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Schrödinger equation on the Heisenberg group

by

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Abstract. Let L be the full laplacian on the Heisenberg group $\mathbb{H}^n$ of arbitrary dimension n. Then for $f \in L^2(\mathbb{H}^n)$ such that $(I-L)^{s/2} f \in L^2(\mathbb{H}^n)$ for some $s > 1/2$ and for every $\phi \in C_c(\mathbb{H}^n)$ we have

$$\int_{\mathbb{H}^n} |\phi(x)| \sup_{0<t\le 1} |e^{\sqrt{-1}\, tL} f(x)|^2 \, dx \le C_\phi \|f\|^2_{W^s}.$$

Introduction. Let V_t be the Schrödinger unitary group generated by a self-adjoint, positive differential operator L on $\mathbb{R}^d$. The degree of smoothness needed for the almost everywhere convergence of $V_t f$ to f as $t \to 0$ has been extensively studied. In general, the result of Cowling [Cw] says that if $\|(1+L)^{s/2} f\|_{L^2} < \infty$ for some $s > 1$, then

$$(*) \qquad \lim_{t\to 0} V_t f(x) = f(x) \quad \text{a.e.}$$

This does not depend on any other properties of L.

For $-L$ being the Laplace operator on $\mathbb{R}^d$, $s > 1/2$ suffices for all d, and for $d = 2$, $s > 1/2 - \delta$ is also sufficient. See [B], [Mo]. In our previous paper [Z] the Laplace operator L on the Heisenberg group $\mathbb{H}^n$ has been studied from this point of view, and we have proved that $s > 3/4$ implies $(*)$. In this paper we simplify the proof of the result in [Z] and decrease the needed regularity of f to $f \in W^s, s > 1/2$.

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0. Preliminaries. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and consequently $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. Denote by $S(\mathbf{z},\mathbf{w}) = 2\Im(\mathbf{z}\cdot\overline{\mathbf{w}})$ the standard symplectic form on $\mathbb{R}^{2n}$.

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For $m = 0, 1, 2, \ldots$ let

$$L_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-x)^k}{k!}$$

be the Laguerre polynomial of degree m and for $a \neq 0$ let

$$l_{m,a}(z) = e^{-|a||z|^2} L_m(2|a|\,|z|^2)$$

be the corresponding Laguerre function.

Let $\mathbf{m} = (m_1, \ldots, m_n)$ and $\mathbf{z} = (z_1, \ldots, z_n)$. We write

$$q_{\mathbf{m},a}(\mathbf{z}) = l_{m_1,a}(z_1) l_{m_2,a}(z_2) \ldots l_{m_n,a}(z_n).$$

It is well known that the $q_{\mathbf{m},1}$ form an orthonormal basis of the space of polyradial functions on $\mathbb{C}^n$.

We denote by $d\mathbf{z}$ the Lebesgue measure on $\mathbb{C}^n$ and for $a \neq 0$ we define the twisted convolution

$$f \times_a g(\mathbf{z}) = \int f(\mathbf{z} - \mathbf{w}) g(\mathbf{w}) e^{iaS(\mathbf{z},\mathbf{w})}\, d\mathbf{w}, \qquad f, g \in C_{\mathrm{c}}^\infty(\mathbb{C}^n).$$

We have the following orthogonality relation for the Laguerre functions (cf. [M]):

$$|a|^n q_{\mathbf{k},a} \times_a q_{\mathbf{m},a}(\mathbf{z}) = \delta_{\mathbf{k},\mathbf{m}} q_{\mathbf{m},a}(\mathbf{z}). \tag{0.1}$$

Fix a real $a \neq 0$ and let

$$Q_{\mathbf{m},a} f(\mathbf{z}) = |a|^n q_{\mathbf{m},a} \times_a f(\mathbf{z}). \tag{0.2}$$

It follows from (0.1) that for a fixed $a \neq 0$ the operators $Q_{\mathbf{m},a}$ are mutually orthogonal projectors. Moreover $\sum_{\mathbf{m}} Q_{\mathbf{m},a} = \mathrm{Id}$ (cf. [M]).

We introduce a separate notation for the operators $Q_{\mathbf{m},a}$ in the case $\mathbf{m} = m \in \mathbb{N}$, i.e. $n = 1$. We then write

$$Q_{\mathbf{m},a} f = P_{m,a} f = |a| l_{m,a} \times_a f.$$

The Heisenberg group $\mathbb{H}^n$ is defined as $\mathbb{C}^n \times \mathbb{R}$, with the group product $(\mathbf{z}, s)(\mathbf{w}, t) = (\mathbf{z} + \mathbf{w}, s + t + 2\Im(\mathbf{z} \cdot \overline{\mathbf{w}}))$ where $\mathbf{z} = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$. Then the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ is the Haar measure on $\mathbb{H}^n$.

Let

$$X_i = \partial_{x_i} + 2y_i \partial_t, \qquad Y_i = \partial_{y_i} - 2x_i \partial_t \quad \text{ for } 1 \leq i \leq n, \qquad T = \partial_t,$$

and let

$$L = \sum_{i=1}^{n} (X_i^2 + Y_i^2) + T^2$$

be the *elliptic* laplacian on $\mathbb{H}^n$. The closure of L on $C_{\mathrm{c}}^\infty(\mathbb{H}^n)$ is a self-adjoint operator (see [NS]). Therefore iL generates a group $\{V_t\}_{t \in \mathbb{R}}$ of unitary operators on $L^2(\mathbb{H}^n)$. We will use the following formula for V_t, valid for $f \in S(\mathbb{H}^n)$

(cf. [M]):

$$V_t f(\mathbf{z},u)=\sum_{\mathbf{m}}\int_{\mathbb{R}} e^{iua}e^{it\lambda_{|\mathbf{m}|}(a)}Q_{\mathbf{m},a}f^a(\mathbf{z})\,da, \tag{0.3}$$

where $\lambda_{|\mathbf{m}|}(a)=(2|\mathbf{m}|+n)|a|+a^2$, $|\mathbf{m}|=m_1+\ldots+m_k$ and f^a denotes the Fourier transform with respect to the central variable.

Let $s\ge 0$. We define a scale of Sobolev spaces by putting

$$\|f\|_{W^s}=\|(I-L)^{s/2}f\|_{L^2}.$$

Since $Q_{\mathbf{m},a}$ are mutually orthogonal projectors and

$$Lf(\mathbf{z},u)=\int_{\mathbb{R}}\sum_{\mathbf{m}} e^{iua}\lambda_{|\mathbf{m}|}(a)Q_{\mathbf{m},a}f^a(\mathbf{z})\,da$$

the Plancherel theorem applied to the variable u implies

$$\|f\|_{W^s}^2=\sum_{\mathbf{m}}\int_{\mathbb{R}}(1+\lambda_{|\mathbf{m}|}(a))^s\|Q_{\mathbf{m},a}f^a\|_{L^2(\mathbb{C}^n)}^2\,da. \tag{0.4}$$

For a more detailed exposition of the preliminary facts we refer the reader to [M] and [Z].

1. Basic lemmas. Let $0<\alpha<1$. The *fractional derivative* of order α is defined by

$$\partial^\alpha f(s)=\int_{\mathbb{R}}(f(s-t)-f(s))|t|^{-(1+\alpha)}\,dt.$$

LEMMA 1 (Sobolev). *Let $\gamma>0$ be a Schwartz function and $1/2<\alpha<1$. Then*

$$\sup_{-1\le t\le 1}|f(t)|^2\le C_\alpha\Big(\int_{\mathbb{R}}|\partial^\alpha f(t)|^2\gamma(t)\,dt+\int_{\mathbb{R}}|f(t)|^2\gamma(t)\,dt\Big).$$

For a function ϕ, let M_ϕ denote the operator of multiplication by ϕ. Set $B(r)=\{z:|z|\le r\}$.

Fix $\phi\in C_c^\infty(\mathbb{C})$ with $\operatorname{supp}\phi\subset B(1)$ and $|\phi(z)|\le 1$, and define

$$T_{m,a}f(z)=M_\phi P_{m,a}f(z)=\phi(z)|a|l_{m,a}\times_a f(z).$$

Since $P_{m,a}$ is an orthogonal projector we have $\|T_{m,a}\|_{L^2\to L^2}\le 1$. The following two lemmas have been proved in [Z].

LEMMA 2. *For $4\le |a|\le m+1$ we have*

$$\|T_{m,a}\|_{L^2\to L^2}^2\le C\bigg(\frac{|a|}{m+1}\bigg)^{1/2}.$$

LEMMA 3. *For $|a|\le 4$ we have*

$$\|T_{m,a}\|^2\le C\bigg(\frac{|a|}{m+1}\bigg)^{1/2}.$$

For the reader's convenience we include the proofs of Lemmas 2 and 3. To do this we need a number of consequences of the classical estimates for Laguerre functions, collected below.

LEMMA 4.

$$L_m(\lambda x) = \sum_{k=0}^{m} \binom{m}{k} L_k(x)\lambda^k(1-\lambda)^{m-k}. \tag{1.1}$$

LEMMA 5. *Let* $1 \le |z| \le (m+1)^{1/2}$. *Then*

$$|l_{m,1}(z)| \le C(m+1)^{-1/4}|z|^{-1/2}.$$

Proof. Let $0 < \varepsilon \le \varphi \le \pi/2 - \varepsilon(m+1)^{-1/2}$. Then by a theorem of Szegő [Sz], for $x = (4m+2)\cos^2\varphi$, we have

$$\begin{aligned} e^{-x/2}L_m(x) = (-1)^m(\pi\sin\varphi)^{-1/2}(&\sin((m+1/2)(\sin 2\varphi - 2\varphi) + 3\pi/4) \\ &\times (x(m+1))^{-1/4} + (x(m+1))^{-1/2}O(1)). \end{aligned}$$

LEMMA 6. *Let* $|z| \le 1$. *Then*

$$l_{m,1}(z) = J_0(2^{1/2}|z|(m+1/2)^{1/2}) + O((m+1)^{-3/4}),$$

where J_0 *is the zero Bessel function.*

Proof. Follows from an asymptotic formula for the Laguerre polynomials (cf. [Sz]):

$$e^{-x/2}L_m(x) = J_0((2x(m+1/2))^{1/2}) + O((m+1)^{-3/4}).$$

LEMMA 7. *There is a constant* C *such that for* $A \ge 1$ *we have*

$$\int |l_{m,1}(z)|^2 e^{-|z|^2/A^2}\,dz \le CA(m+1)^{-1/2}.$$

Proof. By Lemma 5, we obtain

$$\begin{aligned} \int\limits_{1\le|z|\le(m+1)^{1/2}} &|l_{m,1}(z)|^2 e^{-|z|^2/A^2}\,dz \\ &\le C\int \frac{1}{|z|(m+1)^{1/2}}\, e^{-|z|^2/A^2}\,dz \le CA(m+1)^{-1/2}. \end{aligned}$$

Also

$$\int\limits_{|z|\ge(m+1)^{1/2}} |l_{m,1}(z)|^2 e^{-|z|^2/A^2}\,dz \le e^{-m/A^2}\int |l_{m,1}(z)|^2\,dz \le CA(m+1)^{-1/2}.$$

On the other hand, by Lemma 6, using the estimate $|J_0(x)| \le C(1+|x|)^{-1/2}$ for the Bessel function (see [Sz]) we obtain

$$|l_{m,1}(z)| \le C(1+|z|^{1/2}(m+1)^{1/4})^{-1}.$$

Hence

$$\int\limits_{|z|\le 1} |l_{m,1}(z)|^2 e^{-|z|^2/A^2}\,dz \le C(m+1)^{-1/2}.$$

Proof of Lemma 2. Since $P_{m,a}$ is an orthogonal projector, $T_{m,a}T^*_{m,a} = M_\phi P_{m,a} M_\phi$. Hence, the kernel K of $T_{m,a}T^*_{m,a}$ is given by the formula

$$K(z_1, z_2) = \phi(z_1)|a|l_{m,a}(z_1 - z_2)e^{-iaS(z_1,z_2)}\phi(z_2). \tag{1.1}$$

We write

$$1 = e^{-|z_1-z_2|^2}e^{|z_1-z_2|^2} = e^{-|z_1-z_2|^2}\sum_\alpha c_\alpha z_1^{\alpha_1}\overline{z}_1^{\alpha_3}z_2^{\alpha_2}\overline{z}_2^{\alpha_4}.$$

Thus

$$\begin{aligned} &K(z_1, z_2) \\ &\quad= \sum_\alpha c_\alpha z_1^{\alpha_1}\overline{z}_1^{\alpha_3}\phi(z_1)e^{-|z_1-z_2|^2}|a|l_{m,a}(z_1 - z_2)e^{-iaS(z_1,z_2)}\phi(z_2)z_2^{\alpha_2}\overline{z}_2^{\alpha_4}. \end{aligned} \tag{1.2}$$

Consequently, the operator $T_{m,a}T^*_{m,a}$ is the sum over α of operators

$$c_\alpha M_\phi M_{z_1^{\alpha_1}\overline{z}_1^{\alpha_3}} T_{K_1} M_{z_2^{\alpha_2}\overline{z}_2^{\alpha_4}} M_\phi,$$

where

$$T_{K_1} = f \times_a K_1, \qquad K_1(z) = e^{-|z|^2}|a|l_{m,a}(z).$$

Since c_α converges to zero faster than exponentially, it suffices to estimate the norm of T_{K_1}. Dilating we see that the norm of T_{K_1} is the same as the norm of the 1-twisted convolution operator by

$$\mathcal{K}(z) = e^{-|z|^2|a|^{-1}}l_{m,1}(z).$$

The radial function $\mathcal{K}(z)$ has a decomposition

$$\mathcal{K}(z) = \sum_{k=0}^\infty c_{k,m,a}l_{k,1}(z),$$

where

$$c_{k,m,a} = \int e^{-|z|^2|a|^{-1}}l_{k,1}(z)l_{m,1}(z)\,dz. \tag{$*$}$$

So

$$\mathcal{K}(z) \times_1 f(z) = \sum_{k=0}^\infty c_{k,m,a}P_{m,1}f(z).$$

Since $P_{m,1}$, $m = 0, 1, \ldots$, are mutually orthogonal projectors the norm of the operator $f \mapsto \mathcal{K} \times_1 f$ is equal to

$$\sup_k |c_{k,m,a}|.$$

By the Schwarz inequality, we obtain

$$|c_{k,m,a}| \le \|e^{-|z|^2/2|a|}l_{k,1}(z)\|_{L^2}\|e^{-|z|^2/2|a|}l_{m,1}(z)\|_{L^2}.$$

Now, by Lemma 7, if $10k \ge m$, then

$$|c_{k,m,a}| \le C\left(\frac{|a|}{m+1}\right)^{1/4}\left(\frac{|a|}{k+1}\right)^{1/4} \le C\left(\frac{|a|}{m+1}\right)^{1/2}.$$

It remains to estimate the coefficients $c_{k,m,a}$ for $10k \le m$. Observe that by the definition of $l_{m,1}(z)$, for $\lambda = (1+(2|a|)^{-1})^{-1}$, $(*)$ turns into

$$c_{k,m,a} = C\int_0^\infty e^{-\lambda^{-1}x} L_m(x)L_k(x)\,dx.$$

Then

$$c_{k,m,a} = C\lambda\int_0^\infty e^{-x} L_m(x\lambda)L_k(x\lambda)\,dx,$$

whence, in virtue of (1.1), because the L_k form an orthonormal basis with the weight e^{-x}, we obtain

$$\begin{aligned} c_{k,m,a} &= C\lambda\sum_{s_1=0}^{m}\sum_{s_2=0}^{k}\binom{m}{s_1}\binom{k}{s_2}\lambda^{(s_1+s_2)}(1-\lambda)^{m+k-(s_1+s_2)} \\ &\quad\times\int_0^\infty e^{-x}L_{s_1}(x)L_{s_2}(x)\,dx \\ &= C\lambda\sum_{s=0}^{k}\binom{m}{s}\binom{k}{s}\lambda^{2s}(1-\lambda)^{m+k-2s}. \end{aligned}$$

Now, if $|a| \ge 4$ then $2/3 \le \lambda \le 1$ so for $10k \le m$ we have

$$\begin{aligned} |c_{k,m,a}| &\le \sum_{s=0}^{k} 2^m 2^k (1-\lambda)^{m+k-2k} \le k2^m2^k3^{-m-k} \\ &\le k\left(\frac{2}{3}\right)^{m+k} \le 2^{-\varepsilon m} \le 2^{-\varepsilon m}|a| \end{aligned}$$

for some positive constant ε.

Proof of Lemma 3. In order to estimate the norm of $T_{m,a}T^*_{m,a}$ we use (1.1) and the asymptotic formula for the Laguerre functions given in Lemma 6.

Let $|a| \le 4$. By the Taylor series expansion for $e^{iaS(z_1,z_2)}$ we have

$$\begin{aligned} K(z_1,z_2) &= \sum_\alpha z_1^{\alpha_1}\overline{z}_1^{\alpha_3}\phi(z_1)|a|l_{m,a}(z_1-z_2)\phi(z_2)z_2^{\alpha_2}\overline{z}_2^{\alpha_4}a_\alpha|a|^{|\alpha|/2} \\ &= \sum_\alpha a_\alpha|a|^{|\alpha|/2}K_\alpha(z_1,z_2). \end{aligned}$$

Since the a_α's decay faster than exponentially, and the norms of the operators $M_\phi M_{z^\alpha}$ grow at most exponentially, it suffices to estimate the norm of the operator K given by the kernel

$$A(z_1,z_2)=\psi(z_1)|a|l_{m,a}(z_1-z_2)\psi(z_2),\quad \text{where} \quad \psi\in C_{\mathrm{c}}^{\infty} \text{ with } \psi(z)=1 \text{ on } \operatorname{supp}\phi.$$

Now using Lemma 6 we obtain

$$\psi(z_1)|a|l_{m,a}(z_1-z_2)\psi(z_2)=C\psi(z_1)|a|J_0(2|a|^{1/2}|z_1-z_2|(2m+1)^{1/2})\psi(z_2) + \psi(z_1)\psi(z_2)O(|a|(m+1)^{-3/4}).$$

Observe that the error term in the last formula gives an operator with norm of order $|a|(m+1)^{-3/4}$, so it is negligible.

Hence, for a function $\widetilde{\phi}\in S(\mathbb{C})$ with $\widetilde{\phi}=1$ on $\operatorname{supp}\psi-\operatorname{supp}\psi$ we write

$$\begin{aligned}\psi(z_1)|a|J_0(|a|^{1/2}|z_1-z_2|(2m+1)^{1/2})\psi(z_2)\\ =\widetilde{\phi}(z_1-z_2)\psi(z_1)|a|J_0(|a|^{1/2}|z_1-z_2|(2m+1)^{1/2})\psi(z_2).\end{aligned}$$

Thus we may drop $\psi(z_1),\psi(z_2)$ and we estimate the norm of the convolution operator by the function

$$R=\widetilde{\phi}(z)|a|J_0(|a|^{1/2}|z|(2m+1)^{1/2}).$$

By definition, J_0 is the Fourier transform of the normalized Lebesgue measure supported on the unit circle. Hence

$$\widehat{R}=\widehat{\widetilde{\phi}}*|a|\mu,$$

where μ is the normalized Lebesgue measure supported by the circle of radius $|a(2m+1)|^{1/2}$. We write (using a smooth resolution of identity $1=\sum_{j\in\mathbb{Z}^2}k(z-j)$ with $\operatorname{supp}k\subset B(2)$)

$$\widehat{\widetilde{\phi}}=\sum_j\alpha_j\phi_j,$$

where $\sum_j|\alpha_j|<\infty$, $\|\phi_j\|_{L^\infty}\le 1$ and the support of ϕ_j is contained in the disc of radius two. A trivial geometric argument shows that for $|(2m+1)a|\ge 1$, $\|\phi_j*\mu\|_{L^\infty}\le C|(2m+1)a|^{-1/2}$. These imply that the L^∞ norm of $\widehat{R}$ is bounded by $C|a|^{1/2}|(m+1)|^{-1/2}$. If $|(2m+1)a|\le 1$ then $\|\phi_j*\mu\|_{L^\infty}\le C$ and consequently $\|\widehat{R}\|_{L^\infty}\le C|a|\le C|(m+1)|^{-1/2}|a|^{1/2}$. This proves the lemma.

2. Main theorem. For a fixed $\phi\in C_{\mathrm{c}}^{\infty}(\mathbb{H}^n)$ we define the local maximal function of the group V_t by

$$Mf(\mathbf{z},u)=\phi(\mathbf{z},u)\sup_{0\le t\le 1}|V_tf(\mathbf{z},u)|.$$

We have

THEOREM 1. *Let $s > 1/2$ and $f \in W^s$. Then*

$$\|Mf\|_{L^2} \le C\|f\|_{W^s}.$$

Proof. Let $f \in L^2(\mathbb{H}^n)$. To estimate $\|Mf\|_{L^2(\mathbb{H}^n)}$ we introduce a family of projections P_α. Then we write

$$\|Mf\|_{L^2(\mathbb{H}^n)} \le \sum_\alpha \|MP_\alpha f\|_{L^2(\mathbb{H}^n)}$$

and we estimate each $\|MP_\alpha f\|_{L^2(\mathbb{H}^n)}$ separately.

We will use the abbreviation

$$s \approx 2^k \quad \text{iff} \quad 2^k \le s < 2^{k+1}.$$

For $k, l \in \mathbb{N}$ let

$$\begin{aligned} P_{k,l} f(\mathbf{z}, u) &= \sum_{\{\mathbf{m}\,:\,|\mathbf{m}|\approx 2^k\}} \int_{\{|a|\approx 2^l\}} e^{iua} Q_{\mathbf{m},a} f^a(\mathbf{z})\, da, \\ P_0 f(\mathbf{z}, u) &= \sum_{\mathbf{m}} \int_{\{|a|\le 1\}} e^{iua} Q_{\mathbf{m},a} f^a(\mathbf{z})\, da. \end{aligned}$$

Then obviously

$$P_0 + \sum_{k,l} P_{k,l} = \mathrm{Id}\,.$$

The maximal function of the theorem splits into the maximal functions

$$\begin{aligned} S_{k,l} f(\mathbf{z}, u) &= \sup_{0\le t\le 1} |\psi(\mathbf{z})\phi(u) P_{k,l} V_t f(\mathbf{z}, u)|, \\ S_0 f(\mathbf{z}, u) &= \sup_{0\le t\le 1} |\psi(\mathbf{z}) P_0 V_t f(\mathbf{z}, u)|, \end{aligned} \tag{2.2}$$

where $\psi \in C_c^\infty(\mathbb{C}^n)$, $\widehat{\phi} \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\widehat{\phi} \subset B(1)$.

We are going to estimate the norms $\|S_{k,l}\|_{W^{1/2+\varepsilon}\to L^2}$ and $\|S_0\|_{W^{1/2+\varepsilon}\to L^2}$. Then we sum up the estimates. With no loss of generality we may consider only the $\mathbf{m}$'s in $I_1 = \{\mathbf{m} : m_1 = \max(m_1, \ldots, m_n)\}$.

Let $A = \{(\mathbf{m}, \mathbf{r}) : m_2 = r_2, \ldots, m_n = r_n,\ \mathbf{m}, \mathbf{r} \in I_1\}$. We fix a and we note that $|\mathbf{m}| = |\mathbf{r}|$ and $(\mathbf{m}, \mathbf{r}) \in A$ imply $\mathbf{m} = \mathbf{r}$. By the orthogonality relations (0.1) for $P_{m,a}$ we have

$$\begin{aligned} &\int Q_{\mathbf{m},a} f(\mathbf{z})\, \overline{Q_{\mathbf{r},a} f(\mathbf{z})}\, dz_2 \ldots dz_n \\ &\quad = \int P_{m_1,a} P_{m_2,a} \ldots P_{m_n,a} f(\mathbf{z})\, \overline{P_{r_1,a} P_{r_2,a} \ldots P_{r_n,a} f(\mathbf{z})}\, dz_2 \ldots dz_n = 0 \end{aligned}$$

if $(m_2, \ldots, m_n) \ne (r_2, \ldots, r_n)$. In the formula above $P_{m_i,a}$ acts on the variable z_i.

We begin by estimating the norm of S_0, making use of the Sobolev lemma. We have

$$|S_0 f(\mathbf{z},u)|^2 \le C\Big(\int_{\mathbb{R}} |\partial_t^{1/2+\varepsilon} V_t P_0 f(\mathbf{z},u)|^2\gamma(t)\,dt + \int_{\mathbb{R}} |V_t P_0 f(\mathbf{z},u)|^2\gamma(t)\,dt\Big)\psi(\mathbf{z}).$$

In what follows we assume that $\widehat{\gamma}$ is supported in the interval $[-1,1]$. Integrating with respect to $d\mathbf{z}du$, by the Plancherel theorem applied to the Fourier transform in the central variable, we have

$$\begin{aligned}
\int |S_0 f(\mathbf{z},u)|^2\,d\mathbf{z}\,du &\le \int |\partial_t^{1/2+\varepsilon} P_0 V_t f(\mathbf{z},u)|^2 \psi(\mathbf{z})\gamma(t)\,d\mathbf{z}\,du\,dt + C\|f\|_{L^2}^2 \\
&= C\iiint \Big|\sum_m I_{\{0\le|a|\le 1\}}(a) Q_{\mathbf{m},a} f^a(\mathbf{z})\partial_t^{1/2+\varepsilon} e^{i\lambda_{|\mathbf{m}|}(a)t}\Big|^2 da\,\gamma(t)\,dt\,\psi(\mathbf{z})\,d\mathbf{z} \\
&\quad + C\|f\|_{L^2}^2 \\
&\le C\iiint \Big|\sum_{\mathbf{m}} I_{\{C/|\mathbf{m}|\le|a|\le 1\}}(a)(\lambda_{\mathbf{m}}(a))^{1/2+\varepsilon} \\
&\quad \times Q_{\mathbf{m},a} f^a(\mathbf{z}) e^{i\lambda_{|\mathbf{m}|}(a)t}\Big|^2 da\,\gamma(t)\,dt\,\psi(\mathbf{z})\,d\mathbf{z} + \|f\|_{L^2}^2.
\end{aligned}$$

In the last inequality we have used the fact that for $|a| \le C|\mathbf{m}|^{-1}$, we have $\lambda_{|\mathbf{m}|}(a) \le C$.

In the above sum the multiindices $\mathbf{m}$ belong to I_1. We enlarge the last expression by replacing the $\psi(\mathbf{z})$ by $\psi(z_1)$, $\psi \in C_{\mathrm{c}}^\infty(\mathbb{C})$. Thus

$$\begin{aligned}
\iiint &\Big|\sum_{\mathbf{m}} I_{\{C/|\mathbf{m}|\le|a|\le 1\}}(a)(\lambda_{|\mathbf{m}|}(a))^{1/2+\varepsilon} Q_{\mathbf{m},a} f^a(\mathbf{z}) e^{i\lambda_{|\mathbf{m}|}(a)t}\Big|^2 da\,\gamma(t)\,dt\,\psi(\mathbf{z})\,d\mathbf{z} \\
&= \iint \sum_{\mathbf{m}\in I_1}\sum_{\mathbf{r}\in I_1} I_{\{C/|\mathbf{m}|\le|a|\le 1\}}(a) I_{\{C/|\mathbf{r}|\le|a|\le 1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{1/2+\varepsilon} \\
&\quad \times Q_{\mathbf{m},a} f^a(\mathbf{z})\,\overline{Q_{\mathbf{r},a} f^a(\mathbf{z})}\,\widehat{\gamma}(\lambda_{|\mathbf{m}|}(a) - \lambda_{|\mathbf{r}|}(a))\,da\,\psi(z_1)\,d\mathbf{z}.
\end{aligned}$$

By orthogonality of $P_{m,a}$ the last expression is equal to

$$\begin{aligned}
\int &\sum_{(\mathbf{m},\mathbf{r})\in A} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\le|a|\le 1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{1/2+\varepsilon} \\
&\quad \times \int Q_{\mathbf{m},a} f^a(\mathbf{z})\,\overline{Q_{\mathbf{r},a} f^a(\mathbf{z})}\,\psi(z_1)\,d\mathbf{z}\,\widehat{\gamma}(\lambda_{|\mathbf{m}|}(a) - \lambda_{|\mathbf{r}|}(a))\,da \\
&= \int \sum_{(\mathbf{m},\mathbf{r})\in A} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\le|a|\le 1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{1/2+\varepsilon} \\
&\quad \times \int Q_{\mathbf{m},a} f^a(\mathbf{z})\,\overline{Q_{\mathbf{r},a} f^a(\mathbf{z})}\,\psi(z_1)\,d\mathbf{z}\,\widehat{\gamma}(2m_1|a| - 2r_1|a|)\,da
\end{aligned}$$

$$\leq \int \sum_{(\mathbf{m},\mathbf{r})\in A} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\leq|a|\leq 1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{1/2+\varepsilon}$$

$$\times\left(\frac{|a|}{|\mathbf{m}|+1}\,\frac{|a|}{|\mathbf{r}|+1}\right)^{1/4}$$

$$\times\|Q_{\mathbf{m},a}f^a\|\,\|Q_{\mathbf{r},a}f^a\|\widehat{\gamma}(2(m_1-r_1)|a|)\,da.$$

To verify the last inequality we use Lemma 3. The last expression is bounded by

$$S=C\int \sum_{(\mathbf{m},\mathbf{r})\in A} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\leq|a|\leq 1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{1/2+\varepsilon}$$

$$\times\left(\left(\frac{|a|}{|\mathbf{m}|+1}\right)^{1/2}\|Q_{\mathbf{m},a}f^a\|^2+\left(\frac{|a|}{|\mathbf{r}|+1}\right)^{1/2}\|Q_{\mathbf{r},a}f^a\|^2\right)\widehat{\gamma}(2(m_1-r_1)|a|)\,da.$$

For fixed $\mathbf{r}$ we have

$$\sum_{\{\mathbf{m}\,:\,(\mathbf{m},\mathbf{r})\in A\}} I_{\{C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\leq|a|\leq 1\}}(a)(\lambda_{|\mathbf{m}|}(a)\lambda_{|\mathbf{r}|}(a))^{1/2+\varepsilon} \times\left(\frac{|a|}{|\mathbf{r}|+1}\right)^{1/2}\widehat{\gamma}((m_1-r_1)|a|)\leq C(\lambda_{|\mathbf{r}|}(a))^{1/2+2\varepsilon}. \tag{2.3}$$

In order to verify (2.3) we observe that for $\mathbf{m}$, $\mathbf{r}$, and a as in (2.3) one can write

$$c\lambda_{|\mathbf{m}|}(a)\leq(|\mathbf{m}|+1)|a|\leq C\lambda_{|\mathbf{m}|}(a),\qquad c\lambda_{|\mathbf{r}|}(a)\leq(|\mathbf{r}|+1)|a|\leq C\lambda_{|\mathbf{r}|}(a),$$
$$c|\mathbf{m}|\leq|\mathbf{r}|\leq C|\mathbf{m}|.$$

To show the last inequality we observe that the conditions $\widehat{\gamma}((m_1-r_1)|a|)\neq 0$, $(\mathbf{m},\mathbf{r})\in A$ and $C\max\{|\mathbf{m}|^{-1},|\mathbf{r}|^{-1}\}\leq|a|\leq 1$ imply that $\big||\mathbf{m}|-|\mathbf{r}|\big|\leq C\min\{|\mathbf{r}|,|\mathbf{m}|\}$. Also

$$\sharp\{\mathbf{m}:(\mathbf{m},\mathbf{r})\in A,\ |r_1-m_1|\,|a|\in\operatorname{supp}\widehat{\gamma}\}\leq C/|a|.$$

Now (2.3) follows by an easy calculation.

By (2.3), S is dominated by

$$2\int\sum_{\mathbf{r}} I_{\{C\mathbf{r}^{-1}\leq|a|\leq 1\}}(a)(\lambda_{|\mathbf{r}|}(a))^{1/2+\varepsilon}\|Q_{\mathbf{r},a}f^a\|^2da\leq\|f\|^2_{W^{1/2+\varepsilon}}.$$

We are going to estimate $\|S_{k,l}f(\mathbf{z},u)\|_{L^2}$ in a similar way. Without loss of generality, we can consider only

$$S^1_{k,l}f(\mathbf{z},u)=\sup_{0\leq t\leq 1}|\psi(z_1)\phi(u)P^1_{k,l}V_tf(\mathbf{z},u)|,$$

where

$$P^1_{k,l}f(\mathbf{z},u)=\sum_{\{\mathbf{m}\in I_1\,:\,|\mathbf{m}|\approx 2^k\}}\int_{\{|a|\approx 2^l\}} e^{iua}Q_{\mathbf{m},a}f^a(\mathbf{z})\,da.$$

Again by the Sobolev lemma, the norm $\|S^1_{k,l}f(\mathbf{z},u)\|^2_{L^2}$ is controlled by

$$\iiint \Big| \sum_{\{\mathbf{m}\in I_1\,:\,|\mathbf{m}|\approx 2^k\}} \int_{\{|a|\approx 2^l\}} e^{iua+it\lambda_{|\mathbf{m}|}(a)} \lambda_{|\mathbf{m}|}^{1/2+\varepsilon}(a) Q_{\mathbf{m},a} f^a(\mathbf{z})\, da\Big|^2$$
$$\times\, \psi(z_1)\phi(u)\, d\mathbf{z}\, du\, \gamma(t)\, dt$$
$$= \iiint \sum_{\{\mathbf{m},\mathbf{r}\,:\,|\mathbf{m}|,|\mathbf{r}|\approx 2^k,\,\mathbf{r}\in I_1\}} \iint e^{iu(a_1-a_2)+it(\lambda_{|\mathbf{m}|}(a_1)-\lambda_{|\mathbf{r}|}(a_2))}$$
$$\times\, I_{\{|a|\approx 2^l\}}(a_1) I_{\{|a|\approx 2^l\}}(a_2) (\lambda_{|\mathbf{m}|}(a_1)\lambda_{|\mathbf{r}|}(a_2))^{1/2+\varepsilon}$$
$$\times\, Q_{\mathbf{m},a_1} f^{a_1}(\mathbf{z}) Q_{\mathbf{r},a_2} f^{a_2}(\mathbf{z}) \psi(z_1)\, d\mathbf{z}\, \phi(u)\, du\, \gamma(t)\, dt\, da_1\, da_2$$
$$= \sum_{\{\mathbf{m},\mathbf{r}\,:\,|\mathbf{r}|\approx 2^k,\,\mathbf{m},\mathbf{r}\in I_1\}} \iint I_{\{|a|\approx 2^l\}}(a_1) I_{\{|a|\approx 2^l\}}(a_2) (\lambda_{|\mathbf{m}|}(a_1)\lambda_{|\mathbf{r}|}(a_2))^{1/2+\varepsilon}$$
$$\times\, Q_{\mathbf{m},a_1} f^{a_1}(\mathbf{z}) Q_{\mathbf{r},a_2} f^{a_2}(\mathbf{z}) \psi(z_1)\, d\mathbf{z}$$
$$\times\, \widehat{\phi}(a_1-a_2)\widehat{\gamma}(\lambda_{|\mathbf{m}|}(a_1)-\lambda_{|\mathbf{r}|}(a_2))\, da_1\, da_2$$
$$= \sum_{\{(\mathbf{m},\mathbf{r})\in A\,:\,\mathbf{m},\mathbf{r}\in I_1,\,|\mathbf{m}|,|\mathbf{r}|\approx 2^k\}} \iint I_{\{|a|\approx 2^l\}}(a_1) I_{\{|a|\approx 2^l\}}(a_2)$$
$$\times\, (\lambda_{|\mathbf{m}|}(a_1)\lambda_{|\mathbf{r}|}(a_2))^{1/2+\varepsilon} \int Q_{\mathbf{m},a} f^a(\mathbf{z}) Q_{\mathbf{r},a} f^a(\mathbf{z}) \psi(z_1)\, d\mathbf{z}$$
$$\times\, \widehat{\phi}(a_1-a_2)\widehat{\gamma}(\lambda_{|\mathbf{m}|}(a_1)-\lambda_{|\mathbf{r}|}(a_2))\, da_1\, da_2$$
$$\le 2\int \sum_{\{\mathbf{r}\in I_1\,:\,|\mathbf{r}|\approx 2^k\}} \int \sum_{\{\mathbf{m}\,:\,(\mathbf{m},\mathbf{r})\in A,\,|\mathbf{m}|\approx 2^k\}} I_{\{|a|\approx 2^l\}}(a_1) I_{\{|a|\approx 2^l\}}(a_2)$$
$$\times\, (\lambda_{|\mathbf{m}|}(a_1)\lambda_{|\mathbf{r}|}(a_2))^{1/2+\varepsilon} \widehat{\gamma}(\lambda_{|\mathbf{m}|}(a_1)-\lambda_{|\mathbf{r}|}(a_2)) \widehat{\phi}(a_1-a_2)\, da_1$$
$$\times \int |Q_{\mathbf{r},a_2} f^{a_2}(\mathbf{z})|^2 \psi(z_1)\, d\mathbf{z}\, da_2 = J.$$

For fixed $\mathbf{r}$ and a_2 we have

$$\tag{2.4} \int \sum_{\{\mathbf{m}\,:\,(\mathbf{m},\mathbf{r})\in A,\,|\mathbf{m}|\approx 2^k\}} I_{\{|a|\approx 2^l\}}(a_1) I_{\{|a|\approx 2^l\}}(a_2)$$
$$\times\, (\lambda_{|\mathbf{m}|}(a_1)\lambda_{|\mathbf{r}|}(a_2))^{1/2+\varepsilon} \widehat{\phi}(a_1-a_2)\widehat{\gamma}(\lambda_{|\mathbf{m}|}(a_1)-\lambda_{|\mathbf{r}|}(a_2))\, da_1$$
$$\le (2^k+2^l)^{(1+2\varepsilon)} 2^{2l\varepsilon}.$$

To see (2.4) we observe that $\frac{d}{da_1}\lambda_{|\mathbf{m}|}(a_1) = ((2|\mathbf{m}|+n)+2|a_1|)\mathrm{sgn}(a_1)$. So the measure of $\{a_1 : \lambda_{|\mathbf{m}|}(a_1)-\lambda_{|\mathbf{r}|}(a_2) \in \operatorname{supp}\widehat{\gamma}\}$ is dominated by $C/(2^k+2^l)$. Hence

$$\tag{2.5} \int \widehat{\gamma}(\lambda_{|\mathbf{m}|}(a_1)-\lambda_{|\mathbf{r}|}(a_2))\, da_1 \le \frac{C}{2^k+2^l}.$$

Also

$$\sharp\{\mathbf{m}\approx 2^k : (\mathbf{m},\mathbf{r})\in A \text{ and } \exists_{a_1\approx 2^l}\lambda_{|\mathbf{m}|}(a_1)-\lambda_{|\mathbf{r}|}(a_2)\in \operatorname{supp}\widehat{\gamma} \\ \text{and } a_1-a_2\in\operatorname{supp}\widehat{\phi}\}\le C\max\{1,|\mathbf{r}|/2^l\}. \tag{2.6}$$

Combining (2.5) and (2.6) gives (2.4).

Hence by Lemma 2 and (2.4) we obtain the desired estimate for J:

$$J\le \int \sum_{\{\mathbf{r}\in I_1\,:\,|\mathbf{r}|\approx 2^k\}} 2^{2l\varepsilon}\left(\frac{2^l}{2^k+2^l}\right)^{1/2}(2^k+2^l)^{1+2\varepsilon}\|Q_{\mathbf{r},a}f^a\|_{L^2}^2\,da$$
$$\le \int \sum_{\{\mathbf{r}\in I_1\,:\,|\mathbf{r}|\approx 2^k\}} (2^k2^l+2^{2l})^{1/2+2\varepsilon}\|Q_{\mathbf{r},a}f^a\|_{L^2}^2 da\le C\|f\|_{W^{1/2+8\varepsilon}}2^{-(k+l)\varepsilon}$$

Summing up the estimates for S_0 and $S_{k,l}$ we get the theorem.

Remark. The above theorem combined with the estimates obtained in [Z] allows one to state a slightly sharper result. This requires a different definition of the scale of Sobolev spaces. We do not go into details here.

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