alpha\| \\ & \le \limsup_\alpha \|R_n r_\alpha - T_{n+1} R_n r_\alpha\| + \lim_\alpha \|T_{n+1} R_n r_\alpha - r_\alpha\| \\ & = \limsup_\alpha \|R_n r_\alpha - R_n T_{n+1} R_n r_\alpha\| \le \lim_\alpha \|r_\alpha - T_{n+1} R_n r_\alpha\| = 0. \end{aligned}$$

In the same manner we can see that for every $m \le n$,

$$\begin{aligned} \limsup_\alpha \|T_m r_\alpha - r_\alpha\| & \le \limsup_\alpha \|T_m r_\alpha - T_m R_n r_\alpha\| + \limsup_\alpha \|T_m R_n r_\alpha - r_\alpha\| \\ & \le \lim_\alpha \|r_\alpha - R_n r_\alpha\| + \lim_\alpha \|R_n r_\alpha - r_\alpha\| = 0. \end{aligned}$$

Since T_m is w^*-continuous, this easily yields

$$T_m R_{n+1} x = R_{n+1} x,$$

and consequently

$$(5) \qquad R_{n+1} x \in \operatorname{Fix} \mathcal{S}_n.$$

Finally, by using (3) and (4), we get

$$\begin{aligned} \limsup_\alpha \|T_{n+1} r_\alpha - r_\alpha\| & \le \limsup_\alpha \|T_{n+1} r_\alpha - T_{n+1} R_n r_\alpha\| \\ & \quad + \lim_\alpha \|T_{n+1} R_n r_\alpha - r_\alpha\| \\ & \le \lim_\alpha \|r_\alpha - R_n r_\alpha\| = 0. \end{aligned}$$

Then, from the w^*-continuity of T_{n+1},

$$T_{n+1} R_{n+1} x = R_{n+1} x,$$

which combined with (5) gives

$$R_{n+1} x \in \operatorname{Fix} \mathcal{S}_{n+1}.$$

That is, $\operatorname{Fix} \mathcal{S}_{n+1}$ is nonempty and R_{n+1} acts onto it, which completes the proof. ■

We are now in a position to prove our main theorem.

THEOREM 3.6. *Suppose that C is as in Theorem* 3.1 *and $\mathcal{S}$ is an arbitrary family of commuting w^*-CN self-mappings on C. Then* $\operatorname{Fix} \mathcal{S}$ *is a nonexpansive retract of C.*

Proof. If $\mathcal{S}$ is finite, we can use Lemma 3.5. So assume that $\mathcal{S}$ is infinite. First notice that

$$\operatorname{Fix} T = (T - I)^{-1}\{0\}$$

is closed in the w^*-topology for every $T \in \mathcal{S}$. Let

$$\Lambda = \{\alpha \subset \mathcal{S} : \#\alpha < \infty\}$$

be a directed set with the inclusion relation $\leq$. Denote by R_α the nonexpansive retraction from C to $\operatorname{Fix}_\alpha = \bigcap_{T\in\alpha} \operatorname{Fix} T$ (a more convenient way of writing $\operatorname{Fix}\alpha$) whose existence is guaranteed by Lemma 3.5. Then we have a net $(R_\alpha)_{\alpha\in\Lambda}$, and we can select a subnet $(R_{\alpha_\gamma})_{\gamma\in\Gamma}$, w^*-convergent for any $x \in C$. Define

$$Rx = w^*\text{-}\lim_{\gamma} R_{\alpha_\gamma} x.$$

For a fixed $T \in \mathcal{S}$, take γ_0 such that $\alpha_\gamma \geq \{T\}$ for every $\gamma \geq \gamma_0$. It exists, directly from the definition of subnet. Then

$$\forall_{\gamma\geq\gamma_0}\ R_{\alpha_\gamma} x \in \operatorname{Fix}_{\alpha_\gamma} \subset \operatorname{Fix}_{\alpha_{\gamma_0}} \subset \operatorname{Fix} T$$

and hence $R_{\alpha_\gamma} x$ lies eventually in the w^*-closed set $\operatorname{Fix} T$. Therefore, $Rx \in \operatorname{Fix} T$ for every $T \in \mathcal{S}$, which implies $Rx \in \operatorname{Fix}\mathcal{S}$. It is easy to see that R is nonexpansive. Also, for every α,

$$x \in \operatorname{Fix}\mathcal{S} \ \Rightarrow\ x \in \operatorname{Fix}_\alpha \ \Rightarrow\ R_\alpha x = x,$$

which yields

$$Rx = x, \qquad x \in \operatorname{Fix}\mathcal{S}. \tag{6}$$

Thus R is a nonexpansive retraction from C onto $\operatorname{Fix}\mathcal{S}$. ■

REMARK 3.7. In particular, the set $\operatorname{Fix}\mathcal{S}$ is non-empty. Thus Theorem 3.6 answers affirmatively [10, Question 1] in the case of commutative semigroups.

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