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Polaroid type operators under perturbations

by

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Abstract. A bounded operator T defined on a Banach space is said to be polaroid if every isolated point of the spectrum is a pole of the resolvent. The "polaroid" condition is related to the conditions of being left polaroid, right polaroid, or a-polaroid. In this paper we explore all these conditions under commuting perturbations K. As a consequence, we give a general framework from which we obtain, and also extend, recent results concerning Weyl type theorems (generalized or not) for $T + K$, where K is an algebraic or a quasi-nilpotent operator commuting with T.

1. Introduction and preliminaries. Polaroid operators on infinite-dimensional complex Banach spaces have recently been extensively investigated, together with the related conditions for an operator of being left, right polaroid or a-polaroid ([21], [20], [19], [3], [6]). Although the polaroid conditions are neither necessary nor sufficient for an operator to satisfy Weyl type theorems, almost all of the commonly considered classes of operators satisfy Weyl type theorems since they are polaroid type and have the single valued extension property (SVEP) (see [3]). In [3] it has also been proved that if T is polaroid, or left polaroid, or a-polaroid, then some Weyl type theorems in their classical form, or in their generalized form, are equivalent. Since the SVEP is transferred to $T+K$ in the case where K is a commuting algebraic operator [8, Theorem 2.14], it is of interest to consider the problem of preserving the polaroid conditions from T to $T + K$. This is what we do in the second section. In the last section we apply the results obtained to the study of Weyl type theorems (generalized or not) for $T + K$.

We start by explaining the relevant terminology. Let $L(X)$ be the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X and, if $T \in L(X)$, denote by $\alpha(T)$ the dimension of the kernel $\ker T$ and by $\beta(T)$ the codimension of the range $T(X)$. Recall that the operator $T \in L(X)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $\alpha(T) < \infty$ and the range $T(X)$ is closed, and *lower semi-Fredholm*, $T \in$

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$\Phi_-(X)$, if $\beta(T) < \infty$. If T is either upper or lower semi-Fredholm then it is said to be a *semi-Fredholm operator*, while if T is both upper and lower semi-Fredholm then it is said to be a *Fredholm operator*.

If T is semi-Fredholm then the *index* of T is defined to be $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. A bounded operator $T \in L(X)$ is said to be a *Weyl operator*, $T \in W(X)$, if T is a Fredholm operator having index 0. The classes of *upper semi-Weyl* and *lower semi-Weyl* operators are defined, respectively, by

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \le 0\},$$
$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \ge 0\}.$$

Clearly, $W(X) = W_+(X) \cap W_-(X)$. The *Weyl spectrum* is

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\},$$

the *upper semi-Weyl spectrum* is

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\},$$

and the *lower semi-Weyl spectrum* is

$$\sigma_{\mathrm{lw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\}.$$

Let $p(T)$ and $q(T)$ denote the *ascent* and the *descent* of $T \in L(X)$. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Moreover, for $\lambda \in \mathbb{C}$ the condition $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ is equivalent to λ being a pole of the resolvent of T (see [22, Prop. 50.2]). An operator $T \in L(X)$ is said to be *Browder* (resp. *upper semi-Browder*; *lower semi-Browder*) if T is Fredholm and $p(T) = q(T) < \infty$ (resp. T is upper semi-Fredholm and $p(T) < \infty$; T is lower semi-Fredholm and $q(T) < \infty$). Denote by $B(X)$, $B_+(X)$ and $B_-(X)$ the classes of Browder operators, upper semi-Browder operators and lower semi-Browder operators, respectively. Clearly, $B(X) \subseteq W(X)$, $B_+(X) \subseteq W_+(X)$ and $B_-(X) \subseteq W_-(X)$. Let

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}$$

denote the *Browder spectrum*, $\sigma_{\mathrm{ub}}(T)$ the *upper semi-Browder spectrum*,

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\},$$

and $\sigma_{\mathrm{lb}}(T)$ the *lower semi-Browder spectrum*,

$$\sigma_{\mathrm{lb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\}.$$

Then $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$, $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{lw}}(T) \subseteq \sigma_{\mathrm{lb}}(T)$.

Recall that $T \in L(X)$ is said to be a *Riesz operator* if $\lambda I - T \in \Phi(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Classical examples of Riesz operators are all compact and quasi-nilpotent operators. By a result of Rakočević ([28]), the semi-Browder operators are stable under commuting Riesz perturbations, i.e., if $R \in L(X)$ is a Riesz operator for which $RT = TR$, then

$$T \text{ is Browder} \Leftrightarrow T + R \text{ is Browder}, \tag{1}$$

and

(2) T is upper semi-Browder $\Leftrightarrow$ $T+R$ is upper semi-Browder.

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory [18]. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index) if $p(T)=q(T)<\infty$, which is equivalent to saying that $T=T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent (see [24, Corollary 2.2] and [23, Prop. A]).

DEFINITION 1.1. $T \in L(X)$ is said to be *left Drazin invertible* if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed, and *right Drazin invertible* if $q := q(T) < \infty$ and $T^q(X)$ is closed.

Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible. In fact, if $0 < p := p(T) = q(T)$ then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection associated with the spectral set $\{0\}$ (see [22, Proposition 50.2]).

2. Polaroid type operators under commuting perturbations. The concepts of left or right Drazin invertibility lead to the concepts of left or right pole. Let us denote by $\sigma_{\mathrm{a}}(T)$ the classical *approximate point spectrum* and by $\sigma_{\mathrm{s}}(T)$ the *surjectivity spectrum*. It is well known that $\sigma_{\mathrm{a}}(T') = \sigma_{\mathrm{s}}(T)$, where T' denotes the dual of T, and $\sigma_{\mathrm{s}}(T') = \sigma_{\mathrm{a}}(T)$. Evidently, $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{a}}(T)$.

DEFINITION 2.1. Let $T \in L(X)$, X a Banach space. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{a}}(T)$ then λ is said to be a *left pole* of the resolvent of T. A left pole λ is said to have *finite rank* if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$ then λ is said to be a *right pole* of the resolvent of T. A right pole λ is said to have *finite rank* if $\beta(\lambda I - T) < \infty$.

Evidently, λ is a pole for T if and only if λ is both a left and a right pole for T. Moreover, λ is a pole for T if and only if λ is a pole for T'. In the case of Hilbert space operators, λ is a pole for T' if and only if $\overline{\lambda}$ is a pole for T^*.

DEFINITION 2.2. Let $T \in L(X)$. Then T is said to be

(i) *left polaroid* if every isolated point of $\sigma_{\mathrm{a}}(T)$ is a left pole of the resolvent of T;
(ii) *right polaroid* if every isolated point of $\sigma_{\mathrm{s}}(T)$ is a right pole of the resolvent of T;
(iii) *a-polaroid* if every $\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T)$ is a pole of the resolvent of T.

If T is a Hilbert space operator, we denote by T^* the Hilbert adjoint of T. The concepts of left and right polaroid are dual to each other:

Theorem 2.3 ([3]). *If $T \in L(X)$, X a Banach space, then the following equivalences hold:*

(i) *T is left polaroid if and only if T' is right polaroid.*
(ii) *T is right polaroid if and only if T' is left polaroid.*
(iii) *T is polaroid if and only if T' is polaroid.*

It should be noted that if T is a Hilbert space operator then in the equivalences (i)–(iii), T' may be replaced by T^*. Moreover, T' is a-polaroid if and only if T^* is a-polaroid. This easily follows from the equality $\sigma_{\mathrm{a}}(T^*) = \overline{\sigma_{\mathrm{a}}(T')}$, so if T' is a-polaroid and $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T^*)$, then $\overline{\lambda} \in \operatorname{iso}\sigma_{\mathrm{a}}(T')$ and hence $\overline{\lambda}$ is a pole for T', or equivalently λ is a pole for T^*.

The *quasi-nilpotent part* of $T \in L(X)$ is the set

$$H_0(T) := \Big\{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\Big\}.$$

Clearly, $\ker T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$. The *analytic core* of T is $K(T) := \{x \in X$: there exist $c > 0$ and a sequence $(x_n)_{n\geq 1} \subseteq X$ such that $Tx_1 = x$, $Tx_{n+1} = x_n$ for all $n \in \mathbb{N}$, and $\|x_n\| \leq c^n\|x\|$ for all $n \in \mathbb{N}\}$. Note that $T(K(T)) = K(T)$ (see [1, Theorem 1.21]).

Theorem 2.4 ([6, Theorem 2.2]). *Let $T \in L(X)$.*

(i) *T is polaroid if and only if there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \textit{for all } \lambda \in \operatorname{iso}\sigma(T). \tag{3}$$

(ii) *If T is left polaroid then there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \textit{for all } \lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T). \tag{4}$$

In [3, Theorem 2.6] it has been observed that if T is *both* left and right polaroid then T is polaroid. The following theorem shows that this is true if T is *either* left or right polaroid.

Theorem 2.5. *For $T \in L(X)$ the following implications hold:*

$$T \text{ } a\text{-polaroid} \Rightarrow T \text{ left polaroid} \Rightarrow T \text{ polaroid}.$$

Furthermore, if T is right polaroid then it is polaroid.

Proof. The first implication is clear, since a pole is always a left pole. Assume that T is left polaroid and let $\lambda \in \operatorname{iso}\sigma(T)$. It is known that the boundary of the spectrum lies in $\sigma_{\mathrm{a}}(T)$, in particular so does every isolated point of $\sigma(T)$, thus $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ and hence λ is a left pole of the resolvent of T. By Theorem 2.4(ii) there exists $\nu := \nu(\lambda I - T) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^\nu$. But λ is isolated in $\sigma(T)$ so, by 2.4(i), λ is is a pole of the resolvent, i.e. T is polaroid.

To show the last assertion suppose that T is right polaroid. By Theorem 2.3, T' is left polaroid, and hence, by the first part, T' is polaroid, or equivalently T is polaroid. ∎

The following property plays a relevant role in local spectral theory: see the recent monographs by Laursen and Neumann [25] and [1].

Definition 2.6. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}$ about λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$.

An operator is said to have SVEP if it has SVEP at every point of $\mathbb{C}$.

Evidently, every operator has SVEP at each isolated point of its spectrum.

We also have

$$p(\lambda I - T) < \infty \;\Rightarrow\; T \text{ has SVEP at } \lambda, \tag{5}$$

and dually, if T' denotes the dual of T,

$$q(\lambda I - T) < \infty \;\Rightarrow\; T' \text{ has SVEP at } \lambda \tag{6}$$

(see [1, Theorem 3.8]). Furthermore, from the definition of localized SVEP it easily seen that

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \;\Rightarrow\; T \text{ has SVEP at } \lambda, \tag{7}$$

and dually,

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \;\Rightarrow\; T' \text{ has SVEP at } \lambda. \tag{8}$$

Note that generally $H_0(T)$ is not closed and (see [1, Theorem 2.31])

$$H_0(\lambda I - T) \text{ closed } \;\Rightarrow\; T \text{ has SVEP at } \lambda. \tag{9}$$

Remark 2.7. The converses of the implications (1)–(5) hold if $\lambda I - T$ is semi-Fredholm (see [1, Chapter 3]).

In [3] it has been observed that if T' has SVEP (respectively, T has SVEP) then all polaroid type conditions for T (respectively, for T') are equivalent. Actually, we have a more precise result.

Theorem 2.8. *Let $T \in L(X)$.*

(i) *If T' has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$ then the properties of being polaroid, a-polaroid and left polaroid for T are all equivalent.*
(ii) *If T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ then the properties of being polaroid, a-polaroid and left polaroid for T' are all equivalent.*

Proof. (i) Note first that $\sigma_{\mathrm{a}}(T) = \sigma(T)$. In fact, suppose that $\lambda \notin \sigma_{\mathrm{a}}(T)$. Then $p(\lambda I - T) = 0$ and $\lambda I - T \in W_+(X)$, so $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and hence by assumption T' has SVEP at λ. By Remark 2.7 it then follows that $q(\lambda I - T) < \infty$ and hence $p(\lambda I - T) = q(\lambda I - T) = 0$, i.e. $\lambda \notin \sigma(T)$. This proves the equality $\sigma_{\mathrm{a}}(T) = \sigma(T)$. The equivalence of the polaroid conditions is now clear: every polaroid operator is a-polaroid, since $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \operatorname{iso}\sigma(T)$. Therefore, by Theorem 2.5 the equivalence is proved.

(ii) By using dual arguments to those of the proof of (i), if T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ then $\sigma_{\mathrm{s}}(T) = \sigma(T)$, and hence $\sigma_{\mathrm{a}}(T') = \sigma(T')$ by duality. Therefore, if T' is polaroid then T' is a-polaroid, so the equivalence is proved. ∎

The following result is well known.

Lemma 2.9. *If $T \in L(X)$ and N is a nilpotent operator commuting with T then $H_0(T+N) = H_0(T)$.*

Lemma 2.10. *If $T \in L(X)$ and N is a nilpotent operator commuting with T, then λ is a pole of the resolvent of T if and only if λ is a pole of the resolvent of $T+N$.*

Proof. If λ is a pole for T then $\lambda \in \operatorname{iso}\sigma(T)$ and since the spectrum is invariant under nilpotent commuting perturbations we have $\lambda \in \operatorname{iso}\sigma(T+N)$. Now, set $p := p(\lambda I - T) = q(\lambda I - T)$ and suppose that $N^\nu = 0$. By [1, Theorem 3.74] we have $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Set $m := p\nu$. Then $\ker(\lambda I - (T+N))^m \subseteq H_0(\lambda I - (T+N))$.

We show the opposite inclusion. Let $x \in H_0(\lambda I - (T+N))$. By Lemma 2.9 we have $x \in H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Then can write

$$(\lambda I - (T+N)^m x = \sum_{j=m-p+1} \mu_{j,m}(\lambda I - T)^{m-j} N^{j-\nu} N^\nu x = 0,$$

with suitable binomial coefficients $\mu_{j,m}$. Hence

$$H_0(\lambda I - (T+N)) = \ker(\lambda I - T)^p \subseteq \ker(\lambda I - (T+N))^m.$$

Consequently,

$$H_0(\lambda I - (T+N)) = \ker(\lambda I - (T+N))^m.$$

Since $\lambda \in \operatorname{iso}\sigma(T+N)$ it then follows, by Theorem 2.4, that λ is a pole of the resolvent of $T+N$.

Conversely, if if λ is a pole for $T+N$ then it is a pole for $(T+N)-N = T$, since $T+N$ commutes with N. ∎

Theorem 2.11. *Suppose that $T \in L(X)$ and let N be a nilpotent operator which commutes with T. Then*

(i) *T is polaroid if and only if $T+N$ is polaroid.*
(ii) *$T+N$ is a-polaroid if and only if $T+N$ is a-polaroid.*

Proof. (i) Suppose T is polaroid. If $\lambda \in \operatorname{iso}\sigma(T+N)$ then $\lambda \in \operatorname{iso}\sigma(T)$, so it is a pole for T. By Lemma 2.10 then λ is a pole for $T+N$, i.e. $T+N$ is polaroid. Conversely, if $T+N$ is polaroid then by the first part of the proof $T = (T+N) - N$ is also polaroid.

(ii) It is well known that the approximate point spectrum is invariant under nilpotent commuting perturbations. Suppose that T is a-polaroid and $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T+N) = \operatorname{iso}\sigma_{\mathrm{a}}(T)$. Since T is a-polaroid, λ is a pole for T and hence for $T+N$, by Lemma 2.10. Therefore, $T+N$ is a-polaroid. The converse follows by symmetry: if $T+N$ is a-polaroid then $(T+N)-N=T$ is a-polaroid. ∎

If T is left polaroid then T is polaroid, so $T+N$ is polaroid for any nilpotent commuting perturbation N. The next result shows that assuming SVEP, $T+N$ is then also left polaroid.

Corollary 2.12. *Let $T \in L(X)$ and let N be a nilpotent operator which commutes with T.*

(i) *If T' has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and T is left polaroid then $T+N$ is left polaroid.*
(ii) *If T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ and T is right polaroid then $T+N$ is right polaroid.*

Proof. (i) Suppose that T is left polaroid. Then, by Theorem 2.8, T is a-polaroid and hence $T+N$ is a-polaroid by Theorem 2.11. Consequently, $T+N$ is left polaroid.

(ii) If T is right polaroid then T' is left polaroid and hence, again by Theorem 2.8, T' is a-polaroid. Since N' is also nilpotent, Theorem 2.11 shows that $T'+N'$ is a-polaroid and hence left polaroid. By Theorem 2.3 it then follows that $T+N$ is right polaroid. ∎

It is not known to the authors if the results of Corollary 2.12 hold without assuming SVEP. The following result shows that the answer is positive for Hilbert space operators:

Theorem 2.13. *Suppose that $T \in L(H)$, H a Hilbert space, and let N be a nilpotent operator which commutes with T. Then T is left polaroid (respectively, right polaroid) if and only if $T+N$ is left polaroid (respectively, right polaroid).*

Proof. Suppose that T is left polaroid and $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T+N) = \operatorname{iso}\sigma_{\mathrm{a}}(T)$. We can suppose that $\lambda = 0$. Then 0 is a left pole of the resolvent of T and this is equivalent, by Theorem 2.5 of [11], to saying that there exists a decomposition $H = H_0 \oplus H_1$, with H_0 and H_1 closed T-invariant subspaces of H, such that $T_0 := T|H_0$ is bounded below and $T_1 := T|H_1$ is nilpotent. Since $TN = NT$, both H_0 and H_1 are invariant under N. Write $N = N_0 \oplus N_1$ where $N_0 := N|H_0$ and $N_1 := N|H_1$. Clearly, both N_0 and N_1 are nilpotent, $T_0 + N_0$ is bounded below, and $T_1 + N_1$ is nilpotent. From

$$T + N = (T_0 + N_0) \oplus (T_1 + N_1)$$

we then deduce, again by [11, Theorem 2.5], that $T+N$ is left polaroid. The converse follows by symmetry: $T = (T+N) - N$, so by the first part, T is left polaroid whenever $T+N$ is.

If T is right polaroid then the Hilbert adjoint T^* is left polaroid and this, by the first part, is equivalent to saying that $(T+N)^* = T^* + N^*$ is left polaroid, or equivalently, by Theorem 2.3, $T+N$ is right polaroid. ∎

Recall that a bounded operator $T \in L(X)$ is said to be *algebraic* if there exists a nonconstant polynomial h such that $h(T) = 0$. Trivially, every nilpotent operator is algebraic and it is well known that every finite-dimensional operator is algebraic. It is also known that every algebraic operator has a finite spectrum.

Definition 2.14. An operator $T \in L(X)$ is said to be *hereditarily polaroid* if the restriction of T to every closed T-invariant subspace is polaroid.

Theorem 2.15. *Let $T \in L(X)$ and let $K \in L(X)$ be an algebraic operator which commutes with T.*

(i) *Suppose that T has SVEP (respectively, T' has SVEP), and T is hereditarily polaroid. Then $T+K$ is polaroid and $T'+K'$ is a-polaroid (respectively, $T+K$ is a-polaroid and $T'+K'$ is polaroid).*
(ii) *Suppose that T' has SVEP (respectively, T has SVEP) and T' is hereditarily polaroid operator. Then $T'+K'$ is polaroid and $T+K$ is a-polaroid (respectively, $T'+K'$ is a-polaroid and $T+K$ is polaroid).*

Proof. (i) An algebraic operator has a finite spectrum. Let $\sigma(K) = \{\lambda_1, \ldots, \lambda_n\}$ and denote by P_j the spectral projection associated with K and the spectral set $\{\lambda_j\}$. Set $Y_j := P_j(X)$ and $Z_j := \ker P_j$. From the classical spectral decomposition we know that $X = Y_j \oplus Z_j$, where Y_j and Z_j are closed subspaces invariant under T and K. Moreover, if $K_j := K|Y_j$ and $T_j := T|Y_j$ then K_j and T_j commute, $\sigma(K_j) = \{\lambda_j\}$ and

$$\sigma(T+K) = \bigcup_{j=1}^{n} \sigma(T_j + K_j).$$

We claim that $N_j := \lambda_j I - K_j$ is nilpotent for every $j = 1, \ldots, n$. To see this, let h be a nontrivial polynomial such that $h(K) = 0$. Then $h(K_j) = h(K)|Y_j = 0$, and since

$$h(\{\lambda_j\}) = h(\sigma(K_j)) = \sigma(h(K_j)) = \{0\},$$

it then follows that $h(\lambda_j) = 0$. Write

$$h(\mu) = (\lambda_j - \mu)^\nu q(\mu) \quad \text{with } q(\lambda_j) \neq 0.$$

Then

$$0 = h(K_j) = (\lambda_j - K_j)^\nu q(K_j)$$

with $q(K_j)$ invertible. Therefore, $(\lambda_j - K_j)^\nu = 0$ and hence $N_j := \lambda_j - K_j$ is nilpotent for every $j = 1, \dots, n$, as desired.

We claim that $T + K$ is polaroid. Let $\lambda \in \operatorname{iso}\sigma(T + K)$. Then $\lambda \in \operatorname{iso}\sigma(T_j + K_j)$ for some $j = 1, \dots, n$ and hence $\lambda - \lambda_j \in \operatorname{iso}\sigma(T_j + K_j - \lambda_j I)$. The restriction T_j is polaroid, by assumption, and as proved before, $\lambda_j I - K_j$ is nilpotent. By Theorem 2.11, $T_j + K_j - \lambda_j I$ is then polaroid. Therefore $\lambda - \lambda_j$ is a pole of the resolvent of $T_j + K_j - \lambda_j I$ and by [1, Theorem 3.74] there exists a $\nu_j \in \mathbb{N}$ such that

$$H_0((\lambda - \lambda_j)I - (T_j + K_j + \lambda_j I)) = H_0(\lambda I - (T_j + K_j)) = \ker(\lambda I - (T_j + K_j))^{\nu_j}.$$

Taking into account that $H_0(\lambda I - (T_j + K_j) = \{0\}$ if $\lambda \notin \sigma(T_j + K_j)$ it then follows that

$$H_0(\lambda I - (T + K)) = \bigoplus_{j=1}^{n} H_0((\lambda I - (T_j + K_j)) = \bigoplus_{j=1}^{n} \ker(\lambda I - (T_j + K_j))^{\nu_j}.$$

If we put $\nu := \max\{\nu_1, \dots, \nu_n\}$ we obtain

$$H_0(\lambda I - (T + K)) = \ker(\lambda I - (T + K))^{\nu}.$$

As λ is an isolated point of $\sigma(T + K)$, it then follows, by Theorem 2.4, that $T + K$ is polaroid, as desired.

Now, assume first that T has SVEP. By duality we know that $T' + K'$ is polaroid. Since T has SVEP, so does $T + K$ (see [8, Theorem 2.14]). By Theorem 2.8 it then follows that $T' + K'$ is a-polaroid.

Suppose now that T' has SVEP. Obviously, K' is algebraic and commutes with T'. Therefore $T' + K'$ has SVEP, always by [8, Theorem 2.14], and hence $T + K$ is a-polaroid, by Theorem 2.8.

(ii) By (i) we know that $T' + K'$ is polaroid, or equivalently, $T + K$ is polaroid. If T' has SVEP then $T' + K'$ has SVEP, so $T + K$ is a-polaroid. Suppose that T has SVEP. Then $T + K$ has SVEP, so $T' + K'$ is a-polaroid. ∎

Remark 2.16. In the case of Hilbert space operators, the assertions of Theorem 2.15 are still valid if T' is replaced with T^*.

The next simple example shows that the result of Corollary 2.12, as well as the result of Theorem 2.11, cannot be extended to quasi-nilpotent operators Q commuting with T.

Example 2.17. Let $Q \in L(\ell^2(\mathbb{N}))$ is defined by

$$Q(x_1, x_2, \dots) = (x_2/2, x_3/3, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then Q is quasi-nilpotent and if $e_n := (0, \dots, 1, 0, \dots)$, where 1 is the nth term and all others are 0, then $e_{n+1} \in \ker Q^{n+1}$ while $e_{n+1} \notin \ker Q^n$, so that

$p(Q) = \infty$. If we take $T = 0$, the null operator, then T is both left polaroid and a-polaroid, while $T + Q = Q$ is neither left polaroid, nor a-polaroid.

However, the following theorem shows that $T + Q$ is polaroid in a very special case. Recall first that if $\alpha(T) < \infty$ then $\alpha(T^n) < \infty$ for all $n \in \mathbb{N}$.

THEOREM 2.18. *Suppose that $Q \in L(X)$ is a quasi-nilpotent operator which commutes with $T \in L(X)$ and suppose that all eigenvalues of T have finite multiplicity.*

(i) *If T is polaroid then $T + Q$ is polaroid.*
(ii) *If T is left polaroid then $T + Q$ is left polaroid.*
(iii) *If T is a-polaroid then $T + Q$ is a-polaroid.*

Proof. (i) Let $\lambda \in \operatorname{iso}\sigma(T + Q)$. It is well known that the spectrum is invariant under commuting quasi-nilpotent perturbations, thus $\lambda \in \operatorname{iso}\sigma(T)$ and hence is a pole of the resolvent of T (consequently, an eigenvalue of T). Therefore, $p := p(\lambda I - T) = q(\lambda I - T) < \infty$ and since by assumption $\alpha(\lambda I - T) < \infty$ we then have $\alpha(\lambda I - T) = \beta(\lambda I - T)$ (see [1, Theorem 3.4]), so $\lambda I - T$ is Browder. By (1) we then infer that $\lambda I - (T + Q)$ is Browder, hence λ is a pole of the resolvent of $T + Q$, thus $T + Q$ is polaroid.

(ii) Let $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T + Q)$. As $\sigma_{\mathrm{a}}(T)$ is invariant under commuting quasi-nilpotent perturbations, we have $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ and hence, since T is left polaroid, λ is a left pole of the resolvent of T. Therefore, $p := p(\lambda I - T) < \infty$ and $(\lambda I - T)^{p+1}(X)$ is closed. Now, either $\lambda I - T$ is injective or λ is an eigenvalue of T. In both cases we have $\alpha(\lambda I - T) < \infty$ and hence $\alpha((\lambda I - T)^{p+1}) < \infty$. Thus, $(\lambda I - T)^{p+1} \in \Phi_+(X)$ and this implies that $\lambda I - T \in \Phi_+(X)$. Consequently, $\lambda I - T \in B_+(X)$ and by (2), $\lambda I - (T + Q)$ is upper-Browder. This implies that $p' = p(\lambda I - (T + Q)) < \infty$ and since $(\lambda I - (T + Q))^{p'+1}$ is upper semi-Browder, we deduce that $\lambda I - (T + Q))^{p'+1}(X)$ is closed and hence $\lambda I - (T + Q)$ is left Drazin invertible. Since $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T + Q)$ it then follows that λ is a left pole for $T + Q$ and hence $T + Q$ is left polaroid.

(iii) The proof is analogous to that of (i). In fact, if $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T+Q)$ then $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ and hence, since T is a-polaroid, λ is a pole of the resolvent of T. By assumption, $\alpha(\lambda I - T) < \infty$, so proceeding as in (i) we find that λ is a pole for $T + Q$, thus $T + Q$ is a-polaroid. ∎

The argument of the proof of Theorem 2.18(i) also works if we assume that every isolated point of $\sigma(T)$ is a finite rank pole (in this case T is said to be *finitely polaroid*). This is the case, for instance, of Riesz operators having infinite spectrum. Evidently, $T + Q$ is also finitely polaroid, since for every $\lambda \in \operatorname{iso}\sigma(T + Q)$ we have $\alpha(\lambda I - (T + Q)) < \infty$.

3. Weyl type theorems. In this section we apply the results of the previous section in order to establish Weyl type theorems for perturbations of polaroid type operators. For $T \in L(X)$ define

$$E(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\},$$
$$E^a(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T)\}.$$

Evidently, $E^0(T) \subseteq E(T) \subseteq E^a(T)$ for every $T \in L(X)$. Define

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$
$$\pi_{00}^a(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Let $p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, i.e. $p_{00}(T)$ is the set of all poles of the resolvent of T.

DEFINITION 3.1. An operator $T \in L(X)$ is said to satisfy *Weyl's theorem*, in symbols (W), if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$; T is said to satisfy *a-Weyl's theorem*, in symbols (aW), if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^a(T)$; and T is said to have property (w) if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$.

Recall that $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, and *a-Browder's theorem* if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$. Weyl's theorem for T entails Browder's theorem for T, while a-Weyl's theorem entails a-Browder's theorem. Either a-Weyl's theorem or property (w) entails Weyl's theorem. Property (w) and a-Weyl's theorem are independent (see [10]).

The concept of semi-Fredholm operators has been generalized by Berkani ([14], [17]) in the following way: for every $T \in L(X)$ and a nonnegative integer n denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). Then $T \in L(X)$ is said to be *semi-B-Fredholm* (resp. *B-Fredholm, upper semi-B-Fredholm*, *lower semi-B-Fredholm*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is semi-Fredholm (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([17]). This enables one to define the index of a semi-B-Fredholm as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$.

A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi-B-Weyl, lower semi-B-Weyl*) if for some integer $n \geq 0$, $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl, lower semi-Weyl).

In an obvious way all the classes of operators generate spectra, for instance the *B-Weyl spectrum* $\sigma_{\mathrm{bw}}(T)$ and the *upper B-Weyl spectrum* $\sigma_{\mathrm{ubw}}(T)$.

Analogously, a bounded operator $T \in L(X)$ is said to be *B-Browder* (respectively, *upper semi-B-Browder, lower semi-B-Browder*) if for some integer $n \geq 0$, $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Browder, lower semi-Browder). The *B-Browder spectrum* is denoted by $\sigma_{\mathrm{bb}}(T)$, and the *upper semi-B-Browder spectrum* by $\sigma_{\mathrm{ubb}}(T)$. Note that $\sigma_{\mathrm{ubb}}(T)$ coincides with the left Drazin spectrum $\sigma_{\mathrm{ld}}(T)$ ([4]).

The generalized versions of Weyl type theorems are defined as follows:

Definition 3.2. An operator $T \in L(X)$ is said to satisfy *generalized Weyl's theorem*, in symbols (gW), if $\sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = E(T)$; $T \in L(X)$ is said to satisfy *generalized a-Weyl's theorem*, in symbols (gaW), if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T) = E^a(T)$; and $T \in L(X)$ is said to satisfy *generalized property* (w), in symbols (gw), if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T) = E(T)$.

Recall that $T \in L(X)$ is said to satisfy *generalized Browder's theorem* if $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{bw}}(T)$, and *generalized a-Browder's theorem* if $\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{ubw}}(T)$. Browder's theorem and generalized Browder's theorem are equivalent, as also are a-Browder's theorem and generalized a-Browder's theorem (see [12]). a-Browder theorems entail Browder theorems and if T, or T', has SVEP then a-Browder's theorem holds for T. Generalized a-Weyl's theorem, as well as generalized property (w), entails generalized a-Browder's theorem.

In the following diagrams we summarize the relationships between all Weyl type theorems:

$$(\mathrm{gw}) \Rightarrow (\mathrm{w}) \Rightarrow (\mathrm{W}), \quad (\mathrm{gaW}) \Rightarrow (\mathrm{aW}) \Rightarrow (\mathrm{W})$$

(see [15, Theorem 2.3], [10] and [16]). Generalized property (w) and generalized a-Weyl's theorem are also independent (see [15]). Furthermore,

$$(\mathrm{gw}) \Rightarrow (\mathrm{gW}) \Rightarrow (\mathrm{W}), \quad (\mathrm{gaW}) \Rightarrow (\mathrm{gW}) \Rightarrow (\mathrm{W})$$

(see [15] and [16]). The converse of none of these implications holds in general. Furthermore, by [2, Theorem 3.1],

$$(\mathrm{W}) \text{ holds for } T \Leftrightarrow \text{Browder's theorem holds for } T \text{ and } p_{00}(T) = \pi_{00}(T).$$

The following equivalences have been proved in [3, Theorem 3.7 and Corollary 3.8]:

Theorem 3.3. *Let $T \in L(X)$.*

(i) *If T is polaroid then* (W) *and* (*gW*) *are equivalent for T.*
(ii) *If T is left polaroid then* (aW) *and* (gaW) *are equivalent for T.*
(iii) *If T is a-polaroid then* (aW), (gaW), *and* (gw) *are equivalent for T.*

Theorem 3.4. *Let $T \in L(X)$ be polaroid and suppose that either T' has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$, or T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$. Then both T and T' satisfy Weyl's theorem.*

Proof. Each of the assumptions on the SVEP ensures that T, or equivalently T', satisfies Browder's theorem. In fact, if T' has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$ then a-Browder's theorem (and hence Browder's theorem) holds for T, while if T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ then a-Browder's theorem (and hence Browder's theorem) holds for T' (see [5, Theorem 2.3]). The polaroid condition for T entails that $p_{00}(T) = \pi_{00}(T)$, so Weyl's theorem holds for T. If T is polaroid then T' is polaroid and hence $p_{00}(T') = \pi_{00}(T')$, so Weyl's theorem also holds for T'. ∎

For a bounded operator $T \in L(X)$, define $\Pi^a(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$. It is clear that $\Pi^a_{00}(T)$ is the set of all left poles of the resolvent.

THEOREM 3.5. *Let $T \in L(X)$ be left polaroid and suppose that either T or T' has SVEP. Then T satisfies generalized a-Weyl's theorem.*

Proof. T satisfies a-Browder's theorem and the left polaroid condition entails that $\Pi^a(T) = E^a(T)$. By [9, Theorem 2.18], (gaW) holds for T. ∎

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is nonconstant on each component of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}_{nc}(\sigma(T))$. We now apply the results of the second section in order to obtain a general framework for establishing Weyl type theorems for perturbations of hereditarily polaroid operators.

THEOREM 3.6. *Let $T \in L(X)$ and let $K \in L(X)$ be an algebraic operator commuting with $T \in L(X)$.*

(i) *If $T \in L(X)$ has SVEP (respectively, T' has SVEP) and T is hereditarily polaroid, then $f(T + K)$ (respectively, $f(T' + K')$) satisfies* (gW)*, while $f(T' + K')$ (respectively, $f(T + K)$) satisfies every Weyl type theorem (generalized or not) for every $f \in \mathcal{H}_{nc}(\sigma(T + K))$.*
(ii) *If $T' \in L(X)$ has SVEP (respectively, T has SVEP) and T' is hereditarily polaroid, then $f(T' + K')$ (respectively, $f(T + K)$) satisfies* (gW)*, while $f(T + K)$ (respectively, $f(T' + K')$) satisfies every Weyl type theorem (generalized or not) for every $f \in \mathcal{H}_{nc}(\sigma(T + K))$.*

Proof. (i) Suppose first that T has SVEP. As we know, the SVEP for T entails that $T + K$ has SVEP, and hence, by [1, Theorem 2.40], also $f(T + K)$ has SVEP. Moreover, by Theorem 2.15(i), $T + K$ is polaroid and consequently also $f(T + K)$ is polaroid (see [3, Lemma 3.11]). Therefore, by Theorem 3.4, $f(T + K)$ satisfies Weyl's theorem and this, by Theorem 3.3, is equivalent to $f(T + K)$ satisfying generalized Weyl's theorem.

To show the second assertion, note that, by Theorem 2.15, $T' + K'$ is a-polaroid, in particular left polaroid. Again from [3, Lemma 3.11] we deduce that $f(T' + K') = [f(T + K)]'$ is left polaroid. The SVEP of $f(T + K)$ entails, by Theorem 2.8(ii), that $f(T' + K')$ is a-polaroid, so, by Theorem 3.5, (gaW) holds for $f(T' + K')$. Equivalently, by Theorem 3.3, (gw) holds for $f(T' + K')$.

The case where T' has SVEP uses similar arguments.

(ii) Assume that T' has SVEP. The first assertion is obvious, by (i). As in (i), $f(T'+K')$ satisfies SVEP. Furthermore, by Theorem 2.15(ii), $T+K$ is a-polaroid and hence left polaroid. Again [3, Lemma 3.11] entails that $f(T+K)$ is left polaroid, or equivalently, by Theorem 2.8, $f(T+K)$ is a-polaroid. Combining Theorems 3.5 and 3.3(iii) we conclude that $f(T+K)$ satisfies every generalized Weyl type theorem for every $f \in \mathcal{H}_{nc}(\sigma(T+K))$.

The case where T has SVEP uses similar arguments. ∎

Part of statement (i) of Theorem 3.6 has been proved by Duggal [20, Theorem 3.6] by using different methods.

Remark 3.7. In the case of Hilbert space operators, in Theorem 3.3 the assertions hold if T' is replaced by the Hilbert adjoint T^*.

The class of hereditarily polaroid operators is rather large. It contains the $H(p)$ operators introduced by Oudghiri in [26], where $T \in L(X)$ is said to belong to the class $H(p)$ if there exists a natural $p := p(\lambda)$ such that

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{10}$$

From the implication (9) we see that every operator T which belongs to $H(p)$ has SVEP. Moreover, every $H(p)$ operator T is polaroid. Furthermore, if T is $H(p)$ then every part of T is $H(p)$ [26, Lemma 3.2], so T is hereditarily polaroid. Property $H(p)$ is satisfied by every generalized scalar operator (see [25] for details), and in particular for p-hyponormal, log-hyponormal or M-hyponormal operators on Hilbert spaces (see [26]). Therefore, algebraically p-hyponormal or algebraically M-hyponormal operators are $H(p)$.

Another important class of hereditarily polaroid operators is given by *paranormal* operators on Hilbert spaces, satisfying $\|Tx\|^2 \le \|T^2x\|\,\|x\|$ for all $x \in H$. In fact, these operators have SVEP, are polaroid and obviously any of their parts are still paranormal (see [7]). Weyl's theorem for $T+K$ in the case that T is $H(p)$ has been proved by Oudghiri [27], while Weyl's theorem for $T+K$ in the case that T is paranormal has been proved in [7]. Theorem 3.6 extends and subsumes both results. Theorem 3.6 also extends the results of [8, Theorems 2.15 and 2.16], since every algebraically paranormal operator is polaroid and has SVEP.

Other examples of hereditarily polaroid operators are given by *completely hereditarily normaloid* operators on Banach spaces. In particular, all (p,k)-*quasihyponormal* operators on Hilbert spaces are hereditarily polaroid (see for details [20]). Also *algebraically quasi-class A operators* on a Hilbert space, considered in [13], are hereditarily polaroid. In fact, every part of an algebraically quasi-class A operator T is algebraically quasi-class A and every algebraically quasi-class A operator is polaroid [13, Lemma 2.3].

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