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Almost periodicity of C-semigroups, integrated semigroups and C-cosine functions

by

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Abstract. We investigate the characterization of almost periodic C-semigroups, via the Hille–Yosida space Z_0, in case of $R(C)$ being non-dense. Analogous results are obtained for C-cosine functions. We also discuss the almost periodicity of integrated semigroups.

0. Introduction. Characterizations of almost periodic semigroups and groups of class C_0 were studied by Bart and Goldberg [1] in 1978. Later, Cioranescu [3], Piskarev [14, 15] and others discussed the almost periodicity of strongly continuous cosine functions. Recently, Zheng and Liu [21] studied the almost periodicity of C-semigroups and C-cosine functions under the assumption that $R(C)$ is dense.

In this paper, we investigate the situation where $R(C)$ is allowed to be non-dense. We characterize the generator of an almost periodic C-semigroup, A, via the Hille–Yosida space, Z_0, which is a maximal continuously imbedded subspace of X on which A generates a strongly continuous semigroup. Kantorovitz [13] first introduced the Hille–Yosida space for a closed operator A with $(0,\infty) \subset \varrho(A)$, on which the restriction of A generates a semigroup of class C_0. R. deLaubenfels [8] extended it to more general cases that A has no eigenvalues in $(0,\infty)$, and used it to connect C-semigroups with semigroups of class C_0. Similarly, Cioranescu [2] constructed the Hille–Yosida space of cosine functions. For the extensive literature on this subject, we refer to [19].

Let $\mathcal{I} := \operatorname{span}\{x \in D(A) : Ax = irx \text{ for some } r \in \mathbb{R}\}$. We show in Theorem 2.4 that if A has no eigenvalues in $(0,\infty)$ and $C^{-1}AC = A$, then A generates an almost periodic C-semigroup if and only if the image of C is contained in $(Z_0)_a$, the closure of $\mathcal{I}$ in Z_0, the Hille–Yosida space for A; and $(Z_0)_a$ is proved to be a maximal continuously imbedded subspace of X on

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which A generates an almost periodic semigroup of class C_0 of contractions (Theorem 2.6). The key fact here is that a solution of the abstract Cauchy problem is almost periodic in Z_0 if and only if it is almost periodic in X. The same method applies to the case of asymptotic almost periodicity of C-semigroups; but this is the subject of another paper ([18]). Theorem 4.2 gives the analogous result for C-cosine functions. We also consider the periodicity (Theorems 2.8 and 4.3). Our results generalize the corresponding ones in [21].

If $\sigma(A) \cap i\mathbb{R}$ is at most countable, then a C-semigroup $T(t)$ is almost periodic if and only if $e^{-\lambda t}T(t)x$ has uniformly convergent means for $\lambda \in \sigma(A) \cap i\mathbb{R}$, $x \in X$. This is proved in Theorem 2.9.

In Section 3 the almost periodicity of integrated semigroups is discussed. Theorem 3.3 asserts that, if A generates a bounded $(r-A)^{-1}$-semigroup $T(t)$ and a bounded integrated semigroup $S(t)$, then $T(t)$ is almost periodic if and only if $S(t)$ is almost periodic. Theorem 3.3 relates almost periodicity of bounded $(r-A)^{-1}$-groups and bounded integrated groups to uniformly convergent means.

Throughout this paper, X will be a Banach space, the dual space will be denoted by X^*. All operators are linear. The space of all bounded linear operators on X will be denoted by $B(X)$. $C \in B(X)$ will be injective. For an operator A, we will write $D(A)$ for its domain, $R(A)$ for its range. Finally, $J = \mathbb{R}$ or $\mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$.

1. Preliminaries. First, we recall the definition and basic properties of C-semigroups or groups.

DEFINITION 1.1. A strongly continuous family $T(t)\,(t \in J) \subset B(X)$ is called a *C-semigroup* ($J = \mathbb{R}^+$) or a *C-group* ($J = \mathbb{R}$) if $T(t+s)C = T(t)T(s)$ for $t, s \in J$ and $T(0) = C$. The *generator* A is defined by

$$D(A) = \{x \in X : \lim_{J \ni t \to 0} t^{-1}(T(t)x - Cx) \text{ exists and belongs to } R(C)\}$$

with

$$Ax = C^{-1}(\lim_{J \ni t \to 0} t^{-1}(T(t)x - Cx)) \quad \text{for } x \in D(A).$$

The complex number λ is in $\varrho_C(A)$, the *C-resolvent set of* A, if $\lambda - A$ is injective and $R(C) \subseteq R(\lambda - A)$; we set $\sigma_C(A) := C \setminus \varrho_C(A)$.

LEMMA 1.2 ([8]). *Let $T(t)$ $(t \in J)$ be a C-semigroup or C-group with generator A. Then*

(a) *A is closed and $R(C) \subset \overline{D(A)}$;*

(b) *$\int_0^t T(s)x\,ds \in D(A)$ with $A\int_0^t T(s)x\,ds = T(t)x - Cx$ for all $x \in X$ and $t \in J$;*

(c) $T(t)x \in D(A)$ *with* $AT(t)x = T(t)Ax$, *and* $\int_0^t T(s)Ax\,ds = T(t)x - Cx$ *for all* $x \in D(A)$ *and* $t \in J$;

(d) *if* $T(t)$ *is uniformly bounded*, *then* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \in J\backslash\{0\}\} \subset \varrho_C(A)$ *and* $(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}T(t)x\,dt$ *for all* $x \in X$ *and* $\operatorname{Re}\lambda > 0$.

Next, we need to introduce the Hille–Yosida space for an operator; for the details we refer to [8].

Definition 1.3. Suppose A has no eigenvalues in $(0, \infty)$ and is a closed linear operator. The *Hille–Yosida space for* A, Z_0, is defined by

$$Z_0 = \{x \in X : \text{the Cauchy problem } u'(t) = Au(t), u(0) = x \text{ has a bounded uniformly continuous mild solution } u(\cdot, x)\}$$

with

$$\|x\|_{Z_0} = \sup\{\|u(t,x)\| : t \geq 0\} \quad \text{for } x \in Z_0.$$

Lemma 1.4 ([8]). *Let A generate a bounded strongly uniformly continuous C-semigroup $T(t)$. Then $R(C) \subset Z_0$ and $A|_{Z_0}$ generates a contraction semigroup of class C_0 given by $S(t) = C^{-1}T(t)$ and*

$$Z_0 = \{x : t \to C^{-1}T(t)x \textit{ is bounded and uniformly continuous}\}$$

with

$$\|x\|_{Z_0} = \sup_{t \geq 0} \|C^{-1}T(t)x\|.$$

Now we introduce the notion of a mild C-existence family, which is more general than C-semigroup.

Definition 1.5. The family of operators $\{T(t)\}_{t \geq 0} \subseteq B(X)$ is a *mild C-existence family for A* if

(a) the map $t \mapsto T(t)x$, from $[0, \infty)$ into X, is continuous, for all $x \in X$;

(b) for all $x \in X$ and $t > 0$, $\int_0^t T(s)x\,ds \in D(A)$ with $A(\int_0^t T(s)x\,ds) = T(t)x - Cx$.

Definition 1.6. (a) A function $f \in C(J, X)$ is *almost periodic*, written $f \in \mathrm{AP}(J, X)$, if for every $\varepsilon > 0$, there exists $l > 0$ such that every subinterval of J of length l contains at least one τ satisfying $\|f(t + \tau) - f(t)\| \leq \varepsilon$ for all $t \in J$.

(b) Let $F(t) \in B(X)$ $(t \in J)$ be a strongly continuous operator family. Then $F(t)$ is *almost periodic* if for every $x \in X$, $F(\cdot)x$ is almost periodic; $F(t)$ is *periodic* with period p if $F(t + p) = F(t)$ for all $t \in J$.

We collect some basic results on vector-valued almost periodic functions in the following lemma (see [21]).

Lemma 1.7. *Let $f \in \mathrm{AP}(\mathbb{R}, X)$. Then*

(a) *$f(t)$ is bounded, i.e., $\sup_{t\in\mathbb{R}} \|f(t)\| < \infty$;*

(b) *if $g \in \mathrm{AP}(\mathbb{R}, X)$, $h \in \mathrm{AP}(\mathbb{R}, C)$, then $f+g, hf \in \mathrm{AP}(\mathbb{R}, X)$;*

(c) *$a_r(f) := \lim_{t\to\infty} t^{-1} \int_0^t e^{-irs} f(s)\, ds$ exists and*

$$a_r(f) = \lim_{t\to\infty} \frac{1}{t} \int_{\alpha}^{\alpha+t} e^{-irs} f(s)\, ds \quad \text{for all } r, \alpha \in \mathbb{R};$$

(d) *if $a_r(f) = 0$ for all $r \in \mathbb{R}$, then $f(t) = 0$ for all $t \in \mathbb{R}$;*

(e) *$\sigma(f) := \{r \in \mathbb{R} : a_r(f) \neq 0\}$ is at most countable;*

(f) *if $X \not\supset c_0$ (that is, X does not contain an isomorphic copy of c_0, where c_0 is the space of all numerical sequences converging to 0), and $g(t) = \int_0^t f(s)\, ds$ ($t \in \mathbb{R}$) is bounded, then $g \in \mathrm{AP}(\mathbb{R}, X)$;*

(g) *if $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{AP}(\mathbb{R}, X)$ and $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f, then $f \in \mathrm{AP}(\mathbb{R}, X)$;*

(h) *if $f'(t)$ exists and is uniformly continuous, then $f' \in \mathrm{AP}(\mathbb{R}, X)$.*

The following lemma follows immediately from Lemmas 1.4 and 1.7.

Lemma 1.8. *Suppose $T(t)$ is an almost periodic C-semigroup with generator A. Then*

(a) *$T(t)$ is bounded and strongly uniformly continuous;*

(b) *$R(C) \subset Z_0$, the Hille–Yosida space for A, and $T(t) = e^{tA|_{Z_0}} C$.*

2. Almost periodic C-semigroups and C-groups. In this section, we discuss the almost periodicity of C-semigroups and C-groups. The following is the main result of this section.

Theorem 2.1. *Let $T(t)$ be a C-semigroup on X with generator A. Then $T(t)$ is almost periodic if and only if $R(C) \subset (Z_0)_a$, the closure of $\mathcal{I}$ in Z_0.*

Proof. Sufficiency. Since $R(C) \subset (Z_0)_a$, for fixed $x \in X$ and $\varepsilon > 0$, there exist finitely many points $r_k \in \mathbb{R}$ and $x_k \in \ker(ir_k - A)$ such that $\|Cx - \sum \alpha_k x_k\|_{Z_0} \le \varepsilon$. Thus $\|e^{tA|_{Z_0}} Cx - \sum \alpha_k e^{tA|_{Z_0}} x_k\|_{Z_0} \le \varepsilon$. But $Ax_k = ir_k x_k$, so $e^{tA|_{Z_0}} x_k = e^{ir_k t} x_k \in \mathrm{AP}(\mathbb{R}^+, X)$, i.e.,

$$\Big\| e^{tA|_{Z_0}} Cx - \sum \alpha_k e^{ir_k t} x_k \Big\|_{Z_0} \le \varepsilon.$$

So we have

$$\Big\| e^{tA|_{Z_0}} Cx - \sum \alpha_k e^{ir_k t} x_k \Big\| \le \Big\| e^{tA|_{Z_0}} Cx - \sum \alpha_k e^{ir_k t} x_k \Big\|_{Z_0} \le \varepsilon \quad \text{for } t \ge 0.$$

Hence $T(t)x = e^{tA|_{Z_0}} Cx \in \mathrm{AP}(\mathbb{R}^+, X)$, and so $T(t)$ is almost periodic.

Necessity. Define $P_r x = \lim_{t\to\infty} t^{-1} \int_0^t e^{-irs} T(s)x\, ds$ for each $r \in \mathbb{R}$ and $x \in X$. Then by Lemma 1.7(c) and from the proof of [21, Theorem 2.1], we know that $P_r x$ exists and belongs to $D(A)$ with $AP_r x = irP_r x$. Thus,

$$T(t)P_r x = \lim_{s\to\infty} \frac{1}{s}\int_0^s e^{-ir\tau}T(t+\tau)Cx\,d\tau = C\lim_{s\to\infty}\frac{1}{s}\int_t^{t+s} e^{-ir(\tau-t)}T(\tau)x\,d\tau$$
$$= Ce^{irt}\lim_{s\to\infty}\frac{1}{s}\int_t^{t+s} e^{-ir\tau}T(\tau)x\,d\tau = e^{irt}CP_r x.$$

Hence, $T(t)P_r x \in R(C)$ and $C^{-1}T(t)P_r x = e^{irt}P_r x$ is bounded, and uniformly continuous. This implies $P_r x \in Z_0$ and $\{P_r x : r \in \mathbb{R},\, x \in X\} \subset D(A|_{Z_0})$ with $A|_{Z_0}P_r x = irP_r x$.

For every $x \in X$, since $t \mapsto T(t)x$ is bounded and uniformly continuous, we see that $T(t)x \in Z_0$ for $t \ge 0$. Next, we show $T(t)x \in \mathrm{AP}(\mathbb{R}^+, Z_0)$. Since $T(t)x \in \mathrm{AP}(\mathbb{R}^+, X)$, for every $\varepsilon > 0$, there exists $l > 0$ such that every subinterval of $\mathbb{R}^+$ of length l contains at least one τ satisfying $\sup_{t\in\mathbb{R}^+}\|T(t+\tau)x - T(t)x\| \le \varepsilon$. Then

$$\begin{aligned}\sup_{t\ge0}\|T(t+\tau)x - T(t)x\|_{Z_0} &= \sup_{t,s\ge0}\|C^{-1}T(s)T(t+\tau)x - C^{-1}T(s)T(t)x\| \\ &= \sup_{t,s\ge0}\|T(t+s+\tau)x - T(t+s)x\| \\ &\le \sup_{t\ge0}\|T(t+\tau)x - T(t)x\| \le \varepsilon,\end{aligned}$$

i.e., $T(t)x \in \mathrm{AP}(\mathbb{R}^+, Z_0)$. If $f \in Z_0^*$ is such that $f(P_r x) \equiv 0$ for all $x \in X$ and $r \in \mathbb{R}$, then $\lim_{t\to\infty} t^{-1}\int_0^t e^{-irs}f(T(s)x)\,ds = f(P_r x) \equiv 0$. But $f(T(t)x) \in \mathrm{AP}(\mathbb{R}^+, \mathbf{C})$. Thus by Lemma 1.7(d), we get $f(T(t)x) \equiv 0$ for all $t \in \mathbb{R}^+$ and $x \in X$. In particular, $f(Cx) \equiv 0$. Therefore, $\{P_r x : r \in \mathbb{R},\, x \in X\}^\perp \subset R(C)^\perp$, i.e.,

$$\begin{aligned}R(C) \subset {}^\perp(R(C)^\perp) &\subset {}^\perp(\{P_r x : r \in \mathbb{R},\, x \in X\}^\perp) \\ &= \overline{\mathrm{span}}\{P_r x : r \in \mathbb{R},\, x \in X\} \subset (Z_0)_a = \overline{\mathcal{I}},\end{aligned}$$

where all the closures are taken in Z_0. ∎

Now we have the following result ([21, Theorem 2.1]) as a corollary.

Corollary 2.2. *If $\overline{R(C)} = X$, then $T(t)$ is an almost periodic C-semigroup with generator A if and only if $T(t)$ is bounded and $X = X_a$, where X_a is the closure of $\mathcal{I}$ in X.*

Proof. The sufficiency is obvious. For the converse, since $Z_0 \hookrightarrow X$, a Cauchy sequence in Z_0 is also a Cauchy sequence in X, so that $(Z_0)_a \subseteq X_a$. By Theorem 2.1, $R(C) \subset (Z_0)_a$, hence $R(C) \subset X_a$; taking closure on both sides yields $X = X_a$. ∎

By Definition 1.5 and combining Theorem 2.1 with [8, Theorem 5.16], we have

THEOREM 2.3. *Suppose A has no eigenvalues in $(0,\infty)$. Then there exists an almost periodic mild C-existence family for A if and only if $R(C) \subset (Z_0)_a$.*

Moreover, combining Theorem 2.1 with [8, Theorem 5.17] and [10, Corollary 3.14] gives

THEOREM 2.4. *Suppose A is closed and has no eigenvalues in $(0,\infty)$, and $C^{-1}AC = A$. Then A generates an almost periodic C-semigroup if and only if $R(C) \subset (Z_0)_a$.*

Now we investigate a special case.

COROLLARY 2.5. *If $C = (r-A)^{-n}$ for some $n \in \mathbb{N}$, and $T(t)$ is a bounded strongly uniformly continuous C-semigroup generated by A, then $T(t)$ is almost periodic if and only if $S(t) := e^{tA|_{Z_0}}$ is almost periodic.*

Proof. From the proof of Theorem 2.1, we see that $T(t)$ almost periodic on X implies $T(t) = S(t)(r-A)^{-n}$ is almost periodic on Z_0. Applying Lemma 1.7(h) n times, we deduce that $S(t)$ is almost periodic. The converse holds since $T(t) = S(t)C$ and $Z_0 \hookrightarrow X$. ∎

The following theorem clarifies the relations between almost periodic C-semigroups and semigroups of class C_0.

THEOREM 2.6. *Let $T(t)$ be an almost periodic C-semigroup with generator A. Then there exists a maximal continuously imbedded subspace W of X such that $A|_W$ generates a contraction almost periodic semigroup of class C_0 on W and $R(C) \subset W$; W is maximal-unique in the sense that if $Y \hookrightarrow X$ and $A|_Y$ generates a contraction almost periodic semigroup of class C_0 on Y, then $Y \hookrightarrow W$.*

Proof. Let $S(t)$ be the semigroup of class C_0 generated by $A|_{Z_0}$. Since $S(t)x = e^{irt}x$, for $Ax = irx$, $S(t)$ clearly takes $\mathcal{I}$ to itself, therefore, since $S(t)$ is continuous, it takes the closure of $\mathcal{I}$ to itself, that is to say, $S(t)(Z_0)_a \subset (Z_0)_a$. Set $W = (Z_0)_a$; the first half of the result follows.

Now suppose $Y \hookrightarrow X$ and $A|_Y$ generates a contraction almost periodic semigroup of class C_0. Then $Y \hookrightarrow Z_0$, since Z_0 is maximal (cf. [8, Theorem 5.5]). It follows that $(Z_0)_a$ contains the closure of $\operatorname{span}\{x \in D(A|_Y) : Ax = irx \text{ for some } r \in \mathbb{R}\}$ in Y, which is exactly Y, so that $Y \hookrightarrow W = (Z_0)_a$. ∎

REMARK 2.7. We can consider $(Z_0)_a$ for any closed operator A with $s - A$ injective for $s > 0$. The results of Theorem 2.5 are also true, and it is not hard to see that $(Z_0)_a$ equals the set of all almost periodic orbits.

THEOREM 2.8. *Assume that A generates a C-group $T(t)$. Then $T(t)$ is a periodic C-group with period p if and only if $\sigma_C(A) \subset (2\pi i/p)\mathbb{Z}$ and $R(C) \subset (Z_0)_a$.*

Proof. Necessity. By Lemma 1.2(b) and the fact that $T(p) = C$,

$$(\lambda - A)\int_0^p e^{-\lambda s} T(s)x\,ds = (1 - e^{-\lambda p})Cx \quad \text{for all } x \in X.$$

Combining this with $T(s)Ax = AT(s)x$ for every $x \in D(A)$, we get $\sigma_C(A) \subset (2\pi i/p)\mathbb{Z}$, while $R(C) \subset (Z_0)_a$ follows from Theorem 2.1.

Sufficiency. If $x \in \ker(2\pi ik/p - A)$ for some $k \in \mathbb{Z}$, then $T(t)x = e^{2\pi ikt/p}Cx$, which implies $T(t+p)x = T(t)x$ for $t \in \mathbb{R}$; the same holds for every $x \in \mathcal{I}$. Since $T(t)$ is continuous in Z_0, we have $T(t+p)x = T(t)x$ for all $x \in (Z_0)_a$; in particular, $T(t+p)Cx = T(t)Cx$ for all $x \in X$ by our assumption $R(C) \subset (Z_0)_a$, therefore, since C is injective, we obtain $T(t+p) = T(t)$. ∎

It is shown in [1] that every almost periodic semigroup of class C_0 can be extended to an almost periodic group; from [21, Theorem 3.1], we know that every almost periodic C-semigroup can also be extended to an almost periodic C-group. So we can assume that A generates an almost periodic C-group.

Applying [17, Theorem 4.4] and the Hille–Yosida space, we obtain the following result, where we say that a function u has *uniformly convergent means* if

$$\lim_{R\to\infty} \frac{1}{R} \int_{a-R}^{a+R} u(s)\,ds$$

exists, uniformly in $a \in \mathbb{R}$.

THEOREM 2.9. *Suppose $T(t)$ is a bounded strongly uniformly continuous C-group with generator A such that $\sigma(A) \cap i\mathbb{R}$ is at most countable. Then the following assertions are equivalent.*

(a) *$T(t)$ is almost periodic.*
(b) *For $\lambda \in \sigma(A) \cap i\mathbb{R}, x \in X, e^{-\lambda t}T(t)x$ has uniformly convergent means.*

Proof. By [21, Theorem 3.1], A and $-A$ generate C-semigroups $T(t)$ and $T(-t)$ $(t \ge 0)$, respectively, so that the Cauchy problem $u'(t) = Au(t)$ has a bounded uniformly continuous mild solution $T(t)x$ on $\mathbb{R}$.

Suppose $S(t)$ is the semigroup of class C_0 generated by $A|_{Z_0}$. From the proof of Theorem 2.1, we know $T(t)x$ is almost periodic if and only if $S(t)Cx$ is almost periodic in Z_0. To see that (b) implies (a), by [17, Theorem 4.4], we only need to show that $e^{-\lambda t}S(t)Cx$ has uniformly convergent means in Z_0 for $\lambda \in \sigma(A|_{Z_0}) \cap i\mathbb{R}$. This can be achieved by a small modification of [9, Theorem 4].

(a)⇒(b) is trivial, since $T(t)x$ and $e^{-\lambda t}T(t)x$ $(\lambda \in i\mathbb{R})$ are almost periodic. ∎

3. Almost periodicity of integrated semigroups. An *integrated semigroup* is a strongly continuous family $S(t)$ such that $S(0)=0$ and

$$(1)\qquad S(t)S(s)=\int_t^{s+t} S(r)\,dr-\int_0^s S(r)\,dr$$

for all $s,t\ge 0$.

Let $r\in\varrho(A)\neq\emptyset$. From [8, Theorem 18.3], we know that A generates an $(r-A)^{-1}$-semigroup $T(t)$ if and only if A generates an integrated semigroup $S(t)$, and $T(t)x=\frac{d}{dt}S(t)(r-A)^{-1}x$.

Suppose $T(t)$ and $S(t)$ are bounded, and strongly uniformly continuous.

If $S(t)$ is almost periodic, then $S(t)(r-A)^{-1}x$ is almost periodic, and $T(t)x=\frac{d}{dt}S(t)(r-A)^{-1}x$ is uniformly continuous, so that $T(t)x$ is almost periodic.

Conversely, suppose $T(t)$ is almost periodic, and X does not contain an isomorphic copy of c_0. Since

$$(2)\qquad S(t)x=(r-A)\int_0^t T(s)x\,ds=r\int_0^t T(s)x\,ds-T(t)x+(r-A)^{-1}x$$

is bounded, we conclude that $\int_0^t T(s)x\,ds$ is bounded; by Lemma 1.7(f), $\int_0^t T(s)x\,ds$ is almost periodic, therefore so is $S(t)x$.

Combining the above with Theorem 2.1, we have

THEOREM 3.1. *Suppose $r\in\varrho(A)\neq\emptyset$, A generates a bounded strongly uniformly continuous $(r-A)^{-1}$-semigroup $T(t)$ and a bounded integrated semigroup $S(t)$, and suppose X does not contain an isomorphic copy of c_0. Then the following statements are equivalent.*

(a) *$T(t)$ is almost periodic.*
(b) *$S(t)$ is almost periodic.*
(c) *$D(A)\subset(Z_0)_a$.*

REMARK 3.2. (a) If A generates an almost periodic $(r-A)^{-1}$-semigroup $T(t)$, then A also generates an integrated semigroup $S(t)$. However, the almost periodicity of $T(t)$ does not guarantee the almost periodicity of $S(t)$. In fact, if $T(t)$ is periodic with period p, and $\int_0^p T(t)x\,dt\neq 0$, then $\int_0^t T(s)x\,ds$ is not bounded, so that $S(t)x$ is not bounded. So the assumption that $S(t)$ is bounded in Theorem 3.1 is necessary.

(b) The assumption that $X\not\supset c_0$ is not needed for the implication (b)⇒(a) of Theorem 3.1; the same holds for (b)⇒(a) of Theorem 3.3.

Let $C=(r-A)^{-1}$. Suppose A generates a C-group $T(t)$. Then A and $-A$ generate C-semigroups $T(t)$ and $T(-t)$ $(t\ge 0)$, respectively. Hence A and $-A$ also generate integrated semigroups $S(t)$ and $S(-t)$ $(t\ge 0)$ such that $T(t)=\frac{d}{dt}S(t)(r-A)^{-1}, T(-t)=\frac{d}{dt}S(-t)(r-A)^{-1}$, respectively. It is easy to verify that (1) holds for all $t, s \in \mathbb{R}$. So we call $S(t)$ $(t \in \mathbb{R})$ an *integrated group*.

THEOREM 3.3. *Let $r \in \varrho(A) \neq \emptyset$. Suppose A generates a bounded strongly uniformly continuous $(r - A)^{-1}$-group $T(t)$ and a bounded integrated group $S(t)$ such that $\sigma(A) \cap i\mathbb{R}$ is at most countable, and X does not contain an isomorphic copy of c_0. Then the following statements are equivalent.*

(a) *$T(t)$ is almost periodic.*

(b) *$S(t)$ is almost periodic.*

(c) *For $\lambda \in \sigma(A) \cap i\mathbb{R}$ and $x \in X$, $e^{-\lambda t}T(t)x$ has uniformly convergent means.*

(d) *For $\lambda \in \sigma(A) \cap i\mathbb{R}$ and $x \in X$, $e^{-\lambda t}S(t)x$ has uniformly convergent means.*

Proof. We only need to show (c)⇔(d).

(c)⇒(d). By (c) and Theorem 3.1, $S(t)$ is almost periodic, thus $S(t)$ has uniformly convergent means, i.e., (d) holds for $\lambda = 0$.

Now suppose $\lambda \in \sigma(A) \cap i\mathbb{R}\backslash\{0\}$. Fix $\varepsilon > 0$. Then by the assumption of (c), there exists T_ε such that

$$\left\| \frac{1}{T} \int_{h-T}^{h+T} e^{-\lambda t} T(t)x\,dt - \frac{1}{S} \int_{h-S}^{h+S} e^{-\lambda t} T(t)x\,dt \right\| < \varepsilon$$

for all $T, S > T_\varepsilon$ and $h \in \mathbb{R}$.

To prove $e^{-\lambda t}S(t)x$ has uniformly convergent means, by (2), it suffices to show $e^{-\lambda t}\int_0^t T(s)x\,ds$ has uniformly convergent means. Suppose $\|\int_0^t T(s)x\,ds\| \leq M$ and $T, S > 1/|\lambda\varepsilon|$. Then

$$\begin{aligned}
&\left\| \frac{1}{T} \int_{h-T}^{h+T} e^{-\lambda t} \int_0^t T(\tau)x\,d\tau\,dt - \frac{1}{S} \int_{h-S}^{h+S} e^{-\lambda t} \int_0^t T(\tau)x\,d\tau\,dt \right\| \\
&\quad = \left\| \frac{1}{\lambda T} \int_{h-T}^{h+T} e^{-\lambda t} T(t)x\,dt - \frac{1}{\lambda S} \int_{h-S}^{h+S} e^{-\lambda t} T(t)x\,dt \right. \\
&\qquad - \frac{1}{\lambda T} e^{-\lambda(h+T)} \int_0^{h+T} T(t)x\,dt + \frac{1}{\lambda T} e^{-\lambda(h-T)} \int_0^{h-T} T(t)x\,dt \\
&\qquad \left. + \frac{1}{\lambda S} e^{-\lambda(h+S)} \int_0^{h+S} T(t)x\,dt - \frac{1}{\lambda S} e^{-\lambda(h-S)} \int_0^{h-S} T(t)x\,dt \right\| \\
&\quad < \varepsilon + 4M\varepsilon;
\end{aligned}$$

the result then follows.

(d)⇒(c). Given $\varepsilon > 0$ and $x \in X$, there exists T_ε such that

$$\left\| \frac{1}{K} \int\limits_{h-K}^{h+K} e^{-\lambda t} S(t)(r-A)^{-1} x\, dt - \frac{1}{L} \int\limits_{h-L}^{h+L} e^{-\lambda t} S(t)(r-A)^{-1} x\, dt \right\| < \varepsilon$$

for all $K, L > T_\varepsilon$ and $h \in \mathbb{R}$. Suppose $\|S(t)(r-A)^{-1}x\| \le M$ and $K, L > 1/\varepsilon$. Then

$$\begin{aligned}
&\left\| \frac{1}{K} \int\limits_{h-K}^{h+K} e^{-\lambda t} T(t) x\, dt - \frac{1}{L} \int\limits_{h-L}^{h+L} e^{-\lambda t} T(t) x\, dt \right\| \\
&\quad = \left\| \frac{1}{K} \int\limits_{h-K}^{h+K} e^{-\lambda t} \frac{d}{dt} S(t)(r-A)^{-1} x\, dt - \frac{1}{L} \int\limits_{h-L}^{h+L} e^{-\lambda t} \frac{d}{dt} S(t)(r-A)^{-1} x\, dt \right\| \\
&\quad = \left\| \frac{\lambda}{K} \int\limits_{h-K}^{h+K} e^{-\lambda t} S(t)(r-A)^{-1} x\, dt - \frac{\lambda}{L} \int\limits_{h-L}^{h+L} e^{-\lambda t} S(t)(r-A)^{-1} x\, dt \right. \\
&\qquad + \frac{1}{K} e^{-\lambda(K+h)} S(h+K)(r-A)^{-1} x - \frac{1}{K} e^{-\lambda(h-K)} S(h-K)(r-A)^{-1} x \\
&\qquad \left. - \frac{1}{L} e^{-\lambda(h+L)} S(h+L)(r-A)^{-1} x + \frac{1}{L} e^{-\lambda(h-L)} S(h-L)(r-A)^{-1} x \right\| \\
&\quad < \lambda\varepsilon + 4M\varepsilon;
\end{aligned}$$

thus we get (c). ∎

4. Almost periodic C-cosine functions. A *C-cosine function* $C(t)$ is a strongly continuous operator family such that $C(0) = C$ and $2C(t)C(s) = C(t+s)C + C(s-t)C$ for all $t, s \in \mathbb{R}$. The corresponding *C-sine function*, $S(t)$, is defined by $S(t) = \int_0^t C(s)\, ds$. The generator A of $C(t)$ is defined by

$$D(A) = \left\{ x \in X : \lim_{\mathbb{R} \ni t \to 0} \frac{2}{t^2}(C(t)x - Cx) \text{ exists and is in } R(C) \right\},$$

$$Ax = C^{-1}\left(\lim_{R \ni t \to 0} \frac{2}{t^2}(C(t)x - Cx) \right) \quad \text{for } x \in D(A).$$

For more details on cosine and C-cosine functions, we refer to [11, 19, 21].

First we introduce the interpolation space for C-cosine functions (cf. [19, Theorem 1.2.5]).

LEMMA 4.1. *Suppose A generates a strongly uniformly continuous and uniformly bounded C-cosine function. Then there exists a Banach space Y such that*

(1) *$A|_Y$ generates a bounded strongly continuous cosine function $G(t)$, with corresponding sine function $H(t)$;*

(2) *$R(C) \subset Y \hookrightarrow X$;*

(3) *$C(t) = G(t)C$, $S(t) = H(t)C$, Y may be chosen as*

$$Y = \{x \in X : t \to C^{-1}C(t)x \text{ is bounded and uniformly continuous}\}$$

and

$$\|x\|_Y = \sup_{t\in\mathbb{R}} \|C^{-1}C(t)x\|.$$

Using the above results and arguments similar to those in Section 2, we can prove the following theorem on the almost periodicity of C-cosine functions.

THEOREM 4.2. (a) *A C-cosine function $C(t)$ is almost periodic if and only if $C(t)$ is bounded and $R(C) \subset Y_b := \overline{\operatorname{span}}\{x \in D(A|_Y) : Ax = -r^2x$ for some $r \in \mathbb{R}\}$, the closure taken in Y, where Y is as in Lemma* 4.1.

(b) *$S(t)$ is almost periodic if and only if $S(t)$ is bounded*, $0 \notin P_\sigma(A)$ *and $R(C) \subset Y_b$.*

We can also derive [21, Theorem 4.1] from Theorem 4.2, as in the proof of Corollary 2.2.

Finally, we characterize the periodicity of C-cosine functions.

THEOREM 4.3. *A C-cosine function $C(t)$ is periodic with period p if and only if $C(t)$ is bounded*, $\sigma_C(A) \subset \{-4\pi^2k^2/p^2 : k \in \mathbb{N}\}$ *and $R(C) \subset Y_b$.*

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References

[1] H. Bart and S. Goldberg, *Characterizations of almost periodic strongly continuous groups and semigroups*, Math. Ann. 236 (1978), 105–116.

[2] I. Cioranescu, *On the second order Cauchy problem associated with a linear operator*, J. Math. Anal. Appl. 154 (1991), 238–242.

[3] —, *Characterizations of almost periodic strongly continuous cosine operator functions*, ibid. 116 (1986), 222–229.

[4] E. B. Davies and M. M. Pang, *The Cauchy problem and a generalization of the Hille–Yosida theorem*, Proc. London Math. Soc. 55 (1987), 181–208.

[5] R. deLaubenfels, *Integrated semigroups, C-semigroups and the abstract Cauchy problem*, Semigroup Forum 42 (1990), 83–95.

[6] —, *C-semigroups and the Cauchy problem*, J. Funct. Anal. 111 (1993), 44–61.

[7] —, *C-semigroups and strongly continuous semigroups*, Israel J. Math. 81 (1993), 227–255.

[8] R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations*, Lecture Notes in Math. 1570, Springer, 1994.

[9] R. deLaubenfels and Vũ Quôc Phóng, *Stability and almost periodicity of solutions of ill-posed abstract Cauchy problems*, Proc. Amer. Math. Soc. 125 (1997), 235–241.

[10] R. deLaubenfels, G. Z. Sun and S. W. Wang, *Regularized semigroups, existence families and the abstract Cauchy problem*, J. Differential Integral Equations 8 (1995), 1477–1496.

[11] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, Elsevier, Amsterdam, 1985.

[12] A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Math. 377, Springer, Berlin, 1974.

[13] S. Kantorovitz, *The Hille–Yosida space of an arbitrary operator*, J. Math. Anal. Appl. 136 (1988), 107–111.

[14] S. I. Piskarev, *Periodic and almost periodic cosine functions*, Math. USSR-Sb. 46 (1982), 391–402.

[15] —, *Almost periodic solutions of second-order differential equations*, Siberian Math. J. 25 (1984), 451–460.

[16] W. M. Ruess and W. H. Summers, *Ergodic theorems for semigroup of operators*, Proc. Amer. Math. Soc. 114 (1992), 423–432.

[17] W. M. Ruess and Vũ Quôc Phóng, *Asymptotically almost periodic solutions of evolution equations in Banach spaces*, J. Differential Equations 122 (1995), 282–301.

[18] L. H. Xie, M. Li and F. L. Huang, *Asymptotic almost periodicity of C-semigroups*, preprint.

[19] J. Z. Zhang, *Regularized operator families and applications to differential operators*, Dissertation, Huazhong Univ. of Science and Technology, 1996.

[20] Q. Zheng, *Strongly Continuous Semigroups of Linear Operators*, Huazhong University of Science and Technology, Wuhan, 1994 (in Chinese).

[21] Q. Zheng and L. P. Liu, *Almost periodic regularized groups, semigroups and cosine functions*, J. Math. Anal. Appl. 197 (1996), 90–112.

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