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On the Kunen–Shelah properties in Banach spaces

by

Antonio S. Granero (Madrid), Mar Jiménez (Madrid),
Alejandro Montesinos (Madrid), José P. Moreno (Madrid) and
Anatolij Plichko (Kirovograd)

Abstract. We introduce and study the Kunen–Shelah properties KS_i, $i = 0, 1, \ldots, 7$. Let us highlight some of our results for a Banach space X: (1) X^* has a w^*-nonseparable equivalent dual ball iff X has an *ω_1-polyhedron* (i.e., a bounded family $\{x_i\}_{i<\omega_1}$ such that $x_j \notin \overline{\mathrm{co}}(\{x_i : i \in \omega_1 \setminus \{j\}\})$ for every $j \in \omega_1$) iff X has an *uncountable bounded almost biorthogonal system* (UBABS) of type η for some $\eta \in [0, 1)$ (i.e., a bounded family $\{(x_\alpha, f_\alpha)\}_{1\le\alpha<\omega_1} \subset X \times X^*$ such that $f_\alpha(x_\alpha) = 1$ and $|f_\alpha(x_\beta)| \le \eta$ if $\alpha \ne \beta$); (2) if X has an uncountable ω-independent system then X has an UBABS of type η for every $\eta \in (0, 1)$; (3) if X does not have the property (C) of Corson, then X has an ω_1-polyhedron; (4) X has no ω_1-polyhedron iff X has no *convex right-separated ω_1-family* (i.e., a bounded family $\{x_i\}_{i<\omega_1}$ such that $x_j \notin \overline{\mathrm{co}}(\{x_i : j < i < \omega_1\})$ for every $j \in \omega_1$) iff every w^*-closed convex subset of X^* is w^*-separable iff every convex subset of X^* is w^*-separable iff $\mu(X) = 1$, $\mu(X)$ being the Finet–Godefroy index of X (see [1]).

1. Introduction. If X is a Banach space and θ an ordinal, a family $\{x_\alpha : \alpha < \theta\} \subset X$ is said to be a *θ-basic sequence* if there exists $1 \le K < \infty$ such that for every $n < m$ in $\mathbb{N}$, any $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, m$, and $\alpha_1 < \ldots < \alpha_m < \theta$ we have $\|\sum_{i=1}^n \lambda_i x_{\alpha_i}\| \le K \|\sum_{i=1}^m \lambda_i x_{\alpha_i}\|$. A family $\{x_i\}_{i\in I} \subset X$ is a *basic sequence* if it is a θ-basic sequence for some ordinal θ. If $K = 1$ the basic sequence is said to be *monotone.* A *biorthogonal system* in X is a family $\{(x_i, x_i^*) : i \in I\} \subset X \times X^*$ such that $x_i^*(x_i) = 1$ and $x_i^*(x_j) = 0$ for $i, j \in I$, $i \ne j$. A *Markushevich system* (for short, an M-system) in X is a biorthogonal system $\{(x_i, x_i^*) : i \in I\}$ in X such that $\{x_i^* : i \in I\}$ is total on $\overline{[\{x_i : i \in I\}]}$ (see [14]).

It is well known (see [14, p. 599]) that if the density of a Banach space X satisfies $\mathrm{Dens}(X) \ge \aleph_1$, then X has a monotone ω_1-basic sequence. Also if $\mathrm{Dens}(X) > \mathfrak{c}$, then X has a monotone ω_1-basic sequence, because in

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this case an easy calculation shows that w^*-Dens$(X^*) \geq \aleph_1$. However, if $\aleph_1 \leq$ Dens$(X) \leq \mathfrak{c}$ and w^*-Dens$(X^*) \leq \aleph_0$, then X can fail to have an uncountable basic sequence, even an uncountable biorthogonal system. Indeed, under the axiom $\diamondsuit_{\aleph_1}$ (which implies the continuum hypothesis (CH)), Shelah [13] constructed a nonseparable Banach space S that fails to have an uncountable biorthogonal system. Later Kunen [8, p. 1123] constructed under (CH) a Hausdorff compact space K such that $C(K)$ is nonseparable and has no uncountable biorthogonal system, among other interesting pathological properties.

A Banach space X is said to have the *Kunen–Shelah property* KS_0 (resp. KS_1) if X has no uncountable basic sequence (resp. uncountable Markushevich system). A Banach space X is said to have the *Kunen–Shelah property* KS_2 if X has no uncountable biorthogonal system. Clearly, $\mathrm{KS}_2 \Rightarrow \mathrm{KS}_1 \Rightarrow \mathrm{KS}_0$.

The first example of a Banach space X such that $X \in \mathrm{KS}_0$ but $X \notin \mathrm{KS}_2$ was given in [9]; it is the space of Johnson–Lindenstrauss JL_2 (see [5]). The properties KS_2 and KS_1 were separated in [2] (see also [1]), where it was proved that if a Banach space X has the property (C) of Corson and w^*-Dens$(X^*) \leq \aleph_0$, then $X \in \mathrm{KS}_1$.

QUESTION 1. Does there exist a Banach space X such that $X \in \mathrm{KS}_0$ but $X \notin \mathrm{KS}_1$?

In this paper we study some structures similar to uncountable biorthogonal systems, namely: uncountable ω-independent families, ω_1-polyhedrons, uncountable bounded almost biorthogonal systems (UBABS), etc. The lack of these structures defines the Kunen–Shelah properties $\mathrm{KS}_3, \mathrm{KS}_4$, etc.

In Section 2 we prove that a Banach space X has an ω_1-polyhedron iff X has an UBABS iff X^* has a w^*-nonseparable dual equivalent ball. Section 3 deals with uncountable ω-independent families. In Section 4 it is proved that X has no ω_1-polyhedron iff every w^*-closed convex subset of X^* is w^*-separable. In Section 5 we answer some questions posed by Finet and Godefroy [1] concerning the index $\mu(X)$. In Section 6 we prove that a space X has no convex right-separated ω_1-family iff every w^*-closed convex subset of X^* is w^*-separable. Finally, in Section 7 we show that X has an ω_1-polyhedron iff X has a convex right-separated ω_1-family, whence every w^*-closed convex subset of X^* is w^*-separable iff every convex subset of X^* is so.

Let us introduce some notation. ω_1 is the first uncountable ordinal, $|A|$ the cardinality of the set A, and $\mathfrak{c} = |\mathbb{R}|$. If X is a Banach space, X^* denotes its dual, $B(X)$ and $S(X)$ the closed unit ball and sphere of X, resp., and $B(x,r)$ the closed ball with radius r and center x. If $A \subset X$ we denote by $[A]$ the linear subspace spanned by A. Recall that a Banach space X is said to have the *property* (C) of Corson (for short, $X \in$ (C)) if $\bigcap_{i\in I} C_i \neq \emptyset$ whenever $\{C_i : i \in I\}$ is a family of closed bounded convex subsets of X with the countable intersection property, i.e., $\emptyset \neq \bigcap_{i\in J} C_i$ for every countable subset $J \subset I$.

2. UBABS and ω_1-polyhedrons. If X is a Banach space, a bounded family $\{(x_\alpha, f_\alpha)\}_{1\le\alpha<\omega_1} \subset X \times X^*$ is said to be an *uncountable bounded almost biorthogonal system* (for short, an UBABS) if there exists a real number $0 \le \eta < 1$ such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) \le \eta$ if $\alpha \neq \beta$. If in addition $|f_\alpha(x_\beta)| \le \eta$ for $\alpha \neq \beta$, then the UBABS is said to be of *type* η. Define the index $\tau(X)$ as follows:

$$\tau(X) = \inf\{0 \le \eta < 1 : X \text{ has an UBABS of type } \eta\},$$

where $\inf\{\emptyset\} = 1$. Clearly, $\tau(X)$ is invariant under isomorphisms and: (1) if X has an uncountable biorthogonal system, then $\tau(X) = 0$; (2) $\tau(X) < 1$ iff X has an UBABS.

If τ is a cardinal, a bounded family $\{x_i\}_{i\in\tau}$ in a Banach space X is said to be a *τ-polyhedron* iff $x_j \notin \overline{\mathrm{co}}(\{x_i\}_{i\in\tau\setminus\{j\}})$ for every $j \in \tau$. In a dual Banach space X^* one can define a w^*-τ-polyhedron in an analogous way, using the w^*-topology instead of the w-topology.

PROPOSITION 2.1. *A Banach space X has an ω_1-polyhedron iff X^* has a w^*-ω_1-polyhedron.*

Proof. Let $\{x_\alpha\}_{\alpha<\omega_1} \subset B(X)$ be an ω_1-polyhedron. By the Hahn–Banach Theorem there exists $f_\alpha \in S(X^*)$ such that

$$f_\alpha(x_\alpha) > \sup\{f_\alpha(x_i) : i \in \omega_1 \setminus \{\alpha\}\} =: e_\alpha.$$

By passing to a subsequence, we can suppose that there exist $0 < \varepsilon < \infty$ and $r \in \mathbb{R}$ such that $f_\alpha(x_\alpha) - e_\alpha \ge \varepsilon > 0$ and $|r - f_\alpha(x_\alpha)| \le \varepsilon/4$ for all $\alpha < \omega_1$. Hence, if $\alpha, \beta < \omega_1$ with $\alpha \neq \beta$, we have

$$f_\alpha(x_\alpha) \ge r - \varepsilon/4 > r - 3\varepsilon/4 \ge f_\beta(x_\beta) - \varepsilon \ge e_\beta \ge f_\beta(x_\alpha),$$

which implies that $\{f_\alpha\}_{\alpha<\omega_1}$ is a w^*-ω_1-polyhedron in X^*.

The converse implication is analogous. ∎

In the following proposition we give the relation between ω_1-polyhedrons and UBABS.

PROPOSITION 2.2. *For a Banach space X the following are equivalent*:

(1) *X has an ω_1-polyhedron.*
(2) *X has an UBABS of type η for some $0 \le \eta < 1$.*
(3) *X has an UBABS.*

Proof. (1)⇒(2). If w^*-Dens$(X^*) \ge \aleph_0$, then X has an uncountable biorthogonal system and so X has an UBABS of type 0. Now assume that

w^*-Dens$(X^*) \le \aleph_0$. Let $\{x_\alpha\}_{1\le\alpha<\omega_1} \subset X$ be an ω_1-polyhedron. Assume that $x_1 = 0$ and $\|x_\alpha\| \le 1$. For each $1 \le \alpha < \omega_1$ consider $f_\alpha \in S(X^*)$ such that

$$1 \ge f_\alpha(x_\alpha) > \sup\{f_\alpha(x_i) : 1 \le i < \omega_1,\ i \ne \alpha\} =: \varrho_\alpha.$$

Observe that $\varrho_\alpha \ge 0$ if $\alpha \ne 1$. By passing to an uncountable subsequence, it can be assumed that there are real numbers $0 < \varepsilon, r \le 1$ such that $f_\alpha(x_\alpha) - \varrho_\alpha \ge \varepsilon$ and $|r - f_\alpha(x_\alpha)| < \varepsilon/8$ for every $2 \le \alpha < \omega_1$. Since w^*-Dens$(X^*) \le \aleph_0$, by passing again to a subsequence, we assume that there exists $z \in X^*$ such that $z(x_\alpha) > 0$ and $|z(x_\beta)/z(x_\alpha) - 1| < \varepsilon/8$ for every $2 \le \alpha, \beta < \omega_1$. Then, if $g_\alpha = f_\alpha + z/z(x_\alpha)$, $2 \le \alpha < \omega_1$, we have

$$\begin{aligned} g_\alpha(x_\alpha) &= f_\alpha(x_\alpha) + 1 \ge r - \varepsilon/8 + 1 > r - 6\varepsilon/8 + 1 \ge f_\alpha(x_\alpha) - 7\varepsilon/8 + 1 \\ &\ge \sup\{g_\alpha(x_\beta) : 2 \le \beta < \omega_1,\ \beta \ne \alpha\} \\ &\ge \inf\{g_\alpha(x_\beta) : 2 \le \beta < \omega_1,\ \beta \ne \alpha\} \ge -\varepsilon/8. \end{aligned}$$

Define $h_\alpha = g_\alpha/g_\alpha(x_\alpha)$. Then, for $2 \le \alpha, \beta < \omega_1$, $\alpha \ne \beta$, we have $h_\alpha(x_\alpha) = 1$ and

$$-\frac{\varepsilon/8}{r - \varepsilon/8 + 1} \le -\frac{\varepsilon/8}{g_\alpha(x_\alpha)} \le h_\alpha(x_\beta) = \frac{g_\alpha(x_\beta)}{g_\alpha(x_\alpha)} \le \frac{r + 1 - 6\varepsilon/8}{r + 1 - \varepsilon/8}.$$

So, $\{(x_\alpha, h_\alpha) : 2 \le \alpha < \omega_1\} \subset X \times X^*$ is an UBABS of type η such that

$$0 \le \eta = \max\left\{\frac{\varepsilon/8}{r - \varepsilon/8 + 1}, \frac{r + 1 - 6\varepsilon/8}{r + 1 - \varepsilon/8}\right\} < 1.$$

(2)⇒(3) is obvious and (3)⇒(1) is clear, because if $\{(x_\alpha, f_\alpha)\}_{1\le\alpha<\omega_1} \subset X \times X^*$ is an UBABS, then $\{x_\alpha\}_{\alpha<\omega_1}$ is an ω_1-polyhedron. ∎

Let us consider some results on representation of elements in polyhedrons, which we need later. If $\{x_i\}_{i\in I}$ is a w^*-τ-polyhedron in a dual Banach space X^* with $\tau = \mathrm{card}(I)$ and $K = \overline{\mathrm{co}}^{w^*}(\{x_i\}_{i\in I})$, then the *core* of K is the set

$$K_0 = \mathrm{core}(K) = \bigcap\{\overline{\mathrm{co}}^{w^*}(\{x_i\}_{i\in I\setminus A}) : A \subset I,\ A \text{ finite}\}.$$

Define the function $\lambda : K \to [0,1]$ as follows: for $k \in K$,

$$\lambda(k) = \sup\{\lambda \in [0,1] : \exists u \in K, \exists i \in I \text{ such that } k = \lambda x_i + (1-\lambda)u\}.$$

Let $H = \{x \in K : \lambda(x) = 0\}$. Since for every finite subset $A \subset I$, each $x \in K$ has the expression $x = \sum_{i\in A} \lambda_i x_i + (1-\mu)u$ with $u \in \overline{\mathrm{co}}^{w^*}(\{x_i\}_{i\in I\setminus A})$, $\lambda_i \in [0,1]$, $i \in A$, $\mu = \sum_{i\in A} \lambda_i \le 1$, it can be easily seen that $H \subset K_0$.

LEMMA 2.3. *Let $\{x_i\}_{i\in I}$ be a w^*-τ-polyhedron in the dual Banach space X^*, $\tau = \mathrm{card}(I)$, $K = \overline{\mathrm{co}}^{w^*}(\{x_i\}_{i\in I})$, $K_0 = \mathrm{core}(K)$ and $H = \{x \in K : \lambda(x) = 0\}$. If $x \in K$, then there exist a sequence $\{\mu_n\}_{n\ge1}$ of positive numbers with $0 \le \sum_{n\ge1} \mu_n = \mu \le 1$, a sequence $\{i_n\}_{n\ge1} \subset I$ of indices (not necessarily distinct) and $u \in H$ such that $x = \sum_{n\ge1} \mu_n x_{i_n} + (1-\mu)u$.*

Proof. Clearly the statement is true if $x \in H$. Assume that $x \notin H$, i.e., $\lambda(x) > 0$. Choose $0 < \frac{1}{2}\lambda(x) \le \lambda_1 \le 1$, $i_1 \in I$, and $u_1 \in \overline{\mathrm{co}}^{w^*}(\{x_i\}_{i\in I\setminus\{i_1\}})$ such that $x = \lambda_1 x_{i_1} + (1-\lambda_1)u_1$. If $u_1 \in H$, we are done. Otherwise, $\lambda(u_1) > 0$ and we choose $0 < \frac{1}{2}\lambda(u_1) \le \lambda_2 \le 1$, $i_2 \in I$, and $u_2 \in \overline{\mathrm{co}}^{w^*}(\{x_i\}_{i\in I\setminus\{i_2\}})$ such that $u_1 = \lambda_2 x_{i_2} + (1-\lambda_2)u_2$. By reiteration, there are two possibilities:

(A) $u_m \in H$ for some $m \in \mathbb{N}$. Then we obtain the representation

$$x = \sum_{k=1}^{m} \lambda_k P_{k-1} x_{i_k} + P_m u_m, \qquad P_n = \prod_{k=1}^{n}(1-\lambda_k), \qquad P_0 = 1. \tag{1}$$

(B) Always $u_m \notin H$. As P_m decreases in (1), the limit $\lim_{m\ge1} P_m = P \in [0,1]$ exists. We have two cases:

• $P > 0$. Observe that this happens iff $\sum_{k\ge1}\lambda_k < \infty$. In consequence, the series $\sum_{k\ge1}\lambda_k P_{k-1}x_{i_k}$ converges and $u_m \to u \in K$ as $m \to \infty$. So $x = \sum_{k\ge1}\lambda_k P_{k-1}x_{i_k} + Pu$. We claim that $\lambda(u) = 0$. Indeed, suppose that $\mu := \lambda(u) > 0$ and pick $q \in \mathbb{N}$ such that $P/P_q > 1/2$, $\lambda_{q+1} < \mu/8$. Then

$$u_q = \frac{1}{P_q}\Big(\sum_{j\ge1}\lambda_{q+j}P_{q+j-1}x_{q+j} + Pu\Big),$$

which implies that $\lambda(u_q) \ge (P/P_q)\lambda(u) = (P/P_q)\mu > \mu/2$. Since $0 < \frac{1}{2}\lambda(u_q) \le \lambda_{q+1} \le 1$, we obtain $\mu/8 > \lambda_{q+1} \ge \mu/4$, a contradiction.

• $P = 0$. In this case $P_m u_m \to 0$ as $m \to \infty$ and we obtain the representation $x = \sum_{k\ge1}\lambda_k P_{k-1}x_{i_k}$ with $\sum_{k\ge1}\lambda_k P_{k-1} = 1$. ∎

In order to connect the existence of an UBABS in a Banach space X with the w^*-nonseparability of dual equivalent unit balls of X^*, we introduce the index $\sigma(X)$. If $K \subset X^*$ is a disc (i.e., a convex symmetric subset of X^*), define

$$\sigma(K) = \max\{0 \le t \le 1 : \exists A \subset K,\ |A| \le \aleph_0,\ tK \subseteq \overline{\mathrm{co}}^{w^*}(A \cup (-A))\}.$$

Observe that $0 \le \sigma(K) < 1$ iff K is w^*-nonseparable and that there exists a countable subset $A \subset K$ such that $\sigma(K)\cdot K \subset \overline{\mathrm{co}}^{w^*}(A \cup (-A))$.

Lemma 2.4. *Let X be a Banach space, $K \subset X^*$ a w^*-nonseparable disc and $\sigma(K) < \varrho \le 1$. Then there exists $\varepsilon = \varepsilon(\varrho) > 0$ (depending on ϱ) such that for every countable subset $A \subset K$ there exists $k \in K$ satisfying* $\mathrm{dist}(\varrho k, \overline{\mathrm{co}}^{w^*}(A \cup (-A))) \ge \varepsilon$.

Proof. In the contrary case, there exist a sequence of real numbers $\varepsilon_n \downarrow 0$ and a sequence of countable subsets $A_n \subset K$, $n \ge 1$, such that every $k \in K$ satisfies $\mathrm{dist}(\varrho k, \overline{\mathrm{co}}^{w^*}(A_n \cup (-A_n))) < \varepsilon_n$. So, if $A = \bigcup_{n\ge1} A_n$ we have $\varrho K \subset \overline{\mathrm{co}}^{w^*}(A \cup (-A))$, a contradiction. ∎

Define the index $\sigma(X)$, X a Banach space, as follows:

$$\sigma(X) = \inf\{\sigma(K) : K \subset X^* \text{ a dual equivalent ball of } X^*\}.$$

It is clear that $\sigma(X)$ is invariant under isomorphisms.

PROPOSITION 2.5. *For a Banach space X we have*

$$\sigma(X) = \inf\{\sigma(K) : K \subset X^* \text{ a } w^*\text{-compact disc}\}.$$

Proof. Obviously $\sigma(X) \geq \inf\{\sigma(K) : K \subset X^* \text{ a } w^*\text{-compact disc}\}$. In order to prove the opposite inequality, it is enough to see that $\sigma(X) \leq \sigma(K)$ for any w^*-compact disc $K \subset X^*$. Assume that such a K is w^*-nonseparable, pick $\sigma(K) < \varrho < 1$ and let $\varepsilon = \varepsilon(\varrho) > 0$ be given by Lemma 2.4. For $0 < \delta < \varepsilon$ such that $\varrho + \delta < 1$ consider $H_\delta = K + \delta B(X^*)$, which is an equivalent dual ball of X^*. We claim that $\sigma(H_\delta) \leq \varrho + \delta$. Indeed, let $\varrho + \delta < t \leq 1$ and $A \subset H_\delta$ be a countable subset. Then $A \subset A_1 + A_2$, where $A_1 \subset K$ and $A_2 \subset \delta B(X^*)$ are countable. Assume that $tH_\delta \subset \overline{\text{co}}^{w^*}(A \cup (-A))$. As $\overline{\text{co}}^{w^*}(A \cup (-A)) \subset \overline{\text{co}}^{w^*}(A_1 \cup (-A_1)) + \delta B(X^*)$, we get

$$tK \subset tH_\delta \subset \overline{\text{co}}^{w^*}(A_1 \cup (-A_1)) + \delta B(X^*),$$

which implies that $\text{dist}(tk, \overline{\text{co}}^{w^*}(A_1 \cup (-A_1))) \leq \delta$ for all $k \in K$. But by Lemma 2.4 there exists $k \in K$ such that $\text{dist}(\varrho k, \overline{\text{co}}^{w^*}(A_1 \cup (-A_1))) \geq \varepsilon$. Thus $\text{dist}(tk, \overline{\text{co}}^{w^*}(A_1 \cup (-A_1))) > \delta$, a contradiction. Therefore, we have $tH_\delta \not\subseteq \overline{\text{co}}^{w^*}(A \cup (-A))$ and $\sigma(H_\delta) \leq \varrho + \delta$ for $0 < \delta < \varepsilon$. Hence, $\sigma(X) \leq \varrho$ for every $\sigma(K) < \varrho < 1$, and we conclude that $\sigma(X) \leq \sigma(K)$. ∎

PROPOSITION 2.6. *If X is a Banach space then $\sigma(X) \leq \tau(X)$.*

Proof. Assume that $\tau(X) < \eta < 1$ and choose an UBABS $\{(x_\alpha, f_\alpha)\}_{\alpha<\omega_1} \subset X \times X^*$ of type η such that $\|f_\alpha\| = 1$ and $\|x_\alpha\| \leq M$ for all $\alpha < \omega_1$, for some $0 < M < \infty$. Clearly, $\{\pm f_\alpha\}_{\alpha<\omega_1}$ is a w^*-ω_1-polyhedron. Define $K = \overline{\text{co}}^{w^*}(\{\pm f_\alpha\}_{\alpha<\omega_1})$, $K_0 = \text{core}(K)$ and $H = \{z \in K : \lambda(z) = 0\}$. It is easy to see that $|z(x_\alpha)| \leq \eta$ for every $z \in K_0$ and $\alpha < \omega_1$. We claim that $\sigma(K) \leq \eta$. Indeed, let $A \subset K$ be countable. By Lemma 2.3 there exists $\gamma < \omega_1$ such that

$$A \subset \overline{\text{co}}(\{\pm f_\alpha\}_{\alpha\leq\gamma} \cup H) \subset \overline{\text{co}}^{w^*}(\{\pm f_\alpha\}_{\alpha\leq\gamma} \cup H).$$

Clearly, $\overline{\text{co}}^{w^*}(A \cup (-A)) \subset \overline{\text{co}}^{w^*}(\{\pm f_\alpha\}_{\alpha\leq\gamma} \cup H)$ and for every $\gamma < \varrho < \omega_1$ and every $z \in \overline{\text{co}}^{w^*}(\{\pm f_\alpha\}_{\alpha\leq\gamma} \cup H)$ we have $|z(x_\varrho)| \leq \eta$.

Hence, for every $\gamma < \varrho < \omega_1$ and $\eta < t \leq 1$ we have $tf_\varrho \notin \overline{\text{co}}^{w^*}(A \cup (-A))$. So $\sigma(K) \leq \eta$ and we conclude that $\sigma(X) \leq \tau(X)$. ∎

Now we prove for a Banach space X that $\sigma(X) = 1$ iff $\tau(X) = 1$.

PROPOSITION 2.7. *A Banach space X has an UBABS of type η for some $\eta \in [0, 1)$ iff X^* has a w^*-nonseparable equivalent dual unit ball. So, $\sigma(X) = 1$ iff $\tau(X) = 1$.*

Proof. Firstly, if X has an UBABS of type η for some $\eta \in [0, 1)$ (i.e., $\tau(X) < 1$), then by Proposition 2.6 we have $\sigma(X) < 1$ (i.e., X^* has a w^*-nonseparable equivalent dual unit ball).

Assume now that X is a Banach space with $\sigma(X) < 1$ equipped with an equivalent norm such that $\sigma(B(X^*)) < 1$. Fix $\varrho > 0$ with $\sigma(B(X^*)) < \varrho < 1$. If $A \subset S(X)$ and $\varepsilon \geq 0$ we put

$$(A,\varepsilon)^{\perp} = \{z \in X^* : |z(x)| \leq \varepsilon, \forall x \in A\}, \qquad S((A,\varepsilon)^{\perp}) = S(X^*) \cap (A,\varepsilon)^{\perp}.$$

Clearly $\varepsilon B(X^*) + A^{\perp} \subset (A,\varepsilon)^{\perp}$.

CLAIM 0. *If $A \subset S(X)$ and $A^{\perp} \neq \{0\}$, then $\varepsilon S(X^*) \subset \operatorname{co}(S((A,\varepsilon)^{\perp}))$ for $0 \leq \varepsilon < 1$.*

Indeed, let $u \in \varepsilon S(X^*)$ and $v \in A^{\perp} \setminus \{0\}$. We can find $\lambda, \mu > 0$ such that $u + \lambda v, u - \mu v \in S(X^*)$. Thus, $u + \lambda v, u - \mu v \in S((A,\varepsilon)^{\perp})$. Let $t \in (0,1)$ be such that $t\lambda + (1-t)(-\mu) = 0$. Then $u = t(u+\lambda v) + (1-t)(u - \mu v) \in \operatorname{co}(S((A,\varepsilon)^{\perp}))$.

CLAIM 1. *For any countable subsets $A \subset S(X)$ and $F \subset S(X^*)$ there exists $f \in S((A,\sqrt{\varrho})^{\perp})$ such that $\sqrt{\varrho}\, f \notin \overline{\operatorname{co}}^{w^*}(F \cup (-F))$.*

The opposite means that $\sqrt{\varrho}\, S((A,\sqrt{\varrho})^{\perp}) \subset \overline{\operatorname{co}}^{w^*}(F \cup (-F))$. By Claim 0 we have $\sqrt{\varrho}\, S(X^*) \subset \operatorname{co}(S((A,\sqrt{\varrho})^{\perp}))$. So

$$\varrho B(X^*) \subset \overline{\operatorname{co}}^{w^*}(\varrho S(X^*)) \subset \overline{\operatorname{co}}^{w^*}(\sqrt{\varrho}\, S((A,\sqrt{\varrho})^{\perp})) \subset \overline{\operatorname{co}}^{w^*}(F \cup (-F)),$$

a contradiction because $\sigma(B(X^*)) < \varrho$. So, Claim 1 holds.

CLAIM 2. *There exist $0 \leq \delta < \varepsilon \leq 1 - \sqrt{\varrho}$ such that for any countable subsets $A \subset S(X)$ and $F \subset S(X^*)$ there exist $f_0 \in S((A,\sqrt{\varrho})^{\perp})$ and $x_0 \in S(X)$ such that $f_0(x_0) \geq 1 - \delta$ and $f(x_0) \leq 1 - \varepsilon$ for all $f \in F$.*

Define $\mathcal{R} = \{r = (r_1, r_2) \in \mathbb{Q} \times \mathbb{Q} : 0 < r_1 < r_2 \leq 1 - \sqrt{\varrho}\}$. As $\mathcal{R}$ is countable, we can put $\mathcal{R} = \{r_n\}_{n\geq 1}$. If Claim 2 is false, for every pair $r_n = (r_{n1}, r_{n2}) \in \mathcal{R}$ we can choose countable subsets $A_n \subset S(X)$, $F_n \subset S(X^*)$, $n \geq 1$, such that for every $g \in S((A_n,\sqrt{\varrho})^{\perp})$ and every $x \in S(X)$, either $g(x) < 1 - r_{n1}$ or there exists $f \in F_n$ with $f(x) > 1 - r_{n2}$. Let $A = \bigcup_{n\geq 1} A_n$, $F = \bigcup_{n\geq 1} F_n$. By Claim 1 there exists $f_0 \in S((A,\sqrt{\varrho})^{\perp})$ such that $\sqrt{\varrho}\, f_0 \notin \overline{\operatorname{co}}^{w^*}(F \cup (-F))$. By the Hahn–Banach Theorem there exists $y \in S(X)$ such that

$$\sqrt{\varrho}\, f_0(y) > \sup\{|f(y)| : f \in F\} =: \gamma_0 \geq 0.$$

Choose a sequence $\{z_n\}_{n\geq 1} \subset S(X)$ such that

$$\text{(2)} \qquad \lim_{n\to\infty} f_0(z_n) = \|f_0\| = 1, \qquad 1 - f_0(z_n) < \frac{1}{n}\,(f_0(y) - \gamma_0), \qquad n \geq 1.$$

Then

$$f_0\bigg(\frac{z_n + \frac{1}{n} y}{\|z_n + \frac{1}{n} y\|}\bigg) = 1 - \delta_n$$

with

$$0 \le \delta_n = \frac{\|z_n + \frac{1}{n}y\| - f_0(z_n) - \frac{1}{n}f_0(y)}{\|z_n + \frac{1}{n}y\|} \le \frac{1 - f_0(z_n) + \frac{1}{n}(1 - f_0(y))}{\|z_n + \frac{1}{n}y\|}.$$

Hence, $\lim_{n\to\infty} \delta_n = 0$. On the other hand, for every $f \in F$,

$$f\left(\frac{z_n + \frac{1}{n}y}{\|z_n + \frac{1}{n}y\|}\right) \le \frac{1 + \frac{1}{n}\gamma_0}{\|z_n + \frac{1}{n}y\|} = 1 - \varepsilon_n,$$

where

$$\varepsilon_n = \frac{\|z_n + \frac{1}{n}y\| - 1 - \frac{1}{n}\gamma_0}{\|z_n + \frac{1}{n}y\|} \le \frac{1 + \frac{1}{n} - 1 - \frac{1}{n}\gamma_0}{\|z_n + \frac{1}{n}y\|} = \frac{\frac{1}{n}(1 - \gamma_0)}{\|z_n + \frac{1}{n}y\|}$$

and

$$\varepsilon_n > \frac{\|z_n + \frac{1}{n}y\| - f_0(z_n) - \frac{1}{n}f_0(y)}{\|z_n + \frac{1}{n}y\|} = \delta_n \ge 0$$

by (2). Pick any $n \in \mathbb{N}$ such that $\frac{1}{n}(1-\gamma_0)/\|z_n + \frac{1}{n}y\| \le 1 - \sqrt{\varrho}$. Then $0 \le \delta_n < \varepsilon_n \le 1 - \sqrt{\varrho}$ and there is some $m \in \mathbb{N}$ such that $\delta_n \le r_{m1} < r_{m2} \le \varepsilon_n$. Let $x_0 = (z_n + \frac{1}{n}y)/\|z_n + \frac{1}{n}y\| \in S(X)$ and observe that $f_0 \in S((A_m, \sqrt{\varrho})^\perp)$, $f_0(x_0) \ge 1 - \delta_n$ and $f(x_0) \le 1 - \varepsilon_n$ for all $f \in F$. Then $f_0 \in S((A_m, \sqrt{\varrho})^\perp)$, $f_0(x_0) \ge 1 - r_{m1}$ and $f(x_0) \le 1 - \varepsilon_n \le 1 - r_{m2}$ for all $f \in F_m$, a contradiction. So, Claim 2 holds.

Let $0 \le \delta < \varepsilon \le 1 - \sqrt{\varrho}$ be from Claim 2. We will construct a transfinite sequence $\{(x_\alpha, f_\alpha)\}_{\alpha<\omega_1} \subset S(X) \times S(X^*)$ so that for every $\alpha < \omega_1$,

(3) $$f_\alpha(x_\alpha) \ge 1 - \delta,$$

(4) $$f_\alpha(x_\beta) \le 1 - \varepsilon \quad \text{if } \alpha \ne \beta.$$

In the first step, we take $x_1 \in S(X)$ and $f_1 \in S(X^*)$ such that $f_1(x_1) = 1$. Let $1 < \alpha_0 < \omega_1$ and suppose we have constructed a family $\{(x_\alpha, f_\alpha) : \alpha < \alpha_0\}$ satisfying (3) and (4). Apply Claim 2, with $F = \{f_\alpha : \alpha < \alpha_0\}$ and $A = \{x_\alpha : \alpha < \alpha_0\}$. Denote the resulting elements x_0 and f_0 by x_{α_0} and f_{α_0}. The inequality (3) for $\alpha = \alpha_0$ is satisfied by construction. The inequality (4) for $\alpha = \alpha_0$ and $\beta < \alpha_0$ holds because $f_0 \in S((A, \sqrt{\varrho})^\perp)$ and $\varepsilon \le 1 - \sqrt{\varrho}$. For $\beta = \alpha_0$ and $\alpha < \alpha_0$, it follows because $\sup\{f(x_0) : f \in F\} \le 1 - \varepsilon$. Now the set $\{(\overline{x}_\alpha, \overline{f}_\alpha)\}_{\alpha<\omega_1}$, where $\overline{x}_\alpha = x_\alpha$, $\overline{f}_\alpha = f_\alpha/f_\alpha(x_\alpha)$, $1 \le \alpha < \omega_1$, is an uncountable bounded (by $(1-\delta)^{-1}$) almost biorthogonal system. ∎

PROPOSITION 2.8. *Let X be a Banach space such that $\sigma(X) < 1/3$. Then $\tau(X) \le 2\sigma(X)/(1 - \sigma(X))$. So, for every Banach space X*:

(1) $\sigma(X) = 0$ *iff* $\tau(X) = 0$.

(2) $\sigma(X) = 0$ *whenever X has an uncountable biorthogonal system.*

Proof. (A) Let $\|\cdot\|$ be an equivalent norm on X such that the corresponding dual unit ball $B(X^*)$ satisfies $\sigma(B(X^*)) < 1/3$. It is enough to prove that for every $\sigma(B(X^*)) < a < 1/3$ there exists in X an UBABS of type $\eta \le 2a/(1-a)$. So, fix such an a. By induction we choose a family $\{(x_\alpha, f_\alpha)\}_{\alpha<\omega_1} \subset S(X) \times S(X^*)$ such that

$$f_\alpha(x_\alpha) > \frac{1-a}{2}, \qquad |f_\alpha(x_\beta)| < a \quad \text{if } \alpha \ne \beta. \tag{5}$$

Pick $(x_1, f_1) \in S(X) \times S(X^*)$ satisfying $f_1(x_1) = 1$. Let $\alpha < \omega_1$ and assume that we have chosen $\{(x_\beta, f_\beta)\}_{\beta<\alpha} \subset S(X) \times S(X^*)$ satisfying (5). Set

$$A_\alpha = \overline{[\{x_\beta : \beta < \alpha\}]}, \qquad F_\alpha = \overline{\mathrm{co}}^{w^*}(\{\pm f_\beta : \beta < \alpha\} \cup G_0),$$

where $G_0 \subset B(X^*)$ is a countable symmetric subset 1-norming on A_α. By [15, Lemma 4.3] there exists $x_\alpha \in S(X)$ such that $\sup\{|f(x_\alpha)| : f \in F_\alpha\} < a$. We claim that $\mathrm{dist}(x_\alpha, A_\alpha) > (1-a)/2$. Indeed, pick $z \in A_\alpha$ and observe that if $\|z\| < (1+a)/2$, then clearly $\|z - x_\alpha\| > (1-a)/2$, and if $\|z\| \ge (1+a)/2$, then

$$\begin{aligned}\|z - x_\alpha\| &\ge \sup\{f(z - x_\alpha) : f \in F_\alpha\}\\ &\ge \|z\| - \sup\{f(x_\alpha) : f \in F_\alpha\} > \frac{1+a}{2} - a = \frac{1-a}{2}.\end{aligned}$$

This means that if $Q : X \to X/A_\alpha$ is the canonical quotient mapping, then $\|Q(x_\alpha)\| > (1-a)/2$. So, as $(X/A_\alpha)^* = A_\alpha^\perp$ there exists $f_\alpha \in S(X^*) \cap A_\alpha^\perp$ such that $f_\alpha(x_\alpha) > (1-a)/2$. Thus we have chosen the pair (x_α, f_α), and this completes the induction.

Now put $\widetilde{f}_\alpha = f_\alpha/f_\alpha(x_\alpha)$, consider the family $\mathfrak{F} = \{(x_\alpha, \widetilde{f}_\alpha)\}_{\alpha<\omega_1}$ and observe that:

(a) $\mathfrak{F}$ is bounded because $\|x_\alpha\| = 1$ and

$$\|\widetilde{f}_\alpha\| = \frac{\|f_\alpha\|}{|f_\alpha(x_\alpha)|} < \frac{1}{(1-a)/2} = \frac{2}{1-a} < \frac{2}{1-1/3} = 3.$$

(b) $\widetilde{f}_\alpha(x_\alpha) = 1$ and

$$|\widetilde{f}_\alpha(x_\beta)| = \frac{|f_\alpha(x_\beta)|}{f_\alpha(x_\alpha)} < \frac{a}{(1-a)/2} = \frac{2a}{1-a} < 1 \quad \text{if } \alpha \ne \beta.$$

So, $\mathfrak{F}$ is an UBABS of type $\eta \le 2a/(1-a)$.

(B) (1) follows from (A) and Proposition 2.6; (2) follows from the definition of $\tau(X)$ and (1). ∎

3. On ω-independence. The Kunen–Shelah property KS_3**.** A family $\{x_i\}_{i\in I}$ in a Banach space X is said to be *ω-independent* if for every sequence $(i_n)_{n\ge1} \subset I$ of distinct indices, and every sequence $(\lambda_n)_{n\ge1} \subset \mathbb{R}$, the series $\sum_{n=1}^\infty \lambda_n x_{i_n}$ converges (in norm) to 0 iff $\lambda_n = 0$ for every $n \ge 1$ (see [6], [12]). A Banach space X is said to have the *Kunen–Shelah property* KS_3

if X has no uncountable ω-independent family. Of course, every biorthogonal family is ω-independent (i.e., $\mathrm{KS}_3 \Rightarrow \mathrm{KS}_2$), but there are ω-independent families which are not merely biorthogonal systems. Here is an example: $X = C([0,1]^{\omega_1})$ and $\{f_\alpha^n\}_{\alpha<\omega_1, n\geq 1}$ defined as

$$f_\alpha^n((t_\gamma)_{\gamma<\omega_1}) = t_\alpha^n$$

for every $x = (t_\gamma)_{\gamma<\omega_1} \in [0,1]^{\omega_1}$. This family is ω-independent but not a biorthogonal system by the Theorem of Müntz–Szasz (see [11, Th. 15.26]).

Question 2. Does a Banach space have an uncountable biorthogonal system whenever it has an uncountable ω-independent family?

Unfortunately, the indices $\sigma(X), \tau(X)$ do not separate the properties KS_2 and KS_3, because as we prove in the following, if $X \in \mathrm{KS}_3$, then $\sigma(X) = 0$.

Lemma 3.1. *Let X be a Banach space, $\{x_i\}_{1\leq i<\omega_1} \subset X$ an uncountable bounded ω-independent family, $H \subset X$ a closed separable subspace and $N \in \mathbb{N}$. Then there exist ordinal numbers $\varrho < \gamma < \omega_1$ such that $x_\varrho \notin \overline{\mathrm{co}}(H \cup \{\pm N x_i\}_{\gamma\leq i<\omega_1})$.*

Proof. Without loss of generality suppose that $\|x_i\| \leq 1$ for all $i < \omega_1$. Assume that for every pair of ordinal numbers ϱ, γ such that $\varrho < \gamma < \omega_1$ we have $x_\varrho \in \overline{\mathrm{co}}(H \cup \{\pm N x_i\}_{\gamma\leq i<\omega_1})$. For $n \in \mathbb{N}$ and $\varrho < \gamma < \omega_1$, define $D_\gamma = \mathrm{co}(\{\pm N x_i\}_{\gamma\leq i<\omega_1})$ and

$$H(\varrho,\gamma,n) = \left\{(u,\lambda) \in H \times (0,1] : \exists v \in D_\gamma \text{ with } \|\lambda u + (1-\lambda)v - x_\varrho\| < \frac{1}{2n}\right\}.$$

If $\varrho < \gamma < \gamma' < \omega_1$ and $n \geq 1$, then by hypothesis and definition, we have $H(\varrho,\gamma,n) \neq \emptyset$ and $H(\varrho,\gamma,n+1) \subset H(\varrho,\gamma,n) \supset H(\varrho,\gamma',n)$.

For $\beta < \omega_1$ and $n \geq 1$ define

$$H(\beta,n) = \mathrm{cl}\left(\bigcup\{H(\varrho,\gamma,n) : \beta \leq \varrho < \gamma < \omega_1\}\right)$$

where "cl" means closure in $H \times (0,1]$. Clearly, for $\beta < \beta'$ and $n \geq 1$ we have

$$\emptyset \neq H(\beta',n) \subset H(\beta,n) \supset H(\beta,n+1).$$

Since $H \times (0,1]$ is hereditarily Lindelöf, for each $n \geq 1$ there exists $\beta_n < \omega_1$ such that for every $\beta_n \leq \beta < \omega_1$ we have $H(\beta,n) = H(\beta_n,n)$. So, for every $(u,\lambda) \in H(\beta_n,n)$ and every $\beta_n \leq \beta < \omega_1$ we have $(u,\lambda) \in H(\beta,n)$, which implies that there exist $\beta \leq \varrho < \gamma < \omega_1$ and $v \in D_\gamma$ such that

$$\|x_\varrho - (\lambda u + (1-\lambda)v)\| < 1/n.$$

Let $\beta_0 = \sup_{n\geq 1} \beta_n$ and fix $\beta_0 \leq \varrho < \gamma < \omega_1$ and $n \geq 1$. Pick $(u,\mu) \in H(\varrho,\gamma,n)$ and $w \in D_\gamma$ such that $\|x_\varrho - (\mu u + (1-\mu)w)\| < 1/(2n)$. Since

$(u,\mu)\in H(\beta_0,n)=H(\gamma,n)$, there exist $\gamma\le\sigma<\theta<\omega_1$ and $v\in D_\theta$ such that $\|x_\sigma-(\mu u+(1-\mu)v)\|<1/n$.

Set $T=x_\sigma-(\mu u+(1-\mu)v)$. Then $\mu u=x_\sigma-T-(1-\mu)v$ and

$$\|x_\varrho-(x_\sigma-T-(1-\mu)v+(1-\mu)w)\|<\frac{1}{2n}.$$

Since $\|T\|<1/n$, we obtain

$$\begin{aligned}\|x_\varrho&-(x_\sigma-(1-\mu)v+(1-\mu)w)\|\\ &=\|x_\varrho-(x_\sigma-T-(1-\mu)v+(1-\mu)w)-T\|\\ &\le\|x_\varrho-(x_\sigma-T-(1-\mu)v+(1-\mu)w)\|+\|T\|<\frac{1}{2n}+\frac{1}{n}=\frac{3}{2n}.\end{aligned}$$

Since $x_\sigma,v,w\in E_\gamma:=\overline{[\{x_i\}_{\gamma\le i<\omega_1}]}$, letting $n\to\infty$ (with ϱ,γ fixed) we deduce that $x_\varrho\in E_\gamma$ (in particular, this implies that $E_{\beta_0}=E_\beta$ for all $\beta_0\le\beta<\omega_1$). Set $S=x_\varrho-(x_\sigma-(1-\mu)v+(1-\mu)w)$. Then

$$x_\varrho=S+\mu v+(1-\mu)w+x_\sigma-v.$$

Taking into account that $\mu v+(1-\mu)w,-v\in D_\gamma,\ x_\sigma\in(1/N)D_\gamma$ and that $\|S\|<3/(2n)$, we finally get $x_\varrho\in\mathrm{cl}((1+1/N)D_\gamma+D_\gamma)=\mathrm{cl}((2+1/N)D_\gamma)$. So, x_ϱ is an accumulation point of $F_\gamma:=(2+1/N)D_\gamma$ (because $x_\varrho\in\overline{F}_\gamma\setminus F_\gamma$).

In consequence, we can conclude that every x_i, $\beta_0\le i<\omega_1$, is an accumulation point of every F_γ for $\gamma<\omega_1$.

Let $(a_n)_{n\ge1}$ be a sequence of positive numbers such that $\lim_{n\to\infty}a_n=0$, $\sum_{n\ge1}a_n=\infty$, and let $b_n=\sup_{m>n}a_m$. Fix $\beta_0<\tau<\omega_1$. Using the proof of [6, Th. 3], as in [12], we can construct inductively a sequence $\{\varepsilon_n\}_{n\ge1}$ of signs, a sequence $\{\lambda_r^n\}_{n\ge1,\,1\le r\le k(n)}$ of real numbers and a sequence $\{\gamma_r^n\}_{n\ge1,\,1\le r\le k(n)}$ of ordinals such that:

(1) $\sum_{r=1}^{k(n)}|\lambda_r^n|\le 2N+1$ for every $n\ge1$.
(2) $\tau<\gamma_1^n<\ldots<\gamma_{k(n)}^n<\gamma_1^{n+1}<\ldots<\omega_1$ for every $n\ge1$.
(3) $x_\tau+\sum_{n\ge1}a_n\varepsilon_ny_n=0$, where $y_n=\sum_{r=1}^{k(n)}\lambda_r^nx_{\gamma_r^n}$.

Let us see the first two steps of this argument. Set $K=\{x_i\}_{\tau<i<\omega_1}$.

Step 1. By the proof of [6, Th. 3] we can find $p_1\in\mathbb{N}$, a finite sequence $\{h_n\}_{1\le n\le p_1}$ of (not necessarily distinct) elements of K and a finite sequence $\{\varepsilon_n\}_{1\le n\le p_1}$ of signs such that

$$\Big\|x_\tau+\sum_{n=1}^{p_1}a_n\varepsilon_nh_n\Big\|<2^{-1},$$

$$\Big\|x_\tau+\sum_{n=1}^{j}a_n\varepsilon_nh_n\Big\|<b_1+1+2^{-1}\quad\text{ for } 1\le j\le p_1.$$

Since $h_n \in \mathrm{cl}(F_\beta)$ for $\beta_0 \le \beta < \omega_1$, we can find, for $1 \le n \le p_1$, real numbers $\{\lambda_r^n\}_{1\le r\le k(n)}$ with $\sum_{r=1}^{k(n)} |\lambda_r^n| \le 2N+1$, and ordinals $\{\gamma_r^n\}_{r=1}^{k(n)}$ such that:

(a) $\tau < \gamma_1^n < \ldots < \gamma_{k(n)}^n < \gamma_1^{n+1} < \ldots < \omega_1$.

(b) $\|x_\tau + \sum_{n=1}^{p_1} a_n\varepsilon_n(\sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n})\| < 2^{-1}$.

(c) $\|x_\tau + \sum_{n=1}^{j} a_n\varepsilon_n(\sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n})\| < b_1 + 1 + 2^{-1}$ for $1 \le j \le p_1$.

Step 2. Let $u_1 = x_\tau + \sum_{n=1}^{p_1} a_n\varepsilon_n(\sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n})$. By the proof of [6, Th. 3] we can find $p_1 < p_2 \in \mathbb{N}$, a finite sequence $\{h_n\}_{p_1+1\le n\le p_2}$ of (not necessarily distinct) elements of K and a finite sequence $\{\varepsilon_n\}_{p_1+1\le n\le p_2}$ of signs such that

$$\Big\|u_1 + \sum_{n=p_1+1}^{p_2} a_n\varepsilon_n h_n\Big\| < 2^{-2},$$

$$\Big\|u_1 + \sum_{n=p_1+1}^{j} a_n\varepsilon_n h_n\Big\| < b_{p_1} + 2^{-1} + 2^{-2} \quad \text{for } p_1+1 \le j \le p_2.$$

Since $h_n \in \mathrm{cl}(F_\beta)$ for $\beta_0 \le \beta < \omega_1$, we can find, for $p_1 < n \le p_2$, real numbers $\{\lambda_r^n\}_{1\le r\le k(n)}$ with $\sum_{r=1}^{k(n)} |\lambda_r^n| \le 2N+1$, and ordinals $\{\gamma_r^n\}_{r=1}^{k(n)}$ such that:

(a) $\gamma_{k(p_1)}^{p_1} < \gamma_1^n < \ldots < \gamma_{k(n)}^n < \gamma_1^{n+1} < \ldots < \omega_1$.

(b) $\|u_1 + \sum_{n=p_1+1}^{p_2} a_n\varepsilon_n(\sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n})\| < 2^{-2}$.

(c) $\|u_1 + \sum_{n=p_1+1}^{j} a_n\varepsilon_n(\sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n})\| < b_{p_1} + 2^{-1} + 2^{-2}$ for $p_1 < j \le p_2$.

Now by reiteration we obtain the complete construction. It is easy to see that the series $x_\tau + \sum_{n\ge1} a_n\varepsilon_n(\sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n})$ converges to zero. This proves that $\{x_i\}_{i<\omega_1}$ is not ω-independent, a contradiction. So, we can choose $\varrho < \gamma < \omega_1$ such that $x_\varrho \notin \overline{\mathrm{co}}(H \cup \{\pm N x_i\}_{\gamma\le i<\omega_1})$. ∎

Proposition 3.2. *Let a Banach space X have an uncountable ω-independent family $\{x_\alpha\}_{1\le\alpha<\omega_1}$. Then for every $0 < \eta < 1$, there exist an uncountable subsequence $\{\alpha_i\}_{i<\omega_1} \subset \omega_1$ and an UBABS $\{(z_i, f_i)\}_{i<\omega_1} \subset X \times X^*$ of type η such that $z_i = x_{\alpha_i}$ and $f_i(z_j) = 0$ for $j < i < \omega_1$. So, $\tau(X) = 0$ and X has an ω_1-polyhedron.*

Proof. Let $\{x_i\}_{1\le i<\omega_1} \subset X$ be an uncountable ω-independent family and suppose, without loss of generality, that $\|x_i\| \le 1$ for every $i < \omega_1$. Let $N \in \mathbb{N}$ be such that $1/N \le \eta$. In the following we choose by induction two subsequences $\{i_\alpha, j_\alpha\}_{\alpha<\omega_1}$ of ordinal numbers, with $i_\alpha < j_\alpha \le i_\beta < j_\beta < \omega_1$ for $\alpha < \beta < \omega_1$, such that

$$x_{i_\alpha} \notin \overline{\mathrm{co}}(\overline{[\{x_{i_\beta} : \beta < \alpha\}]} \cup \{\pm N x_j\}_{j_\alpha \le j < \omega_1}). \tag{6}$$

Indeed, let $\alpha < \omega_1$ and assume that we have chosen $\{i_\beta, j_\beta\}_{\beta<\alpha}$ satisfying (6). Put $H = \overline{[\{x_{i_\beta}\}_{\beta<\alpha}]}$ and $\nu = \sup_{\beta<\alpha}\{j_\beta\}$ (if $\alpha = 1$, put $H = \{0\}$ and $\nu = 1$). By Lemma 3.1 there exist $\nu \le \varrho < \gamma < \omega_1$ such that $x_\varrho \notin \overline{\mathrm{co}}\,(H \cup \{\pm N x_i\}_{\gamma\le i<\omega_1})$. So, we put $i_\alpha = \varrho,\ j_\alpha = \gamma$, and this completes the induction. Let $z_\alpha = x_{i_\alpha}$ for $\alpha < \omega_1$. By (6) we have $z_\alpha \notin \overline{\mathrm{co}}(\overline{[\{z_\beta : \beta < \alpha\}]} \cup \{\pm N z_j\}_{\alpha<j<\omega_1})$. So, by the Hahn–Banach Theorem there exists $f_\alpha \in X^*$ such that

$$1 = f_\alpha(z_\alpha) > \sup\{f_\alpha(x) : x \in \overline{\mathrm{co}}(\overline{[\{z_\beta : \beta < \alpha\}]} \cup \{\pm N z_j\}_{\alpha<j<\omega_1})\}.$$

Clearly, $f_\alpha(z_\beta) = 0$ if $\beta < \alpha$, and $|f_\alpha(N z_\beta)| < 1$, i.e., $|f_\alpha(z_\beta)| < 1/N$, if $\alpha < \beta < \omega_1$. Finally, if we choose an uncountable subsequence $A \subset \omega_1$ with $\{\|f_\alpha\| : \alpha \in A\}$ bounded, then $\{(z_\alpha, f_\alpha) : \alpha \in A\}$ is the UBABS of type η we are looking for. ■

4. The Kunen–Shelah property KS_4**.** A Banach space X is said to have the *Kunen–Shelah property* KS_4 if X has no ω_1-polyhedron. The implication $\mathrm{KS}_4 \Rightarrow \mathrm{KS}_3$ was proved in [3]. It also follows from Proposition 3.2 and from Proposition 7.3 and a result of Sersouri [12].

PROPOSITION 4.1. *Let Z be a Banach space and $X \subset Z$ a closed subspace such that Z/X is separable. Then the following are equivalent*:

(a) $Z \in \mathrm{KS}_4$.

(b) $X \in \mathrm{KS}_4$.

Proof. (a)⇒(b). This is obvious.

(b)⇒(a). Assume that $Z \notin \mathrm{KS}_4$; we will prove that $X \notin \mathrm{KS}_4$. By Proposition 2.2 there exists in Z an UBABS $\{(z_\alpha, f_\alpha) : \alpha < \omega_1\}$ of type $\eta \in [0,1)$ with $\|f_\alpha\| \le M$ for all $\alpha < \omega_1$, for some $0 < M < \omega_1$. Set $\varepsilon := 1 - \eta$. Since Z/X is separable, there exists an uncountable subset $I \subset \omega_1$ such that if $Q : Z \to Z/X$ is the canonical quotient mapping, then $\|Qz_\alpha - Qz_\beta\| < \varepsilon/(4M)$ for every $\alpha, \beta \in I$. Fix $\tau \in I$ and define $y_\alpha = z_\alpha - z_\tau$ for $\alpha \in I$. Since $\|Qy_\alpha\| < \varepsilon/(4M)$, there exists $x_\alpha \in X$ such that $\|x_\alpha - y_\alpha\| < \varepsilon/(4M)$ for all $\alpha \in I$. Then for any $\alpha, \beta \in I,\ \alpha \ne \beta$, we have

$$\begin{aligned} f_\alpha(x_\alpha) &= f_\alpha(y_\alpha) + f_\alpha(x_\alpha - y_\alpha) \ge f_\alpha(y_\alpha) - M\,\frac{\varepsilon}{4M} = f_\alpha(z_\alpha) - f_\alpha(z_\tau) - \frac{\varepsilon}{4} \\ &= 1 - f_\alpha(z_\tau) - \frac{\varepsilon}{4} > \eta - f_\alpha(z_\tau) + \frac{\varepsilon}{4} \ge f_\alpha(z_\beta) - f_\alpha(z_\tau) + \frac{\varepsilon}{4} \\ &= f_\alpha(y_\beta) + \frac{\varepsilon}{4} = f_\alpha(y_\beta) + M\,\frac{\varepsilon}{4M} \ge f_\alpha(x_\beta), \end{aligned}$$

which implies that $\{x_\alpha : \alpha \in I\}$ is an uncountable polyhedron in X, i.e., $X \notin \mathrm{KS}_4$. ■

In the following we obtain some characterizations of the property KS_4. We first prove some lemmas.

LEMMA 4.2. *Let X be a locally convex topological space, $\tau = \sigma(X, X^*)$, $f \in X^* \setminus \{0\}$, $C \subset f^{-1}(1)$ a bounded convex subset and $B = \mathrm{co}(C \cup (-C))$. Then C is τ-separable iff B is τ-separable.*

Proof. Clearly, B is τ-separable whenever C is. For the converse, suppose that B is τ-separable and choose a countable subset $A \subset C$ such that $D := \{tx - (1-t)y : x, y \in A,\, t \in [0,1]\}$ is τ-dense in B. Now it is an easy exercise to prove that $C \subset \tau\text{-cl}(A)$, i.e., C is τ-separable. ∎

LEMMA 4.3. *Let X be a locally convex topological space, $\tau = \sigma(X, X^*)$, and $C \subset X$ a convex subset such that for some $f \in X^*$ there exists a countable subset $\mathcal{R} \subset \mathbb{R}$ satisfying*:

(1) $\emptyset \neq (\inf\{f(x) : x \in C\}, \sup\{f(x) : x \in C\}) \subset \overline{\mathcal{R}}$.

(2) $C_r := \{x \in C : f(x) = r\}$ *is τ-separable for each $r \in \mathcal{R}$.*

Then C is τ-separable.

Proof. By hypothesis $\inf\{f(x) : x \in C\} < \sup\{f(x) : x \in C\}$. For each $r \in \mathcal{R}$, choose a countable subset $A_r \subset C_r$ such that $C_r \subset \tau\text{-cl}(A_r)$. Let $A = \bigcup_{r \in \mathcal{R}} A_r$, a countable subset of C. We claim that A is τ-dense in C. Indeed, pick $z_0 \in C$ arbitrarily and let U be a τ-neighborhood of z_0 in C. By hypothesis, there exists some $r \in \mathcal{R}$ such that $C_r \cap U \neq \emptyset$. So, $A_r \cap U \neq \emptyset$, whence $A \cap U \neq \emptyset$. ∎

PROPOSITION 4.4. *Let X be a Banach space. The following are equivalent*:

(1) $X \in \mathrm{KS}_4$.

(2) *$K \subset X^*$ is w^*-separable whenever K is a w^*-compact convex symmetric subset such that $\|\cdot\|\text{-int}(K) \neq \emptyset$.*

(3) *$K \subset X^*$ is w^*-separable whenever K is a w^*-compact convex symmetric subset, i.e., $\sigma(X) = 1 = \tau(X)$.*

(4) *$K \subset X^*$ is w^*-separable whenever K is a w^*-closed convex symmetric subset.*

(5) *$K \subset X^*$ is w^*-separable whenever K is a w^*-closed convex subset.*

Proof. (1)⇒(2). This follows from Propositions 2.7 and 2.2, because if $K \subset X^*$ is a w^*-compact convex symmetric subset such that $\|\cdot\|\text{-int}(K) \neq \emptyset$, then K is the dual unit ball of X^* when X is equipped with the equivalent norm $|\cdot|$ such that $|x| = \sup\{x^*(x) : x^* \in K\}$ for every $x \in X$.

(2)⇒(3). Let $K \subset X^*$ be a w^*-compact convex symmetric subset and set $K_n = K + \frac{1}{n} B(X^*)$, which is a w^*-compact convex symmetric subset of X^* with nonempty interior. By (2) there is a countable family $\{x_{n,m}\}_{m \geq 1} \subset K_n$

such that $K_n = \overline{\{x_{n,m} : m \geq 1\}}^{w^*}$ for every $n \geq 1$. Pick $k_{n,m} \in K$ such that $\|k_{n,m} - x_{n,m}\| \leq 1/n$. Then it is easy to see that $K = \overline{\{k_{n,m} : n, m \geq 1\}}^{w^*}$.

(3)⇒(4). Let $K \subset X^*$ be a w^*-closed convex symmetric subset and define $K_n = K \cap nB(X^*)$. By (3), K_n is w^*-separable and hence so is K, because $K = \bigcup_{n\geq 1} K_n$.

(4)⇒(5). It is enough to prove that if $K \subset X^*$ is a w^*-compact convex subset, then K is w^*-separable. Without loss of generality, assume that $0 \notin K$. Let $f \in X$ be such that $0 < \min\{f(k) : k \in K\} \leq \max\{f(k) : k \in K\} < \infty$. If $t \in [\min\{f(k) : k \in K\}, \max\{f(k) : k \in K\}]$, define $K_t = \{k \in K : f(k) = t\}$ and $C_t = \overline{\mathrm{co}}^{w^*}(K_t \cup (-K_t))$. By (4) and Lemma 4.2 each C_t is w^*-separable. So, from Lemma 4.3 we conclude that K is w^*-separable.

(5)⇒(1). Suppose that there exists in X a bounded ω_1-polyhedron $\{x_i\}_{i<\omega_1}$. By Proposition 2.2, there exists in X an UBABS $\{(x_\alpha, f_\alpha)\}_{\alpha<\omega_1} \subset X \times X^*$ such that $\|f_\alpha\| = 1$, $\|x_\alpha\| \leq M$, $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) \leq 1 - \varepsilon$ for every $\alpha, \beta < \omega_1$, $\alpha \neq \beta$, and some $1 \geq \varepsilon > 0$, $1 \leq M < \infty$. Let $K = \overline{\mathrm{co}}^{w^*}(\{f_\alpha : \alpha < \omega_1\})$. Consider the w^*-open slices $U_\alpha = \{k \in K : k(x_\alpha) > 1 - \varepsilon/3\}$ for all $\alpha < \omega_1$. Then U_α is a w^*-open neighborhood of f_α in K and we can easily see that $U_\alpha \cap U_\beta = \emptyset$ whenever $\alpha \neq \beta$. Thus K is w^*-nonseparable, a contradiction to (5). So, $X \in \mathrm{KS}_4$. ∎

Question 3. Let X be a Banach space. If $\tau(X) < 1$, is $\tau(X) = 0$? If $\tau(X) = 0$, does X have an uncountable ω-independent family?

5. The Finet–Godefroy indices. If X is a Banach space, the *Finet–Godefroy indices* $d_\infty(X)$ and $\mu(X)$ were introduced in [1] and defined as follows:

$$d_\infty(X) = \inf\{d(X,Y) : Y \text{ a subspace of } \ell_\infty(\mathbb{N})\},$$

where $d(X,Y)$ is the Banach–Mazur distance. Clearly, $d_\infty(X)$ depends upon the norm $\|\cdot\|$ of X and we see easily that: (i) $d_\infty(X) \in [1,\infty]$; (ii) $d_\infty(X) < \infty$ iff X is isomorphic to a subspace of $\ell_\infty(\mathbb{N})$; (iii) $d_\infty(X, \|\cdot\|) = 1$ iff $(X, \|\cdot\|)$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$ iff the dual unit ball $B(X^*)$ is w^*-separable. The corresponding isomorphic invariant index is

$$\mu(X) = \sup\{d_\infty(X, |\cdot|)\},$$

where the supremum is computed over the set of equivalent norms on X.

Proposition 5.1. *Let X be a Banach space. Then*:

(1) $\mu(X) = \sigma(X)^{-1}$ $(0^{-1} = \infty)$.

(2) *If X has an uncountable ω-independent system*, *then* $\mu(X) = \infty$.

Proof. (1) This follows from [1, Lemma III.1] and a simple calculation.

(2) By Proposition 3.2 and 2.8 we find that $\sigma(X) = 0$. Now apply (1). ∎

The following questions are proposed in [1]:

(1) It is clear that $\mu(X) = 1$ if X is separable. Is the converse true?

(2) Does there exist a nonseparable Banach space X such that every quotient of X is isometric to a subspace of $\ell_\infty(\mathbb{N})$?

In the following we answer these questions.

Proposition 5.2. *Let X be a Banach space. The following are equivalent*:

(1) $X \in \mathrm{KS}_4$.

(2) *Every quotient of* $(X, |\cdot|)$ *is isometric to a subspace of* $\ell_\infty(\mathbb{N})$, *for every equivalent norm* $|\cdot|$ *on* X.

(3) $\mu(X) = 1$.

(4) *Every quotient of X has the property* KS_4.

Proof. (1)⇒(2). Let $|\cdot|$ be an equivalent norm on X, $Y \subset X$ a closed subspace and $Z = (X/Y, |\cdot|)$ the corresponding quotient space. Clearly, $(B(Z^*), w^*) = (B(Y^\perp), w^*)$. But $(B(Y^\perp), w^*)$ is w^*-separable by Proposition 4.4. So, Z is isometric to a subspace of $\ell_\infty(\mathbb{N})$.

(2)⇒(3). By (2), $d_\infty(X, |\cdot|) = 1$ for every equivalent norm $|\cdot|$ on X. So, $\mu(X) = 1$.

(3)⇒(4). Since $\mu(X/Y) \le \mu(X)$ for every quotient X/Y (see [1, Th. III-2]), (3) implies that $\mu(X/Y) = 1$, i.e., $\sigma(X/Y) = 1$. So, by Proposition 4.4 we infer that $X/Y \in \mathrm{KS}_4$.

(4)⇒(1). This is obvious. ■

Corollary 5.3. *If X is either the space $C(K)$, under CH and K being the Kunen compact space, or the space S of Shelah, under* $\diamondsuit_{\aleph_1}$, *then X is nonseparable*, $\mu(X) = 1$ *and every quotient of* $(X, |\cdot|)$ *is isometric to a subspace of* $\ell_\infty(\mathbb{N})$, *for every equivalent norm* $|\cdot|$ *of* X.

Proof. This follows from Proposition 5.2 since in both cases $X \in \mathrm{KS}_4$ (see Section 6). ■

Remarks. (1) The fact that every quotient of $(X, |\cdot|)$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$ for every equivalent norm $|\cdot|$ of X, when $X = C(K)$, K being the Kunen compact, was shown in [4, Cor. 4.5].

(2) In [1] it is asked if $\mu(X) = \infty$ whenever the Banach space X satisfies $\mu(X) > 1$. In fact, no Banach space X with $1 < \mu(X) < \infty$ is known. Observe that $1 < \mu(X) < \infty$ implies that $X \in \mathrm{KS}_3$ but $X \notin \mathrm{KS}_4$, because: (i) $1 < \mu(X) < \infty$ iff $1 > \sigma(X) > 0$ by Proposition 5.1; (ii) $1 > \sigma(X)$ iff $X \notin \mathrm{KS}_4$ by Proposition 4.4; and (iii) $\sigma(X) > 0$ implies $X \in \mathrm{KS}_3$ by Propositions 3.2 and 2.8.

6. The Kunen–Shelah property KS_5**.** Let θ be an ordinal. A *convex right-separated θ-family* in a Banach space X is a bounded family $\{x_i\}_{i<\theta} \subset X$ such that $x_j \notin \overline{\mathrm{co}}(\{x_i : j < i < \theta\})$ for every $j \in \theta$. A family $\{C_\alpha\}_{\alpha<\theta}$ of convex closed bounded subsets in X is said to be a *contractive* (resp. *expansive*) *θ-onion* iff $C_\alpha \subsetneq C_\beta$ (resp. $C_\beta \subsetneq C_\alpha$) whenever $\beta < \alpha < \theta$. It is easy to prove that X has a contractive θ-onion iff X has a convex right-separated θ-family. In the dual Banach space X^* one can define a contractive (resp. expansive) w^*-θ-onion in an analogous way, using the w^*-topology instead of the w-topology.

A Banach space X is said to have the *Kunen–Shelah property* KS_5 if X has no contractive uncountable onion. If X has a τ-polyhedron $\{x_\alpha : \alpha < \tau\}$, it is clear that $\{C_\alpha : \alpha < \tau\}$, where $C_\alpha = \overline{\mathrm{co}}(\{x_\beta : \alpha < \beta < \tau\})$, is a contractive τ-onion. So, the property KS_5 implies KS_4, whence by Proposition 3.2 we get $\mathrm{KS}_5 \Rightarrow \mathrm{KS}_3$, a result proved by Sersouri in [12].

PROPOSITION 6.1. *Let X be a Banach space. Then*:

(1) *X has a contractive ω_1-onion iff X^* has an expansive w^*-ω_1-onion.*
(2) *X has an expansive ω_1-onion iff X^* has a contractive w^*-ω_1-onion.*
(3) *X is nonseparable iff X^* has a contractive w^*-ω_1-onion.*

Proof. (1) Assume that X has a contractive ω_1-onion, i.e., there exists a sequence $\{x_\alpha\}_{\alpha<\omega_1} \subset B(X)$ such that $x_\alpha \notin \overline{\mathrm{co}}(\{x_\beta\}_{\alpha<\beta<\omega_1})$. By the Hahn–Banach Theorem there exists $f_\alpha \in X^*$ such that

$$f_\alpha(x_\alpha) > \sup\{f_\alpha(x_\beta) : \alpha < \beta < \omega_1\} =: e_\alpha.$$

By passing to a subsequence, we can suppose that there exist $0 < \varepsilon, M < \infty$ and $r \in \mathbb{R}$ such that $\|f_\alpha\| \le M$, $f_\alpha(x_\alpha) - e_\alpha \ge \varepsilon > 0$ and $|r - f_\alpha(x_\alpha)| \le \varepsilon/4$ for all $\alpha < \omega_1$. Hence, if $\beta < \alpha < \omega_1$, we have

$$f_\alpha(x_\alpha) \ge r - \varepsilon/4 > r - 3\varepsilon/4 \ge f_\beta(x_\beta) - \varepsilon \ge e_\beta \ge f_\beta(x_\alpha),$$

which implies that $f_\alpha \notin \overline{\mathrm{co}}^{w^*}(\{f_\beta : \beta < \alpha\}) =: K_\alpha$, i.e., $\{K_\alpha : \alpha < \omega_1\}$ is an expansive w^*-ω_1-onion in X^*.

The converse implication is analogous.

(2) Use the same argument as in (1).

(3) Apply (2) and the fact that X has an expansive ω_1-onion iff X is nonseparable. ∎

A Banach space has the *property* HL(1) (for short, $X \in \mathrm{HL}(1)$) whenever for every family $\{U_i\}_{i\in I}$ of open semi-spaces of X there exists a countable subset $\{i_n\}_{n\ge1} \subset I$ such that $\bigcup_{n\ge1} U_{i_n} = \bigcup_{i\in I} U_i$, i.e., every closed convex subset of X is the intersection of a countable family of closed semi-spaces of X.

Proposition 6.2. *Let X be a Banach space. Then the following are equivalent*:

(1) $X \in \mathrm{KS}_5$.
(2) *Every convex subset of* X^* *is* w^**-separable.*
(3) $X \in \mathrm{HL}(1)$.

Proof. (1)⇔(2). By Proposition 6.1, X has no contractive uncountable onion iff X^* has no expansive uncountable w^*-onion, and it is trivial to prove that this occurs iff every convex subset of X^* is w^*-separable.

(2)⇒(3). Suppose that $X \notin \mathrm{HL}(1)$ and let $\mathfrak{F} = \{U_i\}_{i<\omega_1}$ be an uncountable family of open semi-spaces of X such that $\mathfrak{F}$ has no countable subcover. Assume that $U_i = \{x \in X : x_i^*(x) < a_i\}$ with $a_i \neq 0$ for all $i < \omega_1$ (if $a_i = 0$ for some $i < \omega_1$, we replace U_i by the family $U_{in} = \{x \in X : x_i^*(x) < -1/n\}$, $n \geq 1$). Dividing by $|a_i|$, we can suppose that each U_i has the expression $U_i = \{x \in X : y_i^*(x) < \varepsilon_i\}$ with $\varepsilon_i = \pm 1$ and $y_i^* = x_i^*/|a_i|$. Set $\mathfrak{F}_1 = \{U_i \in \mathfrak{F} : \varepsilon_i = +1\}$ and $\mathfrak{F}_2 = \{U_i \in \mathfrak{F} : \varepsilon_i = -1\}$. It is clear that either $\mathfrak{F}_1$ or $\mathfrak{F}_2$ has no countable subcover.

Assume that $\mathfrak{F}_1$ does not admit a countable subcover (the argument for $\mathfrak{F}_2$ is similar). So, there exists an uncountable family $\{V_\alpha : \alpha < \omega_1\} \subset \mathfrak{F}_1$, $V_\alpha = \{x \in X : z_\alpha^*(x) < 1\}$, such that there exist $x_\alpha \in V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$ for each $\alpha < \omega_1$. Put $A = \mathrm{co}\{z_i^*\}_{i<\omega_1}$, which is w^*-separable by hypothesis. Thus, we can find $\varrho < \omega_1$ such that $A \subset \overline{\mathrm{co}}^{w^*}(\{z_i^*\}_{i\leq\varrho})$. Pick $\varrho < \alpha < \omega_1$. As $x_\alpha \in V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$, we see that $z_\alpha^*(x_\alpha) < 1$ and $z_\beta^*(x_\alpha) \geq 1$ for every $\beta < \alpha$. Let $C = \{x^* \in X^* : x^*(x_\alpha) \geq 1\}$, which is a convex w^*-closed subset of X^*. Since $z_i^* \in C$ for all $i \leq \varrho$, it follows that $A \subset C$. So, $z_\alpha^* \notin C$ and $z_\alpha^* \in A$, a contradiction which proves (3).

(3)⇒(1). Suppose that X has a contractive ω_1-onion $\{C_\alpha\}_{\alpha<\omega_1}$. We choose vectors $x_\alpha \in C_\alpha \setminus C_{\alpha+1}$ and a sequence $\{U_\alpha\}_{\alpha<\omega_1}$ of open semi-spaces such that $x_\alpha \in U_\alpha$ and $U_\alpha \cap C_{\alpha+1} = \emptyset$. Clearly, no countable subfamily of $\{U_\alpha\}_{\alpha<\omega_1}$ covers $\{x_\alpha\}_{\alpha<\omega_1}$, which contradicts (3). ∎

Remark. If X is a Banach space, we write $X \in \mathrm{L}(1)$ if from every cover of X by open semi-spaces we can choose a countable subcover. Clearly, X has the property (C) of Corson iff $X \in \mathrm{L}(1)$. Since $X \in \mathrm{HL}(1) \Rightarrow X \in \mathrm{L}(1)$, we find that $X \in \mathrm{KS}_5$ implies $X \in (\mathrm{C})$.

Proposition 6.3. *If X is either the space $C(K)$, under CH and K being the Kunen compact space*, *or the space S of Shelah*, *under $\diamondsuit_{\aleph_1}$*, *then* $X \in \mathrm{KS}_5$.

Proof. The space $C(K)$, K being the Kunen compact space, satisfies $C(K) \in \mathrm{KS}_5$ because for every uncountable family $\{x_i : i \in I\} \subset C(K)$, there exists $j \in I$ such that $x_j \in \mathrm{wcl}(\{x_i : i \in I\setminus\{j\}\})$ (wcl = weak closure). It is clear that a space with this property cannot have an ω_1-onion.

The space S of Shelah has the property (see [13, Lemma 5.2]) that if $\{y_i\}_{i<\omega_1} \subset S$ is an uncountable sequence, then for every $\varepsilon > 0$ and $n \geq 1$, there exist $i_0 < i_1 < \ldots < i_n < \omega_1$ such that

$$\left\| y_{i_0} - \frac{1}{n}(y_{i_1} + \ldots + y_{i_n}) \right\| \leq \frac{1}{n} \|y_{i_0}\| + \varepsilon. \tag{7}$$

Assume that S has an ω_1-onion $\{C_\alpha : 1 \leq \alpha < \omega_1\}$, with $C_1 \subset B(S)$. Choose $x_\alpha \in C_\alpha \setminus C_{\alpha+1}$ and let $\eta_\alpha := \operatorname{dist}(x_\alpha, C_{\alpha+1})$, which satisfies $\eta_\alpha > 0$. By passing to a subsequence, it can be assumed that $\eta_\alpha \geq \eta > 0$ for all $\alpha < \omega_1$. Let $m \in \mathbb{N}$ satisfy $1/m < \eta/2$. By (7) there exist $i_0 < i_1 < \ldots < i_m < \omega_1$ such that

$$\left\| x_{i_0} - \frac{1}{m}(x_{i_1} + \ldots + x_{i_m}) \right\| \leq \frac{1}{m} \|x_{i_0}\| + \frac{\eta}{2} < \eta.$$

Since $\frac{1}{m}(x_{i_1} + \ldots + x_{i_m}) \in C_{i_0+1}$ and $\operatorname{dist}(x_{i_0}, C_{i_0+1}) \geq \eta$, we get a contradiction which proves that $S \in \mathrm{KS}_5$. ■

7. KS_4 and KS_5 are equivalent. If X is Asplund or has the property (C) of Corson, it is easy to prove that $X \in \mathrm{KS}_4 \Leftrightarrow X \in \mathrm{KS}_5$. In the following we prove the equivalence $\mathrm{KS}_5 \Leftrightarrow \mathrm{KS}_4$ in general. A sequence $\{C_\alpha : \alpha < \omega_1\}$ of convex closed bounded subsets of a Banach space X is said to be a *generalized ω_1-onion* if $\emptyset \neq C_\alpha \subset C_\beta$ for $\beta < \alpha$, and there exists a subsequence $\{\alpha_\beta\}_{\beta<\omega_1} \subset \omega_1$, with $\alpha_{\beta_1} < \alpha_{\beta_2}$ if $\beta_1 < \beta_2$, such that $C_{\alpha_{\beta_1}} \neq C_{\alpha_{\beta_2}}$, i.e., $\{C_{\alpha_\beta} : \beta < \omega_1\}$ is an ω_1-onion. For $C \subset X$, denote by $\operatorname{cone}(C)$ the closed convex cone generated by C. Observe that if C is convex, then $\operatorname{cone}(C) = \operatorname{cl}(\bigcup_{\lambda\geq 0} \lambda C)$.

LEMMA 7.1. *Let X be a Banach space, $C \subset X$ a convex closed separable subset and $\{C_\alpha : 1 \leq \alpha < \omega_1\}$ a generalized ω_1-onion in X.*

(1) *If $\operatorname{dist}(C, C_\alpha) = 0$ for every $\alpha < \omega_1$, then for every $\varepsilon > 0$ there exists $c_\varepsilon \in C$ such that $\operatorname{dist}(c_\varepsilon, C_\alpha) \leq \varepsilon$ for every $\alpha < \omega_1$.*

(2) *There are two mutually exclusive alternatives: either*

- (A) *there exist two ordinals $\beta < \alpha < \omega_1$ and $z \in C_\beta$ such that $z \notin \overline{\mathrm{co}}([C] \cup \operatorname{cone}(C_\alpha))$ or*
- (B) *for every pair of ordinals $\beta < \alpha < \omega_1$ we have $C_\beta \subset \overline{\mathrm{co}}([C] \cup \operatorname{cone}(C_\alpha))$. In this case,*

$$\overline{\mathrm{co}}([C] \cup \operatorname{cone}(C_\alpha)) = \overline{\mathrm{co}}([C] \cup \operatorname{cone}(C_\beta)), \qquad \forall \alpha, \beta < \omega_1,$$

and for every $\varepsilon > 0$ there exists $c_\varepsilon \in X$ such that $\operatorname{dist}(c_\varepsilon, C_\alpha) \leq \varepsilon$ for every $\alpha < \omega_1$.

Proof. (1) For every $\alpha < \omega_1$ and $n \geq 1$ consider $C(\alpha, n) = \{x \in C : \operatorname{dist}(x, C_\alpha) \leq 1/n\}$. Then $\{C(\alpha, n) : \alpha < \omega_1\}$ is a family of nonempty closed convex subsets such that $C(\alpha, n) \supset C(\beta, n)$ if $\alpha < \beta$, with the countable intersection property. Since C is separable, we conclude that $\bigcap_{\alpha<\omega_1} C(\alpha, n) \neq \emptyset$ for every $n \geq 1$. So, if for every $n \geq 1$ we pick $c_n \in \bigcap_{\alpha<\omega_1} C(\alpha, n)$, then $\operatorname{dist}(c_n, C_\alpha) \leq 1/n$ for every $\alpha < \omega_1$.

(2) Clearly, the alternatives (A) and (B) are mutually exclusive. Suppose that (B) holds. Since $[C]$ is separable there exist two ordinals $\beta_0 < \alpha_0 < \omega_1$ and $z_0 \in C_{\beta_0} \setminus C_{\alpha_0}$ such that $z_0 \notin \overline{[C]}$ but $z_0 \in \overline{\operatorname{co}}([C] \cup \operatorname{cone}(C_\alpha))$ for every $\alpha < \omega_1$.

Claim. *If* $H = \overline{[C \cup \{z_0\}]}$, *then* $\operatorname{dist}(H, C_\alpha) = 0$ *for every* $\alpha < \omega_1$.

Indeed, let $\varepsilon_0 = \operatorname{dist}(z_0, \overline{[C]})$ and $n_0 \geq 1$ be such that $2/n_0 < \varepsilon_0$. Observe that for every $\alpha < \omega_1$ and $\varepsilon > 0$ we can choose $\lambda \in [0, 1)$, $\mu > 0$, $w \in [C]$ and $v \in C_\alpha$ such that

$$\|\lambda w + (1-\lambda)\mu v - z_0\| \leq \varepsilon. \tag{8}$$

Let $M > 0$ be such that $C_1 \subset B(0, M)$. We claim that if we pick $\alpha < \omega_1$, $n \geq n_0$, $\lambda \in [0, 1)$, $\mu > 0$, $w \in [C]$ and $v \in C_\alpha$ satisfying (8) with $\varepsilon = 1/n$, then $(1-\lambda)\mu \geq 1/(n_0 M)$. Indeed, otherwise

$$\begin{aligned}\varepsilon_0 &\leq \|\lambda w - z_0\| = \|\lambda w + (1-\lambda)\mu v - z_0 - (1-\lambda)\mu v\| \\ &\leq \|\lambda w + (1-\lambda)\mu v - z_0\| + \|(1-\lambda)\mu v\| \\ &\leq \frac{1}{n_0} + \frac{1}{n_0} < \varepsilon_0,\end{aligned}$$

which is a contradiction. So, for every α, n, λ, μ, w and v as above we have

$$\left\| \frac{z_0}{(1-\lambda)\mu} - \frac{\lambda}{(1-\lambda)\mu} w - v \right\| \leq \frac{1}{(1-\lambda)\mu n} \leq \frac{n_0 M}{n},$$

and this proves that $\operatorname{dist}(H, C_\alpha) = 0$ for every $\alpha < \omega_1$.

As H is separable, given $\varepsilon > 0$, applying (1) we can choose $c_\varepsilon \in X$ such that $\operatorname{dist}(c_\varepsilon, C_\alpha) \leq \varepsilon$ for every $\alpha < \omega_1$, and this completes the proof. ■

Proposition 7.2. *Let X be a Banach space without the property* (C) *of Corson. Then there exists a sequence $\{(y_\alpha, y_\alpha^*) : \alpha < \omega_1\} \subset X \times X^*$ such that $y_\alpha^*(y_\alpha) = 1$ for all $\alpha < \omega_1$ but $y_\alpha^*(y_\beta) = 0$ if $\beta < \alpha$, and $y_\alpha^*(y_\beta) \leq 0$ if $\beta > \alpha$. So, X has an ω_1-polyhedron and $X \notin \mathrm{KS}_4$.*

Proof. Since X fails (C), it is easy to see that there exists in X an ω_1-onion $\{C_\alpha : \alpha < \omega_1\}$ such that $\bigcap_{\alpha<\omega_1} C_\alpha = \emptyset$. Using transfinite induction with ω_1 steps we construct:

(1) A sequence $\{n_\alpha : \alpha < \omega_1\} \subset \{0,1\}$ such that if $p(\alpha) = |\{\beta \le \alpha : n_\beta = 1\}|$ then $p(\alpha) < \aleph_0$.

(2) Two sequences $\{\varrho_\gamma, \tau_\gamma : \gamma < \omega_1\}$ of ordinals such that $1 \le \varrho_\gamma < \tau_\gamma \le \varrho_\beta < \omega_1$ if $\gamma < \beta < \omega_1$.

(3) For each $\alpha < \omega_1$, a generalized ω_1-onion $\{C_\beta^{(\alpha)} : \varrho_\alpha \le \beta < \omega_1\}$ such that $C_\gamma \supset C_\gamma^{(\alpha)} \supset C_\gamma^{(\beta)} \neq \emptyset$ if $\alpha \le \beta < \omega_1$ and $\varrho_\beta \le \gamma < \omega_1$.

(4) For each α with $n_\alpha = 0$, an element $y_\alpha \in C_{\varrho_\alpha}^{(\alpha)}$ such that if $H_\alpha = \overline{[\{y_\beta : \beta < \alpha, n_\beta = 0\}]}$ then $y_\alpha \notin \overline{\mathrm{co}}(H_\alpha \cup \mathrm{cone}(C_{\tau_\alpha}^{(\alpha)}))$. Also, in this case we demand that $C_\gamma^{(\alpha)} = \bigcap_{\beta<\alpha} C_\gamma^{(\beta)}$ for every $\varrho_\alpha \le \gamma < \omega_1$.

(5) For each α with $n_\alpha = 1$, a vector $a_{p(\alpha)} \in X$ such that $C_\beta^{(\alpha)} \subset B(a_{p(\alpha)}, 2^{-p(\alpha)})$ for every $\tau_\alpha \le \beta < \omega_1$, which will imply that

$$\mathrm{diam}(C_\beta^{(\alpha)}) \le 2^{-p(\alpha)+1}, \quad \mathrm{dist}(a_{p(\alpha)}, C_\beta^{(\alpha)}) \le 2^{-p(\alpha)}, \quad \forall \tau_\alpha \le \beta < \omega_1.$$

Step 1. We choose $n_1 = 0$, $\varrho_1 = 1$, $\tau_1 = 2$, $C_\beta^{(1)} = C_\beta$ for every $1 \le \beta < \omega_1$, $y_1 \in C_1 \setminus C_2$ arbitrary and $H_1 = \{0\}$.

Step $\alpha + 1 < \omega_1$. Suppose all the steps $\beta \le \alpha$ satisfying the above requirements are constructed. By hypothesis $\{C_\beta^{(\alpha)} : \tau_\alpha \le \beta < \omega_1\}$ is a generalized ω_1-onion. By Lemma 7.1 there are two mutually exclusive alternatives:

(A) There exist two ordinals $\tau_\alpha \le \beta_0 < \alpha_0 < \omega_1$ and a vector $z_0 \in C_{\beta_0}^{(\alpha)}$ such that $z_0 \notin \overline{\mathrm{co}}(H_\alpha \cup \mathrm{cone}(C_{\alpha_0}^{(\alpha)}))$. Then we set $\varrho_{\alpha+1} = \beta_0$, $\tau_{\alpha+1} = \alpha_0$, $n_{\alpha+1} = 0$, $y_{\alpha+1} = z_0$ and $C_\beta^{(\alpha+1)} = C_\beta^{(\alpha)}$ for every $\varrho_{\alpha+1} \le \beta < \omega_1$.

(B) If (A) does not hold, there exists $c \in X$ such that $\mathrm{dist}(c, C_\beta^{(\alpha)}) \le 2^{-(p(\alpha)+2)}$ for every $\tau_\alpha \le \beta < \omega_1$. In this case we set $n_{\alpha+1} = 1$, $p(\alpha+1) = p(\alpha) + 1$, $\varrho_{\alpha+1} = \tau_\alpha$, $\tau_{\alpha+1} = \tau_\alpha + 1$, $a_{p(\alpha+1)} = c$ and $C_\beta^{(\alpha+1)} = B(a_{p(\alpha+1)}, 2^{-p(\alpha+1)}) \cap C_\beta^{(\alpha)}$ for every $\varrho_{\alpha+1} \le \beta < \omega_1$. Since $n_{\alpha+1} = 1$ we do not choose $y_{\alpha+1}$.

Step $\alpha < \omega_1$, *α a limit ordinal.* Let $\alpha < \omega_1$ be a limit ordinal, and suppose all the steps $\beta < \alpha$ satisfying the above requirements are constructed.

Claim. $|\{\beta < \alpha : n_\beta = 1\}| < \aleph_0$.

Indeed, otherwise we would have a sequence of ordinals $\{\beta_m\}_{m\ge1} \uparrow \alpha$, with $\beta_m < \beta_{m+1} < \alpha$, such that $n_{\beta_m} = 1$ for every $m \ge 1$. Obviously $p(\beta_m) \uparrow +\infty$ as $m \to \infty$. The sequence $\{a_{p(\beta_m)}\}_{m\ge1}$ is a Cauchy sequence. Indeed, if $r < s$ are two integers, then for every $\tau_{\beta_s} \le \beta < \omega_1$, since $C_\beta^{(\beta_s)} \subset C_\beta^{(\beta_r)}$, we have

$$\operatorname{dist}(a_{p(\beta_r)}, a_{p(\beta_s)}) \le \operatorname{dist}(a_{p(\beta_r)}, C_\beta^{(\beta_r)}) + \operatorname{diam}(C_\beta^{(\beta_r)}) + \operatorname{dist}(a_{p(\beta_s)}, C_\beta^{(\beta_r)})$$
$$\le 2^{-p(\beta_r)} + 2^{-p(\beta_r)+1} + 2^{-p(\beta_s)} \xrightarrow{r,s\to\infty} 0.$$

Let $a_0 := \lim_{m\to\infty} a_{p(\beta_m)}$ and $\gamma_0 = \sup\{\tau_\beta : \beta < \alpha\}$. Then $a_0 \in C_\gamma$ for every $\gamma_0 \le \gamma < \omega_1$ because

$$\operatorname{dist}(a_0, C_\gamma) \le \operatorname{dist}(a_0, a_{p(\beta_m)}) + \operatorname{dist}(a_{p(\beta_m)}, C_\gamma^{(\beta_m)}) \xrightarrow{m\to\infty} 0.$$

Hence $\bigcap_{\alpha<\omega_1} C_\alpha \ne \emptyset$, a contradiction which proves the Claim.

Define as above $\gamma_0 = \sup\{\tau_\beta : \beta < \alpha\}$ and let $D_\gamma := \bigcap_{\beta<\alpha} C_\gamma^{(\beta)}$ for every $\gamma_0 \le \gamma < \omega_1$. By the Claim and the construction of the previous steps we have:

(a) There exists an ordinal $\delta_0 < \alpha$ such that $n_\delta = 0$ for every $\delta_0 \le \delta < \alpha$. So, $p(\delta) = p(\delta_0)$ for every $\delta \in [\delta_0, \alpha)$.

(b) For every $\gamma_0 \le \gamma < \omega_1$ we have $D_\gamma = C_\gamma^{(\delta_0)}$, which by the induction hypothesis implies that $\{D_\gamma : \gamma_0 \le \gamma < \omega_1\}$ is a generalized ω_1-onion.

If $H_\alpha := \overline{[\{y_\beta : \beta < \alpha,\ n_\beta = 0\}]}$, by Lemma 7.1 we have the following mutually exclusive alternatives:

(A) There are two ordinals $\gamma_0 \le \beta_0 < \alpha_0 < \omega_1$ and a vector $z_0 \in D_{\beta_0}$ such that $z_0 \notin \overline{\operatorname{co}}(H_\alpha \cup \operatorname{cone}(D_{\alpha_0}))$. In this case we set $\varrho_\alpha = \beta_0$, $\tau_\alpha = \alpha_0$, $n_\alpha = 0$, $y_\alpha = z_0$ and $C_\beta^{(\alpha)} = D_\beta$ for every $\varrho_\alpha \le \beta < \omega_1$.

(B) If (A) does not hold, there exists $c \in X$ such that $\operatorname{dist}(c, D_\gamma) \le 2^{-p(\delta_0)+2}$ for every $\gamma_0 \le \gamma < \omega_1$. In this case we set $n_\alpha = 1$, $p(\alpha) = p(\delta_0) + 1$, $\varrho_\alpha = \gamma_0$, $\tau_\alpha = \varrho_\alpha + 1$, $a_{p(\alpha)} = c$ and $C_\gamma^{(\alpha)} = B(a_{p(\alpha)}, 2^{-p(\alpha)}) \cap D_\gamma$ for $\gamma_0 \le \gamma < \omega_1$. Since $n_\alpha = 1$ we do not choose y_α.

This completes the induction.

Obviously, there exists $\varrho < \omega_1$ such that $n_\alpha = 0$ for every $\varrho \le \alpha < \omega_1$, which gives us the sequence $\{y_\alpha : \varrho \le \alpha < \omega_1\}$ such that

$$y_\alpha \notin \overline{\operatorname{co}}(\overline{[\{y_\beta : \varrho \le \beta < \alpha\}]} \cup \operatorname{cone}(\{y_\beta : \alpha < \beta < \omega_1\})) =: K_\alpha$$

for every $\varrho \le \alpha < \omega_1$. Therefore, by the Hahn–Banach Theorem there exists $y_\alpha^* \in X^*$ such that $y_\alpha^*(y_\alpha) = 1$ but $\sup\{y_\alpha^*(y) : y \in K_\alpha\} < 1$. In particular, $y_\alpha^*(y_\beta) = 0$ if $\varrho \le \beta < \alpha$, and $y_\alpha^*(y_\beta) \le 0$ if $\alpha < \beta < \omega_1$. ■

PROPOSITION 7.3. *Let X be a Banach space. We have*:

(1) *If $X \in \mathrm{KS}_4$, then $X \in$ (C).*
(2) *$X \in \mathrm{KS}_4$ iff $X \in \mathrm{KS}_5$.*

Proof. (1) This follows from Proposition 7.2 where it is proved that if $X \notin$ (C) then X has an ω_1-polyhedron.

(2) Clearly, $X \in \mathrm{KS}_5$ implies $X \in \mathrm{KS}_4$. Assume that $X \in \mathrm{KS}_4$. By (1) we see that $X \in (\mathrm{C})$. In order to prove that $X \in \mathrm{KS}_5$, by Proposition 6.2 it is enough to prove that every convex subset $C \subset X^*$ is w^*-separable. Since $X \in \mathrm{KS}_4$, $\overline{C}^{w^*}$ is w^*-separable by Proposition 4.4. So, there exists a countable family $\{z_n : n \geq 1\} \subset \overline{C}^{w^*}$ w^*-dense in $\overline{C}^{w^*}$. Since $X \in (\mathrm{C})$, by [10, p. 147] there exists a countable family $\{z_{nm} : n, m \geq 1\} \subset C$ such that $z_n \in \overline{\mathrm{co}}^{w^*}(\{z_{nm} : m \geq 1\})$ for every $n \geq 1$. So, C is w^*-separable. ∎

Remarks. A nonseparable Banach space X has the *Kunen–Shelah property* KS_6 if for every uncountable family $\{x_i\}_{i \in I} \subset X$ there exists $j \in I$ such that $x_j \in \mathrm{wcl}(\{x_i\}_{i \in I \setminus \{j\}})$. Clearly, $\mathrm{KS}_6 \Rightarrow \mathrm{KS}_5$. It seems that the only known example of a Banach space X such that $X \in \mathrm{KS}_6$ is the space $X = C(K)$, K being the Kunen compact space ([8, p. 1123]) constructed by Kunen under CH. This space $C(K)$ of Kunen has more interesting pathological properties. For example, $((C(K))^n, w^n)$ is hereditarily Lindelöf for every $n \in \mathbb{N}$.

In view of this situation, we can introduce the property KS_7. A Banach space X is said to have the *Kunen–Shelah property* KS_7 if (X^n, w^n) is hereditarily Lindelöf for every $n \in \mathbb{N}$. It can be easily proved that $\mathrm{KS}_7 \Rightarrow \mathrm{KS}_6$. We know neither if the Shelah space S has the property KS_6 nor if the properties KS_5, KS_6 and KS_7 are inequivalent.

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Departamento Análisis Matemático
Facultad de Matemáticas
Universidad Complutense
Madrid 28040, Spain
E-mail: granero@mat.ucm.es
marjim@mat.ucm.es
montesinos@mat.ucm.es

Department of Mathematics
Pedagogical University
Shevchenko st. 1
Kirovograd 2506, Ukraine
E-mail: aplichko@kspu.kr.ua

Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid
Madrid 28049, Spain
E-mail: josepedro.moreno@uam.es

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