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On asymptotically symmetric Banach spaces

by

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Dedicated to Haskell Rosenthal on the occasion of his 65th birthday

Abstract. A Banach space X is *asymptotically symmetric* (a.s.) if for some $C < \infty$, for all $m \in \mathbb{N}$, for all bounded sequences $(x_j^i)_{j=1}^\infty \subseteq X$, $1 \le i \le m$, for all permutations σ of $\{1, \ldots, m\}$ and all ultrafilters $\mathcal{U}_1, \ldots, \mathcal{U}_m$ on $\mathbb{N}$,

$$\lim_{n_1,\mathcal{U}_1} \cdots \lim_{n_m,\mathcal{U}_m} \Big\| \sum_{i=1}^m x_{n_i}^i \Big\| \le C \lim_{n_{\sigma(1)},\mathcal{U}_{\sigma(1)}} \cdots \lim_{n_{\sigma(m)},\mathcal{U}_{\sigma(m)}} \Big\| \sum_{i=1}^m x_{n_i}^i \Big\|.$$

We investigate a.s. Banach spaces and several natural variations. X is weakly a.s. (w.a.s.) if the defining condition holds when restricted to weakly convergent sequences $(x_j^i)_{j=1}^\infty$. Moreover, X is w.n.a.s. if we restrict the condition further to normalized weakly null sequences.

If X is a.s. then all spreading models of X are uniformly symmetric. We show that the converse fails. We also show that w.a.s. and w.n.a.s. are not equivalent properties and that Schlumprecht's space S fails to be w.n.a.s. We show that if X is separable and has the property that every normalized weakly null sequence in X has a subsequence equivalent to the unit vector basis of c_0 then X is w.a.s. We obtain an analogous result if c_0 is replaced by ℓ_1 and also show it is false if c_0 is replaced by ℓ_p, $1 < p < \infty$.

We prove that if $1 \le p < \infty$ and $\| \sum_{i=1}^n x_i \| \sim n^{1/p}$ for all $(x_i)_{i=1}^n \in \{X\}_n$, the nth asymptotic structure of X, then X contains an asymptotic ℓ_p, hence w.a.s. subspace.

0. Introduction. In their fundamental paper [KM] J.-L. Krivine and B. Maurey introduced the notion of a stable Banach space and proved that such spaces must contain almost isometric copies of ℓ_p for some $1 \le p < \infty$. A space X is *stable* if for all ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ on $\mathbb{N}$ and all bounded sequences (x_n) and (y_n) in X,

$$\lim_{n,\mathcal{U}_1} \lim_{m,\mathcal{U}_2} \|x_n + y_m\| = \lim_{m,\mathcal{U}_2} \lim_{n,\mathcal{U}_1} \|x_n + y_m\|. \tag{0.1}$$

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A major application of [KM] was to deduce Aldous' theorem [A] that every infinite-dimensional subspace of L_p $(1 \le p < 2)$ contains almost isometric copies of ℓ_r for some $p \le r \le 2$, by proving that L_p is stable. The starting point of our investigation is a result on noncommutative L_p spaces. We first formulate the problem which led to that research.

PROBLEM 0.1. Let $1 \le p \le 2$ and let X be an infinite-dimensional subspace of $L_p(N)$, the noncommutative L_p space associated with a von Neumann algebra N. Does X contain an isomorph of ℓ_r for some $p \le r \le 2$?

In general [M-N] noncommutative L_p spaces fail to be stable. However, if $1 < p < \infty$ these spaces $L_p(N)$ satisfy the following inequality for some universal $C < \infty$ (see [JR]). For all m, all permutations σ of $\{1, \ldots, m\}$, all bounded sequences $(x_n^i)_{n=1}^{\infty}$ in $L_p(N)$ for $i \le m$, and all ultrafilters $\mathcal{U}_1, \ldots, \mathcal{U}_m$ on $\mathbb{N}$,

$$\lim_{n_1, \mathcal{U}_1} \ldots \lim_{n_m, \mathcal{U}_m} \Big\| \sum_{i=1}^{m} x_{n_i}^i \Big\| \le C \lim_{n_{\sigma(1)}, \mathcal{U}_{\sigma(1)}} \ldots \lim_{n_{\sigma(m)}, \mathcal{U}_{\sigma(m)}} \Big\| \sum_{i=1}^{m} x_{n_i}^i \Big\|. \tag{0.2}$$

Every stable space X satisfies (0.2) with $C = 1$ and if Y is isomorphic to a stable space, then Y satisfies (0.2) for some C.

X is called *asymptotically symmetric* (a.s.) if it satisfies (0.2) for some $C < \infty$ where the sequences $(x_n^i) \subseteq X$ (see [JR]). It is easy to check that the Tsirelson space T is a.s. and so a.s. need not imply that a space contains an isomorph of some ℓ_p or c_0. But the following problem remains open.

PROBLEM 0.2. Let X be an infinite-dimensional asymptotically symmetric Banach space. Does X contain an asymptotic ℓ_p subspace for some p?

A Banach space X with a basis (e_i) is called *asymptotically* ℓ_p (see [MT]) if for some $C < \infty$, for all n, every normalized block basis $(x_i)_{i=1}^n$ of $(e_i)_{i=n}^{\infty}$ is C-equivalent to the unit vector basis of ℓ_p^n. In this case (e_i) is called an *asymptotic* ℓ_p *basis* for X.

If X is a.s. then every spreading model $(\widetilde{x}_i)$ of a normalized basic sequence (x_i) in X is C-symmetric for some fixed $C < \infty$. As we shall see in §2, a.s. says much more than this.

THEOREM 0.3. *There is a reflexive Banach space X such that all spreading models of X are C-symmetric for some fixed C yet X is not asymptotically symmetric.*

In fact given $1 < p < \infty$ we can construct the space X in Theorem 0.3 to have the property that every normalized weakly null sequence in X has a subsequence 4-equivalent to the unit vector basis of ℓ_p. We show that if ℓ_p is replaced by c_0 (or ℓ_1 under an appropriate restatement) then one has positive results.

In §3 we prove

THEOREM 0.4. *There exists a reflexive space Y which is not asymptotically symmetric and yet for some $C < \infty$, (0.2) holds for all normalized weakly null sequences $(x_n^i)_{n=1}^\infty$, $i \le m$, in Y.*

Y is the space $Ti(2;1/2)$, a subsymmetric version of a space invented by L. Tzafriri [Tz], as presented in [CS]. We show that Y contains an asymptotic ℓ_2 subspace and hence is not minimal. More generally, we prove that if X satisfies $K^{-1}n^{1/p} \le \|\sum_{i=1}^n x_i\| \le Kn^{1/p}$ for some $1 \le p < \infty$, $K < \infty$, all $n \in \mathbb{N}$ and all $(x_i)_{i=1}^n \in \{X\}_n$, then X contains an asymptotic ℓ_p, hence w.a.s., subspace. $\{X\}_n$ is the nth asymptotic structure of a space X, defined in [MMT].

In §4 we show that Tsirelson's space T is not iteration stable. This notion, due to H. Rosenthal, is another weakening of the definition of stability.

§1 contains the definitions of certain variants of a.s. and the relations between them as well as certain preliminaries. The authors wish to thank H. Rosenthal for many enlightening discussions.

1. Preliminaries. The definition of an asymptotically symmetric Banach space X can also be formulated in this way: X is *asymptotically symmetric* if for some $C < \infty$, for all $m \in \mathbb{N}$, for all bounded sequences $(x_n^i)_{n=1}^\infty \subseteq X$, $1 \le i \le m$, and for all permutations σ of $\{1, \ldots, m\}$,

$$\lim_{n_1\to\infty} \cdots \lim_{n_m\to\infty} \Big\| \sum_{i=1}^m x_{n_i}^i \Big\| \le C \lim_{n_{\sigma(1)}\to\infty} \cdots \lim_{n_{\sigma(m)}\to\infty} \Big\| \sum_{i=1}^m x_{n_i}^i \Big\|, \tag{1.1}$$

provided that these iterated limits exist.

Just as stability was weakened to "weak stability" to handle the case of spaces like c_0 (cf. [ANZ]), it is useful to consider certain variants of a.s. We will say that X is *weakly asymptotically symmetric* (w.a.s.) if (1.1) holds when restricted to sequences $(x_n^i)_{n=1}^\infty \subseteq X$ which are weakly null. This is actually equivalent to restricting to weakly convergent sequences, although the constant C could vary. One could restrict (1.1) to normalized sequences in X. However, we show in Proposition 1.3 that this is the same as a.s. Further, X is said to be *weakly null normalized asymptotically symmetric* (w.n.a.s.) if (1.1) holds for all normalized weakly null sequences $(x_n^i)_{n=1}^\infty \subseteq X$. In §3 we show that w.a.s. and w.n.a.s. are not equivalent properties.

If X has a basis (e_i) then X is *block asymptotically symmetric* (b.a.s.) with respect to (e_i) if (1.1) holds for all bounded block bases $(x_n^i)_{n=1}^\infty$, $i \le m$, of (e_i).

Recall that $(\widetilde{x}_i)$ is a *spreading model* of a normalized basic sequence (x_i) if for all $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ so that for all $n \le i_1 < \cdots < i_k$

and scalars $(a_i)_{i=1}^k$ with each $|a_i| \le 1$,

$$\Big| \Big\| \sum_{i=1}^k a_i x_{n_i} \Big\| - \Big\| \sum_{i=1}^k a_i \widetilde{x}_i \Big\| \Big| < \varepsilon. \tag{1.2}$$

Using Ramsey's theorem it is easy to show that every normalized basic sequence admits a subsequence (x_i) generating a spreading model (for more on spreading models see [BL]).

If $(\widetilde{x}_i)$ is a spreading model of (x_i) then $(\widetilde{x}_i)$ is 1-*subsymmetric*, i.e.,

$$\Big\| \sum_{i=1}^n a_i \widetilde{x}_i \Big\| = \Big\| \sum_{i=1}^n a_i \widetilde{x}_{k_i} \Big\|$$

for all $(a_i)_{i=1}^n$ and $k_1 < \cdots < k_n$. (Some authors call this 1-*spreading* and reserve the notion of subsymmetric for unconditional spreading sequences.) The fact that if X is a.s. with constant C then every spreading model $(\widetilde{x}_i)$ of X generated by (x_i) is C-*symmetric*, i.e., $\|\sum_{i=1}^m a_i \widetilde{x}_i\| \le C \|\sum_{i=1}^m a_i \widetilde{x}_{\sigma(i)}\|$ for all permutations σ of $\mathbb{N}$, follows easily from (1.1) by setting $x_n^i = a_i x_n$ for all i, n. (Some authors call this C-exchangeable and define the C-symmetric constant via $\|\sum_{i=1}^m \pm a_i \widetilde{x}_i\|$, over all choices of signs. A C-symmetric basis is unconditional, but not necessarily C-unconditional.)

There are several known means of joining the infinite- and finite-dimensional structure of a Banach space in a conceptual manner. In addition to spreading models, we have the theory of asymptotic structure [MMT] (see §3), which, in particular, gives rise to the definition of asymptotic ℓ_p bases presented above. A similar definition can be made for *asymptotically* c_0 *bases*. A third concept is an *asymptotic model* of X (see [HO]). These are generated much like spreading models except that one uses a certain infinite array of normalized basic sequences in X, $(x_j^i)_{j=1}^\infty$, $i \in \mathbb{N}$, and one replaces $\|\sum_{i=1}^k a_i x_{n_i}\|$ in (1.2) by $\|\sum_{i=1}^k a_i x_{n_i}^i\|$. The definition and variants of being asymptotically symmetric constitute a fourth way to join the finite and infinite geometries of a space. We gather together some of the relationships between these four concepts in our next proposition. For the sake of completeness we include some of the above observations. The "moreover" statement requires two of our main results below.

PROPOSITION 1.1. *Let X be an infinite-dimensional Banach space.*

(a) *If X is a.s. then the spreading models of X are uniformly symmetric. Similarly, if X is b.a.s. with respect to (e_i) then the spreading models of block bases of (e_i) are uniformly symmetric.*
(b) *X is a.s. $\Rightarrow$ (X is b.a.s. if X has a basis) $\Rightarrow$ X is w.a.s. $\Rightarrow$ X is w.n.a.s.*

(c) *If* X *is reflexive, then* X *is a.s.* $\Leftrightarrow$ X *is w.a.s* $\Leftrightarrow$ (X *is b.a.s. if* X *has a basis*).

(d) *If* (e_i) *is a boundedly complete basis for* X *then* X *is a.s.* $\Leftrightarrow$ X *is b.a.s. with respect to* (e_i).

(e) *If* X *is a.s. and not reflexive, then* ℓ_1 *is isomorphic to a spreading model of* X.

(f) *If all asymptotic models of* X *are symmetric then* X *is w.a.s.* (*and b.a.s. if it has a basis*).

(g) *If* X *has an asymptotically* ℓ_p *or* c_0 *basis* (e_i), *then* X *is b.a.s. with respect to* (e_i).

Moreover, all of the converses of the one-sided implications in (a)–(g) *are false, in general.*

Proof. (a)–(c) follow easily from our previous remarks and (d), which holds by standard gliding hump arguments. To see (e) we note that a nonreflexive X contains a non-weakly null normalized basic sequence (x_i) having a spreading model $(\widetilde{x}_i)$, which thus satisfies, for some $\delta > 0$, $\|\sum a_i\widetilde{x}_i\| \geq \delta \sum a_i$ if the a_i's are nonnegative. If X is a.s. then $(\widetilde{x}_i)$ is symmetric, hence unconditional, hence equivalent to the unit vector basis of ℓ_1.

(f) If all asymptotic models of X are symmetric, then it follows easily from Krivine's theorem [K] that for some $C < \infty$, if (x_i) is an asymptotic model of X generated by either a weakly null array or a block basis array (if X has a basis), then (x_i) is C-equivalent to the unit vector basis of c_0 or ℓ_p for some fixed $p \in [1, \infty)$ (independent of (x_i); see [HO, 4.7.4]). Thus X is w.a.s. (or b.a.s.) by much the same easy argument that yields (g).

The converses in (a) are shown to fail in §2. By considering the summing basis for c_0, which is subsymmetric but not symmetric, we see that the second converse in (b) fails. c_0 also provides a counterexample to the first converse since it is not a.s. but it is b.a.s. with respect to the unit vector basis. The third converse is proved false in §3. Every normalized unconditional basic sequence in L_p is equivalent to an asymptotic model of L_p, $1 < p < \infty$ (see [HO]), and thus the converses in (f) also fail. L_p also provides a converse to (g). ∎

One can also state variations of Problem 0.2 using these alternate asymptotic notions.

PROBLEM 1.2. Assume that for some $C < \infty$ and $1 \leq p < \infty$, all spreading models of X (or even all asymptotic models of X) are C-equivalent to the unit vector basis of ℓ_p. Does X contain an asymptotic ℓ_p basic sequence?

Replacing normalized weakly null sequences by seminormalized weakly null sequences does not lead to a new class of spaces:

Proposition 1.3. *Let X be w.n.a.s. with constant C. Then X is semi-normalized weakly null asymptotically symmetric. More precisely, let $0 < a < b$. Then there exists a constant $0 < C(a,b) < \infty$ such that for all m and all weakly null sequences (y_n^i), $i = 1,\dots,m$, with $\|y_n^i\| \in [a,b]$, $n \in \mathbb{N}$, $1 \le i \le m$, the inequality*

$$\lim_{n_1\to\infty}\cdots\lim_{n_m\to\infty}\Big\|\sum_{i=1}^m y_{n_i}^i\Big\| \le C(a,b)\lim_{n_{\sigma(1)}\to\infty}\cdots\lim_{n_{\sigma(m)}\to\infty}\Big\|\sum_{i=1}^m y_{n_i}^i\Big\| \tag{1.3}$$

holds for any permutation σ of $\{1,\dots,m\}$ provided these limits exist.

Proof. Let $(x_i^n)_{n=1}^\infty$ be weakly null. Let $0 \le \lambda_i \le 1$. We shall show that

$$\lim_{n_1\to\infty}\cdots\lim_{n_m\to\infty}\Big\|\sum_i \lambda_i x_{n_i}^i\Big\| \le \lim_{n_1\to\infty}\cdots\lim_{n_m\to\infty}\Big\|\sum_i x_{n_i}^i\Big\| \tag{1.4}$$

provided both limits exist. Then (1.3) follows immediately. Indeed, we may consider $x_n^i = by_n^i/\|y_n^i\|$. Then two applications of (1.4) yield

$$\begin{aligned}\lim_{n_1\to\infty}\cdots\lim_{n_m\to\infty}\Big\|\sum_{i=1}^m y_{n_i}^i\Big\| &\le \lim_{n_1\to\infty}\cdots\lim_{n_m\to\infty}\Big\|\sum_{i=1}^m x_{n_i}^i\Big\| \\ \le C\lim_{n_{\sigma(1)}\to\infty}\cdots\lim_{n_{\sigma(m)}\to\infty}\Big\|\sum_{i=1}^m x_{n_i}^i\Big\| &= \frac{Cb}{a}\lim_{n_{\sigma(1)}\to\infty}\cdots\lim_{n_{\sigma(m)}\to\infty}\Big\|\sum_{i=1}^m \frac{a}{b}\,x_{n_i}^i\Big\| \\ \le \frac{Cb}{a}\lim_{n_{\sigma(1)}\to\infty}\cdots\lim_{n_{\sigma(m)}\to\infty}\Big\|\sum_{i=1}^m y_{n_i}^i\Big\|.\end{aligned}$$

(If the middle terms do not converge we could use a limit along a free ultrafilter for which (1.4) also holds.)

By an easy extreme point argument it suffices to show (1.4) in the special case

$$\lim_{n_1\to\infty}\cdots\lim_{n_m\to\infty}\Big\|\sum_{i\in A} x_{n_i}^i\Big\| \le \lim_{n_1\to\infty}\cdots\lim_{n_m\to\infty}\Big\|\sum_i x_{n_i}^i\Big\| \tag{1.5}$$

where $A \subset \{1,\dots,m\}$ is an arbitrary subset. For each $(n_1,\dots,n_m)$ we may choose $x^*_{(n_i)_{i\in A}}$ in the unit sphere such that

$$x^*_{(n_i)_{i\in A}}\Big(\sum_{i\in A} x_{n_i}^i\Big) = \Big\|\sum_{i\in A} x_{n_i}^i\Big\|.$$

By passing to a subsequence in n, which does not affect the above limit, we may assume that for $k \notin A$ we have

$$\lim_{n_1\to\infty}\cdots\lim_{n_m\to\infty} x^*_{(n_i)_{i\in A}}(x_{n_k}^k) = 0. \tag{1.6}$$

Indeed, consider $i_1 < k < i_2$ where i_1 and i_2 are maximal and minimal, respectively, with this property (an analogous argument holds if no such i_1

or i_2 exist). Using the weak* compactness of B_{X^*} and Ramsey's theorem we can pass to a subsequence so that

$$y^*_{n_1,\ldots,n_{i_1}} = w^*\text{-}\lim_{n_{i_2}\to\infty}\ldots\lim_{n_m\to\infty} x^*_{(n_i)_{i\in A}}$$

exists. Thus we get $\lim_{n_{i_2}\to\infty}\ldots\lim_{n_m\to\infty} x^*_{(n_i)_{i\in A}}(x^k_{n_k}) = y^*_{n_1,\ldots,n_{i_1}}(x^k_{n_k})$ for all k. Since $(x^k_{n_k})$ is weakly null we obtain (1.6). Therefore we get

$$\lim_{n_1\to\infty}\ldots\lim_{n_m\to\infty}\Big\|\sum_{i\in A} x^i_{n_i}\Big\| = \lim_{n_1\to\infty}\ldots\lim_{n_m\to\infty} x^*_{(n_i)_{i\in A}}\Big(\sum_{i=1}^m x^i_{n_i}\Big)$$
$$\le \lim_{n_1\to\infty}\ldots\lim_{n_m\to\infty}\Big\|\sum_{i=1}^m x^i_{n_i}\Big\|,$$

which is (1.5). ∎

We end this section with a result promised earlier. Recall that X is normalized asymptotically symmetric with constant C if (1.1) holds for all $m\in\mathbb{N}$ and all normalized sequences $(x^i_n)_{n=1}^\infty$, $i\le m$.

PROPOSITION 1.4. *Let X be normalized asymptotically symmetric with constant C. Then X is a.s. with constant $D = D(C)$.*

Proof. Let F be a two-dimensional subspace of X. We can find a normalized basis (e_0,e_1) of F and a subspace Y of X so that $X = F\oplus Y$ and $|a_0|\le\|a_0e_0+a_1e_1+y\|$, $\|f\|\le 2\|f+y\|$, $\|y\|\le 3\|f+y\|$ for all real a_0, a_1 and all $y\in Y$, $f\in F$. This can be easily done by taking (e_0,e_1) to be an Auerbach basis for F and extending the dual functionals by Hahn–Banach.

We first observe that it suffices to show Y is a.s. with constant C. Indeed, let $(x^i_n)_{n=1}^\infty\subseteq B_X$, the unit ball of X, for $i\le m$. Write $x^i_n = f^i_n + y^i_n$. By passing to a subsequence in n we may assume that $f^i_n\to f^i$ as $n\to\infty$ for $i\le m$, and hence that $x^i_n = f^i + y^i_n$ for all n. Furthermore, there exists an absolute constant K such that

$$\lim_{n_1\to\infty}\ldots\lim_{n_m\to\infty}\Big\|\sum_{i=1}^m x^i_{n_i}\Big\| \overset{K}{\sim} \max\Big(\Big\|\sum_{i=1}^m f^i\Big\|, \lim_{n_1\to\infty}\ldots\lim_{n_m\to\infty}\Big\|\sum_{i=1}^m y^i_{n_i}\Big\|\Big)$$

(as usual, we may assume both limits exist). A similar result holds if we change the order of the limits and this verifies the observation.

Now let $(y^i_n)_{n=1}^\infty\subseteq B_Y$ for $i\le m$. For any n and $i\le m$ choose $a^i_n\ge 0$ and $b^i_n\le 0$ with

$$1 = \|a^i_ne_0 + y^i_n\| = \|b^i_ne_0 + y^i_n\|.$$

Note that $|a^i_n|, |b^i_n|\le 1$ for all i and n. Thus, by passing to a subsequence, we may assume $a^i_n\to a^i\ge 0$ and $b^i_n\to b^i\le 0$ as $n\to\infty$. Choose $c^i\in\{a^i,b^i\}$ so that $|\sum_{i=1}^m c^i|\le 1$. For any n and $i\le m$ set $z^i_n = c^ie_0 + y^i_n\in S_X$. Note that if $\sum_{i=1}^m c^i = 0$, then our earlier observations would yield the desired inequality.

Otherwise we choose $c^{m+1} \in \{1,-1\}$ so that $|\sum_{i=1}^{m+1} c^i| > 1$. Set $z_n^{m+1} = c^{m+1}e_0$ for all n. Then there exists an $f \in S_F$ with $\|\sum_{i=1}^{m+1} c^i e_0 + f\| = 1$. Indeed, assume for convenience $\sum_{i=1}^{m} c^i > 1$. We shall parametrize the unit sphere of F. For $0 \le \theta \le \pi$ let $f(\theta)$ be the unit vector in F which has counterclockwise angle θ with respect to the basis vector e_0. Define the continuous function $g(\theta) = \|\sum_{i=1}^{m+1} c^i e_0 + f(\theta)\|$. Our claim follows since $g(0) > 2$ and $g(\pi) = \|\sum_{i=1}^{m} c^i e_0\| \le 1$.

Set $z_n^{m+2} = f$ and $z_n^{m+3} = -(\sum_{i=1}^{m+1} c^i e_0 + f)$ for all n. We have

$$\lim_{n_1\to\infty} \cdots \lim_{n_{m+3}\to\infty} \Big\| \sum_{i=1}^{m+3} z_{n_i}^i \Big\| = \lim_{n_1\to\infty} \cdots \lim_{n_m\to\infty} \Big\| \sum_{i=1}^{m} y_{n_i}^i \Big\|.$$

The analogous result holds for any permutation σ and so

$$\lim_{n_1\to\infty} \cdots \lim_{n_m\to\infty} \Big\| \sum_{i=1}^{m} y_{n_i}^i \Big\| \le C \lim_{n_{\sigma(1)}\to\infty} \cdots \lim_{n_{\sigma(m)}\to\infty} \Big\| \sum_{i=1}^{m} y_{n_i}^i \Big\|. \ \blacksquare$$

Remark 1.5. A similar argument to the above implies that in the definition of w.a.s. we may replace weakly null sequences by normalized weakly convergent sequences.

2. Spreading models and asymptotic symmetry. The following theorem easily yields Theorem 0.3.

Theorem 2.1. *Let $1 < p < \infty$. There exists a reflexive infinite-dimensional Banach space X which is not asymptotically symmetric and yet satisfies the following.*

(2.1) *Every normalized basic sequence in X admits a subsequence that is 4-equivalent to the unit vector basis of ℓ_p.*

Proof. We shall define spaces X_k for $k \in \mathbb{N}$ and set $X = (\sum_{k=1}^{\infty} X_k)_p$. Each X_k will be reflexive and satisfy (2.1) with 4 replaced by $3+\varepsilon$ for any $\varepsilon > 0$. Gliding hump arguments then yield (2.1) for X.

Fix $1 < q < p < r$ with $1/q + 1/r = 1$ and let $k \in \mathbb{N}$. Each X_k will have a normalized 1-unconditional basis $\{e_j^i : 1 \le i \le k,\ j \ge i\}$ which we visualize as k infinite rows of an upper triangular array. We will define the norm on X_k so that if $n_1 < \cdots < n_k$ then $\|\sum_{i=1}^{k} e_{n_i}^i\| = k^{1/r}$ and we shall say that the collection $(e_{n_i}^i)_{i=1}^k$ is *permissible*. In addition we will have $\|e_{n_k}^1 + \cdots + e_{n_1}^k\| = 1$. Thus

$$\lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \|e_{n_1}^1 + \cdots + e_{n_k}^k\| = k^{1/r} \lim_{n_k\to\infty} \cdots \lim_{n_1\to\infty} \|e_{n_1}^1 + \cdots + e_{n_k}^k\|.$$

Hence X is not a.s.

Let $(a_i)_{i=1}^k \in B_{\ell_q^k}$, the unit ball of ℓ_q^k. Let $x \in \operatorname{span}\{e_j^i : i \le k,\ j \ge i\}$, say $x = \sum b_j^i e_j^i$. We define

$$|x|_{(a_i)_{i=1}^k} = \sup \Big(\sum_{j=1}^{\infty} \Big(\sum_{i=1}^{k} |a_i b_{n_j^i}^i| \Big)^p \Big)^{1/p}$$

where the "sup" is taken over all lexicographically ordered integers $n_1^1 < n_1^2 < \cdots < n_1^k < n_2^1 < n_2^2 < \cdots$. Thus each collection $(b_{n_j^i}^i)_{i=1}^k$ is the coordinates of x with respect to the permissible collection $(e_{n_j^i}^i)_{i=1}^k$, and these collections move to the right in our array picture as j increases.

The space X_k is the completion of $(\operatorname{span}\{e_j^i : i \le k,\ j \ge i\}, \|\cdot\|)$ where

$$\|x\| = \sup\{|x|_{(a_i)_{i=1}^k} : (a_i)_{i=1}^k \in B_{\ell_q^k}\}.$$

If $(e_{n_i}^i)_{i=1}^k$ is a permissible collection and $x = \sum_{i=1}^k b_i e_{n_i}^i$, then $\|x\| = (\sum_{i=1}^k |b_i|^r)^{1/r}$. Indeed, the lower estimate is immediate and suppose that $\|x\| = |x|_{(a_i)_{i=1}^k}$ for some $(a_i)_{i=1}^k \in B_{\ell_q^k}$. We may thus write

$$\|x\| = \Big[\Big(\sum_{i \in I_1} |a_i b_i| \Big)^p + \cdots + \Big(\sum_{i \in I_t} |a_i b_i| \Big)^p \Big]^{1/p}$$

where $I_1, \ldots, I_t$ are disjoint subsets of $\{1, \ldots, k\}$. Thus, since $1/q + 1/r = 1$,

$$\|x\| \le \sum_{i=1}^k |a_i b_i| \le \|(b_i)_{i=1}^k\|_r.$$

Next we let $y = \sum_{i=1}^k e_{n_i}^i$ where $n_k < n_{k-1} < \cdots < n_1$. Let $\|y\| = |y|_{(a_i)_{i=1}^k}$. Since any permissible collection of e_j^i's will intersect the support of y in at most one coordinate we have

$$|y|_{(a_i)_{i=1}^k} \le \Big(\sum_{i=1}^k |a_i|^p \Big)^{1/p} \le \|(a_i)_{i=1}^k\|_q \le 1.$$

This completes the proof of our assertions which imply that X is not a.s.

The basis for X_k is boundedly complete, and so once we prove that X_k satisfies (2.1) with constant 3 and for all normalized block bases, it will follow that X_k is reflexive and satisfies (2.1) for $3+\varepsilon$ and for all normalized basic sequences.

Let $\varepsilon > 0$ and let (x_i) be a normalized block basis of $\{e_j^i : i \le k,\ j \ge i\}$. Passing to a subsequence we may assume that for all m,

$$\max\{j : \exists i \le k, e_j^i \in \operatorname{supp} x_m\} + 2k < \min\{j : \exists i \le k, e_j^i \in \operatorname{supp} x_{m+1}\} \tag{2.2}$$

and for some $(a_i)_{i=1}^k \in B_{\ell_q^k}$,

$$|x_m|_{(a_i)_{i=1}^k} > 1 - \varepsilon. \tag{2.3}$$

Since (2.2) spaces the x_m's to have at least $2k$ "columns" between successive supports, we deduce using (2.3) that for all scalars (c_m),

$$\Big\|\sum_m c_m x_m\Big\| \ge (1-\varepsilon)\Big(\sum_m |c_m|^p\Big)^{1/p}.$$

Indeed, one can string together the lexicographically ordered lists that yield each norm $|x_m|_{(a_i)_{i=1}^k}$, inserting extra elements as needed into the gaps.

It remains to prove that $\|\sum_m c_m x_m\| \le 3(\sum_m |c_m|^p)^{1/p}$. Let $x = \sum_m c_m x_m$ and suppose $\|x\| = |x|_{(a_i)_{i=1}^k}$. We let $x = \sum b_j^i e_j^i$ and choose $n_1^1 < \cdots < n_1^k < n_2^1 < n_2^2 < \cdots$ so that

$$\|x\| = \Big(\sum_{j=1}^\infty \Big(\sum_{i=1}^k |a_i b^i_{n_j^i}|\Big)^p\Big)^{1/p}.$$

For each j let $A_j = \{e^i_{n_j^i}\}_{i=1}^k$ be the corresponding permissible collection.

Let

$$J_1 = \{j : A_j \text{ intersects the support of exactly one } x_m\},$$
$$J_0 = \{j : A_j \text{ intersects the support of more than one } x_m\},$$

and let $J_0 = J_2 \cup J_3$ where J_2 contains every other integer in J_0 and $J_3 = J_0 \setminus J_2$. Thus if j_1 and j_2 are distinct integers in J_2 (or J_3) then A_{j_1} and A_{j_2} cannot both intersect the support of the same x_m.

By the triangle inequality in ℓ_p,

$$\|x\| \le \sum_{l=1}^3 \Big(\sum_{j\in J_l} \Big(\sum_{i=1}^k |a_i b^i_{n_j^i}|\Big)^p\Big)^{1/p}.$$

We shall show that each of these three terms is bounded above by $\|(c_m)\|_p$.

For $m \in \mathbb{N}$ let $I_m = \{j \in J_1 : A_j \cap \operatorname{supp} x_m \ne \emptyset\}$. Thus $I_m \cap I_{m'} = \emptyset$ if $m \ne m'$. Since $|c_m| = \|c_m x_m\|$ we have

$$\Big(\sum_{j\in J_1} \Big(\sum_{i=1}^k |a_i b^i_{n_j^i}|\Big)^p\Big)^{1/p} = \Big(\sum_m \sum_{j\in I_m} \Big(\sum_{i=1}^k |a_i b^i_{n_j^i}|\Big)^p\Big)^{1/p} \le \|(c_m)\|_p.$$

Now, we estimate the J_2 sum (the J_3 estimate is identical). The J_2 sum is an ℓ_p sum of terms of the form

$$Q_j = \sum_{t=1}^s |c_{m_t}| \sum_{i\in I_t} |a_i d_{t,i}|$$

where $I_1 < \cdots < I_s$ are subsets of $\{1,\dots,k\}$ and $m_1 < \cdots < m_s$. Note that $x_{m_1},\dots,x_{m_s}$ are those x_i's for which $A_j \cap \operatorname{supp} x_i \ne \emptyset$, and the $d_{t,i}$'s are the corresponding relevant coordinates of x_{m_t}. The sequence $(x_{m_i})_{i=1}^s$ depends of course upon j but these families are disjoint for different j's in J_2. Thus it suffices to prove that

$$Q_j \le \Big(\sum_{t=1}^{s} |c_{m_t}|^p\Big)^{1/p}.$$

Now for t fixed $(d_{t,i})_{i\in I_t}$ are the coordinates of x_{m_t} with respect to a subset (indexed by I_t) of a permissible collection of e^u_v's and these are in turn 1-equivalent to the unit vector basis of $\ell_r^{|I_t|}$ as we have shown. Hence

$$Q_j \le \sum_{t=1}^{s} |c_{m_t}|\, \|(a_i)_{i\in I_t}\|_q \le \|(c_{m_t})_{t=1}^s\|_r \|(a_i)_{i=1}^k\|_q \le \|(c_{m_t})\|_r \le \|(c_{m_t})\|. \ \blacksquare$$

REMARKS 2.2. The space X constructed in Theorem 2.1 satisfies (2.1) and also has the property that ℓ_r and c_0 are asymptotic versions of X (see [MMT]). A different example of this sort of phenomenon is given in [OS] where a reflexive space Z is constructed satisfying (2.1) with 4 replaced by $1+\varepsilon$, $\varepsilon > 0$ arbitrary, yet Z has ℓ_r as an asymptotic version for some $r \ne p$. It would be interesting to see if one could construct X as in Theorem 2.1 to satisfy (2.1) with 4 replaced by $1+\varepsilon$. Another natural question is to ascertain what happens if ℓ_p is replaced by ℓ_1 or c_0. We will show that in these cases one obtains positive results.

THEOREM 2.3. *Let X have a basis (e_i). Assume that for some $K < \infty$ every spreading model of any normalized block basis of (e_i) is K-equivalent to the unit vector basis of ℓ_1. Then X is block asymptotically symmetric with respect to (e_i).*

Proof. By renorming X we may assume that (e_i) is a bimonotone basis. Let $m \in \mathbb{N}$, $(b_i)_{i=1}^m \subseteq [-1,1]$, $(x^i_j)_{j=1}^\infty$ be a normalized block basis of (e_i) for $i \le m$ and σ a permutation of $\{1,\dots,m\}$ so that the iterated limits

$$\lim_{n_1\to\infty}\cdots\lim_{n_m\to\infty}\Big\|\sum_{i=1}^m b_i x^i_{n_i}\Big\|, \qquad \lim_{n_{\sigma(1)}\to\infty}\cdots\lim_{n_{\sigma(m)}\to\infty}\Big\|\sum_{i=1}^m b_i x^i_{n_i}\Big\|$$

both exist, the first being equal to 1.

We visualize $(x^i_j)_{i,j}$ as an array of m infinite rows. Using Ramsey's theorem, by passing to a subsequence of the columns, given $\varepsilon > 0$ we may assume that for all integers $n_1 < \dots < n_m$ and $k_1 < \dots < k_m$, for all $f \in B_{X^*}$ there exists $g \in B_{X^*}$ with $|f(x^i_{n_i}) - g(x^i_{k_i})| \le \varepsilon$ for $i \le m$. This follows by partitioning $[-1,1]$ into finitely many intervals $(I_t)_{t=1}^l$ of length less than ε and thus inducing a finite coloring of $[\mathbb{N}]^m$ as follows: $(n_1,\dots,n_m)$ has color $(I_{t_1},\dots,I_{t_m})$ if there exists $f \in B_{X^*}$ with $f(x^i_{n_i}) \in I_{t_i}$ for $i \le m$.

It follows that up to an arbitrarily small error

$$\Big\|\sum_{i=1}^{m} b_i x^i_{n_i}\Big\| \approx \Big\|\sum_{i=1}^{m} b_i x^i_{k_i}\Big\| \approx 1$$

whenever $n_1 < \cdots < n_m$ and $k_1 < \cdots < k_m$. To avoid trivial but tedious error estimates, in the remainder of the proof we shall assume $\varepsilon = 0$. We may also assume that if $x \in \operatorname{span}(x^i_{j_1})_{i=1}^m$, $y \in \operatorname{span}(x^i_{j_2})_{i=1}^m$ for $j_1 < j_2$ then $\operatorname{supp} x < \operatorname{supp} y$ with respect to (e_n). Finally we assume similar stabilizations for the order induced by σ.

Let $n_1^1 < \cdots < n_1^m < n_2^1 < \cdots < n_2^m < \cdots$ and set $y_j = \sum_{i=1}^m b_i x^i_{n^i_j}$ for $j \in \mathbb{N}$. Then (y_j) is a normalized block basis of (e_i) and thus, passing to a subsequence, we may assume it has a spreading model which is K-equivalent to the unit vector basis of ℓ_1.

Hence for all k there exists $F \subseteq \mathbb{N}$ with $|F| = k$ and $f \in B_{X^*}$ with $f(y_j) \geq 1/K$ for $j \in F$ (up to an arbitrarily small error). Using the pigeonhole principle and ignoring small errors we obtain m of the y_j's (which we relabel as $y_1, \ldots, y_m$), $f \in B_{X^*}$ and $(a_i)_{i=1}^m$ so that $f(x^i_{n^i_j}) = a_i$ for all j and $1 \leq i, j \leq m$.

We shall show that

$$1 \leq K \Big\|\sum_{i=1}^{m} b_{\sigma(i)} x^{\sigma(i)}_{n_{\sigma(i)}}\Big\|$$

provided that $n_{\sigma(1)} < \cdots < n_{\sigma(m)}$. From our stabilizations it suffices to produce $n_{\sigma(1)} < \cdots < n_{\sigma(m)}$ satisfying this. We choose them so that each $x^{\sigma(i)}_{n_{\sigma(i)}}$ is in the support of some y_j, $j \leq m$. It follows that $f(x^{\sigma(i)}_{n_{\sigma(i)}}) = a_{\sigma(i)}$ and the claim follows. ∎

The proof tells us that the b.a.s. constant of X is bounded by a function of K and the basis constant of (e_i).

THEOREM 2.4. *Let X be a separable Banach space such that every spreading model of a normalized weakly null sequence in X is equivalent to the unit vector basis of c_0. Then X is weakly asymptotically symmetric.*

Proof. It follows from [AOST, Proposition 3.2] that for some $C < \infty$ every spreading model of a normalized weakly null sequence in X is C-equivalent to the unit vector basis of c_0.

Let $m \in \mathbb{N}$ and let $(x^i_j)_{j=1}^\infty$, $i \leq m$, be normalized weakly null sequences in X. Let $K < \infty$, $(b_i)_{i=1}^m \subseteq \mathbb{R}$ and assume that for some permutation σ,

$$\lim_{n_1\to\infty} \cdots \lim_{n_m\to\infty} \Big\|\sum_{i=1}^{m} b_i x^i_{n_i}\Big\| = K, \qquad \lim_{n_{\sigma(1)}\to\infty} \cdots \lim_{n_{\sigma(m)}\to\infty} \Big\|\sum_{i=1}^{m} b_i x^i_{n_i}\Big\| = 1.$$

We will prove that $K \leq C$, which will complete the proof of the theorem.

As in the proof of the previous theorem, by passing to a subsequence of the columns and ignoring arbitrarily small errors we may assume that $\|\sum_{i=1}^m b_i x_{n_i}^i\| = K$ if $n_1 < \cdots < n_m$. Moreover, we may assume that if $f \in B_{X^*}$ with $f(x_{n_i}^i) = a_i$ for $i \le m$ and if $k_1 < \cdots < k_m$, then there exists $g \in B_{X^*}$ with $g(x_{k_i}^i) = a_i$ for $i \le m$.

We shall say that z is $((b_i)_{i=1}^m, \sigma)$ *distributed* if

$$z = \sum_{i=1}^m b_{\sigma(i)} x_{n_{\sigma(i)}}^{\sigma(i)}$$

for some $n_{\sigma(1)} < \cdots < n_{\sigma(m)}$ and as above, by Ramsey's theorem, we may assume that for such a vector, $\|z\| = 1$. In addition we may assume that if $(z_i)_{i=1}^m$ are all $((b_i)_{i=1}^m, \sigma)$ distributed with $z_j = \sum_{i=1}^m b_{\sigma(i)} x_{n_{\sigma(i)}^j}^i$ and $n_{\sigma(m)}^j < n_{\sigma(1)}^{j+1}$ for $j < m$, then $\|\sum_{j=1}^m z_j\|$ does not depend upon the particular choice of the $n_{\sigma(i)}^j$'s. Finally, since the rows $(x_j^i)_{j=1}^\infty$ are weakly null, we can assume that the coordinates supporting such a sequence $(z_j)_{j=1}^m$, namely $(x_{n_{\sigma(i)}^j}^i)_{i=1,\,j=1}^{m,m}$ are suppression-1 unconditional. (This argument is used in [HO] and [AOST].) Roughly, if one has $f \in B_{X^*}$ and one considers $z = \sum_{i,j=1}^m a_j^i x_{n_{\sigma(i)}^j}^i$ with its coordinates sufficiently spread out then one can "slide" the coordinates I one wishes to kill, preserving the order, so that $f \approx 0$ on these coordinates. The new vector w, distributed exactly the same as z, hence with $\|w\| = \|z\|$, satisfies

$$f(w) = \Big| f\Big(\sum_{i,j \notin I} a_j^i x_{n_{\sigma(i)}^j}^i \Big) \Big| \le \|w\| = \|z\|.$$

Now let $(z_j)_{j=1}^\infty$ be a "block basis" of (x_j^i) with each z_j having $((b_i)_{i=1}^m, \sigma)$ distribution. As (z_j) is normalized weakly null, passing to a subsequence we may assume it has a spreading model which is C-equivalent to the unit vector basis of c_0. In particular, by relabeling, we may assume that $\|\sum_{i=1}^m z_i\| \le C$. By restricting this vector to a suitable set of coordinates we obtain a vector equal to $\sum_{i=1}^m b_i x_{n_i}^i$ for some $n_1 < \cdots < n_m$. Thus $C \ge \|\sum_{i=1}^m z_i\| \ge \|\sum_{i=1}^m b_i x_{n_i}^i\| = K$, and the proof is complete. ∎

3. Variants of a.s. and Tsirelson-like spaces

THEOREM 3.1. *There exists a reflexive Banach space Y which is w.n.a.s. but not w.a.s.*

The required Y is Tzafriri's space $Ti(2; 1/2)$ ([CS, Section X.D]). We recall the definition. Let c_{00} be the linear space of finitely supported sequences of reals. If $x \in c_{00}$ and $E \subseteq \mathbb{N}$, we set $Ex(i) = x(i)$ if $i \in E$ and 0 otherwise. For sets $E, F \subseteq \mathbb{N}$, $E < F$ denotes $\max E < \min F$. Now, Y is the completion of c_{00} under the norm given by the following implicit equation:

$$\|x\| = \max\Big(\|x\|_\infty, \sup \frac{1}{2\sqrt{n}}\sum_{i=1}^n \|E_i x\|\Big) \tag{3.1}$$

where the "sup" is taken over all $n \in \mathbb{N}$ and $E_1 < \cdots < E_n$.

The space Y is reflexive and the unit vector basis (e_i) is a normalized 1-unconditional 1-subsymmetric basis for Y. We recall two facts from [CS].

Fact 1. *For all $x \in Y$, $\|x\| \le \|x\|_2$.*

Fact 2 ([CS, Lemma X.d.4, p. 109]). *If $(u_i)_{i=1}^n$ is a finite block basis of (e_i) then*

$$\Big\|\sum_{i=1}^n u_i\Big\| \le \sqrt{3}\Big(\sum_{i=1}^n \|u_i\|^2\Big)^{1/2}.$$

From Fact 2 and (3.1) we deduce that if $(y_i)_{i=1}^n$ is a normalized block basis of (e_i) then

$$\sqrt{n}/2 \le \Big\|\sum_{i=1}^n y_i\Big\| \le 3\sqrt{n}\ .$$

This implies that Y is w.n.a.s. with constant 6. That Y is not w.a.s. follows from either the next theorem or [Sa] (see the remarks below).

Theorem 3.2. *c_0 is finitely representable in Y. Moreover, for some $C < \infty$ (equivalently, for all $C > 1$), for all n there exist disjointly supported (with respect to (e_i)) normalized vectors $(x_i)_{i=1}^n$ in Y with $(x_i)_{i=1}^n$ being C-equivalent to the unit vector basis of ℓ_∞^n.*

First we shall show how this theorem completes the proof of Theorem 3.1. $(e_i)_{i=1}^\infty$ is 1-subsymmetric and hence is its own spreading model. But Theorem 3.2 and (3.1) imply that (e_i) is not symmetric, which every spreading model of a w.a.s. reflexive space would be.

Tzafriri [Tz] constructed a symmetric version of Y denoted by $V_{1/2,2}$. The definition of the norm is given by (3.1) where the sup is taken over disjoint sets $(E_i)_{i=1}^n$ in $\mathbb{N}$. The space $V_{1/2,2}$ has finite cotype and hence does not contain ℓ_∞^n's uniformly. Thus, a consequence of Theorem 3.2 is the following corollary which answers a question from [CS].

Corollary 3.3. *The spaces Y and $V_{1/2,2}$ are not isomorphic.*

This question was answered independently by B. Sari [Sa] who used different techniques. In fact Sari has proved that $Y = Ti(2;1/2)$ does not contain a symmetric basic sequence. Moreover, Sari's result also proves that Y is not w.a.s. since (e_i) is not symmetric. We include Theorem 3.2 because it is of separate interest.

Lemma 3.4. *For $n \ge 4$, $\|\sum_{i=1}^n e_i\| = \sqrt{n}/2$.*

Proof. The lower estimate is immediate and the case $n = 4$ is easy using (3.1). For $n > 4$ there exist $E_1 < \dots < E_k$ with

$$\Big\|\sum_{i=1}^{n} e_i\Big\| = \frac{1}{2\sqrt{k}}\sum_{j=1}^{k}\Big\|E_j\Big(\sum_{i=1}^{n} e_i\Big)\Big\|.$$

Let $|E_j \cap \{1,\dots,n\}| = n_j$, hence $\sum_{j=1}^{k} n_j \le n$. By Fact 1 and Cauchy–Schwarz,

$$\Big\|\sum_{i=1}^{n} e_i\Big\| \le \frac{1}{2\sqrt{k}}\sum_{j=1}^{k}\sqrt{n_j} \le \frac{1}{2\sqrt{k}}\Big(\sum_{j=1}^{k} n_i\Big)^{1/2}\sqrt{k} \le \frac{\sqrt{n}}{2}. \ \blacksquare$$

LEMMA 3.5. *Let $\delta = \sqrt{3}/2 < 1$ and let $(u_i)_{i=1}^{n}$ be a block basis of (e_i) with $\|u_i\| \le 1$ for $i \le n$. Let $E_1 < \dots < E_k$ be subsets of $\mathbb{N}$ so that for $i \le n$, $\operatorname{supp} u_i$ intersects at most one E_j. Then*

$$\frac{1}{2\sqrt{k}}\sum_{j=1}^{k}\Big\|E_j\Big(\sum_{i=1}^{n} u_i\Big)\Big\| \le \delta\sqrt{n}.$$

Proof. For $j \le k$ let $n_j = |\{i : \operatorname{supp} u_i \cap E_j \ne \emptyset\}|$. Thus $\sum_{j=1}^{k} n_j \le n$. By Fact 2 and Cauchy–Schwarz,

$$\frac{1}{2\sqrt{k}}\sum_{j=1}^{k}\Big\|E_j\Big(\sum_{i=1}^{n} u_i\Big)\Big\| \le \frac{\sqrt{3}}{2\sqrt{k}}\sum_{j=1}^{k}\sqrt{n_j} \le \frac{\sqrt{3}}{2\sqrt{k}}\Big(\sum_{j=1}^{k} n_j\Big)^{1/2}\sqrt{k} \le \delta\sqrt{n}. \ \blacksquare$$

Proof of Theorem 3.2. Let $m \in \mathbb{N}$. We shall construct disjointly supported normalized vectors $(x_i)_{i=1}^{m}$ in Y so that $\|\sum_{i=1}^{m} x_i\| \le \sum_{i=0}^{\infty}\delta^i + 1$. By the unconditionality of (e_i) this completes the proof. Each (x_i) will be a normalized average of certain basis vectors, the number of which rapidly increases with i and the supports are uniformly mixed (just as the 1 inch marks on a yardstick are uniformly separated by the $\frac{1}{32}$ inch marks and so on). To do this we need some notation. We shall choose below rapidly increasing integers $q_1 < \dots < q_m$. Given these we define $p_i = \prod_{j=1}^{i} q_j$ for $i \le m$ and we then choose natural numbers $r_{t_1,\dots,t_i}$ for each $i \le m$ and $t_j \le q_j$ for $j \le i$ so that $r_{t_1,\dots,t_i} < r_{s_1,\dots,s_j}$ whenever $(t_1,\dots,t_i)$ is less than $(s_1,\dots,s_j)$ lexicographically. For example

$$r_1 < r_{2,1,4} < r_{2,2} < r_{2,2,3} < r_3.$$

We shall say that $(x_i)_{i=1}^{m}$ *corresponds* to $(q_1,\dots,q_m)$ if for $i \le m$,

$$x_i = \frac{2}{\sqrt{p_i}}\sum_{j_1=1}^{q_1}\sum_{j_2=1}^{q_2}\cdots\sum_{j_i=1}^{q_i} e_{r_{j_1,\dots,j_i}}.$$

Since (e_i) is 1-subsymmetric the particular choice of the $r_{j_1,\dots,j_i}$'s does not matter but their order does. By Lemma 3.4, $\|x_i\| = 1$ for $i \le m$.

Let $(\varepsilon_n)_{n=1}^{\infty}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \varepsilon_n < 1$. We shall prove by induction on m that for every integer $q \geq 4$ there exist integers $q < q_2 < \cdots < q_m$ so that for all integers q_1 with $4 \leq q_1 \leq q$, if $(x_i)_{i=1}^{m}$ corresponds to $(q_1, \ldots, q_m)$ then

$$\Big\| \sum_{i=1}^{m} x_i \Big\| \leq \sum_{i=0}^{m-1} \delta^i + \sum_{i=1}^{m} \varepsilon_i \equiv M(m). \tag{3.2}$$

This is obvious for $m = 1$ so assume it holds for some m. Let $q \geq 4$. Choose $d \in \mathbb{N}$ so that

$$\sqrt{q} < \varepsilon_{m+1} \sqrt{d}\,. \tag{3.3}$$

Choose $n \in \mathbb{N}$ so that

$$\frac{2dM(m)}{\sqrt{n}} < \varepsilon_{m+1}. \tag{3.4}$$

Let $q_2 = dn$. By the inductive hypothesis for $q_0 \equiv qq_2$ we can find integers $q_0 < q_3 < \cdots < q_{m+1}$ so that if $4 \leq s \leq q_0$ and if $(y_i)_{i=1}^{m}$ is a sequence corresponding to $(s, q_3, q_4, \ldots, q_{m+1})$, then $\|\sum_{i=1}^{m} y_i\| \leq M(m)$.

Let $4 \leq q_1 \leq q$ and let $(x_i)_{i=1}^{m+1}$ correspond to $(q_1, \ldots, q_{m+1})$. There exists $k \geq 2$ and $E_1 < \cdots < E_k$ so that

$$\begin{aligned}\Big\| \sum_{i=1}^{m+1} x_i \Big\| &= \frac{1}{2\sqrt{k}} \sum_{j=1}^{k} \Big\| E_j\Big(\sum_{i=1}^{m+1} x_i \Big) \Big\| \\ &\leq \frac{1}{2\sqrt{k}} \sum_{j=1}^{k} \|E_j(x_1)\| + \frac{1}{2\sqrt{k}} \sum_{j=1}^{k} \Big\| E_j\Big(\sum_{i=2}^{m+1} x_i \Big) \Big\|.\end{aligned}$$

Case 1: $k \geq d$. Then

$$\|x_1\|_1 = 2\sqrt{q_1} \leq 2\sqrt{q} < 2\varepsilon_{m+1}\sqrt{d}$$

by (3.3). Thus

$$\frac{1}{2\sqrt{k}} \sum_{j=1}^{k} \|E_j(x_1)\| \leq \frac{1}{2\sqrt{k}} \|x_1\|_1 < \frac{\sqrt{d}}{\sqrt{k}} \varepsilon_{m+1} \leq \varepsilon_{m+1}.$$

Also

$$\frac{1}{2\sqrt{k}} \sum_{j=1}^{k} \Big\| E_j\Big(\sum_{i=2}^{m+1} x_i \Big) \Big\| \leq \Big\| \sum_{i=2}^{m+1} x_i \Big\|.$$

Now $(x_i)_{i=2}^{m+1}$ corresponds to the m-tuple $(q_1 q_2, q_3, \ldots, q_{m+1})$ and since $q_1 q_2 \leq qq_2 = q_0$, by the inductive hypothesis we have

$$\Big\| \sum_{i=2}^{m+1} x_i \Big\| \leq M(m).$$

Thus

$$\Big\|\sum_{i=1}^{m+1} x_i\Big\| \le M(m) + \varepsilon_{m+1} < M(m+1).$$

CASE 2: $k < d$. For the x_1 term we use the estimate

$$\frac{1}{2\sqrt{k}} \sum_{j=1}^{k} \|E_j(x_1)\| \le 1.$$

To estimate the $\sum_{i=2}^{m+1} x_i$ term we write $x_i = \sum_{h=1}^{n} x_{i,h}$ for $2 \le i \le m+1$, where $(x_{i,h})_{h=1}^{n}$ is an identically distributed block basis. Thus $|\operatorname{supp} x_{i,h}| = p_i/n = q_1 d q_3 \cdots q_i$.

By Lemma 3.4,

$$\|x_{i,h}\| = \frac{1}{2}\sqrt{\frac{p_i}{n}}\,\frac{2}{\sqrt{p_i}} = \frac{1}{\sqrt{n}} \qquad \text{for } 2 \le h \le m+1.$$

Thus for $h \le n$, $(\sqrt{n}\, x_{i,h})_{i=2}^{m+1}$ corresponds to $(q_1 q_2/n, q_3, \ldots, q_{m+1})$ and since $q_1 q_2/n = q_1 d \le q q_2 = q_0$, by the inductive hypothesis we have

$$\Big\|\sum_{i=2}^{m+1} \sqrt{n}\, x_{i,h}\Big\| \le M(m). \tag{3.5}$$

Set $z_h = \sum_{i=2}^{m+1} x_{i,h}$ for $h \le n$. Then $(z_h)_{h=1}^{n}$ is an identically distributed block basis of (e_i) and hence

$$\|z_1\| = \cdots = \|z_n\| \equiv a \le \frac{M(m)}{\sqrt{n}}$$

by (3.5).

Since $E_1 < \cdots < E_k$, for $j \le k$ there are at most two h's for which $E_j \cap \operatorname{supp} z_h \ne \emptyset$ and $E_{j'} \cap \operatorname{supp} z_h \ne \emptyset$ for some $j' \ne j$.

For $j \le k$ let

$$\widetilde{E}_j = \bigcup\{\operatorname{supp} z_h : h \le n,\ \operatorname{supp} z_h \cap E_j \ne \emptyset \\ \text{and } \operatorname{supp} z_h \cap E_{j'} = \emptyset \text{ if } j \ne j'\}.$$

We let n_j be the cardinality of the set of such h's. Set $z = \sum_{i=2}^{m+1} x_i$. Then since $a \le M(m)/\sqrt{n}$, we have $\|E_j z - \widetilde{E}_j z\| \le 2M(m)/\sqrt{n}$. Hence

$$\sum_{j=1}^{k} \|E_j z\| \le \frac{2M(m)k}{\sqrt{n}} + \sum_{j=1}^{k} \|\widetilde{E}_j z\|.$$

Now

$$\frac{2M(m)k}{\sqrt{n}} \le \frac{2M(m)d}{\sqrt{n}} < \varepsilon_{m+1} \qquad \text{by (3.4).}$$

Also

$$\sum_{j=1}^{k} \|\widetilde{E}_j z\| \le 2a\sqrt{k}\,\delta\Big(\sum_{j=1}^{k} n_j\Big)^{1/2} \le 2a\sqrt{k}\,\delta\sqrt{n} \qquad \text{by Lemma 3.5.}$$

Thus

$$\frac{1}{2\sqrt{k}}\sum_{j=1}^{k} \|E_j(z)\| \le \varepsilon_{m+1} + a\delta\sqrt{n}.$$

Now $a\sqrt{n} \le M(m)$ and $\delta < 1$ so this in turn is

$$< \delta\sum_{i=0}^{m-1} \delta^i + \sum_{i=1}^{m} \varepsilon_i + \varepsilon_{m+1} = \sum_{i=1}^{m} \delta^i + \sum_{i=1}^{m+1} \varepsilon_i.$$

From the x_1 estimate of $1 = \delta^0$ we obtain $\|\sum_{i=1}^{m+1} x_i\| \le M(m+1)$ in Case 2. ∎

REMARKS 3.6. A natural question is whether Y contains an a.s. subspace. This is true by our next theorem. The argument is motivated by arguments given in [KOS]. B. Sari has also used a variation of these arguments to prove the following results. First we give some terminology from [MMT].

Let $(x_i)_{i=1}^{\infty}$ be a basis for a space X. A normalized basic sequence $(d_i)_{i=1}^{n}$ is said to be in the nth *asymptotic structure* of X, and we write $(d_i)_{i=1}^{n} \in \{X\}_n$, if $\forall\varepsilon > 0$ $\forall m_1$ $\exists y_1 \in \operatorname{span}(x_i)_{i\ge m_1}$ $\forall m_2$ $\exists y_2 \in \operatorname{span}(x_i)_{i\ge m_2} \ldots \forall m_n$ $\exists y_n \in \operatorname{span}(x_i)_{i\ge m_n}$ so that $(y_i)_{i=1}^{n}$ is $(1+\varepsilon)$-equivalent to $(d_i)_{i=1}^{n}$.

X is *Asymptotic* ℓ_p if $\exists K < \infty$ $\forall n$ $\forall (d_i)_{i=1}^{n} \in \{X\}_n$, $(d_i)_{i=1}^{n}$ is K-equivalent to the unit vector basis of ℓ_p^n. If X is Asymptotic ℓ_p then X contains an asymptotic ℓ_p basis, as defined above [MMT].

Note that we use a capital letter "A" in the above definition in contrast to the different notion of asymptotic ℓ_p from the introduction. If X is Asymptotic ℓ_p, one can pass to a block basis which is asymptotic ℓ_p (see [MMT]).

X is *Asymptotically unconditional* if $\exists K<\infty$ $\forall n$ $\forall (d_i)_{i=1}^{n} \in \{X\}_n$, $(d_i)_{i=1}^{n}$ is K-unconditional. $A \overset{K}{\sim} B$ means $K^{-1}A \le B \le KA$.

THEOREM ([Sa]).

(1) *Let $1 < p < \infty$. If X is Asymptotically unconditional and for some $K < \infty$, for all $m \le n \in \mathbb{N}$ and for all disjointly supported normalized vectors $(y_i)_{i=1}^{m}$ in* $\operatorname{span}(d_i)_{i=1}^{n}$, $\|\sum_{i=1}^{m} y_i\| \overset{K}{\sim} m^{1/p}$, *then X is Asymptotic ℓ_p.*

(2) *If X is Asymptotically unconditional and for some $K < \infty$, for all $n \in \mathbb{N}$ and for all $(d_i)_{i=1}^{n} \in \{X\}_n$, $\|\sum_{i=1}^{n} d_i\| \ge n/K$, then X is Asymptotic ℓ_1.*

THEOREM 3.7. *Let Z be a Banach space with a basis (z_i). Let $1 \le p < \infty$ and $K < \infty$. Assume that for all $(d_i)_{i=1}^n \in \{Z\}_n$, $\|\sum_{i=1}^n d_i\| \overset{K}{\sim} n^{1/p}$. Then every infinite-dimensional subspace of Z contains an asymptotic ℓ_p basic sequence.*

COROLLARY 3.8. *Every infinite-dimensional subspace of $Y = Ti(2; 1/2)$ contains an asymptotic ℓ_2, hence a.s., subspace.*

Recall that a Banach space X is *minimal* if every subspace of X contains a further subspace isomorphic to X. For instance, Schlumprecht's space S is minimal [S2].

COROLLARY 3.9. *The space Y is not minimal.*

Indeed, Corollary 3.8 tell us that every subspace of Y contains an asymptotic ℓ_2 subspace Z. Since Y does not contain an isomorph of ℓ_2 (cf. [CS]), it follows that Z cannot contain a subsymmetric basic sequence and hence Y does not embed into Z.

We do not know if Y contains a minimal subspace.

Proof of Theorem 3.7. By standard perturbation arguments we need only show that every normalized block basis (x_i) of (z_i) admits a further block basis which is asymptotically ℓ_p. We may assume that (z_i) is bimonotone, by renorming, and that $K > 2$. Furthermore, by passing to a block basis of (x_i) we may assume ([MMT], [KOS]) that given $\varepsilon_n \downarrow 0$, for $X = [(x_i)]$,

(3.6) $\{X\}_n = \{W\}_n$ for all $n \in \mathbb{N}$ and all block bases (w_i) of (x_i); here $W = [(w_i)]$.

(3.7) For $n \in \mathbb{N}$, if $(y_i)_{i=1}^n$ is a normalized block basis of $(x_i)_{i=n}^\infty$, then $(y_i)_{i=1}^n$ is $(1+\varepsilon_n)$-equivalent to some $(d_i)_{i=1}^n \in \{X\}_n$.

Thus, by increasing K, we may assume that

(3.8) for $n \in \mathbb{N}$, if $(y_i)_{i=1}^n$ is a normalized block basis of $(x_i)_{i=n}^\infty$, then

$$\Big\|\sum_{i=1}^n y_i\Big\| \overset{K}{\sim} n^{1/p} .$$

Fix $m \in \mathbb{N}$ and $(d_i)_{i=1}^m \in \{X\}_m$. We will prove that if $\sum_{i=1}^m |a_i|^p = 1$ then

$$\frac{1}{2}\,\frac{1}{(2K)^2} \le \Big\|\sum_{i=1}^m a_i d_i\Big\| \le 2(2K)^2, \tag{3.9}$$

which will complete the proof of the theorem in view of our remarks above.

We choose $\varepsilon, \delta, \delta' > 0$ and $N' \in \mathbb{N}$ to satisfy

$$0 < \varepsilon < \frac{1}{8m} \frac{1}{(2K)^{2p}}, \tag{3.10}$$

$$\delta = \frac{\varepsilon}{4Km}, \tag{3.11}$$

$$N' > 2m\left(\frac{2K}{\delta}\right)^p, \tag{3.12}$$

$$0 < \delta' < \frac{\delta}{2KN'}. \tag{3.13}$$

Let $(w_i)_{i=1}^{\infty}$ be a normalized block basis of $(x_i)_{i=N'}^{\infty}$ so that for all i, if $w_i = \sum b_{i,j} x_j$ then $\sup_j |b_{i,j}| < \delta'$. Such a (w_i) exists by virtue of (3.8); one can let w_i be a suitably long average of x_j's. Given $\eta > 0$ we can thus find, by (3.6), a normalized block basis $(y_i)_{i=1}^m$ of (w_i) which is $(1+\eta)$-equivalent to $(d_i)_{i=1}^m$. We will prove that (3.9) holds with $(d_i)_{i=1}^m$ replaced by $(y_i)_{i=1}^m$ and thus obtain (3.9).

Let $(a_i)_{i=1}^m \subseteq \mathbb{R}$ with $\sum_{i=1}^m |a_i|^p = 1$. From our construction we can write $a_i y_i = \sum_{j=1}^{n_i+1} y_{i,j}$ for $i \le m$, where $n_i \ge 0$ and $(y_{i,j})_{j=1}^{n_i+1}$ is a block basis of $(x_i)_{i=N'}^{\infty}$, $\delta \le \|y_{i,j}\| < \delta + \delta'$ if $j \le n_i$ and $\|y_{i,n_i+1}\| < \delta$. Set

$$N = \sum_{\substack{i \le m \\ |a_i| \ge \varepsilon}} n_i.$$

It follows from (3.10) and (3.11) that $N \ge 1$.

Now we prove that

$$\text{if } i \le m \text{ and } |a_i| \ge \varepsilon \text{ then } 1 \le n_i \le (2K/\delta)^p. \tag{3.14}$$

Indeed, suppose that $n_i \ge n_0 \equiv [\![(2K/\delta)^p]\!] + 1$. Then $1 \ge \|a_i y_i\| \ge (\delta/K) n_0^{1/p} - n_0 \delta'$ from (3.8). (We shrink each of n_0 successive y_{ij}'s to have norm exactly δ at a loss of at most δ'. Note $n_0 < N'$ so (3.8) applies.) By (3.10) we get $n_i \ge 1$.

Now $n_0 \delta' < (\delta/2K) n_0^{1/p}$ since this is equivalent to $\delta' < \delta/2K n_0^{1/q}$ (where $1/p + 1/q = 1$) and we have

$$\delta' \overset{(3.13)}{<} \frac{\delta}{2KN'} \overset{(3.12)}{<} \frac{\delta \delta^p}{2K2m(2K)^p} < \frac{\delta}{2K n_0^{1/q}}$$

where the last inequality holds since $n_0^{1/q} \le n_0 < 2m(2K)^p/\delta^p$. Thus $1 \ge (\delta/2K) n_0^{1/p}$ and so $n_0 \le (2K/\delta)^p$, a contradiction.

Therefore, by (3.14) and (3.12),

$$N \le m\left(\frac{2K}{\delta}\right)^p < N', \tag{3.15}$$

$$\text{if } i \le m \text{ and } |a_i| \ge \varepsilon \text{ then } |a_i| = \|a_i y_i\| > \frac{\delta}{2K}\, n_i^{1/p}. \tag{3.16}$$

Indeed, since $n_i \le N < N'$ we can argue as in (3.14) to get

$$\|a_i y_i\| > \frac{\delta}{K}\, n_i^{1/p} - n_i\delta'.$$

Now $n_i\delta' < (\delta/2K)n_i^{1/p}$ is equivalent to $\delta' < \delta/2Kn_i^{1/q}$. But by (3.13), $\delta' < \delta/2KN' < \delta/2Kn_i$ and so putting this together we obtain (3.16).

We show in turn that

$$\text{if } i \le m \text{ and } |a_i| \ge \varepsilon \text{ then } |a_i| = \|a_i y_i\| < 2\delta K n_i^{1/p}. \tag{3.17}$$

Again we have $\|a_i y_i\| < \delta K n_i^{1/p} + n_i\delta' + \delta$ by shrinking each $y_{i,j}$, $j \le n_i$, to have norm exactly δ at a cost of δ', and using (3.8) and adding δ for the term $\|y_{i,n_i+1}\|$.

We claim that $n_i\delta' + \delta < \delta K n_i^{1/p}$. Since $n_i\delta' + \delta < 2\delta$, by $n_i < N'$ and (3.13) and since $2\delta < \delta K n_i^{1/p}$ this yields (3.17).

Let

$$\sum_{i=1}^{m}{}' a_i y_i = \sum_{\substack{|a_i|\ge\varepsilon\\ 1\le i\le m}} a_i y_i, \qquad \sum_{i=1}^{m}{}'' a_i y_i = \sum_{\substack{|a_i|\ge\varepsilon\\ 1\le i\le m}} \sum_{j=1}^{n_i} y_{i,j}.$$

We claim that

$$\Big\|\sum_{i=1}^{m}{}'' a_i y_i\Big\| \ge \frac{\delta}{2K}\, N^{1/p}. \tag{3.18}$$

Indeed, by our now familiar method,

$$\Big\|\sum_{i=1}^{m}{}'' a_i y_i\Big\| \ge \frac{\delta}{K}\, N^{1/p} - N\delta';$$

but $N\delta' < (\delta/2K)N^{1/p}$ since

$$N\delta' < N'\delta' \overset{(3.13)}{<} \frac{\delta}{2K} < \frac{\delta}{2K} N^{1/p}.$$

Thus (3.18) is proved.

From (3.18) we have

$$\Big\|\sum_{i=1}^{m}{}'' a_i y_i\Big\|^p > \Big(\frac{\delta}{2K}\Big)^p N = \Big(\frac{\delta}{2K}\Big)^p \sum_{i=1}^{m}{}' n_i \overset{(3.17)}{>} \frac{1}{(2K)^p}\sum_{i=1}^{m}{}' \frac{|a_i|^p}{(2K)^p}$$
$$= \frac{1}{(2K)^{2p}}\sum_{i=1}^{m}{}' |a_i|^p > \frac{1-m\varepsilon}{(2K)^{2p}}.$$

Thus

$$\Big\|\sum_{i=1}^{m} a_i y_i\Big\| > \Big\|\sum_{i=1}^{m}{}'' a_i y_i\Big\| - m\varepsilon - m\delta > (1-m\varepsilon)^{1/p}\frac{1}{(2K)^2} - 2m\varepsilon \overset{(3.10)}{>} \frac{1}{2}\frac{1}{(2K)^2}.$$

Next we show

$$\Big\|\sum_{i=1}^{m}{}' a_i y_i\Big\| < 2\delta K N^{1/p}. \tag{3.19}$$

As usual, $\|\sum_{i=1}^{\prime m} a_i y_i\| < \delta K N^{1/p} + N\delta' + m\delta$ and $N\delta' + m\delta < (m+1)\delta$. We claim that $2m < KN^{1/p}$, which will complete the proof of (3.19). First note that if $|a_i| \ge \varepsilon$ then by (3.17), $n_i^{1/p} > \varepsilon/2\delta K = 2m$ by (3.11), which yields (3.19).

Now

$$\Big\|\sum_{|a_i|<\varepsilon} a_i y_i\| < m\varepsilon < \frac{1}{2}\Big\|\sum_{i=1}^{m}{}' a_i y_i\Big\|$$

since

$$\frac{1}{2}\Big\|\sum_{i=1}^{m}{}' a_i y_i\Big\| > \frac{1}{4}\frac{1}{(2K)^2} > m\varepsilon.$$

Thus by (3.19), $\|\sum_{i=1}^{m} a_i y_i\| < 3\delta K N^{1/p}$ so

$$\Big\|\sum_{i=1}^{m} a_i y_i\Big\|^p < 3^p\delta^p K^p \sum_{i=1}^{m}{}' n_i \overset{(3.16)}{<} 3^p K^p (2K)^p \sum_{i=1}^{m}{}' |a_i|^p < 3^p K^p (2K)^p,$$

and this completes the proof. ■

We do not know if the modified space $V_{1/2,2}$ or if the modified versions of Schlumprecht's space S (see [S]) are a.s. Since their natural bases are symmetric, our arguments fail. However, we do have the next theorem.

THEOREM 3.10. *Schlumprecht's space S is not w.n.a.s.*

Recall that S is the completion of c_{00} under the norm which satisfies the implicit equation

$$\|x\| = \max\Big\{\|x\|_\infty, \sup \frac{1}{f(k)}\sum_{j=1}^{k}\|E_j x\|\Big\}$$

where the "sup" is taken over all $k \geq 2$ and sets of integers $E_1 < \cdots < E_k$ with $f(k) = \log_2(k+1)$. It was shown in [KL] that c_0 is finitely represented in S and indeed our proof of Theorem 3.2 was modeled after that construction. In [M] an alternative "partially nested" construction was used. We shall follow the proof of Theorem 3.1 in [M] and use his notation to sketch the proof of Theorem 3.10.

Take $p = 1$, $q = \infty$, $\theta_k = 1/f(k)$, $n_k = k$, $1/p_k = 1 - \log_k(1/\theta_k) = 1 - \ln f(k)/\ln k$ and $1/q_k = \ln f(k)/\ln k$.

In the course of the proof, if n is arbitrary, k_0 is chosen so that $1/f(k_0) \leq 1/n$ and then k_1 is chosen with $f(k_0)/f(k_1) \leq 1/n$. If we take k_0 so that $f(k_0)$ has "order n", we need k_1 to satisfy (basically)

$$f(k_1) \geq n^2 \ . \tag{3.20}$$

Also the proof in [M] requires $k_0^{1/q_{k_1}} \leq 2$, which essentially transforms into

$$\frac{n \ln f(k_1)}{\ln k_1} \leq 1. \tag{3.21}$$

Proof of Theorem 3.10 (*sketch*). For an arbitrary integer n let $d = k_1$ satisfy (3.20) and (3.21). To prove that S is not w.n.a.s. we need only check the condition for $m \equiv nd$.

In order to apply Theorem 3.1 of [M] we choose a rapidly increasing sequence of integers $m \ll q_1 \ll \cdots \ll q_n$. We shall employ n different normalized distributions of elements of S. Precisely, for $j \leq n$ set

$$v_j = \frac{f(q_j)}{q_j} \sum_{i=1}^{q_j} e_i$$

where (e_i) is the unit vector basis of S. We then successively repeat each v_j d-times so that altogether we have $m = nd$ vectors to serve as distributions. We shall compare the norms of two different permutations of a block sequence of m vectors with these distributions. Since (e_i) is 1-subsymmetric this will show that S is not w.n.a.s.

Let $(y_{i,j})_{i=1,\, j=1}^{d,n}$ be a block basis of (e_i) in lexicographic order with $y_{i,j}$ equal to v_j in distribution. For $j \leq n$ let $u_j = \sum_{i=1}^{d} y_{i,j}$. Then

$$\|u_j\| = \frac{f(q_j) d q_j}{q_j f(d q_j)} \approx d.$$

The proof of Theorem 3.1 in [M] yields a constant C, independent of m, such that $(d^{-1}u_j)_{j=1}^n$ is C-equivalent to the unit vector basis of ℓ_∞^n. Thus

$$\Big\| \sum_{i=1}^{d} \sum_{j=1}^{n} y_{i,j} \Big\| = \Big\| \sum_{j=1}^{n} u_j \Big\| \leq Cd = \frac{Cm}{n}. \tag{3.22}$$

We next consider a different order. Let $(z_{i,j})_{i=1,\,j=1}^{d,n}$ be a block basis of (e_i), ordered with respect to the lexicographic order (j,i), the reverse coordinates, with $z_{i,j}$ equal in distribution to v_j. Set $w_j = \sum_{i=1}^{d} z_{i,j}$ so that $(w_j)_{j=1}^{d}$ is a block basis of (e_i). As before

$$\Big\|\sum_{i=1}^{d}\sum_{j=1}^{n} z_{i,j}\Big\| = \Big\|\sum_{j=1}^{n} w_j\Big\| \geq \frac{n}{f(n)}\,\frac{d}{2} = \frac{1}{2}\,\frac{m}{f(n)}. \tag{3.23}$$

Since $\frac{m}{f(n)}\big/\frac{m}{n} = \frac{n}{f(n)}$ we infer from (3.22) and (3.23) that S is not w.n.a.s. We conclude more precisely that the constant C_m for m sequences is at least of the order $\sqrt{\ln(m)}/\ln(\ln(m))$. ∎

REMARKS 3.11. Clearly the constant C_m, the w.n.a.s. constant for m normalized weakly null sequences in S, satisfies $C_m \leq f(m)$. We can show that $C_2 = f(2)$ with a different construction. Pei-Kee Lin [L] has pointed out that one can adjust the proof in [KL] to obtain a sequence in S whose spreading model is isometric to ℓ_1. Indeed in [KL] it was shown that for every $\varepsilon > 0$ there exists a rapidly increasing sequence of integers, $(p_k)_{k=1}^{\infty}$, so that if

$$u_j = \frac{f(p_j)}{p_j}\sum_{i=1}^{p_j} e_i$$

then for all n there exist disjointly supported vectors $(v_j)_{j=1}^{n}$ in S, each of the same distribution as u_j, with $\|\sum_{j=1}^{n} v_j\| \leq 1+\varepsilon$. One can choose scalars $(a_k)_{k=1}^{\infty} \subseteq (0,1)$ converging to 1 so that $\|z_n\| < 1$ and $\|z_n\| \to 1$ if $z_n = \sum_{j=1}^{n} a_j v_j$. It follows that any block basis (x_n^1) with distribution $x_n^1 =$ distribution z_n has spreading model 1-equivalent to the unit vector basis of ℓ_1. If we let (x_n^2) be a block basis with distribution $x_n^2 =$ distribution u_n, then the w.n.a.s. constant for these two sequences is $f(2)$. Finally, we note that since S is minimal [S2], no subspace of S is w.n.a.s.

4. Tsirelson's space is not iteration stable. No good criterion is known which forces a Banach space X to be isomorphic to a stable space. Attempting to find such a criterion, H. Rosenthal has asked if it might be true that every asymptotically symmetric and iteration stable space X is isomorphic to a stable space. In this regard he asked us if T is iteration stable. In this section we show that it is not. First we give the relevant definitions.

DEFINITION 4.1. A sequence (x_n) in a Banach space X is *type determining* if for all $x \in X$,

$$\lim_{n\to\infty}\|x + x_n\| \text{ exists.}$$

DEFINITION 4.2 (H. Rosenthal). A Banach space X is *iteration stable* if for all type determining sequences (x_n) and (y_n) in X,

$$\lim_{n\to\infty}\lim_{m\to\infty}\|x_n + y_m\| \text{ exists.}$$

Iteration stability is another softening of the definition of stability.

THEOREM 4.3. *Tsirelson's space T is not iteration stable.*

We recall that T is the completion of c_{00} under the norm satisfying the implicit equation

$$\|x\| = \max\Big(\|x\|_\infty, \sup \frac{1}{2}\sum_{i=1}^{n} \|Ex\|\Big)$$

where the "sup" is taken over all $n \geq 2$ and $n \leq E_1 < \cdots < E_n$. The unit vector basis (e_i) is a normalized 1-unconditional basis for the reflexive space T. For $k \geq 2$ we set

$$\|x\|_k = \sup \frac{1}{2}\sum_{i=1}^{k} \|E_i x\|$$

where the "sup" is taken over all $k \leq E_1 < \cdots < E_k$.

Proof of Theorem 4.3. For $n \geq 2$ let

$$z_n = \frac{2}{n^2}\sum_{i=1}^{n^2} e_{n^3+i}$$

so that $\|z_n\| = 1$ and z_n is an ℓ_1^n-average with constant 1. Precisely, $z_n = (1/n)\sum_{j=1}^{n} z_{n,j}$ where $z_{n,j} = (2/n)\sum_{i=1}^{n} e_{n^3+(j-1)n+i}$ for $j \leq n$. The sequence $(z_{n,j})_{j=1}^{n}$ is 1-equivalent to the unit vector basis of ℓ_1^n. A standard calculation yields

$$\lim_n \|z_n\|_k = \frac{1}{2} \quad \text{for all } k \geq 2. \tag{4.1}$$

For $n \in \mathbb{N}$ we set

$$x_n = \begin{cases} e_{n^3} + \frac{1}{4}e_{n^3+1}, & n \text{ odd},\\ e_{n^3} + \frac{1}{4}z_n, & n \text{ even}.\end{cases}$$

We let $y_m = \frac{1}{2}\sum_{i=1}^{4} e_{m+i}$ for $m \in \mathbb{N}$.

The sequence (y_n) is normalized and clearly type determining. Also $\|x_n\| = \|x_n\|_\infty = 1$ for all n. Moreover for $k \geq 2$, $n \in \mathbb{N}$,

$$\|x_n\|_k \leq \frac{1}{2}\Big(1 + \frac{1}{2}\Big) = \frac{3}{4}. \tag{4.2}$$

Note that

$$\lim_{n \text{ odd}} \lim_m \|x_n + y_m\| = \frac{1}{2}\Big(1 + \frac{1}{4} + 2\Big) = \frac{13}{8},$$

$$\lim_{n \text{ even}} \lim_m \|x_n + y_m\| = \frac{1}{2}\Big(1 + \frac{1}{2} + 2\Big) = \frac{14}{8}.$$

We shall show that (x_n) is type determining and this will complete the proof. To do this we verify by induction on l that $\lim_{n\to\infty} \|x + x_n\|$ exists for all $x \in \operatorname{span}(e_i)$ with $|\operatorname{supp} x| \le l$. The case $l = 0$ holds since $\|x_n\| = 1$ for all n so assume it holds for l and let $|\operatorname{supp} x| = l + 1$.

We may assume that $\operatorname{supp} x < \operatorname{supp} x_n$ and we define

$$S(x + x_n) = \sup_{k \le \max(\operatorname{supp} x)} \sup \frac{1}{2} \sum_{i=1}^{k} \|E_i(x + x_n)\|$$

where the second "sup" is taken over all $2 \le k \le E_1 < \cdots < E_k$ with $E_1 \cap \operatorname{supp} x \ne \emptyset$. We will show that

$$\lim_{n \to \infty} S(x + x_n) \text{ exists.} \tag{4.3}$$

Observe that for any n,

$$\|x + x_n\| = \max\{\|x + x_n\|_\infty, S(x + x_n)\}.$$

Indeed, if $k \le E_1 < \cdots < E_k$ with $\operatorname{supp} x < E_1$, then by (4.2),

$$\frac{1}{2} \sum_{i=1}^{k} \|E_i(x + x_n)\| \le \|x_n\|_k \le \frac{3}{4} < \|x_n\|_\infty.$$

Thus (4.3) will complete the proof.

To prove (4.3) we will show that we can balance the various types of splitting of $x + x_n$ by $(E_i)_{i=1}^k$ between n odd and n even. There is no need to consider splittings such that the last set E_k intersects $\operatorname{supp} x$ since that case is covered by the induction hypothesis. Also any values obtained for $S(x + x_n)$ resulting from splittings for n odd can be treated by a similar splitting for n even, considering $\frac{1}{4} z_n$ as a single element. We need to show that $\overline{\lim}_{n\,\mathrm{even}} S(x + x_n)$ cannot be greater than $\lim_{n\,\mathrm{odd}} S(x + x_n)$.

Take a splitting $(E_i)_{i=1}^k$ giving rise to $S(x + x_n)$ for n even.

Case 1: There exists $i_0 < k$ such that $\max E_{i_0} = n^3$. Thus the support of z_n is split into $k - i_0$ intervals, $k \le \max\operatorname{supp} x$. By (4.1) this presents no problem as we can obtain the same values for n odd (as $n \to \infty$).

Case 2: There exists $i_0 < k$ with $n^3 \in \operatorname{supp} E_{i_0}$ and $E_{i_0} \cap \operatorname{supp} z_n \ne \emptyset$. By the triangle inequality,

$$\begin{aligned}\frac{1}{2} \sum_{i=1}^{k} &\|E_i(x + x_n)\| \\ &\le \frac{1}{2} \sum_{i=1}^{i_0-1} \|E_i x\| + \frac{1}{2} \|E'_{i_0}(x + x_n)\| + \frac{1}{2} \|E''_{i_0} x_n\| + \frac{1}{2} \sum_{i=i_0+1}^{k} \|E_i x_n\|\end{aligned}$$

where $E_{i_0} = E'_{i_0} \cup E''_{i_0}$, $E'_{i_0} < E''_{i_0}$ and $\max E'_{i_0} = n^3$. The new splitting might not be admissible but $j = k - i_0 + 1 \le \max\operatorname{supp} x$, and again by (4.1), the value for the new splitting behaves in the limit as an admissible splitting

$(E_1, \ldots, E_{i_0-1}, E'_{i_0}, E)$ where $E = E''_{i_0} \cup \bigcup_{i=i_0+1}^k E_i$. This last splitting can be mimicked to yield the same value for n odd. ■

REMARK 4.4. Our proof depends upon using vectors whose norm is given by $\|\cdot\|_\infty$. In regard to Rosenthal's question it is of interest to determine if T contains iteration stable subspaces. Accordingly we have the following partial result.

PROPOSITION 4.5. *Let $X \subseteq T$ be an infinite-dimensional subspace so that for all $0 \neq x \in X$, $\|x\| \neq \|x\|_\infty$. If $(x_n), (y_n) \subseteq X$ are weakly null sequences*, $\lim_n \|x_n\|$ *exists and (y_n) is type determining*, *then*

$$\lim_{n\to\infty} \lim_{m\to\infty} \|x_n + y_m\| \quad \textit{exists.} \tag{4.4}$$

Proof. We may assume that $\|x_n\| = 1$ for all n. Also, to prove (4.4) we can freely pass to subsequences of (y_n) and so we may assume that for all k,

$$\lim_m \|y_m\|_k = \lambda(k) \ \text{ exists.} \tag{4.5}$$

By perturbing we may assume that (x_n) and (y_m) are block bases of (e_i).

Since $\|y_m\|_k \le \|y_m\|_{k+1}$ when $\operatorname{supp} y_m > k$, it follows that $\lambda(k) \le \lambda(k+1) \le \lim_m \|y_m\|$ for all k. Let $\lambda(k) \uparrow \lambda$. We prove

$$\lim_n \lim_m \|x_n + y_m\| = (1+\lambda) \vee \lim_m \|y_m\|, \tag{4.6}$$

which yields (4.4).

Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ with $\lambda(k) > \lambda - \varepsilon$. Choose $\overline{t} \in \mathbb{N}$ so that if $t \ge \overline{t}$ and $\sum_{i=1}^t |a_i| = 1$ then there exists $F \subseteq \{1, \ldots, t\}$ with $|F| = k$ and $\sum_{i\in F} |a_i| < \varepsilon$. Let n be such that $\operatorname{supp} x_n > \overline{t}$. Choose m_0 so that for $m \ge m_0$, $\big|\|y_m\|_k - \lambda(k)\big| < \varepsilon$.

We will prove first that

$$\varliminf_m \|x_n + y_m\| > (1 + \lambda - 2\varepsilon) \vee \lim_m \|y_m\|. \tag{4.7}$$

Let $1 = \|x_n\| = \frac{1}{2}\sum_{i=1}^t \|E_i x_n\|$ for some $t \ge \overline{t}$ and $t \le E_1 < \cdots < E_t$. Let $m \ge m_0$ so that $\operatorname{supp} x_n < \operatorname{supp} y_m$ and choose $\min(\operatorname{supp} y_m) \le F_1 < \cdots < F_k$ with $\|y_m\|_k = \frac{1}{2}\sum_{i=1}^k \|F_i y_m\|$. By deleting the smallest k terms from $\frac{1}{2}\sum_{i=1}^t \|E_i x_n\|$ and replacing them by $\frac{1}{2}\sum_{i=1}^k \|F_i y_m\|$ we obtain

$$\|x_n + y_m\| \ge 1 - \varepsilon + \|y_m\|_k > 1 + \lambda - 2\varepsilon$$

and (4.7) follows.

We next prove that

$$\varlimsup_m \|x_n + y_m\| < (1 + \lambda + \varepsilon) \vee \lim_m \|y_m\|, \tag{4.8}$$

which will complete the proof of (4.6). Let $\operatorname{supp} x_n < \operatorname{supp} y_m$ and $\|x_n + y_m\| = \frac{1}{2}\sum_{i=1}^t \|G_i(x_n + y_m)\|$ where $t \ge \overline{t}$ and $t \le G_1 < \cdots < G_t$. Suppose there does not exist i_0 with $G_{i_0} x_n \neq 0$ and $G_{i_0} y_m \neq 0$. Then

$$\|x_n + y_m\| \le \|x_n\| + \|y_m\|_{k(m)}$$

where $k(m) \le t$. If this occurred for infinitely many m's we would obtain $\lim_m \|x_n + y_m\| \le (1+\lambda) \vee \lim_m \|y_m\|$. Suppose such a G_{i_0} exists. Split $G_{i_0} = G'_{i_0} \cup G''_{i_0}$ with $G'_{i_0} < G''_{i_0}$ and $G_{i_0} x_n = G'_{i_0} x_n$. We now have $t+1$ sets in the sum

$$\frac{1}{2} \sum_{i=1}^{i_0-1} \|G_i x_n\| + \|G'_{i_0} x_n\| + \|G''_{i_0} y_m\| + \sum_{i=i_0+1}^{t} \|G_i y_m\|.$$

If we delete the smallest term we have t sets after coordinate t and so for some $l(m) \le \max \operatorname{supp} x_n$,

$$\|x_n + y_m\| \le \|x_n\| + \|y_m\|_{l(m)} + \varepsilon.$$

Again, letting $m \to \infty$, we obtain (4.8). ∎

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