STUDIA MATHEMATICA 161 (2) (2004)

Statistical approximation by positive linear operators

by

O. Duman and C. Orhan (Ankara)

Abstract. Using A-statistical convergence, we prove a Korovkin type approximation theorem which concerns the problem of approximating a function f by means of a sequence $\{T_n(f;x)\}$ of positive linear operators acting from a weighted space C_{ϱ_1} into a weighted space B_{ϱ_2}.

1. Introduction. The sequences of some classical approximation operators tend to converge to the values of the function they approximate. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. There are, however, some exceptions, such as the interpolation operators of Hermite–Fejér [2] that do not converge at points of simple discontinuity. In this case, the matrix summability methods of Cesàro type are applicable to correct the lack of convergence [3]. Statistical convergence, which is a regular non-matrix summability method, is also effective in "summing" divergent sequences [7], [9], [10]. Recently, its use in approximation theory has been considered in [6], [13]. The aim of this paper is to use A-statistical convergence to study Korovkin type approximation of a function f by means of a sequence $\{T_n(f;x)\}$ of positive linear operators from a weighted space C_{ϱ_1} into a weighted space B_{ϱ_2}.

Approximation theory has important applications in various areas of functional analysis, and in numerical solution of differential and integral equations [1], [5], [18].

Before proceeding we recall some notation on statistical convergence. Let $A = (a_{jn})$ be an infinite summability matrix. For a given sequence $x := (x_n)$, the *A-transform* of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn}x_n$, provided the series converges for each j. We say that A is *regular* if $\lim_j (Ax)_j = L$ whenever $\lim_j x_j = L$ (see [14]). Assume now that A is a non-negative regular summability matrix and K is a subset of $\mathbb{N}$, the set of all natural numbers. The *A-density* of K is defined by

2000 *Mathematics Subject Classification*: 41A25, 41A36, 47B38.

Key words and phrases: A-density, A-statistical convergence, sequence of positive linear operators, weight function, weighted space, Korovkin type theorem.

$\delta_A(K) := \lim_j \sum_{n=1}^{\infty} a_{jn}\chi_K(n)$ provided the limit exists, where χ_K is the characteristic function of K. A sequence $x := (x_n)$ is said to be *A-statistically convergent* to a number L if, for every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$; or equivalently

$$\lim_j \sum_{n:\, |x_n - L| \geq \varepsilon} a_{jn} = 0.$$

We denote this limit by $\mathrm{st}_A\text{-}\lim x = L$ ([4], [8], [17], [19]). For $A = C_1$, the Cesàro matrix, A-statistical convergence reduces to statistical convergence ([7], [9], [10]). We note that if $A = (a_{jn})$ is a non-negative regular summability matrix for which $\lim_j \max_n\{a_{jn}\} = 0$, then A-statistical convergence is stronger than convergence [17].

It should be noted that the concept of A-statistical convergence may also be given in normed spaces: Assume $(X, \|\cdot\|)$ is a normed space and $u = (u_n)$ is an X-valued sequence. Then (u_n) is said to be *A-statistically convergent* to $u_0 \in X$ if, for every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : \|u_n - u_0\| \geq \varepsilon\} = 0$ (see [15], [16]). We recall that $x = (x_n)$ is A-statistically convergent to L if and only if there exists a subsequence $\{x_{n(k)}\}$ of x such that $\delta_A\{n(k) : k \in \mathbb{N}\} = 1$ and $\lim_k x_{n(k)} = L$ (see [17], [19]). The same result also holds in normed spaces ([15], [16]).

Now we recall the concepts of weight functions and weighted spaces considered in [11], [12]. Let $\mathbb{R}$ denote the set of real numbers. A real-valued function ϱ is called a *weight function* if it is continuous on $\mathbb{R}$ and

$$\lim_{|x|\to\infty} \varrho(x) = \infty, \qquad \varrho(x) \geq 1 \quad (\text{for all } x \in \mathbb{R}). \tag{1}$$

The space of real-valued functions f defined on $\mathbb{R}$ and satisfying $|f(x)| \leq M_f \varrho(x)$ (for all $x \in \mathbb{R}$) is called the *weighted space* and denoted by B_ϱ, where M_f is a constant depending on the function f. The weighted subspace C_ϱ of B_ϱ is given by

$$C_\varrho := \{f \in B_\varrho : f \text{ is continuous on } \mathbb{R}\}.$$

The spaces B_ϱ and C_ϱ are Banach spaces with the norm

$$\|f\|_\varrho := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{\varrho(x)}.$$

Now let ϱ_1 and ϱ_2 be two weight functions satisfying (1). Assume also that

$$\lim_{|x|\to\infty} \frac{\varrho_1(x)}{\varrho_2(x)} = 0. \tag{2}$$

If T is a positive linear operator from C_{ϱ_1} into B_{ϱ_2}, then the operator norm $\|T\|_{C_{\varrho_1}\to B_{\varrho_2}}$ is given by

$$\|T\|_{C_{\varrho_1}\to B_{\varrho_2}} := \sup_{\|f\|_{\varrho_1}=1} \|Tf\|_{\varrho_2}.$$

The following approximation theorem for a sequence of positive linear operators acting from C_{ϱ_1} into B_{ϱ_2} may be found in [11] and [12].

THEOREM A. *Assume that ϱ_1 and ϱ_2 are weight functions satisfying* (2) *and $\{L_n\}$ is a sequence of positive linear operators from C_{ϱ_1} into B_{ϱ_2}. Then* $\lim_n \|L_n f - f\|_{\varrho_2} = 0$ *for all $f \in C_{\varrho_1}$ if and only if* $\lim_n \|L_n F_v - F_v\|_{\varrho_1} = 0$ *for* $v = 0, 1, 2,$ *where*

$$F_v(x) = \frac{x^v \varrho_1(x)}{1+x^2}, \qquad v = 0, 1, 2.$$

In the present paper, we give an analog of Theorem A with the ordinary limit operator replaced by an A-statistical limit operator. We will also exhibit an example of a sequence of positive linear operators to which Theorem A does not apply but our A-statistical approximation theorem does.

2. Statistical approximation in weighted spaces. In this section we will obtain a Korovkin type approximation theorem for A-statistical convergence of a sequence of positive linear operators acting from C_{ϱ_1} into B_{ϱ_2}.

We require the following lemmas.

LEMMA 1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{T_n\}$ be a sequence of positive linear operators from C_{ϱ_1} into B_{ϱ_2}, where ϱ_1 and ϱ_2 satisfy condition* (2). *Assume that there exists a number $M > 0$ such that*

$$\delta_A\{n \in \mathbb{N} : \|T_n\|_{C_{\varrho_1} \to B_{\varrho_1}} \le M\} = 1. \tag{3}$$

If

$$\operatorname{st}_A\text{-}\lim_n \sup_{\|f\|_{\varrho_1}=1} \sup_{|x|\le s} \frac{|T_n(f;x)|}{\varrho_1(x)} = 0 \qquad \textit{for any } s \in \mathbb{R}, \tag{4}$$

then

$$\operatorname{st}_A\text{-}\lim_n \|T_n\|_{C_{\varrho_1} \to B_{\varrho_2}} = 0.$$

Proof. By (2), given $\varepsilon > 0$, there exists a number s_0 such that $\varrho_1(x) \le (\varepsilon/M)\varrho_2(x)$ for $|x| > s_0$. Also, by the continuity of ϱ_1/ϱ_2, there exists $C > 0$ such that $\varrho_1(x) \le C\varrho_2(x)$ whenever $|x| \le s_0$. Let

$$K := \{n \in \mathbb{N} : \|T_n\|_{C_{\varrho_1} \to B_{\varrho_1}} \le M\}. \tag{5}$$

By (3), $\delta_A(K) = 1$. Then, for all $n \in K$, by (5) we have

$$\begin{aligned} \|T_n\|_{C_{\varrho_1} \to B_{\varrho_2}} &= \sup_{\|f\|_{\varrho_1}=1} \sup_{x\in\mathbb{R}} \frac{|T_n(f;x)|}{\varrho_2(x)} \\ &\le \sup_{\|f\|_{\varrho_1}=1} \sup_{|x|\le s_0} \frac{|T_n(f;x)|}{\varrho_2(x)} + \sup_{\|f\|_{\varrho_1}=1} \sup_{|x|> s_0} \frac{|T_n(f;x)|}{\varrho_2(x)} \end{aligned}$$

$$\leq C \sup_{\|f\|_{\varrho_1}=1} \sup_{|x|\leq s_0} \frac{|T_n(f;x)|}{\varrho_1(x)} + \frac{\varepsilon}{M} \sup_{\|f\|_{\varrho_1}=1} \sup_{x\in\mathbb{R}} \frac{|T_n(f;x)|}{\varrho_1(x)}$$

$$\leq C\varphi_n(s_0) + \frac{\varepsilon}{M}\|T_n\|_{C_{\varrho_1}\to B_{\varrho_1}} \leq C\varphi_n(s_0) + \varepsilon,$$

where

$$\varphi_n(s_0) := \sup_{\|f\|_{\varrho_1}=1} \sup_{|x|\leq s_0} \frac{|T_n(f;x)|}{\varrho_1(x)}.$$

Now for a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. Thus

$$\sum_{n\in K:\, \|T_n\|_{C_{\varrho_1}\to B_{\varrho_2}}\geq r} a_{jn} \leq \sum_{n\in K:\, C\varphi_n(s_0)\geq r-\varepsilon} a_{jn}. \tag{6}$$

Hence, letting $j \to \infty$ in (6) and taking (4) into account, we get the result. ∎

LEMMA 2. *Let $A = (a_{jn})$, ϱ_1 and ϱ_2 be as in Lemma 1. Let $\{T_n\}$ be a sequence of positive linear operators from C_{ϱ_1} into B_{ϱ_2} for which* (3) *holds for some $M > 0$. If, for any $s \in \mathbb{R}$,*

$$\mathrm{st}_A\text{-}\lim_n \sup_{\|f\|_{\varrho_1}=1} \sup_{|x|\leq s} |T_n(f;x) - f(x)| = 0, \tag{7}$$

then

$$\mathrm{st}_A\text{-}\lim_n \|T_nf - f\|_{\varrho_2} = 0 \quad \textit{for all } f \in C_{\varrho_1}.$$

Proof. Let E be the identity operator on C_{ϱ_1} and let $L_n := T_n - E$, $U := \{n \in \mathbb{N} : \|T_n\|_{C_{\varrho_1}\to B_{\varrho_1}} \leq M\}$ and $V := \{n \in \mathbb{N} : \|L_n\|_{C_{\varrho_1}\to B_{\varrho_1}} \leq M+1\}$. Since $\|L_n\|_{C_{\varrho_1}\to B_{\varrho_1}} \leq \|T_n\|_{C_{\varrho_1}\to B_{\varrho_1}} + 1$, we have $U \subseteq V$. Since $\delta_A(U) = 1$, we have $\delta_A(V) = 1$. As $\varrho_1 \geq 1$ on $\mathbb{R}$, we get, for any $s \in \mathbb{R}$,

$$\begin{aligned}\sup_{\|f\|_{\varrho_1}=1} \sup_{|x|\leq s} \frac{|L_n(f;x)|}{\varrho_1(x)} &\leq \sup_{\|f\|_{\varrho_1}=1} \sup_{|x|\leq s} |L_n(f;x)| \\ &= \sup_{\|f\|_{\varrho_1}=1} \sup_{|x|\leq s} |T_n(f;x) - f(x)|.\end{aligned}$$

From (7) it follows that

$$\mathrm{st}_A\text{-}\lim_n \sup_{\|f\|_{\varrho_1}=1} \sup_{|x|\leq s} \frac{|L_n(f;x)|}{\varrho_1(x)} = 0.$$

Hence the sequence $\{L_n\}$ satisfies all the conditions of Lemma 1. So we have

$$\mathrm{st}_A\text{-}\lim_n \|L_n\|_{C_{\varrho_1}\to B_{\varrho_2}} = 0.$$

Combining this with the fact that

$$\|L_nf\|_{\varrho_2} \leq \|L_n\|_{C_{\varrho_1}\to B_{\varrho_2}} \|f\|_{\varrho_1} \quad (\text{for all } f \in C_{\varrho_1}),$$

we immediately conclude that

$$\mathrm{st}_A\text{-}\lim_n \|L_nf\|_{\varrho_2} = \mathrm{st}_A\text{-}\lim_n \|T_nf - f\|_{\varrho_2} = 0.\ \blacksquare$$

Now we present the following main result.

THEOREM 3. *Let $A = (a_{jn})$, ϱ_1 and ϱ_2 be as in Lemma* 1. *Assume that $\{T_n\}$ is a sequence of positive linear operators acting from C_{ϱ_1} into B_{ϱ_2}. Then*

$$\text{st}_A\text{-}\lim_n \|T_n f - f\|_{\varrho_2} = 0 \quad \text{for all } f \in C_{\varrho_1} \tag{8}$$

if and only if

$$\text{st}_A\text{-}\lim_n \|T_n F_v - F_v\|_{\varrho_1} = 0 \quad (v = 0, 1, 2), \tag{9}$$

where

$$F_v(x) = \frac{x^v \varrho_1(x)}{1 + x^2} \quad (v = 0, 1, 2).$$

Proof. Since each F_v belongs to C_{ϱ_1}, it is clear that (8) implies (9). Conversely, assume that (9) holds true. We first prove that (3) holds for some $M > 0$.

By (9), for each $v = 0, 1, 2$, there exists a set $K_v \subseteq \mathbb{N}$ such that $\delta_A(K_v) = 1$ and $\lim_{n \in K_v} \|T_n F_v - F_v\|_{\varrho_1} = 0$, i.e., given $\varepsilon > 0$ there exists $N_v(\varepsilon)$ such that for all $n \in K_v$ and $n \geq N_v(\varepsilon)$ we have $\|T_n F_v - F_v\|_{\varrho_1} < \varepsilon$. Hence there is a positive number M_v such that $\|T_n F_v - F_v\|_{\varrho_1} \leq M_v$ for every $n \in K_v$. Let $K := K_0 \cap K_1 \cap K_2$. Observe that $\delta_A(K) = 1$. So, for every $n \in K$, we have

$$\begin{aligned} \|T_n\|_{C_{\varrho_1} \to B_{\varrho_1}} &= \|T_n \varrho_1\|_{\varrho_1} \leq \|T_n \varrho_1 - \varrho_1\|_{\varrho_1} + 1 \\ &\leq \|T_n F_2 - F_2\|_{\varrho_1} + \|T_n F_0 - F_0\|_{\varrho_1} + 1 \leq M, \end{aligned}$$

where $M := 1 + M_0 + M_2$. This implies that $K \subseteq \{n : \|T_n\|_{C_{\varrho_1} \to B_{\varrho_1}} \leq M\}$, which yields (3).

We now prove that condition (7) holds. To see this we write

$$\begin{aligned} T_n((t-x)^2 F_0(t); x) &= T_n(t^2 F_0(t); x) - 2x T_n(t F_0(t); x) + x^2 T_n(F_0(t); x) \\ &\leq |T_n(F_2(t); x) - F_2(x)| + 2|x|\,|T_n(F_1(t); x) - F_1(x)| \\ &\quad + x^2 |T_n(F_0(t); x) - F_0(x)|. \end{aligned}$$

Hence for any $s \in \mathbb{R}$ and $n \in K$ we get

$$\begin{aligned} u_n &:= \sup_{|x| \leq s} T_n((t-x)^2 F_0(t); x) \\ &\leq B\{\|T_n F_2 - F_2\|_{\varrho_1} + \|T_n F_1 - F_1\|_{\varrho_1} + \|T_n F_0 - F_0\|_{\varrho_1}\}, \end{aligned} \tag{10}$$

where $B := \max\{1, 2\sup_{|x| \leq s} |x| \varrho_1(x), \max_{|x| \leq s} x^2 \varrho_1(x)\}$.

Now let $f \in C_{\varrho_1}$ and let $|x| \leq s$. Since f is continuous on $\mathbb{R}$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ for all t, x with $|t - x| < \delta$. When $|t - x| \geq \delta$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq 2M_f\, \varrho_1(x) \varrho_1(t) = 2M_f\, \varrho_1(x) F_0(t)(1 + t^2) \\ &\leq 4M_f\, \varrho_1(x) F_0(t)(1 + x^2 + (t-x)^2) \end{aligned}$$

$$= 4M_f\,\varrho_1(x)F_0(t)(t-x)^2\left(\frac{1+x^2}{(t-x)^2}+1\right)$$
$$\le K_{\varrho_1}(x)(t-x)^2F_0(t),$$

where $K_{\varrho_1}(x) := 4M_f\varrho_1(x)((1+x^2)/\delta^2+1)$. So, for all $t \in \mathbb{R}$ and $|x| \le s$, we see that

$$|f(t)-f(x)| < \varepsilon + K_{\varrho_1}(x)(t-x)^2F_0(t). \tag{11}$$

It follows from (11) that

$$\begin{aligned}|T_n(f(t);x)-f(x)| &\le T_n(|f(t)-f(x)|;x) + |f(x)|\,|T_n(1;x)-1| \\ &< \varepsilon T_n(1,x) + K_{\varrho_1}(x)T_n((t-x)^2F_0(t);x) \\ &\quad + |f(x)|\,|T_n(1;x)-1|.\end{aligned}$$

This also implies, for any $s \in \mathbb{R}$, that

$$\begin{aligned}v_n &:= \sup_{\|f\|_{\varrho_1}=1}\ \sup_{|x|\le s}|T_n(f(t);x)-f(x)| \\ &< C_1\varepsilon\|T_n(1,x)\|_{\varrho_1} + C_2\sup_{|x|\le s}T_n((t-x)^2F_0(t);x) \\ &\quad + C_3\sup_{|x|\le s}|T_n(1;x)-1|,\end{aligned} \tag{12}$$

where $C_1 := \sup_{|x|\le s}\varrho_1(x)$, $C_2 := \sup_{|x|\le s}K_{\varrho_1}(x)$ and $C_3 := \sup_{|x|\le s}|f(x)|$.

Since $\|T_n(1,x)\|_{\varrho_1} \le \|T_n(\varrho_1,x)\|_{\varrho_1} = \|T_n\|_{C_{\varrho_1}\to B_{\varrho_1}}$, it follows from (12), for all $n \in K$, that

$$v_n \le MC_1\varepsilon + C_2\,u_n + C_3\sup_{|x|\le s}|T_n(1;x)-1|. \tag{13}$$

Since $F_0 \in C_{\varrho_1}$ and

$$F_0(x)|T_n(1;x)-1| \le |T_n(F_0(t);x)-F_0(x)| + |T_n(F_0(t)-F_0(x);x)|,$$

we have, by (11),

$$\begin{aligned}|T_n(1;x)-1| &< \frac{1}{F_0(x)}\,\{|T_n(F_0(t);x)-F_0(x)| + \varepsilon T_n(1;x) \\ &\quad + K_{\varrho_1}(x)T_n((t-x)^2F_0(t);x)\}.\end{aligned}$$

So we conclude, for any $s \in \mathbb{R}$ and all $n \in K$, that

$$\sup_{|x|\le s}|T_n(1;x)-1| \le C_4\{\,\|T_nF_0-F_0\|_{\varrho_1} + \varepsilon M + C_2u_n\}, \tag{14}$$

where $C_4 := \sup_{|x|\le s}\ \varrho_1(x)/F_0(x)$. Taking (10), (13) and (14) into account, for all $n \in K$, we obtain

$$v_n \le C\varepsilon + C\{\|T_nF_0-F_0\|_{\varrho_1} + \|T_nF_1-F_1\|_{\varrho_1} + \|T_nF_2-F_2\|_{\varrho_1}\}, \tag{15}$$

where $C := \max\{M(C_1+C_3C_4), BC_2+C_3C_4+BC_2C_3C_4\}$.

Now for a given $r > 0$, choose $\varepsilon > 0$ such that $C\varepsilon < r$. Define

$$D := \left\{ n \in K : \|T_n F_0 - F_0\|_{\varrho_1} + \|T_n F_1 - F_1\|_{\varrho_1} + \|T_n F_2 - F_2\|_{\varrho_1} \ge \frac{r - C\varepsilon}{C} \right\},$$

$$D_0 := \left\{ n \in K : \|T_n F_0 - F_0\|_{\varrho_1} \ge \frac{r - C\varepsilon}{3C} \right\},$$

$$D_1 := \left\{ n \in K : \|T_n F_1 - F_1\|_{\varrho_1} \ge \frac{r - C\varepsilon}{3C} \right\},$$

$$D_2 := \left\{ n \in K : \|T_n F_2 - F_2\|_{\varrho_1} \ge \frac{r - C\varepsilon}{3C} \right\}.$$

Then it is easy to see that $D \subseteq D_0 \cup D_1 \cup D_2$. Thus (15) yields

$$\sum_{n \in K:\, v_n \ge r} a_{jn} \le \sum_{n \in D} a_{jn} \le \sum_{n \in D_0} a_{jn} + \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn},$$

from which (7) follows. So by Lemma 2, we have

$$\mathrm{st}_A\text{-}\lim_n \|T_n f - f\|_{\varrho_2} = 0 \quad \text{for all } f \in C_{\varrho_1}. \ \blacksquare$$

Note that if we take A to be the identity matrix I, then we immediately get Theorem A.

The next result is a consequence of Theorem 3.

COROLLARY 4. *Let $\{T_n\}$ be a sequence of positive linear operators from C_w into C_w for the weight function w defined by $w(x) = 1 + x^2$ and let $A = (a_{jn})$ be a non-negative regular summability matrix. Also let ϱ_1 and ϱ_2 be weight functions satisfying* (2) *and consider the sequence $\{P_n\}$ of positive linear operators from C_{ϱ_1} into B_{ϱ_2} defined by*

$$P_n(f(t); x) = \frac{\varrho_1(x)}{w(x)}\, T_n\left(\frac{1 + t^2}{\varrho_1(t)}\, f(t); x \right).$$

If

$$\mathrm{st}_A\text{-}\lim_n \|T_n f_v - f_v\|_w = 0,$$

where $f_v(t) = t^v$ $(v = 0, 1, 2)$, then

$$\mathrm{st}_A\text{-}\lim_n \|P_n f - f\|_{\varrho_2} = 0 \quad \textit{for all } f \in C_{\varrho_1}.$$

Proof. By the definition of the operators P_n,

$$P_n(F_v; x) = \frac{\varrho_1(x)}{w(x)}\, T_n(f_v; x) \quad (v = 0, 1, 2),$$

where F_v $(v = 0, 1, 2)$ is as in Theorem 3. Since, for each $v = 0, 1, 2$,

$$P_n(F_v; x) - F_v(x) = \frac{\varrho_1(x)}{w(x)}\, (T_n(f_v; x) - f_v(x)),$$

we have

$$\|P_nF_v - F_v\|_{\varrho_1} = \|T_nf_v - f_v\|_w.$$

So the assertion follows from Theorem 3. ■

Let φ be a continuous increasing function on $\mathbb{R}$. Now we deal with A-statistical approximation in the space C_{ϱ_1} with $\varrho_1(x) = 1 + \varphi^2(x)$.

LEMMA 5. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, let $\{T_n\}$ be a sequence of positive linear operators from C_{ϱ_1} into B_{ϱ_2}, and assume that ϱ_1 and ϱ_2 satisfy* (2)*. If*

$$\text{st}_A\text{-}\lim_n \|T_n\varphi^v - \varphi^v\|_{\varrho_1} = 0 \quad (v = 0, 1, 2), \tag{16}$$

then

$$\text{st}_A\text{-}\lim_n \sup_{\|f\|_{\varrho_1}=1} \sup_{a\le x\le b} |T_n(f; x) - f(x)| = 0$$

for all $a < b$.

Proof. Let $f \in C_{\varrho_1}$. It is shown in [11] that, given $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all $t \in \mathbb{R}$ and all x satisfying $a \le x \le b$ we have

$$|f(t) - f(x)| < \varepsilon + K_{\varrho_1}(x)(\varphi(t) - \varphi(x))^2, \tag{17}$$

where

$$K_{\varrho_1}(x) := 4M_f\, \varrho_1^2(x)\left[\frac{1}{\Delta_\delta^2(\varphi; x)} + 1\right],$$
$$\Delta_\delta(\varphi; x) := \min\{\varphi(x+\delta) - \varphi(x),\, \varphi(x) - \varphi(x-\delta)\}.$$

Now (17) yields

$$\begin{aligned}
|T_n(f(t); x) - f(x)| &\le T_n(|f(t) - f(x)|; x) + |f(x)|\,|T_n(1; x) - 1| \\
&< \varepsilon T_n(1, x) + K_{\varrho_1}(x)T_n((\varphi(t) - \varphi(x))^2; x) + |f(x)|\,|T_n(1; x) - 1| \\
&\le (\varepsilon + |f(x)|)|T_n(1; x) - 1| + \varepsilon + K_{\varrho_1}(x)T_n((\varphi(t) - \varphi(x))^2; x) \\
&\le \varepsilon + (\varepsilon + |f(x)|)|T_n(1; x) - 1| \\
&\quad + K_{\varrho_1}(x)\{|T_n(\varphi^2(t); x) - \varphi^2(x)| + 2|\varphi(x)|\,|T_n(\varphi(t); x) - \varphi(x)| \\
&\quad + \varphi^2(x)|T_n(1; x) - 1|\} \\
&= \varepsilon + \{\varepsilon + |f(x)| + K_{\varrho_1}(x) + \varphi^2(x)\}|T_n(1; x) - 1| \\
&\quad + 2K_{\varrho_1}(x)|\varphi(x)|\,|T_n(\varphi(t); x) - \varphi(x)| + K_{\varrho_1}(x)\,|T_n(\varphi^2(t); x) - \varphi^2(x)|.
\end{aligned}$$

So we get

$$\begin{aligned}
u_n &:= \sup_{\|f\|_{\varrho_1}=1} \sup_{a\le x\le b} |T_n(f(t); x) - f(x)| \\
&\le \varepsilon + C\{\|T_n1 - 1\|_{\varrho_1} + \|T_n\varphi - \varphi\|_{\varrho_1} + \|T_n\varphi^2 - \varphi^2\|_{\varrho_1}\}
\end{aligned} \tag{18}$$

where

$$C := \max\{\sup_{a\le x\le b} \varrho_1(x)(\varepsilon + |f(x)| + K_{\varrho_1}(x) + \varphi^2(x)),$$
$$\sup_{a\le x\le b} 2\varrho_1(x)K_{\varrho_1}(x)\,|\varphi(x)|\}.$$

Now for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Define

$$D := \left\{n : \|T_n 1 - 1\|_{\varrho_1} + \|T_n\varphi - \varphi\|_{\varrho_1} + \|T_n\varphi^2 - \varphi^2\|_{\varrho_1} \ge \frac{r-\varepsilon}{C}\right\},$$
$$D_0 := \left\{n : \|T_n 1 - 1\|_{\varrho_1} \ge \frac{r-\varepsilon}{3C}\right\},$$
$$D_1 := \left\{n : \|T_n\varphi - \varphi\|_{\varrho_1} \ge \frac{r-\varepsilon}{3C}\right\},$$
$$D_2 := \left\{n : \|T_n\varphi^2 - \varphi^2\|_{\varrho_1} \ge \frac{r-\varepsilon}{3C}\right\}.$$

Then it is easy to see that $D \subseteq D_0 \cup D_1 \cup D_2$. By (18) we have

$$\sum_{n:\, u_n\ge r} a_{jn} \le \sum_{n\in D} a_{jn} \le \sum_{n\in D_0} a_{jn} + \sum_{n\in D_1} a_{jn} + \sum_{n\in D_2} a_{jn}. \tag{19}$$

Letting $n \to \infty$ in (19) and using (16) we conclude that

$$\mathrm{st}_A\text{-}\lim_n \sup_{\|f\|_{\varrho_1}=1} \sup_{a\le x\le b} |T_n(f;x) - f(x)| = 0,$$

which completes the proof. ∎

Assume now that $\varrho_1 := 1+\varphi^2$ and ϱ_2 satisfy (2). Then by Lemmas 2 and 5 we get the following A-statistical Korovkin type approximation theorem.

THEOREM 6. *Let $A = (a_{jn})$ and $\{T_n\}$ be as in Lemma* 5. *Then* (8) *holds if and only if $\{T_n\}$ satisfies* (16).

Proof. Since $\varphi^v \in C_{\varrho_1}$ $(v = 0, 1, 2)$, (8) implies (16). Assume now that $\{T_n\}$ satisfies (16). By Lemma 5 we have

$$\mathrm{st}_A\text{-}\lim_n \sup_{\|f\|_{\varrho_1}=1} \sup_{-s\le x\le s} |T_n(f;x) - f(x)| = 0 \tag{20}$$

for any $s \in \mathbb{R}$. Also, as in the proof of Theorem 3 we can find a positive number M such that $\delta_A\{n \in \mathbb{N} : \|T_n\|_{C_{\varrho_1}\to B_{\varrho_1}} \le M\} = 1$. It follows from Lemma 2 that

$$\mathrm{st}_A\text{-}\lim_n \|T_n f - f\|_{\varrho_2} = 0 \qquad \text{for all } f \in C_{\varrho_1}. \ ∎$$

3. Concluding remarks. In this section we deal with an example of a sequence of positive linear operators to which Theorem A does not apply but our Theorem 3 does.

EXAMPLE. Let ϱ_1 and ϱ_2 be weight functions satisfying (2) and let $\{L_n\}$ be a sequence of positive linear operators from C_{ϱ_1} into B_{ϱ_2} satisfying one of the two equivalent properties stated in Theorem A. Assume that $A = (a_{nk})$ is a non-negative regular summability matrix such that $\lim_j \max_n\{a_{jn}\} = 0$; then A-statistical convergence is stronger than convergence. So there is a sequence (u_n) which is A-statistically null but not convergent [17]. Without loss of generality we may assume that (u_n) is non-negative. Now define the sequence $\{T_n\}$ of positive linear operators mapping C_{ϱ_1} into B_{ϱ_2} by $T_n(f) = (1 + u_n)L_n(f)$ for $f \in C_{\varrho_1}$. Observe that $\{u_n L_n(f)\}$ does not tend to zero because $L_n(f) \to f$ for all $f \in C_{\varrho_1}$ and (u_n) is divergent. Hence the sequence $\{\|T_n f - f\|_{\varrho_2}\}$ does not tend to zero either, but it is an A-statistically null sequence for all $f \in C_{\varrho_1}$.

Acknowledgements. The authors wish to thank the referee for his/her valuable suggestions which improved the paper considerably.

References

[1] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and its Applications*, de Gruyter Stud. Math. 17, de Gruyter, Berlin, 1994.

[2] R. Bojanić and F. Cheng, *Estimates for the rate of approximation of functions of bounded variation by Hermite–Fejér polynomials*, in: CMS Conf. Proc. 3, 1983, 5–17.

[3] R. Bojanić and M. K. Khan, *Summability of Hermite–Fejér interpolation for functions of bounded variation*, J. Natur. Sci. Math. 32 (1992), 5–10.

[4] J. S. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull. 32 (1989), 194–198.

[5] R. A. Devore, *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Math. 293, Springer, Berlin, 1972.

[6] O. Duman and C. Orhan, *Statistical approximation in the space of locally integrable functions*, Publ. Math. Debrecen 63 (2003), 133–134.

[7] H. Fast, *Sur la convergence statistique*, Colloq. Math. 2 (1951), 241–244.

[8] A. R. Freedman and J. J. Sember, *Densities and summability*, Pacific J. Math. 95 (1981), 293–305.

[9] J. A. Fridy, *On statistical convergence*, Analysis 5 (1985), 301–313.

[10] J. A. Fridy and C. Orhan, *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. 125 (1997), 3625–3631.

[11] A. D. Gadjiev, *On P. P. Korovkin type theorems*, Mat. Zametki 20 (1976), 781–786 (in Russian).

[12] —, *The convergence problem for a sequence of positive linear operators on unbounded sets, and theorems analogous to that of P. P. Korovkin*, Soviet Math. Dokl. 15 (1974), 1433–1436.

[13] A. D. Gadjiev and C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. 32 (2002), 129–138.

[14] G. H. Hardy, *Divergent Series*, Oxford Univ. Press, London, 1949.

[15] E. Kolk, *Statistically convergent sequences in normed spaces*, in: Reports of Conference, "Methods of Algebra and Analysis", Tartu, 1988, 63–66.

[16] E. Kolk, *The statistical convergence in Banach spaces*, Acta Comment. Tartuensis 928 (1991), 41–52.

[17] —, *Matrix summability of statistically convergent sequences*, Analysis 13 (1993), 77–83.

[18] P. P. Korovkin, *Linear Operators and Theory of Approximation*, Hindustan Publ. Co., Delhi, 1960.

[19] H. I. Miller, *A measure theoretical subsequence characterization of statistical convergence*, Trans. Amer. Math. Soc. 347 (1995), 1811–1819.

Department of Mathematics
Faculty of Science
Ankara University
Tandogan 06100, Ankara, Turkey
E-mail: oduman@science.ankara.edu.tr
orhan@science.ankara.edu.tr

Received March 6, 2003
Revised version July 1, 2003 (5156)