

Lower bounds for norms of products of polynomials on L_p spaces

by

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Abstract. For $1 < p < 2$ we obtain sharp lower bounds for the uniform norm of products of homogeneous polynomials on $L_p(\mu)$, whenever the number of factors is no greater than the dimension of these Banach spaces (a condition readily satisfied in infinite-dimensional settings). The result also holds for the Schatten classes \mathcal{S}_p . For $p > 2$ we present some estimates on the constants involved.

1. Introduction. This work is framed in what is sometimes called the *factor problem* for homogeneous polynomials. Given homogeneous polynomials P_1, \dots, P_n defined on $(\mathbb{C}^N, \|\cdot\|_p)$, our aim is to find the best constant M such that

$$(1.1) \quad \|P_1 \cdots P_n\| \geq M \|P_1\| \cdots \|P_n\|.$$

The constant will necessarily depend on p and on the degrees of the polynomials, but not on the number N of variables. And, of course, we must set what the norm of a polynomial is.

Recall that a mapping $P : X \rightarrow \mathbb{C}$ is a (continuous) *k-homogeneous polynomial* if there exists a (continuous) *k-linear map* $T : X \times \cdots \times X \rightarrow \mathbb{C}$ such that $P(x) = T(x, \dots, x)$ for all $x \in X$. The space of continuous *k-homogeneous polynomials* on a Banach space X is denoted by $\mathcal{P}(^k X)$. It is a Banach space under the uniform norm

$$\|P\|_{\mathcal{P}(^k X)} = \sup_{\|z\|_X=1} |P(z)|.$$

With this norm, inequality (1.1) was studied for polynomials defined on finite- and infinite-dimensional Banach spaces. For instance R. Ryan and B. Turett [RT] gave bounds for the special case where the polynomials $\{P_i\}_{i=1}^n$ are actually continuous linear forms on X . Moreover, C. Benítez,

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Y. Sarantopoulos and A. Tonge [BST] proved that if P_i has degree k_i for $1 \leq i \leq n$, then inequality (1.1) holds with constant

$$M = \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}$$

for any complex Banach space. The authors also showed that this is the best universal constant, since there are polynomials on ℓ_1 for which equality prevails. However, for many spaces it is possible to improve this bound. For instance, for complex Hilbert spaces, the second named author proved in [P] that the optimal constant is

$$(1.2) \quad M = \sqrt{\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}}.$$

In this work we establish the best constant for complex $L_p(\Omega, \mu)$ spaces whenever $1 < p < 2$. We show that in this case inequality (1.1) holds with constant

$$(1.3) \quad M = \sqrt[p]{\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}}.$$

The constant is optimal provided the spaces involved have dimension at least n . This constant also works (and is optimal) for polynomials on the Schatten classes \mathcal{S}_p . For the remaining values of p , we obtain some estimates of the optimal constants.

For the sake of simplicity, we first consider the finite-dimensional situation (Section 2). Then we show in Section 3 how to adapt the ideas to handle the infinite-dimensional L_p spaces and Schatten classes. In Section 4 we address the case $p > 2$.

2. The finite-dimensional setting. We begin with some definitions. If E and F are isomorphic Banach spaces, their *Banach–Mazur distance* (see, for example, [Pi, Chapter 1] or [T]) is defined as

$$d(E, F) = \inf\{\|u\| \|u^{-1}\| \mid u : E \rightarrow F \text{ an isomorphism}\}.$$

Given a Banach space X and $n \in \mathbb{N}$, we define

$$D_n(X) := \sup\{d(E, \ell_2^n) : E \text{ a subspace of } X \text{ with } \dim E = n\}.$$

From Corollary 5 in [L], we obtain

$$(2.1) \quad D_n(L_p(\Omega, \mu)) \leq n^{|1/p-1/2|}$$

whenever $L_p(\Omega, \mu)$ has dimension at least n .

The proof of the following lemma is inspired by Proposition 1 in [RS].

LEMMA 2.1. Let X be a Banach space and let $P_1, \dots, P_n : X \rightarrow \mathbb{C}$ be homogeneous polynomials of degrees k_1, \dots, k_n respectively. Then

$$\|P_1 \cdots P_n\|_{\mathcal{P}(\mathbf{k}X)} \geq \sqrt{\frac{\prod_{i=1}^n k_i^{k_i}}{\mathbf{k}^{\mathbf{k}}}} D_n(X)^{-\mathbf{k}} \|P_1\|_{\mathcal{P}(k_1 X)} \cdots \|P_n\|_{\mathcal{P}(k_n X)},$$

where $\mathbf{k} = \sum_{i=1}^n k_i$.

Proof. Given $\varepsilon > 0$, we can take a set $\{x_1, \dots, x_n\} \subset X$ of norm one vectors such that $|P_j(x_j)| > (1 - \varepsilon)\|P_j\|_{\mathcal{P}(k_j X)}$ for $1 \leq j \leq n$. Let $E \subset X$ be any n -dimensional subspace containing $\{x_1, \dots, x_n\}$ and let $T : \ell_2^n \rightarrow E$ be a norm one isomorphism with $\|T^{-1}\| \leq D_n(X)$. We have

$$\begin{aligned} \|P_1 \cdots P_n\|_{\mathcal{P}(\mathbf{k}X)} &\geq \|P_1 \cdots P_n\|_{\mathcal{P}(\mathbf{k}E)} \geq \|(P_1 \circ T) \cdots (P_n \circ T)\|_{\mathcal{P}(\mathbf{k}\ell_2^n)} \\ &\stackrel{(a)}{\geq} \sqrt{\frac{k_1^{k_1} \cdots k_n^{k_n}}{\mathbf{k}^{\mathbf{k}}}} \|(P_1 \circ T)\|_{\mathcal{P}(k_1 \ell_2^n)} \cdots \|(P_n \circ T)\|_{\mathcal{P}(k_n \ell_2^n)} \\ &\geq \sqrt{\frac{k_1^{k_1} \cdots k_n^{k_n}}{\mathbf{k}^{\mathbf{k}}}} \frac{1}{\|T^{-1}\|_{\mathbf{k}}} \|P_1\|_{\mathcal{P}(k_1 E)} \cdots \|P_n\|_{\mathcal{P}(k_n E)} \\ &> \sqrt{\frac{k_1^{k_1} \cdots k_n^{k_n}}{\mathbf{k}^{\mathbf{k}}}} D_n(X)^{-\mathbf{k}} (1 - \varepsilon)^n \|P_1\|_{\mathcal{P}(k_1 X)} \cdots \|P_n\|_{\mathcal{P}(k_n X)}, \end{aligned}$$

where (a) follows from (1.2). ■

REMARK 2.2. If we restrict ourselves to $L_p(\Omega, \mu)$ spaces and polynomials with the same degree, we can combine Lemma 2.1 with Lewis' result (2.1) to obtain

$$(2.2) \quad \|P_1 \cdots P_n\|_{\mathcal{P}(k_n L_p(\Omega, \mu))} \geq \frac{1}{n^{nk/p}} \|P_1\|_{\mathcal{P}(k L_p(\Omega, \mu))} \cdots \|P_n\|_{\mathcal{P}(k L_p(\Omega, \mu))}$$

for $1 \leq p \leq 2$. For $2 \leq p \leq \infty$ we have

$$\|P_1 \cdots P_n\|_{\mathcal{P}(k_n L_p(\Omega, \mu))} \geq \frac{1}{n^{nk/q}} \|P_1\|_{\mathcal{P}(k L_p(\Omega, \mu))} \cdots \|P_n\|_{\mathcal{P}(k L_p(\Omega, \mu))},$$

where q is the conjugate exponent of p .

Note that (2.2) is precisely (1.1) with the constant M given in (1.3). In order to extend this result to the general case when the polynomials have arbitrary degrees, it is convenient to consider another particular case. We will say that $P, Q : \ell_p^N \rightarrow \mathbb{C}$ depend on different variables if one can find disjoint subsets $I, J \subset \{1, \dots, N\}$ such that

$$P\left(\sum_{i=1}^N a_i e_i\right) = P\left(\sum_{i \in I} a_i e_i\right) \quad \text{and} \quad Q\left(\sum_{i=1}^N a_i e_i\right) = Q\left(\sum_{i \in J} a_i e_i\right),$$

for all $\{a_i\}_{i=1}^N \subset \mathbb{C}$.

For polynomials depending on different variables, (1.1) becomes an equality when M is given by (1.3), as the following lemma shows.

LEMMA 2.3. *Let $\{P_i\}_{i=1}^n$ be homogeneous polynomials of degrees $\{k_i\}_{i=1}^n$ on ℓ_p^N , depending on different variables. Write $\mathbf{k} = k_1 + \dots + k_n$. Then*

$$\|P_1 \cdots P_n\|_{\mathcal{P}(\mathbf{k}\ell_p^N)} = \sqrt[p]{\frac{\prod_{i=1}^n k_i^{k_i}}{\mathbf{k}^{\mathbf{k}}}} \|P_1\|_{\mathcal{P}(k_1\ell_p^N)} \cdots \|P_n\|_{\mathcal{P}(k_n\ell_p^N)}.$$

Proof. First, we prove the equality for two polynomials P and Q of degrees k and l . We may suppose that P depends on the first r variables and Q on the last $N - r$ ones. Given $z \in \ell_p^N$, we can write $z = x + y$, where x and y are the projections of z on the first r and the last $N - r$ coordinates respectively. We then have

$$|P(z)Q(z)| = |P(x)Q(y)| \leq \|P\|_{\mathcal{P}(k\ell_p^N)} \|Q\|_{\mathcal{P}(l\ell_p^N)} \|x\|_p^k \|y\|_p^l.$$

Since $\|z\|_p^p = \|x\|_p^p + \|y\|_p^p$, we can estimate the norm of PQ as follows:

$$\begin{aligned} \|PQ\|_{\mathcal{P}(k+l\ell_p^N)} &= \sup_{\|z\|_p=1} |P(z)Q(z)| \leq \sup_{|a|^p+|b|^p=1} |a|^k |b|^l \|P\|_{\mathcal{P}(k\ell_p^N)} \|Q\|_{\mathcal{P}(l\ell_p^N)} \\ &= \sqrt[p]{\frac{k^k l^l}{(k+l)^{k+l}}} \|P\|_{\mathcal{P}(k\ell_p^N)} \|Q\|_{\mathcal{P}(l\ell_p^N)}, \end{aligned}$$

the last equality being a simple application of Lagrange multipliers. In order to see that this inequality is actually an equality, take norm one vectors x_0 and y_0 at which P and Q respectively attain their norms, each with nonzero entries only in the coordinates on which the corresponding polynomial depends. If we define

$$z_0 = \sqrt[p]{\frac{k}{k+l}} x_0 + \sqrt[p]{\frac{l}{k+l}} y_0,$$

then z_0 is a norm one vector which satisfies

$$|P(z_0)Q(z_0)| = \sqrt[p]{\frac{k^k l^l}{(k+l)^{k+l}}} \|P\|_{\mathcal{P}(k\ell_p^N)} \|Q\|_{\mathcal{P}(l\ell_p^N)}.$$

We prove the general statement by induction on n . We assume the result is valid for $n - 1$ polynomials and we know that it is valid for two. We omit the subscripts in the norms of the polynomials to simplify the notation. We then have

$$\left\| \prod_{i=1}^n P_i \right\| = \sqrt[p]{\frac{k_n^{k_n} (\sum_{i=1}^{n-1} k_i)^{\sum_{i=1}^{n-1} k_i}}{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}} \left\| \prod_{i=1}^{n-1} P_i \right\| \|P_n\|$$

$$\begin{aligned}
 &= \sqrt[p]{\frac{k_n^{k_n} (\sum_{i=1}^{n-1} k_i)^{\sum_{i=1}^{n-1} k_i}}{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}} \sqrt[p]{\frac{\prod_{i=1}^{n-1} k_i^{k_i}}{(\sum_{i=1}^{n-1} k_i)^{\sum_{i=1}^{n-1} k_i}} \left(\prod_{i=1}^{n-1} \|P_i\| \right) \|P_n\|} \\
 &= \sqrt[p]{\frac{\prod_{i=1}^n k_i^{k_i}}{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}} \prod_{i=1}^n \|P_i\|}. \blacksquare
 \end{aligned}$$

Now we are ready to prove our main result.

THEOREM 2.4. *Let $\{P_i\}_{i=1}^n$ be homogeneous polynomials of degrees $\{k_i\}_{i=1}^n$ on ℓ_p^N , $1 \leq p \leq 2$. Write $\mathbf{k} = k_1 + \dots + k_n$. Then*

$$(2.3) \quad \|P_1 \cdots P_n\|_{\mathcal{P}(\mathbf{k}\ell_p^N)} \geq \sqrt[p]{\frac{\prod_{i=1}^n k_i^{k_i}}{\mathbf{k}\mathbf{k}}} \|P_1\|_{\mathcal{P}(k_1\ell_p^N)} \cdots \|P_n\|_{\mathcal{P}(k_n\ell_p^N)}.$$

The constant is optimal provided that $N \geq n$.

Proof. We first prove the inequality for two homogeneous polynomials P and Q of degrees k and l . If $k = l$, the result follows from Remark 2.2. Now suppose $k > l$. Moving to ℓ_p^{N+1} if necessary, we take a norm one polynomial S of degree $d = k - l$, depending on different variables than the polynomials P and Q . An example of such a polynomial is $(e'_{N+1})^d$. In the following, we identify ℓ_p^N with a subspace of ℓ_p^{N+1} in the natural way. We use Lemma 2.3 for (a) and (b) below, and inequality (2.2) for (c), to obtain

$$\begin{aligned}
 \|PQ\|_{\mathcal{P}(k+l\ell_p^N)} &= \|PQ\|_{\mathcal{P}(k+l\ell_p^{N+1})} \|S\|_{\mathcal{P}(d\ell_p^{N+1})} \\
 &\stackrel{(a)}{=} \sqrt[p]{\frac{((k+l)+d)^{(k+l)+d}}{(k+l)^{k+l}d^d}} \|PQS\|_{\mathcal{P}(2k\ell_p^{N+1})} \\
 &\stackrel{(b)}{\geq} \sqrt[p]{\frac{((k+l)+d)^{(k+l)+d}}{(k+l)^{k+l}d^d}} \frac{1}{4^{k/p}} \|P\|_{\mathcal{P}(k\ell_p^{N+1})} \|QS\|_{\mathcal{P}(k\ell_p^{N+1})} \\
 &\stackrel{(c)}{=} \sqrt[p]{\frac{(2k)^{2k}}{(k+l)^{k+l}d^d 4^k}} \sqrt[p]{\frac{l^d d^d}{k^k}} \|P\|_{\mathcal{P}(k\ell_p^{N+1})} \|Q\|_{\mathcal{P}(l\ell_p^{N+1})} \|S\|_{\mathcal{P}(d\ell_p^{N+1})} \\
 &= \sqrt[p]{\frac{k^k l^l}{(k+l)^{k+l}}} \|P\|_{\mathcal{P}(k\ell_p^N)} \|Q\|_{\mathcal{P}(l\ell_p^N)}.
 \end{aligned}$$

The proof of the general case continues by induction on n as in the previous lemma.

To see that the constant is optimal whenever $N \geq n$, consider for each $i = 1, \dots, n$ the polynomial $P_i = (e'_i)^{k_i}$. From Lemma 2.3 we obtain equality in (2.3). \blacksquare

Theorem 2.4 also holds for polynomials on ℓ_p . This is a consequence of the following: if $P \in \mathcal{P}(k\ell_p)$ then

$$\|P\|_{\mathcal{P}(k\ell_p)} = \lim_{N \rightarrow \infty} \|P \circ i_N\|_{\mathcal{P}(k\ell_p^N)}$$

where i_N is the canonical inclusion of ℓ_p^N in ℓ_p . The proof of this fact is rather standard. Anyway, in the next section we will show that Theorem 2.4 holds for $L_p(\mu)$ spaces, which covers ℓ_p as a particular case.

3. L_p spaces and Schatten classes. In this section we show that the results obtained for ℓ_p can be extended to $L_p(\Omega, \mu)$ spaces and to the Schatten classes \mathcal{S}_p for $1 \leq p \leq 2$. We will sometimes omit parts of the proofs which are very similar to those in the previous section.

Let (Ω, μ) be a measure space. From now on, the notation

$$\Omega = A_1 \sqcup \dots \sqcup A_n$$

will mean that Ω is the union of measurable subsets $\{A_i\}_{1 \leq i \leq n}$ such that $\mu(A_i) > 0$ for $1 \leq i \leq n$, and $\mu(A_i \cap A_j) = 0$ for all $1 \leq i < j \leq n$. The following lemma is the analogue to Lemma 2.3 for L_p spaces.

LEMMA 3.1. *Let $P, Q : L_p(\Omega, \mu) \rightarrow \mathbb{C}$ be homogeneous polynomials of degrees k and l respectively. Suppose that $\Omega = A_1 \sqcup A_2$, and that*

$$P(f) = P(f\chi_{A_1}) \quad \text{and} \quad Q(f) = Q(f\chi_{A_2})$$

for all $f \in L_p(\Omega, \mu)$. Then

$$\|PQ\|_{\mathcal{P}(k+l)L_p(\Omega, \mu)} = \sqrt[p]{\frac{k^k l^l}{(k+l)^{k+l}}} \|P\|_{\mathcal{P}(k)L_p(\Omega, \mu)} \|Q\|_{\mathcal{P}(l)L_p(\Omega, \mu)}.$$

Proof. Given $f \in L_p(\Omega, \mu)$ we write $f = f\chi_{A_1} + f\chi_{A_2}$ and then

$$\begin{aligned} |P(f)Q(f)| &= |P(f\chi_{A_1})Q(f\chi_{A_2})| \\ &\leq \|P\|_{\mathcal{P}(k)L_p(\Omega, \mu)} \|Q\|_{\mathcal{P}(l)L_p(\Omega, \mu)} \|f\chi_{A_1}\|_p^k \|f\chi_{A_2}\|_p^l. \end{aligned}$$

Given $\varepsilon > 0$, we can take norm one functions $f_0, g_0 \in L_p(\Omega, \mu)$ such that

$$|P(f_0)| > \|P\|_{\mathcal{P}(k)L_p(\Omega, \mu)} - \varepsilon \quad \text{and} \quad |Q(g_0)| > \|Q\|_{\mathcal{P}(l)L_p(\Omega, \mu)} - \varepsilon.$$

By the hypotheses on P and Q we may assume that $f_0 = f_0\chi_{A_1}$ and $g_0 = g_0\chi_{A_2}$. We clearly have

$$\left\| \sqrt[p]{\frac{k}{k+l}} f_0 + \sqrt[p]{\frac{l}{k+l}} g_0 \right\|_p = 1.$$

Now we can, modulo ε , proceed as in the proof of Lemma 2.3, and then let ε go to zero to obtain the desired result. ■

Combining this lemma with the fact that $D_n(L_p(\mu)) \leq n^{\lfloor 1/p-1/2 \rfloor}$ we obtain the next result.

THEOREM 3.2. *Let $\{P_i\}_{i=1}^n$ be homogeneous polynomials of degrees $\{k_i\}_{i=1}^n$. Write $\mathbf{k} = k_1 + \dots + k_n$. Then*

$$\|P_1 \cdots P_n\|_{\mathcal{P}(k_{L_p}(\Omega, \mu))} \geq \sqrt[p]{\frac{\prod_{i=1}^n k_i^{k_i}}{\mathbf{k}^{\mathbf{k}}}} \|P_1\|_{\mathcal{P}(k_1 L_p(\Omega, \mu))} \cdots \|P_n\|_{\mathcal{P}(k_n L_p(\Omega, \mu))}.$$

If Ω admits a decomposition $\Omega = A_1 \sqcup \dots \sqcup A_n$, then the constant is optimal.

Proof. We prove the result for two polynomials. Let P and Q be homogeneous polynomials of degrees k and l . If $k = l$, the result follows from Remark 2.2.

Now assume $k > l$ and define an auxiliary measure space (Ω', μ') by adding an additional point $\{c\}$ to Ω . The measure μ' in Ω' is given by $\mu'(U) = \mu(U)$ if $U \subseteq \Omega$, and $\mu'(U) = \mu(U \cap \Omega) + 1$ whenever $c \in U$. It is clear that $\Omega' = \Omega \sqcup \{c\}$.

Consider the polynomials P', Q' and S of degree k, l and $d = k - l$ respectively, defined on $L_p(\Omega', \mu')$ by $P'(f) = P(f|_\Omega)$, $Q'(f) = Q(f|_\Omega)$ and $S(f) = (f(c))^d$. Observe that $\|S\|_{\mathcal{P}(d_{L_p}(\Omega', \mu'))} = 1$. The polynomials $P'Q'$ and S satisfy the conditions of Lemma 3.1. Proceeding as in the proof of Theorem 2.4, we have

$$\begin{aligned} \|PQ\|_{\mathcal{P}(k+l_{L_p}(\Omega, \mu))} &= \|P'Q'\|_{\mathcal{P}(k+l_{L_p}(\Omega', \mu'))} \|S\|_{\mathcal{P}(d_{L_p}(\Omega', \mu'))} \\ &= \sqrt[p]{\frac{((k+l)+d)^{(k+l)+d}}{(k+l)^{k+l}d^d}} \|P'Q'S\|_{\mathcal{P}(2k_{L_p}(\Omega', \mu'))} \\ &\geq \sqrt[p]{\frac{((k+l)+d)^{(k+l)+d}}{(k+l)^{k+l}d^d}} \frac{1}{4^{k/p}} \|P'\|_{\mathcal{P}(k_{L_p}(\Omega', \mu'))} \|Q'S\|_{\mathcal{P}(k_{L_p}(\Omega', \mu'))} \\ &= \sqrt[p]{\frac{k^k l^l}{(k+l)^{k+l}}} \|P'\|_{\mathcal{P}(k_{L_p}(\Omega', \mu'))} \|Q'\|_{\mathcal{P}(l_{L_p}(\Omega', \mu'))} \|S\|_{\mathcal{P}(d_{L_p}(\Omega', \mu'))} \\ &= \sqrt[p]{\frac{k^k l^l}{(k+l)^{k+l}}} \|P\|_{\mathcal{P}(k_{L_p}(\Omega, \mu))} \|Q\|_{\mathcal{P}(l_{L_p}(\Omega, \mu))}. \end{aligned}$$

The general case follows by induction exactly as in the proof of Lemma 2.3, and the optimality of the constant is analogous to that of Theorem 2.4. ■

Now we show how the previous proofs can be adapted to obtain the corresponding results for the Schatten classes. Let $\{P_i\}_{i=1}^n$ be k -homogeneous polynomials on $\mathcal{S}_p = \mathcal{S}_p(H)$, the p -Schatten class of operators on the Hilbert space H . In Corollary 2.10 of [T], Tomczak-Jaegermann proved that

$$D_n(\mathcal{S}_p) \leq n^{|1/p-1/2|}.$$

Hence, by Lemma 2.1, we have

$$\|P_1 \cdots P_n\|_{\mathcal{P}(n\mathbf{k}_{\mathcal{S}_p(H)})} \geq \frac{1}{n^{nk/p}} \|P_1\|_{\mathcal{P}(k\mathcal{S}_p(H))} \cdots \|P_n\|_{\mathcal{P}(k\mathcal{S}_p(H))}.$$

Suppose that $H = H_1 \oplus H_2$ (an orthogonal sum) and let $\pi_1, \pi_2 : H \rightarrow H$ be the orthogonal projections onto H_1 and H_2 respectively. If homogeneous polynomials $P, Q : \mathcal{S}_p(H) \rightarrow \mathbb{C}$ satisfy

$$P(s) = P(\pi_1 \circ s \circ \pi_1) \quad \text{and} \quad Q(s) = Q(\pi_2 \circ s \circ \pi_2),$$

for all $s \in \mathcal{S}_p$, we can think of P and Q as depending on different variables. Moreover, for each $s \in \mathcal{S}_p(H)$, it is rather standard that

$$(3.1) \quad \|\pi_1 \circ s \circ \pi_1\|_{\mathcal{S}_p}^p + \|\pi_2 \circ s \circ \pi_2\|_{\mathcal{S}_p}^p = \|\pi_1 \circ s \circ \pi_1 + \pi_2 \circ s \circ \pi_2\|_{\mathcal{S}_p}^p.$$

Also, we have

$$\pi_1 \circ s \circ \pi_1 + \pi_2 \circ s \circ \pi_2 = \frac{1}{2}(s + (\pi_1 - \pi_2) \circ s \circ (\pi_1 - \pi_2)).$$

By the ideal property of Schatten norms, the last operator has norm (in \mathcal{S}_p) not greater than $\|s\|_{\mathcal{S}_p}$, so that

$$(3.2) \quad \|\pi_1 \circ s \circ \pi_1\|_{\mathcal{S}_p}^p + \|\pi_2 \circ s \circ \pi_2\|_{\mathcal{S}_p}^p \leq \|s\|_{\mathcal{S}_p}^p.$$

Now, with (3.1) and (3.2) at hand, we can follow the proof of Lemma 2.3 to obtain the analogous result for Schatten classes.

Finally, the trick of adding a variable in Theorem 2.4 or a singleton in Theorem 3.2 can be performed for Schatten classes by just taking the orthogonal sum of H with a (one-dimensional) Hilbert space. As a consequence, mimicking the proof of Theorem 2.4 we obtain the following.

THEOREM 3.3. *Let P_1, \dots, P_n be polynomials of degrees k_1, \dots, k_n respectively on $\mathcal{S}_p(H)$ with $1 \leq p \leq 2$. Write $\mathbf{k} = k_1 + \dots + k_n$. Then*

$$\|P_1 \cdots P_n\|_{\mathcal{P}(k\mathcal{S}_p(H))} \geq \sqrt[p]{\frac{\prod_{i=1}^n k_i^{k_i}}{\mathbf{k}^{\mathbf{k}}}} \|P_1\|_{\mathcal{P}(k_1\mathcal{S}_p(H))} \cdots \|P_n\|_{\mathcal{P}(k_n\mathcal{S}_p(H))}.$$

The constant is optimal provided that $\dim(H) \geq n$.

4. Remarks on the case $p > 2$. We end this note with some comments on the constant for $p > 2$. If X is a Banach space, let $M(X, k_1, \dots, k_n)$ be the largest value of M such that

$$\|P_1 \cdots P_n\|_{\mathcal{P}(k_X)} \geq M \|P_1\|_{\mathcal{P}(k_1 X)} \cdots \|P_n\|_{\mathcal{P}(k_n X)}$$

for any set of homogeneous polynomials P_1, \dots, P_n on X of degrees k_1, \dots, k_n respectively. From [BST], [P] and Theorem 2.4 we know that

$$M(\ell_p^N, k_1, \dots, k_n) = \sqrt[p]{\frac{\prod_{i=1}^n k_i^{k_i}}{\mathbf{k}^{\mathbf{k}}}},$$

provided that $1 \leq p \leq 2$ and $N \geq n$. In [RS, Proposition 8], the authors show that the best constant for products of linear functionals on an infinite-

dimensional Banach space is worse than the corresponding one for Hilbert spaces. In our notation, they show that

$$M(\ell_2, 1, \dots, 1) \geq M(X, 1, \dots, 1)$$

for every infinite-dimensional Banach space X . Our next theorem, together with Theorems 2.4 and 3.2, shows that the same holds for products of homogeneous polynomials in ℓ_p^N and L_p spaces, provided that the dimension is greater than or equal to the number of factors. That is, the constant for Hilbert spaces is better than the constant for any other L_p space for homogeneous polynomials of any degree, even in the finite-dimensional setting.

THEOREM 4.1. *For $N \geq n$ and $2 \leq p \leq \infty$, we have*

$$\begin{aligned} M(\ell_2^N, k_1, \dots, k_n) &\geq M(\ell_p^N, k_1, \dots, k_n) \\ &\geq (n^{k_1 + \dots + k_n})^{1/p-1/2} M(\ell_2^N, k_1, \dots, k_n). \end{aligned}$$

The same holds for $L_p(\Omega, \mu)$ whenever Ω admits a decomposition as in Theorem 3.2.

Proof. The second inequality is a direct consequence of Lemma 2.1, so let us show the first one. Consider the linear forms on ℓ_p^N defined by the vectors

$$g_j = \left(1, e^{\frac{2\pi i j}{N}}, e^{\frac{2\pi i 2j}{N}}, \dots, e^{\frac{2\pi i (N-1)j}{N}} \right) \quad \text{for } j = 1, \dots, n.$$

These are orthogonal vectors in ℓ_2^N . We can choose an orthogonal coordinate system such that the g_i 's depend on different variables (we are in ℓ_2^N). So by Lemma 2.3, inequality (1.1) is an equality with the constant for Hilbert spaces given in (1.2):

$$M(\ell_2^N, k_1, \dots, k_n) \|g_1^{k_1}\|_{\mathcal{P}(k_1 \ell_2^N)} \cdots \|g_n^{k_n}\|_{\mathcal{P}(k_n \ell_2^N)} = \|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(k \ell_2^N)}.$$

For products of orthogonal linear forms this equality was observed in [A] and for the general case (with arbitrary powers) in Remark 4.2 of [P].

On the other hand, we have

$$\|g_j^{k_j}\|_{\mathcal{P}(k_j \ell_2^N)} = (N^{1/2})^{k_j} \quad \text{and} \quad \|g_l^{k_j}\|_{\mathcal{P}(k_j \ell_p^N)} = (N^{1-1/p})^{k_j}.$$

Combining all this we obtain

$$\begin{aligned} M(\ell_2^N, k_1, \dots, k_n) &= \frac{\|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(k \ell_2^N)}}{\|g_1^{k_1}\|_{\mathcal{P}(k_1 \ell_2^N)} \cdots \|g_n^{k_n}\|_{\mathcal{P}(k_n \ell_2^N)}} \\ &= \frac{\|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(k \ell_2^N)}}{(N^{1/p-1/2})^{k_1} \|g_1^{k_1}\|_{\mathcal{P}(k_1 \ell_p^N)} \cdots (N^{1/p-1/2})^{k_n} \|g_n^{k_n}\|_{\mathcal{P}(k_n \ell_p^N)}} \\ &\geq \frac{\|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(k \ell_p^N)} N^{(1/p-1/2)k}}{(N^{1/p-1/2})^{k_1} \|g_1^{k_1}\|_{\mathcal{P}(k_1 \ell_p^N)} \cdots (N^{1/p-1/2})^{k_n} \|g_n^{k_n}\|_{\mathcal{P}(k_n \ell_p^N)}} \end{aligned}$$

$$= \frac{\|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(\mathbf{k}\ell_p^N)}}{\|g_1^{k_1}\|_{\mathcal{P}(k_1\ell_p^N)} \cdots \|g_n^{k_n}\|_{\mathcal{P}(k_n\ell_p^N)}} \geq M(\ell_p^N, k_1, \dots, k_n).$$

This shows the statement for ℓ_p^N . Since the space $L_p(\Omega, \mu)$, with our assumptions on Ω , contains a 1-complemented copy of ℓ_p^n , the statement for $L_p(\Omega, \mu)$ readily follows. ■

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