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A class of stationary stochastic processes

by

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Abstract. Regular stationary stochastic vector processes whose spectral densities are the boundary values of matrix functions with bounded Nevanlinna characteristic are considered. A criterion for the representability of such processes as output data of linear time invariant dynamical systems is established.

1. Introduction. The theory of stochastic realizations deals with the modeling of random processes. For a given random vector process $y(t) = \{y_j(t)\}_{j=1}^p$, $t \in \mathbb{Z}$, one has to define its representations in simpler terms. In particular, the problem of stochastic realization is to construct models of stationary random processes of the form

$$\begin{cases} x(t+1) = Ax(t) + Kw(t), \\ y(t) = Cx(t) + Lw(t). \end{cases} \tag{1.1}$$

Here $w(t)$ is a *white noise*, i.e. a stationary random process such that

$$E\{w(t)\} = 0, \qquad E\{w(t+\tau)w(t)^*\} = I\delta(\tau),$$

where E is mathematical expectation and δ denotes the Kronecker function. Such a representation is called a *linear realization* of the stationary random process $y(t)$. The properties of all objects in such a realization are defined by the spectral density ρ of the process under consideration (for the definition of the spectral density see e.g. [Roz, p. 30]).

It is worth noting that stochastic realization plays an important role in various applications. In particular, linear realization models are a departure point for the prediction of random processes, controllability and Kalman filters. This leads immediately to stationary Kalman filters. More precisely, stationary random processes with rational spectral densities admit minimal realizations with finite-dimensional space of inner states, which are such filters.

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Linear stochastic realizations originated in works of A. Lindquist, G. Picci, M. Pavon and their colleagues (see [LPa], [LPi1]–[LPi3]). They used a geometrical approach—viz. the method of Markovian splitting subspaces based on Lax–Phillips scattering theory [LPh] and also on various results from prediction theory and from the filtering of random processes [KFA], [Kal], [Kai], [Roz], [WM1]–[WM3]. In particular, real-valued random processes with rational spectral densities naturally arise in applications, so they attract the most attention. On the other hand, realizations of p-dimensional regular wide-sense stationary process $y(t)$, $t \in \mathbb{Z}$, of rank m as an output of a passive linear two-side stable system have been studied in [AR3], [R]. In such a model, the inner states $x(t)$ vanish at $-\infty$. Moreover, the input data of the above mentioned system are the values of an m-dimensional white noise process $w(t)$. Note that the corresponding results are based on the Darlington method [D], [A1] for passive impedance systems with losses of scattering channels [AR1], [AR2]. Thus the set of linear passive realizations was parameterized and special minimal, minimal and optimal, as well as minimal and *-optimal realizations coinciding with stationary Kalman filters [KFA] have been found. Random processes $y(t)$ whose spectral density $\rho(e^{i\mu})$ is the boundary value of a matrix function $\rho(z)$ of rank m with bounded Nevanlinna characteristic have been considered in [R], [AR3].

The present paper is devoted to the representability of stationary random processes as output data of stochastic systems (1.1). We show that the theory of stochastic realizations deals, in fact, with random processes with spectral density of bounded Nevanlinna characteristic. Note that all the stationary stochastic processes considered in [R] and [AR3] are covered by Theorem 3.2 below.

Let us introduce the notation used throughout. First of all, all Hilbert spaces considered are separable and 'subspace' always means a closed subspace. Further, let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc bounded by the unit circle $\mathbb{T}$, and let $\mathbb{D}_e := \{z \in \mathbb{C} : 1 < |z| \leq \infty\}$ be the exterior of $\mathbb{T}$ in the extended complex plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

If $\mathfrak{D}_\alpha \subset \mathfrak{D}$, $\alpha \in A$, is a family of subsets of a vector space $\mathfrak{D}$, then by $\bigvee_{\alpha \in A} \mathfrak{D}_\alpha$ we denote the closed linear span of vectors from $\mathfrak{D}_\alpha$ when α runs through A. In addition, if $\mathcal{X}$ is a subspace of the Hilbert space $\mathcal{H}$, then $P_{\mathcal{X}}$ denotes the orthogonal projection on $\mathcal{X}$, and if A is a linear operator acting on $\mathcal{H}$, then $A|_{\mathcal{X}}$ refers to the restriction of A to $\mathcal{X}$. Also we denote by A^T the transpose of a matrix A.

Further, if $\mathcal{X}$ is a set, then $\mathcal{X}^{p\times q}$ is the set of all $p \times q$ matrices with entries from $\mathcal{X}$. As usual, instead of $\mathcal{X}^{p\times 1}$ we write $\mathcal{X}^p$.

2. Preliminaries. In this section we recall some notions and results of the theory of stochastic realizations of stationary random processes [LPa], [LPi1]–[LPi3], [R], [AR3].

Let $\mathcal{H}$ be a Hilbert space with a unitary operator $U : \mathcal{H} \to \mathcal{H}$. Assume that H^- and H^+ are subspaces of $\mathcal{H}$ such that $H^- \vee H^+$ is invariant with respect to U and U^*, and $U^*H^- \subset H^-$, $UH^+ \subset H^+$.

DEFINITION 2.1. A subspace $X \subset \mathcal{H}$ is called a *splitting subspace* of $\mathcal{H}$ if for any $\alpha \in H^-$ and $\beta \in H^+$,

$$(\alpha - P_X\alpha, \beta - P_X\beta) = 0. \tag{2.1}$$

Condition (2.1) is equivalent to the following two relations:

- $P_{X\vee H^+}\alpha = P_X\alpha$ for all $\alpha \in H^-$,
- $P_{X\vee H^-}\beta = P_X\beta$ for all $\beta \in H^+$.

A splitting subspace X is called *minimal* if it does not contain any other splitting subspace as a proper subset. Note that H^-, H^+ and H are splitting subspaces but they are not minimal.

A subspace $X \subset \mathcal{H}$ is splitting if and only if $X = \mathcal{H}_- \cap \mathcal{H}_+$ for some pair $(\mathcal{H}_-, \mathcal{H}_+)$ of subspaces such that $H^- \subset \mathcal{H}_-$, $H^+ \subset \mathcal{H}_+$ and

$$(\alpha - P_X\alpha, \beta - P_X\beta) = 0, \quad \forall\alpha \in \mathcal{H}_-, \forall\beta \in \mathcal{H}_+.$$

In stochastic realizations such a pair $(\mathcal{H}_-, \mathcal{H}_+)$ of subspaces is called a *scattering pair* of X, due to certain similarities to incoming and outgoing wave-subspaces in Lax–Philips scattering theory [LPh].

Splitting subspaces play an important role in stochastic realizations of stationary random processes. In particular, we will consider Markovian splitting subspaces. Recall that a splitting subspace X is called *Markovian* if there exists a scattering pair $(\mathcal{H}_-, \mathcal{H}_+)$ of X such that

$$U^*\mathcal{H}_- \subset \mathcal{H}_- \quad \text{and} \quad U\mathcal{H}_+ \subset \mathcal{H}_+.$$

Further, let $y(t) = \{y_k(t)\}_{k=1}^p$ be a wide-sense stationary regular random process with spectral density $\rho(\mu)$ of rank m generating the Hilbert space

$$H(y) = \bigvee_{t\in\mathbb{Z},\,1\le k\le p} \{y_k(t)\}.$$

Note that $H(y)$ is a subspace of the space $\mathbb{H}$ of all complex random variables ξ, defined on a probability space Ω, having finite mathematical expectation $E\,|\xi|^2$ and, consequently, finite mean and dispersion. The scalar product in $\mathbb{H}$ is defined by

$$\langle \xi, \eta\rangle = E\{\xi\bar{\eta}\}, \qquad \xi, \eta \in \mathbb{H}.$$

In $\mathbb{H}$, random variables that coincide with probability 1 are identified.

It is easily seen that

$$H(y) = H^-(y) \vee H^+(y),$$

where $H^-(y)$ is the past space constructed from $y(t)$ similarly to the space $H(y)$ but for the random process $\{\dots, y(-2), y(-1)\}$, and $H^+(y)$ is the future space generated by $\{y(0), y(1), \dots\}$.

Definition 2.2 ([LPa], [LPi1]). The triplet $(\mathcal{H}, U, X)$ is said to be a *Markovian representation* of the random process $y(t)$ if X is a Markovian splitting subspace of the Hilbert space $\mathcal{H}$ of random variables and if $U : \mathcal{H} \to \mathcal{H}$ is a unitary shift operator such that

(1) $H(y) \subset \mathcal{H}$ and $H(y)$ is an invariant subspace for U and U^*;
(2) $U|_{H(y)}$ is the natural shift on $H(y)$, i.e.

$$Uy_k(t) = y_k(t+1), \qquad k = 1, \dots, p;$$

(3) $\mathcal{H}$ can be represented as

$$\mathcal{H} = H(y) \vee \bigvee_{t=-\infty}^{\infty} U^t X,$$

and it has a finite number of cyclic generating elements.

Let $\mathcal{H} \supset H(y)$ be a Hilbert space of random variables with a shift U on $H(y)$ satisfying (2) of Definition 2.2, and let X be a subspace of $\mathcal{H}$ such that

$$\mathcal{H} = H(y) \vee \bigvee_{t=-\infty}^{\infty} U^t X.$$

It is known [LPa], [LPi1]–[LPi3] that the triple $(\mathcal{H}, U, X)$ is a Markovian representation of $y(t)$ if and only if there is a pair $(\mathcal{H}_-, \mathcal{H}_+)$ of subspaces of $\mathcal{H}$ satisfying the following conditions:

- $X = \mathcal{H}_- \cap \mathcal{H}_+$;
- $\mathcal{H} = \mathcal{H}_-^\perp \oplus X \oplus \mathcal{H}_+^\perp$;
- $H^-(y) \subset \mathcal{H}_-$ and $H^+(y) \subset \mathcal{H}_+$;
- $U^*\mathcal{H}_- \subset \mathcal{H}_-$ and $U\mathcal{H}_+ \subset \mathcal{H}_+$.

We then write $X \sim (\mathcal{H}, \mathcal{H}_+)$. Let us emphasize that the pair $(\mathcal{H}_-, \mathcal{H}_+)$ is uniquely determined by X in the sense that

$$\mathcal{H}_- = H^-(y) \vee \bigvee_{t=-\infty}^{0} U^t X, \qquad \mathcal{H}_+ = H^+(y) \vee \bigvee_{t=0}^{\infty} U^t X.$$

A Markovian representation $(\mathcal{H}, U, X)$ is called *minimal* if X is a minimal Markovian splitting subspace. This is true if and only if

$$\mathcal{H}_- = H^-(y) \vee \mathcal{H}_+^\perp, \qquad \mathcal{H}_+ = H^+(y) \vee \mathcal{H}_-^\perp.$$

In view of (2.1), in stochastic realization theory, splitting Markovian subspaces are interpreted as *dynamical memory* or *sufficient statistic* which contains information about the past and is needed to predict the future.

The Markovian splitting subspace $X \sim (\mathcal{H}_-, \mathcal{H}_+)$ defines the operators

$$U(X) := P_X U|_X, \quad U(X)^* := P_X U^*|_X.$$

Moreover, the operators

$$U_t(X) := P_X U^t|_X, \quad t \geq 0, \quad U_t(X)^* := P_X (U^*)^t|_X, \quad t \geq 0,$$

have the properties

$$U_s(X)U_t(X) = U_{s+t}(X), \quad U_s(X)^* U_t(X)^* = U_{s+t}(X)^*,$$
$$U_t(X) = U(X)^t, \quad U_t(X)^* = [U(X)^*]^t,$$

and for any $\eta \in X$ and any $t \geq 0$ one has

$$P_{\mathcal{H}_-} U^{-t}\eta = U_t(X)^*\eta, \quad P_{\mathcal{H}_+} U^t \eta = U_t(X)\eta.$$

The Markovian splitting space $X = \mathcal{H}_- \cap \mathcal{H}_+$ is said to be *proper* if the subspaces $\mathcal{H}_-$ and $\mathcal{H}_+$ satisfy the conditions

$$\bigvee_{t=-\infty}^{\infty} U^t \mathcal{H}_-^{\perp} = \mathcal{H}, \quad \bigvee_{t=-\infty}^{\infty} U^t \mathcal{H}_+^{\perp} = \mathcal{H}. \tag{2.2}$$

Since $U^*\mathcal{H}_- \subset \mathcal{H}_-$ and $U\mathcal{H}_+ \subset \mathcal{H}_+$, conditions (2.2) are equivalent to

$$\bigcap_{t=0}^{\infty} (U^*)^t \mathcal{H}_- = \{0\}, \quad \bigcap_{t=0}^{\infty} U^t \mathcal{H}_+ = \{0\}. \tag{2.3}$$

If $X = \mathcal{H}_- \cap \mathcal{H}_+$ is a proper Markovian splitting subspace, then for all $\eta \in X$ and all $t \geq 0$,

$$\begin{aligned} \|U_t(X)\eta\| &= \|P_{\mathcal{H}_-} U^t \eta\| = \|P_{U^{-t}\mathcal{H}_-}\eta\|, \\ \|U_t(X)^*\eta\| &= \|P_{\mathcal{H}_+} U^{-t}\eta\| = \|P_{U^t\mathcal{H}_+}\eta\|. \end{aligned} \tag{2.4}$$

Equations (2.3) and (2.4) imply that both $U(X)^t$ and $[U(X)^*]^t$ converge strongly to zero as $t \to \infty$.

Finally, a Markovian representation $(\mathcal{H}, U, X)$ of the random process $y(t)$ is called *proper* if X is a proper Markovian splitting space.

Let $y(t) = \{y_k(t)\}_{k=1}^p$ be a wide-sense regular stationary random process with spectral density $\rho(\mu)$ of rank $m \leq p$ generating a Hilbert space $H(y)$. Realizations of y as the output of the stochastic systems

$$\begin{cases} x_f(t+1) = Ax_f(t) + Kw_f(t), \\ y(t) = Cx_f(t) + Lw_f(t), \end{cases} \quad t \in \mathbb{Z}, \tag{2.5}$$

and

$$\begin{cases} x_b(t-1) = \widetilde{A}x_b(t) + \widetilde{K}w_b(t), \\ y(t) = \widetilde{C}x_b(t) + \widetilde{L}w_b(t), \end{cases} \quad t \in \mathbb{Z}, \tag{2.6}$$

have been considered in [LPa], [LPi1]–[LPi3], [R], [AR3]. The systems (2.5) and (2.6) evolve forward and backward in time, respectively, where w_f

and w_b are white noise processes of order m with the property

$$H(w_f) = H(w_b). \tag{2.7}$$

Set

$$\mathcal{H} := H(w_f) = H(w_b),$$

and note that $H(y) \subseteq \mathcal{H}$. Moreover, x_f and x_b are inner state processes such that $x_b(t-1) = x_f(t)$ and

$$\lim_{t\to-\infty} x_f(t) = \lim_{t\to\infty} x_b(t) = 0. \tag{2.8}$$

It is also worth noting that the bounded linear operators $A, K, C, L, \widetilde{A}, \widetilde{K}, \widetilde{C}$, and $\widetilde{L}$ act in the Hilbert space $\mathcal{H}$ and satisfy the relations

$$\begin{aligned} &\widetilde{A} = A^*, \quad I = AA^* + KK^* = A^*A + \widetilde{K}\widetilde{K}^*, \\ &\widetilde{C} = CA^* + LK^*, \quad C = \widetilde{C}A + \widetilde{L}\widetilde{K}^*, \end{aligned} \tag{2.9}$$

$$E\{y(0)y(0)^*\} = CC^* + LL^* = \widetilde{C}\widetilde{C}^* + \widetilde{L}\widetilde{L}^*. \tag{2.10}$$

It was shown in [LPi1] that each proper Markovian representation $(\mathcal{H}, U, X)$ with $X = \mathcal{H}_- \cap \mathcal{H}_+$ of a random process $\{y(t)\}$ has a pair of dual stochastic realizations (2.5) and (2.6) satisfying (2.7)–(2.10) and such that

$$\mathcal{H}_+ = H^+(w_f), \qquad \mathcal{H}_- = H^-(w_b).$$

3. Solvability of the stochastic realization problem. In this section we establish a criterion for the solvability of the stochastic realization problem for a stationary random process $y(t)$, $t \in \mathbb{Z}$, with spectral density ρ.

Let $L_2^{p\times q} = L_2^{p\times q}(\mathbb{T})$ be the set of all Lebesgue measurable $p \times q$ matrix functions $f = f(\zeta)$ such that

$$\|f\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{T}} \operatorname{trace}\{f(\zeta)^* f(\zeta)\} \, |d\zeta| < \infty.$$

Consider also the space $H_2^{p\times q} := H_2^{p\times q}(\mathbb{D})$ of all $p \times q$ matrix functions f which are holomorphic in $\mathbb{D}$ and such that

$$\|f\|_2^2 = \sup_{0\le r<1} \int_{\mathbb{T}} \operatorname{trace}\{f(r\zeta)^* f(r\zeta)\} \, |d\zeta| < \infty. \tag{3.1}$$

Other relevant classes of holomorphic functions include the *Carathéodory class* $\mathcal{C}^{p\times p} = \mathcal{C}^{p\times p}(\mathbb{D})$ which consists of $p \times p$ matrix functions $c(z)$ such that $\operatorname{Re} c(z) \ge 0$, $z \in \mathbb{D}$, and the *Schur class* $S^{p\times q} = S^{p\times q}(\mathbb{D})$ of $p \times q$ matrix functions $s(z)$ such that $s(z)^* s(z) \le I_q$, $z \in \mathbb{D}$.

A matrix function $s \in S^{p\times q}$ is *inner* (resp. *-*inner*) if $s(\zeta)^* s(\zeta) = I_q$ (resp. $s(\zeta)s(\zeta)^* = I_p$) for almost all $\zeta \in \mathbb{T}$. The class of inner matrix functions is denoted by $S_{\text{in}}^{p\times q}$, and that of *-inner matrix functions by $S_{\text{in}*}^{p\times q}$. Note that these classes are nonempty if, respectively, $p \ge q$ and $p \le q$. In the case

$p = q$ we have $S_{\rm in}^{p\times p} = S_{\rm in*}^{p\times p}$ [AR1], and the corresponding matrix functions are called *bi-inner*.

Let us introduce the class $N^{p\times q}$ of all meromorphic matrix functions $f(z)$ in $\mathbb{D}$ with bounded Nevanlinna characteristic. This class consists of matrix functions which can be represented as

$$f = h^{-1}g,$$

where g is a holomorphic bounded $p \times q$ matrix function in $\mathbb{D}$ and h is a holomorphic bounded scalar function in $\mathbb{D}$.

Recall that $N^{p\times q}$ contains the classes $H_2^{p\times q}$, $S^{p\times q}$ and also the class $\mathcal{C}^{p\times p}$ if $p = q$. Moreover, each $f \in N^{p\times q}$ has nontangential boundary values $f(\zeta)$ almost everywhere on the unit circle $\mathbb{T}$. Therefore, the limit

$$f(\zeta) = \lim_{r\uparrow 1} f(r\zeta)$$

exists for almost all $\zeta \in \mathbb{T}$, and f is uniquely determined by the boundary values $f(\zeta)$. In fact, $H_2^{p\times q} = \{f(\zeta) \in L_2^{p\times q} : f(\zeta) = \sum_{k=0}^{\infty} \hat{f}_k \zeta^k\}$ can be identified with the Hardy space of matrix valued functions $f(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k$ defined on $\mathbb{D}$ and satisfying (3.1). The orthogonal complement of $H_2^{p\times q}$ in the space $L_2^{p\times q}$ is denoted by $K_2^{p\times q}$. Thus $L_2^{p\times q} = H_2^{p\times q} \oplus K_2^{p\times q}$, where

$$K_2^{p\times q} = \Big\{ f(\zeta) \in L_2^{p\times q} : f(\zeta) = \sum_{k=-\infty}^{-1} \hat{f}_k \zeta^k \Big\}.$$

We also consider the subclass $\Pi^{p\times q}$ of all $f \in N^{p\times q}$ which have meromorphic pseudo-continuation in $\mathbb{D}_e$. Thus, if $f \in \Pi^{p\times q}$, then there is a function f_- meromorphic in $\mathbb{D}_e$ such that $f_-(1/\bar{z})^* \in N^{p\times q}$ and

$$f(\zeta) := \lim_{r\uparrow 1} f(r\zeta) = \lim_{r\downarrow 1} f_-(r\zeta)$$

for almost all $\zeta \in \mathbb{T}$.

For any class $\mathcal{X}^{p\times q}$ of matrix functions, we write $\mathcal{X}^{p\times q}\Pi$ for $\mathcal{X}^{p\times q} \cap \Pi^{p\times q}$.

Now consider a stationary stochastic process $y(t) = \{y_k(t)\}_{k=1}^{p}$ of rank m with spectral density $\rho(e^{i\mu})$. Let

$$R(t) = \{R_{kj}(t)\}_{k,j=1}^{p} := \{E y_k(s+t)\overline{y_j(s)}\}_{k,j=1}^{p}$$

be the correlation function of $y(t)$. In this case [Roz],

$$R(t) = \int_{-\pi}^{\pi} e^{-it\mu} \rho(e^{i\mu})\, d\mu,$$

and the matrix function

$$c_\rho(z) := \frac{1}{2}R(0) + \sum_{k=1}^{\infty} R(k) z^k = \frac{1}{2} \int_{-\pi}^{\pi} \frac{e^{i\mu} + z}{e^{i\mu} - z} \rho(e^{i\mu})\, d\mu \tag{3.2}$$

belongs to the Carathéodory class $\mathcal{C}^{p\times p}$.

Lemma 3.1. *If the spectral density $\rho(\zeta)$ of a stationary stochastic process $y(t) = \{y_k(t)\}_{k=1}^p$ is the nontangential boundary value of a matrix function from the class $N^{p\times p}$, then $c_\rho \in \mathcal{C}^{p\times p}\Pi$.*

Proof. Let the density $\rho(\zeta)$ of the process $y(t)$ be the nontangential boundary value of some matrix function ρ with bounded Nevanlinna characteristic in $\mathbb{D}$. Then $\rho \in N^{p\times p}\Pi$ because $\rho(\zeta) \geq 0$ a.e. on $\mathbb{T}$, and the meromorphic pseudo-continuation of ρ can be determined by the symmetry principle,

$$\rho(z) = \rho(1/\overline{z})^*, \quad z \in \mathbb{D}_e.$$

This and (3.2) imply that the matrix function c_ρ also has a meromorphic pseudo-continuation in $\mathbb{D}_e$, so $c_\rho \in \mathcal{C}^{p\times p}\Pi$,

$$2\operatorname{Re} c_\rho(\zeta) = \rho(\zeta), \quad \zeta = e^{i\mu},$$

and the proof is complete. ∎

Recall that a stationary stochastic process $w(t) = \{w_k(t)\}_{k=1}^m$ with spectral density

$$\rho_w(e^{i\mu}) = \frac{1}{2\pi} I_m$$

is a white noise. Its correlation matrix function is

$$R_w(t) = \begin{cases} I_m & \text{if } t = 0, \\ 0 & \text{if } t \neq 0. \end{cases}$$

The Hilbert space $H(w)$ generated by $w(t)$ has the property

$$H(w) = \bigoplus_{t=-\infty}^{\infty} H_t(w), \quad \text{where} \quad H_t(w) = \bigvee\{w_k(t) : k = 1, \dots, m\}.$$

The process $y(t)$ is called *regular* if

$$\bigcap_{t<0} U^t H^-(y) = \{0\}.$$

This condition is equivalent to the following two conditions:

(i) The spectral function of $y(t)$ is absolutely continuous and the corresponding spectral density $\rho(\zeta)$ has constant rank m a.e. on $\mathbb{T}$.

(ii) There exists a holomorphic matrix function $\psi \in H_2^{p\times m}$ in $\mathbb{D}$ such that

$$\rho(\zeta) = \psi(\zeta)\psi(\zeta)^* \quad \text{a.e. on } \mathbb{T}. \tag{3.3}$$

Such a ψ is called the *spectral factor* of ρ.

The number $m := \dim(H^-(y) \ominus U^{-1}H^-(y))$ is said to be the *rank* of the process $y(t)$. Note that if $y(t)$ satisfies the above condition (i), then it has rank m, and

$$\operatorname{rank}\rho(\zeta) = \operatorname{rank}\psi(\zeta) \quad \text{a.e. on } \mathbb{T}.$$

Assume that for a weak-sense stationary stochastic process $y(t)$ of order p there is a white noise $w(t)$ of order m, stationarily connected with $y(t)$, and such that $H^-(y) \subset H^-(w)$, $H(y) = H(w)$ and on this space the unitary shift operators of y and w coincide. Then $y(t)$ is a regular process of rank m, and there is a spectral factor ψ of rank m of the density ρ such that

$$F_y(d\mu) = \psi(e^{i\mu})F_w(d\mu). \tag{3.4}$$

Here $F_y(d\mu)$ is the spectral random measure of the process $y(t)$, and $F_w(d\mu)$ is the spectral random measure of the white noise $w(t)$. It is known that, using an arbitrary factor ψ of rank m of the density ρ, one can construct a white noise $w(t)$ of order m with the above properties. This construction is used in the proof of Theorem 3.2 below (see also [Roz], [LPi1]).

Denote by C_{00} the set of operators $A : \mathcal{H} \to \mathcal{H}$ such that $\|A\| \le 1$ and

$$\lim_{n\to\infty} A^n h = 0, \qquad \lim_{n\to\infty} (A^*)^n h = 0 \qquad \text{for any } h \in \mathcal{H}.$$

Theorem 3.2. *A stationary stochastic process $y(t) = \{y_k(t)\}_{k=1}^p$ with spectral density $\rho(e^{i\mu})$ can be represented as the output data of stochastic systems* (2.5)–(2.6) *with an operator $A \in C_{00}$ and with the conditions* (2.7)–(2.10) *satisfied if and only if the spectral density ρ is the nontangential boundary value of a function from the class $N^{p\times p}$.*

Proof. Suppose the density $\rho(e^{i\mu})$ of the process $y(t)$ of rank m is the nontangential boundary value of some $\rho \in N^{p\times p}$. By Lemma 3.1 the corresponding matrix function $c_\rho(z)$ defined by (3.2) belongs to $\mathcal{C}^{p\times p}\Pi$. It follows from [AR1, Theorem 1] that there is a representation of $c_\rho(z)$ as a 2×2 block of a $J_{p,m}$-inner matrix function $\theta(z)$ of the form

$$\theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & c_\rho(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}. \tag{3.5}$$

The matrix function θ is $J_{p,m}$-inner in the sense that it is holomorphic and $J_{p,m}$-contractive in $\mathbb{D}$, i.e.

$$\theta(z)^* J_{p,m} \theta(z) \le J_{p,m}, \qquad z \in \mathbb{D},$$

where

$$J_{p,m} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_p \\ 0 & -I_p & 0 \end{bmatrix}.$$

Moreover, θ has $J_{p,m}$-unitary nontangential boundary values a.e. on the unit circle, that is,

$$\theta(\zeta)^* J_{p,m} \theta(\zeta) = J_{p,m}.$$

Recall that θ is called the $J_{p,m}$-*inner dilation* of c_ρ.

If $\theta(z)$ is the $J_{p,m}$-inner dilation of c_ρ, then $\alpha \in S_{in}^{m\times m}$, $\beta \in H_2^{m\times p}\Pi$, $\gamma \in H_2^{p\times m}\Pi$ and for all $z \in \mathbb{D}$,

$$\beta(z)^*\beta(z) \le 2\operatorname{Re} c_\rho(z), \qquad \gamma(z)\gamma(z)^* \le 2\operatorname{Re} c_\rho(z).$$

In addition, almost everywhere on $\mathbb{T}$ the spectral density ρ and the submatrices in (3.5) satisfy

$$\begin{aligned} \rho(\zeta) &= \beta(\zeta)^*\beta(\zeta) = 2\operatorname{Re} c_\rho(\zeta),\\ \rho(\zeta) &= \gamma(\zeta)\gamma(\zeta)^* = 2\operatorname{Re} c_\rho(\zeta),\\ &\quad \gamma(\zeta)\alpha(\zeta)^* = \beta(\zeta)^*. \end{aligned} \tag{3.6}$$

Consider the block γ of the dilation θ. It follows from relations (3.6) that almost everywhere on $\mathbb{T}$ the matrix function γ is a spectral factor for ρ such that

$$\operatorname{rank}\gamma(\zeta) = m.$$

Consequently, the matrix $\gamma(\zeta)$ is left-invertible a.e. on $\mathbb{T}$. If $\gamma_l^{-1}(\zeta)$ denotes a left inverse for $\gamma(\zeta)$, then

$$\gamma_l^{-1}(\zeta)\gamma(\zeta) = I_m \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

Let $F_y(d\mu)$ be the spectral measure of the random process y, so

$$E\{F_y(d\mu)F_y(d\mu)^*\} = \frac{1}{2\pi}\rho d\mu.$$

For a Borel subset $\Delta \subset [-\pi,\pi]$ let

$$F_w(\Delta) = \int_\Delta \gamma_l^{-1}(e^{i\mu})\, F_y(d\mu) \tag{3.7}$$

be the associated m-dimensional random vector. This vector is well-defined because the rows of γ_l^{-1} are square summable with respect to the matrix measure $(1/2\pi)\rho(e^{i\mu})d\mu$ of the process $y(t)$. Thus

$$\begin{aligned} E\{F_w(\Delta_1)F_w(\Delta_2)^*\} &= \frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2} \gamma_l^{-1}(e^{i\mu})\rho(e^{i\mu})(\gamma_l^{-1}(e^{i\mu}))^*\, d\mu\\ &= \frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2} I_m\, d\mu = \frac{1}{2\pi} I_m|\Delta_1\cap\Delta_2|, \end{aligned}$$

where $|\Delta|$ is the Lebesgue measure of $\Delta \subset [-\pi,\pi]$. Hence, F_w is the spectral measure of the m-dimensional white noise process w, and the formula (3.7) can be rewritten as

$$F_w(d\mu) = \gamma_l^{-1}(e^{i\mu})F_y(d\mu).$$

Note that for a full rank process y, the matrix γ is an $m\times m$ spectral factor having the unique left inverse $\gamma_l^{-1} = \gamma^{-1}$. If this is the case, the last relation can be written as

$$F_y(d\mu) = \gamma(e^{i\mu})F_w(d\mu),$$

and the spectral representation of y is

$$y(t) = \int_{-\pi}^{\pi} e^{it\mu}\gamma(e^{i\mu})\, F_w(d\mu). \tag{3.8}$$

On the other hand, for $m < p$ this result is also true. Thus let us show that

$$F_y(d\mu) = \gamma(e^{i\mu})\gamma_l^{-1}(e^{i\mu})F_y(d\mu) = \gamma(e^{i\mu})F_w(d\mu)$$

with probability one or, in other words, I_m and $\gamma\gamma_l^{-1}$ are equal almost everywhere with respect to the measure F_y. Indeed, since $\gamma_l^{-1}(e^{i\mu})\gamma(e^{i\mu}) = I_m$, we have

$$(I_m - \gamma\gamma_l^{-1})\rho(I_m - \gamma\gamma_l^{-1})^* = (I_m - \gamma\gamma_l^{-1})\gamma\gamma^*(I_m - \gamma\gamma_l^{-1})^* = 0$$

a.e. on $\mathbb{T}$.

Further, recall that the white noise process w can also be written as

$$w(t) = \int_{-\pi}^{\pi} e^{it\mu}\gamma_l^{-1}(e^{i\mu})\, F_y(d\mu), \qquad t \in \mathbb{Z}.$$

This shows that y and w are stationarily connected and $w_k(t) \in H(y)$ for all $k = 1, \dots, m$ and $t \in \mathbb{Z}$, so $H(w) \subset H(y)$. However, (3.8) implies that $y_k(t) \in H(w)$ for any $k = 1, \dots, p$ and for any $t \in \mathbb{Z}$, hence $H(y) \subset H(w)$. Therefore, $H(y) = H(w)$.

Let us define a unitary operator $T_w : H(y) \to L_2^m$ (cf. [Roz]). We start with a special representation for elements of $H(y)$. Every $\eta \in H(y) = H(w)$ can be represented as

$$\eta = \sum_{k=-\infty}^{\infty} h_k^T w(k) = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h_k^T e^{ik\mu}\, F_w(d\mu),$$

where $h_k := \{h_{kj}\}_{j=1}^m \in l_2^m$. Set

$$T_w\eta := h, \quad h = h(e^{i\mu}) = \sum_{k=-\infty}^{\infty} h_k e^{ik\mu}.$$

Note that h belongs to the space L_2^m of m-dimensional vector functions square-summable on $\mathbb{T}$ against the measure $(1/2\pi)\, d\mu$. It is clear that

$$T_w(H^-(w)) = K_2^m, \qquad T_w(H^+(w)) = H_2^m,$$

and

$$T_w U = \varUpsilon T_w,$$

where $\varUpsilon$ is the operator of multiplication by $e^{i\mu}$. Due to (3.8), the ith row of the matrix function γ is $T_w y_i(0)$, and

$$T_w y(0) = T_w[y_1(0), \dots, y_p(0)]^T = \gamma(e^{i\mu}).$$

Therefore $T_w^{-1}\gamma = y(0)$.

Recalling that $\gamma \in H_2^{p\times m}$ and using (3.8), one obtains

$$\begin{aligned} y_i(0) &= \int_{-\pi}^{\pi} [\gamma_{i1}(e^{i\mu}), \dots, \gamma_{im}(e^{i\mu})]\,[F_{w_1}(d\mu), \dots, F_{w_m}(d\mu)]^T \\ &= \sum_{j=1}^{m} \int_{-\pi}^{\pi} \gamma_{ij}(e^{i\mu})\, F_{w_j}(d\mu) = \sum_{j=1}^{m} \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} \hat{\gamma}_{ij}(k) e^{ik\mu}\, F_{w_j}(d\mu) \\ &= \sum_{j=1}^{m} \sum_{k=0}^{\infty} \hat{\gamma}_{ij}(k) w_j(k), \end{aligned}$$

so that $y_i(0) \in H^+(w)$, $i = 1, \dots, p$, and $H^+(y) \subset H^+(w)$. If we set $\mathcal{H}_+ := H^+(w)$, then $H^+(y) \subset \mathcal{H}_+$. As a consequence, the subspace $\mathcal{H}_+^\perp = H^-(w)$ generates the whole space $H(y)$, i.e. $\bigvee_{t=-\infty}^{\infty} U^t \mathcal{H}_+^\perp = H(w) = H(y)$.

The inclusions $\alpha \in S_{in}^{m\times m}$, $\beta \in H_2^{m\times p}\varPi$, $\gamma \in H_2^{p\times m}\varPi$ and (3.6) now imply that $\gamma(\zeta)\alpha(\zeta)^* = \beta(\zeta)^*$ is the boundary value of a function $\varphi \in K_2^{p\times m}$. Consequently,

$$\varphi(\zeta)\varphi(\zeta)^* = \gamma(\zeta)\alpha(\zeta)^*\alpha(\zeta)\gamma(\zeta)^* = \rho(\zeta)$$

almost everywhere on $\mathbb{T}$. Thus, φ is also a spectral factor for the density ρ of y and has rank m.

Let F_v be a measure such that

$$F_v(d\mu) = \varphi_l^{-1}(e^{i\mu}) F_y(d\mu).$$

Similarly to the case of $w(t)$ the random process $v(t) = \{v_k(t)\}_{k=1}^m$ given by

$$v(t) = \int_{-\pi}^{\pi} e^{it\mu}\, F_v(d\mu)$$

is a white noise such that $H(v) = H(y)$.

Using v and the procedure in the construction of T_w, one can build another unitary operator $T_v : H(y) \to L_2^m$ such that $UT_v^{-1} = T_v^{-1}\varUpsilon$. Note that $T_v^{-1}\varphi = y(0)$, and since $\varphi \in K_2^{p\times m}$, one has $y_i(0) \in H^-(v)$, $i = 1, \dots, p$. Moreover, for any $t \leq 0$,

$$y_i(t) = U^t y_i(0) \in H^-(v),$$

so that $H^-(y) \subset H^-(v)$. Writing $\mathcal{H}_-$ for $H^-(v)$, one observes that

$$H^-(y) = \bigvee_{t=-\infty}^{-1} U^t \{y_i(0) : i = 1, \dots, p\} \subset H^-(v) = \mathcal{H}_-,$$

and

$$T_v^{-1}(H_2^m) = H^+(v), \quad T_v^{-1}(K_2^m) = H^-(v).$$

Thus $\mathcal{H}_-^\perp = H^+(v)$ generates the space $H(y) = H(v)$.

Consider now the subspace $X = \mathcal{H}_- \cap \mathcal{H}_+$. The previous construction shows that $H^-(y) \subset \mathcal{H}_-$, $H^+(y) \subset \mathcal{H}_+$ and $H(y) = \mathcal{H}_- \vee \mathcal{H}_+$. In addition, X is also Markovian because

$$U^*\mathcal{H}_- = U^*H^-(v) \subset H^-(v) = \mathcal{H}_-,$$
$$U\mathcal{H}_+ = UH^+(w) \subset H^+(w) = \mathcal{H}_+.$$

Let us now show that $\mathcal{H}_-^\perp \subset \mathcal{H}_+$, or equivalently, $H^+(v) \subset H^+(w)$. Writing $G = T_w(H^+(v))$, one gets $G \subset L_2^m$ and for all $t \geq 0$,

$$U^tH^+(v) \subset H^+(v),$$

so that $\Upsilon^t G \subset G$, i.e. G is an invariant subspace for multiplication by $e^{i\mu}$. Since $H^+(v)$ generates $H(y)$, the subset G generates the space L_2^m. Let us also recall that $b = \varphi_l^{-1}\gamma$ and $T_v(H^+(v)) = H_2^m$. Therefore,

$$G = T_wT_v^{-1}(H_2^m) = H_2^m b,$$

and $G \subset H_2^m$ implies $H^+(v) \subset H^+(w)$, and subsequently $\mathcal{H}_-^\perp \subset \mathcal{H}_+$. The relation $\mathcal{H}_+^\perp \subset \mathcal{H}_-$ can be established analogously. Thus X is a proper Markovian splitting subspace, i.e. $H(y) = \mathcal{H}_-^\perp \oplus X \oplus \mathcal{H}_+^\perp$.

Conversely, let $X \sim (\mathcal{H}_-, \mathcal{H}_+)$ be a Markovian splitting subspace. Then there are m-dimensional white noise processes $w(t)$ and $v(t)$ such that $H(y) = H(w) = H(v)$ and $\mathcal{H}_- = H^-(w)$, $\mathcal{H}_+ = H^+(v)$. Consider the above defined unitary operator T_w. If γ is a $p \times m$ matrix function with ith row $T_w y_i(0)$, then

$$y(t) = \int_{-\pi}^{\pi} e^{it\mu}\gamma(e^{i\mu})\, F_w(d\mu).$$

Consequently, γ is a spectral factor of the density ρ of $y(t)$. Analogously, v produces the unitary operator T_v and a spectral factor φ. Since X is a proper Markovian splitting subspace, we have $H^-(y) \subset \mathcal{H}_- = H^-(v)$ and $H^+(y) \subset \mathcal{H}_+ = H^+(w)$. Therefore,

$$T_w\mathcal{H}_+ = T_w(H^+(w)) = H_2^m, \quad T_w(H^-(w)) = K_2^m, \tag{3.9}$$

$$T_v\mathcal{H}_- = T_w(H^-(v)) = K_2^m, \quad T_v(H^+(v)) = H_2^m. \tag{3.10}$$

The first relation in (3.9) (resp. (3.10)) and $H^+(y) \subset H^+(w)$ (resp. $H^-(y) \subset H^-(v)$) imply $\gamma \in H_2^{p\times m}$ (resp. $\varphi \in K_2^{p\times m}$).

Moreover, if $\mathcal{H}_-^\perp \subset \mathcal{H}_+$, then $H^+(v) \subset H^+(w)$. The set $G = T_w(H^+(v))$ generates the space L_2^m and $U(H^+(v)) \subset H^+(v)$, which leads to $\Upsilon G \subset G$. Using now the second relation from (3.10), one obtains

$$G = T_wT_v^{-1}(H_2^m) = H_2^m\varphi_l^{-1}\gamma \subset T_w(H^+(w)) = H_2^m.$$

It remains to note that by the Beurling–Lax Theorem [N] the matrix function $b := \varphi_l^{-1}\gamma$ is inner. Thus the spectral factors φ and γ of the density $\rho(e^{i\mu})$

of the process under consideration are pseudo-extendable and so are the corresponding functions having them as boundary values. ■

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