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A new characteristic property of Mittag-Leffler functions and fractional cosine functions

by

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Abstract. A new characteristic property of the Mittag-Leffler function $E_\alpha(at^\alpha)$ with $1 < \alpha < 2$ is deduced. Motivated by this property, a new notion, named α-order cosine function, is developed. It is proved that an α-order cosine function is associated with a solution operator of an α-order abstract Cauchy problem. Consequently, an α-order abstract Cauchy problem is well-posed if and only if its coefficient operator generates a unique α-order cosine function.

1. Introduction. The two-parameter *Mittag-Leffler functions* are defined by the series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \forall\, \alpha, \beta > 0,\ z \in \mathbb{C},$$

where Γ is the Gamma function [E, P]. For brevity, we denote by E_α the Mittag-Leffler function $E_{\alpha,1}$. For any real number a, the Laplace integral of the Mittag-Leffler function is

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(at^\alpha)\, dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - a}, \qquad \operatorname{Re}\lambda > |a|^{1/\alpha}. \tag{1.1}$$

Let $\alpha > 0$ and $m = [\alpha]$, the smallest integer larger than or equal to α. It is well-known that the Mittag-Leffler function is closely related to the fractional differential equation

$$\begin{cases} {}^C D_0^\alpha u(t) = au(t), & t \ge 0, \\ u(0) = x, \quad u^{(k)}(0) = 0, & k = 1, \ldots, m-1, \end{cases} \tag{1.2}$$

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where ${}^CD_t^\alpha$ is the modified Caputo fractional derivative operator

$$ {}^CD_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\sigma)^{m-1-\alpha} \Big(u(\sigma) - \sum_{k=0}^{m-1} \frac{\sigma^k}{k!} u^{(k)}(0) \Big)\, d\sigma. $$

Concretely, $E_\alpha(at^\alpha)$ is a solution of equation (1.2), that is,

$$ {}^CD_t^\alpha (E_\alpha(at^\alpha)) = aE_\alpha(at^\alpha). \tag{1.3} $$

Furthermore, it is stated in [B] that in the case $1 < \alpha < 2$, the fractional differential equation

$$ \begin{cases} {}^CD_0^\alpha u(t) = au(t), & t \geq 0, \\ u(0) = x, \quad u'(0) = y, \end{cases} $$

has the unique solution

$$ u(t) = E_\alpha(at^\alpha)x + tE_{\alpha,2}(at^\alpha)y. \tag{1.4} $$

The combination of (1.3) and (1.4) implies that

$$ {}^CD_t^\alpha (tE_{\alpha,2}(at^\alpha)) = atE_{\alpha,2}(at^\alpha). \tag{1.5} $$

For $u \in C^{(m)}([0,\infty), X)$, we can calculate

$$ {}^CD_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\sigma)^{m-1-\alpha} u^{(m)}(\sigma)\, d\sigma. $$

For a smooth function f, the following Laplace transform formulas are available:

$$ \widehat{{}^CD_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1} f(0), \qquad 0 < \alpha < 1, \tag{1.6} $$

$$ \widehat{{}^CD_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1} f(0) - \lambda^{\alpha-2} f'(0), \qquad 1 < \alpha < 2, \tag{1.7} $$

$$ \widehat{J_t^\alpha f}(\lambda) = \lambda^{-\alpha} \hat{f}(\lambda). \tag{1.8} $$

Here

$$ J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1} f(t)\, dt \tag{1.9} $$

is the αth integral of f.

Let $(A, D(A))$ be an unbounded linear operator defined in Banach space X. Denote $m = [\alpha]$. Then the *abstract fractional Cauchy problem* corresponding to (1.2) is

$$ \begin{cases} {}^CD_t^\alpha u(t) = Au(t), & t > 0, \\ u(0) = x, \quad u^{(k)}(0) = 0, & k = 1, \dots, m-1. \end{cases} \tag{1.10} $$

It is well-known that the solvability of (1.10) is equivalent to that of the integral equation

$$u(t) = x + J_t^\alpha Au(t), \quad t \geq 0. \tag{1.11}$$

Based on the solution of (1.11) and the related notion in [Pr], Bazhlekova [B] introduced the notion of solution operator for (1.10):

DEFINITION 1.1. A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X is called a *solution operator* for (1.10) if:

(a) $T(t)$ is strongly continuous for $t \geq 0$ and $T(0) = I$,
(b) $T(t)D(A) \subset D(A)$ and $AT(t)x = T(t)Ax$ for all $x \in D(A)$ and $t \geq 0$,
(c) $u(t) = T(t)x$ is a solution of (1.11) for any $x \in D(A)$.

Solution operators for (1.10) have been systematically studied in Bazhlekova's doctoral dissertation [B], and the results obtained generalize some facts of C_0-semigroup and cosine function theory.

Recently, Chen and Li [CL] found a novel characterization of solution operators, by developing a purely algebraic notion, named α-resolvent operator function.

DEFINITION 1.2. Let $\{S(t)\}_{t\geq 0}$ be a family of bounded linear operators on X. Then $\{S(t)\}_{t\geq 0}$ is called an *α-resolvent operator function* if:

- $S(t)$ is strongly continuous and $S(0) = I$.
- $S(s)S(t) = S(t)S(s)$ for all $t, s \geq 0$.
- $S(s)J_t^\alpha S(t) - J_s^\alpha S(s)S(t) = J_t^\alpha S(t) - J_s^\alpha S(s)$ for all $t, s \geq 0$.

Chen and Li proved in [CL, Theorem 3.4] that a family $\{S(t)\}_{t\geq 0}$ is an α-resolvent operator function if and only if it is a solution operator for the fractional Cauchy problem (1.10). This confirms the expectation that the fractional abstract Cauchy problem can be studied by purely algebraic methods, just as we can use the semigroup property to study first-order abstract Cauchy problems.

Since the Mittag-Leffler functions are closely related to fractional differential equations, it seems reasonable to study the fractional abstract Cauchy problem (1.10) through the properties of these functions.

The research on the properties of the Mittag-Leffler functions has been a hot topic ever since it came out. Many publications claimed that the Mittag-Leffler function $E_\alpha(at^\alpha)$ with $0 < \alpha < 1$ had the semigroup property (see, e.g., [J1, (3.10)], [J2, (5.1)]). However, Peng and Li [PL1] pointed out that it is in fact incorrect, and they proved that the Mittag-Leffler function has the following property: for any real number a,

$$\int_0^{t+s} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\tau)^\alpha}\,d\tau - \int_0^{t} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\tau)^\alpha}\,d\tau - \int_0^{s} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\tau)^\alpha}\,d\tau = \alpha\int_0^t\int_0^s \frac{E_\alpha(ar_1^\alpha)E_\alpha(ar_2^\alpha)}{(t+s-r_1-r_2)^{1+\alpha}}\,dr_1\,dr_2, \quad t,s\ge 0. \tag{1.12}$$

Furthermore, motivated by (1.12), Peng and Li derived the following new characteristic property of α-order semigroups [PL2]:

$$\int_0^{t+s} \frac{T(\tau)}{(t+s-\tau)^\alpha}\,d\tau - \int_0^{t} \frac{T(\tau)}{(t+s-\tau)^\alpha}\,d\tau - \int_0^{s} \frac{T(\tau)}{(t+s-\tau)^\alpha}\,d\tau = \alpha\int_0^t\int_0^s \frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}\,dr_1\,dr_2, \quad t,s\ge 0. \tag{1.13}$$

However, in [PL2, Remark 3], the authors mentioned that (1.13) is unavailable for $\alpha > 1$, so they restricted themselves to the case $0 < \alpha < 1$. This means that for $1 < \alpha < 2$, we have to find a new equality. On the other hand, just like [PL2], a natural problem is whether we can derive a new characteristic property of α-order cosine functions with $1 < \alpha < 2$ through the study of the properties of the Mittag-Leffler function $E_\alpha(at^\alpha)$ or not. In fact, we add another Mittag-Leffler function $tE_{\alpha,2}(at^\alpha)$ to the study of $E_\alpha(at^\alpha)$.

The arrangement of the rest of this paper is as follows. In Sec. 2, we will find a characteristic property of the Mittag-Leffler function $E_\alpha(at^\alpha)$ with $1 < \alpha < 2$. Motivated by this property now, in Sec. 3, we propose a new notion, named α-order cosine function with $1 < \alpha < 2$; moreover, it is proved that a solution operator corresponding to an α-order abstract Cauchy problem (1.10) is an α-order cosine function. In Sec. 4 we prove that every α-order cosine function is also a solution operator of (1.10).

2. A property of the Mittag-Leffler function $E_\alpha(at^\alpha)$ with $1 < \alpha < 2$. It is proved in [PL1] that $E_\alpha(at^\alpha)$ has the semigroup property if and only if $a = 0$ or $\alpha = 1$. Moreover, the authors of [PL1] gave a counterexample for $\alpha = 1/2$. Thus, in the case $1 < \alpha < 2$, $E_\alpha(at^\alpha)$ does not have the semigroup property. Next, we will show that $E_\alpha(at^\alpha)$ with $1 < \alpha < 2$ does not have the cosine function property, either.

To do this, we assume that on the one-dimensional space, $\{u(t)\}_{t\ge 0}$ has the cosine function property, that is, $u(0) = 1$ and

$$2u(t)u(s) = u(t+s) + u(t-s), \quad t \ge s \ge 0. \tag{2.1}$$

Then, by the theory of cosine functions [ABHN, G], $\{u(t)\}_{t\geq 0}$ has a unique generator c such that the equation

$$\begin{cases} u''(t) = cu(t), & t \geq 0, \\ u(0) = 1, \quad u'(0) = 0, \end{cases} \tag{2.2}$$

has a unique solution $u \in C^2$. Clearly, the solution of (2.2) is given by

$$u(t) = \begin{cases} \frac{1}{2}(e^{t\sqrt{c}} + e^{-t\sqrt{c}}), & c \geq 0, \\ \cos(t\sqrt{-c}), & c < 0. \end{cases} \tag{2.3}$$

The Laplace transform of $u(t)$ is

$$\int_0^\infty e^{-\lambda t} u(t)\, dt = \frac{\lambda}{\lambda^2 - c}, \qquad \operatorname{Re}\lambda > \sqrt{|c|}.$$

On the other hand, by (1.1),

$$\int_0^\infty e^{-\lambda t} E_\alpha(at^\alpha)\, dt = \frac{\lambda^{\alpha-1}}{\lambda^\alpha - a}, \qquad \operatorname{Re}\lambda > a^{1/\alpha}.$$

So $E_\alpha(at^\alpha)$ has the cosine function property only if $\alpha = 2$ or $a = 0$.

In the rest of this section, we will study a new feature of the Mittag-Leffler function $E_\alpha(at^\alpha)$ with $1 < \alpha < 2$. Since fractional derivative has the memorizing property, it is reasonable to expect that $E_\alpha(at^\alpha)$ also has the memorizing property. Thus the characteristic property of the Mittag-Leffler function $E_\alpha(at^\alpha)$ should be described in the form of integrals. Our proof is mainly based on the Laplace transform technique and the relationship between the fractional differential equation and the Mittag-Leffler function.

Theorem 2.1. *For every real number a and $\alpha \in (1,2)$,*

$$\begin{aligned} &\int_0^{t+s}\int_0^{\sigma} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\sigma)^{\alpha-1}}\, d\tau\, d\sigma - \int_0^{t}\int_0^{\sigma} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\sigma)^{\alpha-1}}\, d\tau\, d\sigma \\ &\quad - \int_0^{s}\int_0^{\sigma} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\sigma)^{\alpha-1}}\, d\tau\, d\sigma \\ &= \int_0^{t}\int_0^{s} \frac{E_\alpha(a\sigma^\alpha)E_\alpha(a\tau^\alpha)}{(t-\sigma)^{\alpha-1}}\, d\tau\, d\sigma + \int_0^{t}\int_0^{s} \frac{E_\alpha(a\sigma^\alpha)E_\alpha(a\tau^\alpha)}{(s-\tau)^{\alpha-1}}\, d\tau\, d\sigma \\ &\quad - \int_0^{t}\int_0^{s} \frac{E_\alpha(a\sigma^\alpha)E_\alpha(a\tau^\alpha)}{(t+s-\sigma-\tau)^{\alpha-1}}\, d\tau\, d\sigma, \qquad t, s \geq 0. \end{aligned} \tag{2.4}$$

Proof. Denote $tE_{\alpha,2}(at^\alpha)$ by $f(t)$ and $E_\alpha(at^\alpha)$ by $g(t)$ for convenience. It follows from (1.5) that

$$ {}^C D_r^\alpha (rE_{\alpha,2}(ar^\alpha))|_{r=t+s} = a(t+s)E_{\alpha,2}(a(t+s)^\alpha). $$

By the definition of Caputo derivative, for all $t, s \geq 0$ we have

$$
\begin{aligned}
(2.5)\qquad {}^C D_t^\alpha f(t+s) &= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\sigma)^{1-\alpha} \frac{d^2 f(\sigma+s)}{d\sigma^2}\, d\sigma \\
&= \frac{1}{\Gamma(2-\alpha)} \int_0^{t+s} (t+s-\sigma)^{1-\alpha} \frac{d^2 f(\sigma)}{d\sigma^2}\, d\sigma \\
&\quad - \frac{1}{\Gamma(2-\alpha)} \int_0^{s} (t+s-\sigma)^{1-\alpha} \frac{d^2 f(\sigma)}{d\sigma^2}\, d\sigma \\
&= af(t+s) - \frac{1}{\Gamma(2-\alpha)} \int_0^{s} (t+s-\sigma)^{1-\alpha} \frac{d^2 f(\sigma)}{d\sigma^2}\, d\sigma.
\end{aligned}
$$

Taking the Laplace transform with respect to t on both sides of (2.5) and using (1.7), we obtain

$$
\begin{aligned}
&\lambda^\alpha \hat{f}_s(\lambda) - \lambda^{\alpha-1} f(s) - \lambda^{\alpha-2} f'(s) \\
&\quad = a\hat{f}_s(\lambda) - \frac{1}{\Gamma(2-\alpha)} \int_0^s (t+\widehat{s-\sigma})^{1-\alpha}(\lambda) \frac{d^2 f(\sigma)}{d\sigma^2}\, d\sigma \\
&\quad = a\hat{f}_s(\lambda) - \frac{1}{\Gamma(2-\alpha)} (t+\widehat{s-\sigma})^{1-\alpha}(\lambda) \frac{df(\sigma)}{d\sigma}\bigg|_{\sigma=0}^{s} \\
&\qquad + \frac{1}{\Gamma(2-\alpha)} \int_0^s \frac{d(t+\widehat{s-\sigma})^{1-\alpha}(\lambda)}{d\sigma} \frac{df(\sigma)}{d\sigma}\, d\sigma \\
&\quad = a\hat{f}_s(\lambda) - \frac{1}{\Gamma(2-\alpha)} \widehat{t^{1-\alpha}}(\lambda) f'(s) + \frac{1}{\Gamma(2-\alpha)} (\widehat{t+s})^{1-\alpha}(\lambda) \\
&\qquad + \frac{\alpha-1}{\Gamma(2-\alpha)} \int_0^s (t+\widehat{s-\sigma})^{-\alpha}(\lambda) \frac{df(\sigma)}{d\sigma}\, d\sigma \\
&\quad = a\hat{f}_s(\lambda) - \lambda^{\alpha-2} f'(s) + \frac{1}{\Gamma(2-\alpha)} (\widehat{t+s})^{1-\alpha}(\lambda) \\
&\qquad + \frac{\alpha-1}{\Gamma(2-\alpha)} \int_0^s (t+\widehat{s-\sigma})^{-\alpha}(\lambda) \frac{df(\sigma)}{d\sigma}\, d\sigma.
\end{aligned}
$$

Here $\hat{f}_s(\lambda)$ and $(t+\widehat{s-\sigma})^{1-\alpha}(\lambda)$ represent respectively the Laplace transforms of $f(t+s)$ and $(t+s-\sigma)^{1-\alpha}$ with respect to t. The fact that $f'(0) = 1$

is used. Then we have

$$\lambda^{\alpha-2}\hat{f}_s(\lambda)=\lambda^{\alpha-2}\frac{\lambda^{\alpha-1}f(s)}{\lambda^\alpha-a}+\frac{\lambda^{\alpha-2}}{\lambda^\alpha-a}\frac{1}{\Gamma(2-\alpha)}\widehat{(t+s)^{1-\alpha}}(\lambda)+\frac{\lambda^{\alpha-2}}{\lambda^\alpha-a}\frac{\alpha-1}{\Gamma(2-\alpha)}\int_0^s\widehat{(t+s-\sigma)^{-\alpha}}(\lambda)\frac{df(\sigma)}{d\sigma}\,d\sigma. \tag{2.6}$$

Taking the inverse Laplace transform on both sides of (2.6), we obtain

$$\begin{aligned}\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-\sigma)^{1-\alpha}f(\sigma+s)\,d\sigma&=\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-\sigma)^{1-\alpha}E_a(a\sigma^\alpha)\,d\sigma f(s)\\&\quad+\frac{1}{\Gamma(2-\alpha)}\int_0^t(t+s-\sigma)^{1-\alpha}f(\sigma)\,d\sigma\\&\quad+\frac{\alpha-1}{\Gamma(2-\alpha)}\int_0^s\Big(\int_0^t(t+s-\sigma-\tau)^{-\alpha}f(\sigma)\,d\sigma\Big)f'(\tau)\,d\tau.\end{aligned}$$

Here the equalities (1.1) and (1.8) are used. Observe that $f'(t)=g(t)$. Thus,

$$\begin{aligned}&\int_0^t\frac{f(\sigma+s)d\sigma}{(t-\sigma)^{\alpha-1}}-\int_0^t\frac{f(\sigma)}{(t+s-\sigma)^{\alpha-1}}\,d\sigma\\&\qquad=\int_0^t\frac{g(\sigma)}{(t-\sigma)^{\alpha-1}}\,d\sigma f(s)+(\alpha-1)\int_0^s\int_0^t\frac{f(\tau)}{(t+s-\sigma-\tau)^\alpha}\,d\tau\,g(\sigma)\,d\sigma.\end{aligned}$$

By integrating by parts, we obtain

$$\begin{aligned}&(\alpha-1)\int_0^s\int_0^t\frac{f(\tau)}{(t+s-\sigma-\tau)^\alpha}\,d\tau g(\sigma)\,d\sigma\\&\qquad=\int_0^s\Big[(t+s-\sigma-\tau)^{1-\alpha}f(\tau)\Big|_{\tau=0}^t-\int_0^t\frac{f'(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\Big]g(\sigma)\,d\sigma\\&\qquad=\int_0^s\frac{f(t)g(\sigma)}{(s-\sigma)^{\alpha-1}}\,d\sigma-\int_0^s\int_0^t\frac{g(\tau)g(\sigma)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma.\end{aligned}$$

Moreover, observe that $f(t)=\int_0^t g(\sigma)\,d\sigma$. We obtain

$$\begin{aligned}&\int_0^{t+s}\int_0^\sigma\frac{g(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma-\int_0^t\int_0^\sigma\frac{g(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma-\int_0^s\int_0^\sigma\frac{g(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma\\&=\int_0^t\int_0^s\frac{g(\sigma)g(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma+\int_0^t\int_0^s\frac{g(\sigma)g(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma-\int_0^t\int_0^s\frac{g(\sigma)g(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma.\end{aligned}$$

The proof is therefore completed by replacing $g(t)$ with $E_\alpha(at^\alpha)$. ■

3. Fractional cosine function. In this section we develop a new notion, named fractional cosine function. Motivated by Theorem 2.1, we define α-order cosine functions and study their properties. Our discussion is restricted to the case $1 < \alpha < 2$.

DEFINITION 3.1. We call a family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X an *α-order cosine function* if:

(i) $T(t)$ is strongly continuous, that is, for any $x \in X$, the mapping $t \mapsto T(t)x$ is continuous over $[0,\infty)$;
(ii) $T(0) = I$ and for all $t, s \geq 0$,

$$\begin{aligned}(3.1)\quad &\int_0^{t+s}\int_0^{\sigma} \frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma - \int_0^{t}\int_0^{\sigma} \frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma \\ &\qquad - \int_0^{s}\int_0^{\sigma} \frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma \\ &= \int_0^{t}\int_0^{s} \frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^{t}\int_0^{s} \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma \\ &\qquad - \int_0^{t}\int_0^{s} \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma,\end{aligned}$$

where the integrals are in the sense of the strong operator topology.

REMARK 3.2. (1) It should be noted that for $\alpha = 2$, the integrals of (2.4) and (3.1) diverge and hence equalities (2.4) and (3.1) are unavailable.

(2) The semigroup property can be obtained by letting $\alpha \to 1^+$. Indeed, in this case (3.1) is converted into

$$(3.2)\quad \int_0^{t+s}\int_0^{\sigma} T(\tau)\,d\tau\,d\sigma - \int_0^{t}\int_0^{\sigma} T(\tau)\,d\tau\,d\sigma - \int_0^{s}\int_0^{\sigma} T(\tau)\,d\tau\,d\sigma = \int_0^{t}\int_0^{s} T(\sigma)T(\tau)\,d\tau\,d\sigma.$$

Taking the derivative with respect to t and s successively on both sides of (3.2), we obtain

$$T(t+s) = T(t)T(s), \qquad t, s \geq 0,$$

which means that $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup.

(3) The cosine function property can be obtained by letting $\alpha \to 2^-$. Indeed, let x belong to some dense subset such that the mapping $t \mapsto T(t)x$ is continuously differentiable on $[0,\infty)$. Below we will take the limits of both sides of (3.1)$\times(2-\alpha)$. The first term of the left side is

$$\lim_{\alpha\to 2^-}(2-\alpha)\int_0^{t+s}\int_0^{\sigma} \frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma = -\lim_{\alpha\to 2^-}\int_0^{t+s}\int_0^{\sigma} T(\tau)\,d\tau\,d(t+s-\sigma)^{2-\alpha}.$$

By integrating by parts, we obtain

$$\lim_{\alpha\to 2^-}(2-\alpha)\int_0^{t+s}\int_0^{\sigma}\frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma=\int_0^{t+s}T(\tau)\,d\tau.$$

Similarly, we derive

$$\lim_{\alpha\to 2^-}(2-\alpha)\int_0^{t}\int_0^{\sigma}\frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma=0,$$
$$\lim_{\alpha\to 2^-}(2-\alpha)\int_0^{s}\int_0^{\sigma}\frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma=0.$$

Thus, the limit of the left side is $\int_0^{t+s}T(\tau)\,d\tau$.

The first term of the right side is calculated as follows:

$$\begin{aligned}\lim_{\alpha\to 2^-}&(2-\alpha)\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma\\ &=\lim_{\alpha\to 2^-}(2-\alpha)\int_0^{t}\frac{T(\sigma)}{(t-\sigma)^{\alpha-1}}\,d\sigma\int_0^{s}T(\tau)\,d\tau\\ &=-\lim_{\alpha\to 2^-}\int_0^{t}T(\sigma)d(t-\sigma)^{2-\alpha}\int_0^{s}T(\tau)\,d\tau\\ &=\int_0^{s}T(\tau)\,d\tau+\int_0^{t}T'(\sigma)\,d\sigma\int_0^{s}T(\tau)\,d\tau=T(t)\int_0^{s}T(\tau)\,d\tau.\end{aligned}$$

Similarly, we derive

$$\lim_{\alpha\to 2^-}(2-\alpha)\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma=\int_0^{t}T(\tau)\,d\tau\,T(s).$$

The third term of the right side is

$$\begin{aligned}-\lim_{\alpha\to 2^-}&(2-\alpha)\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\\ &=\lim_{\alpha\to 2^-}\int_0^{t}T(\sigma)\int_0^{s}T(\tau)d(t+s-\sigma-\tau)^{2-\alpha}\,d\sigma\\ &=\int_0^{t}T(\tau)\,d\tau\,T(s)+\int_0^{t}T(\tau)\,d\tau-\int_0^{t}T(\sigma)\int_0^{s}T'(\tau)\,d\tau\,d\sigma=0.\end{aligned}$$

Therefore, we obtain the formula

$$\int_0^{t+s} T(\tau)\,d\tau = T(t)\int_0^s T(\tau)\,d\tau + \int_0^t T(\tau)\,d\tau\, T(s), \tag{3.3}$$

which means by Lemma 3.3 below that $\{T(t)\}_{t\geq 0}$ is a cosine function.

LEMMA 3.3. *A strongly continuous family of bounded linear operators $\{T(t)\}_{t\geq 0}$ with $T(0) = I$ is a cosine function if and only if $T(t)$ is exponentially bounded and* (3.3) *holds.*

Proof. The necessity is proved in [ABHN, (3.95)]. Now we prove the sufficiency. Assume that $T(t)$ is exponentially bounded and (3.3) holds. Then $T(t)$ is Laplace transformable. Taking the Laplace transform with respect to s and t on the left side of (3.3), we derive

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^{t+s} T(\tau)\,d\tau\,ds\,dt\\
&\quad= \int_0^\infty e^{-\mu t}e^{\lambda t}\int_t^\infty e^{-\lambda s}\int_0^s T(\tau)\,d\tau\,ds\,dt\\
&\quad= \int_0^\infty e^{-\mu t}e^{\lambda t}\Big(\int_0^\infty e^{-\lambda s}\int_0^s T(\tau)\,d\tau\,ds - \int_0^t e^{-\lambda s}\int_0^s T(\tau)\,d\tau\,ds\Big)\,dt\\
&\quad= \int_0^\infty e^{-(\mu-\lambda)t}\frac{\hat{T}(\lambda)}{\lambda}\,dt - \int_0^\infty e^{-\mu t}\int_0^t e^{\lambda(t-s)}\int_0^s T(\tau)\,d\tau\,ds\,dt\\
&\quad= \frac{\hat{T}(\lambda)}{\lambda(\mu-\lambda)} - \frac{\hat{T}(\mu)}{\mu(\mu-\lambda)} = \frac{\lambda\hat{T}(\mu)-\mu\hat{T}(\lambda)}{\lambda\mu(\lambda-\mu)}.
\end{aligned}$$

The Laplace transform with respect to s and t of the right side of (3.3) is

$$\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\Big(T(t)\int_0^s T(\tau)\,d\tau + \int_0^t T(\tau)\,d\tau\,T(s)\Big)\,ds\,dt = \frac{\lambda+\mu}{\lambda\mu}\hat{T}(\mu)\hat{T}(\lambda).$$

Hence,

$$\lambda\hat{T}(\mu) - \mu\hat{T}(\lambda) = (\lambda^2-\mu^2)\hat{T}(\mu)\hat{T}(\lambda).$$

Taking the Laplace transform with respect to s and t of

$$\begin{aligned}
M(t,s) &= T(s)\int_0^t (t-\sigma)T(\sigma)\,d\sigma - \int_0^s (s-\sigma)T(\sigma)\,d\sigma\,T(t)\\
&\quad - \int_0^t (t-\sigma)T(\sigma)\,d\sigma + \int_0^s (s-\sigma)T(\sigma)\,d\sigma,
\end{aligned}$$

we obtain

$$\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}M(t,s)\,ds\,dt = \frac{\hat{T}(\mu)\hat{T}(\lambda)}{\lambda^2} - \frac{\hat{T}(\mu)\hat{T}(\lambda)}{\mu^2} - \frac{\hat{T}(\lambda)}{\lambda^2\mu} + \frac{\hat{T}(\mu)}{\lambda\mu^2}$$
$$= \frac{(\lambda\hat{T}(\mu) - \mu\hat{T}(\lambda)) - (\lambda^2-\mu^2)\hat{T}(\mu)\hat{T}(\lambda)}{\lambda^2\mu^2} = 0.$$

This implies that $M(t,s) = 0$ for any $t,s \geq 0$. By Chen and Li [CL], $\{T(t)\}_{t\geq 0}$ is a cosine function. ∎

Proposition 3.4. *Let $T(t)$ be an α-order cosine function. Then it is commutative, i.e. $T(t)T(s) = T(s)T(t)$ for all $t,s \geq 0$.*

Proof. The symmetry of the left side of (3.1) with respect to t and s implies that

(3.4)
$$\int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma - \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma$$
$$= \int_0^s\int_0^t \frac{T(\sigma)T(\tau)}{(s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^s\int_0^t \frac{T(\sigma)T(\tau)}{(t-\tau)^{\alpha-1}}\,d\tau\,d\sigma$$
$$- \int_0^s\int_0^t \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma.$$

Given any $b > 0$, denote by $g_b(t)$ the truncation of $T(t)$ at b, that is, $g_b(t) = T(t)$ if $0 \leq t \leq b$ and $g_b(t) = 0$ otherwise. Denote

$$R_b(t,s) = \int_0^t\int_0^s \frac{g_b(\sigma)g_b(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^t\int_0^s \frac{g_b(\sigma)g_b(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma$$
$$- \int_0^t\int_0^s \frac{g_b(\sigma)g_b(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma.$$

Denote by $\hat{R}_b(t,\lambda)$ the Laplace transform of $R_b(t,s)$ with respect to s, and by $\hat{R}_b(\mu,\lambda)$ the Laplace transform of $\hat{R}_b(t,\lambda)$ with respect to t. Denote by $G(t,s)$ the left side of (3.4). Thus the right side of (3.4) is $G(s,t)$. Now, (3.4) implies

$$\hat{R}_b(\mu,\lambda) = \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}R_b(t,s)\,ds\,dt = \int_0^b e^{-\mu t}\int_0^b e^{-\lambda s}G(t,s)\,ds\,dt \tag{3.5}$$
$$= \int_0^b e^{-\mu t}\int_0^b e^{-\lambda s}G(s,t)\,ds\,dt = \hat{R}_b(\lambda,\mu).$$

Below we will compute the Laplace transforms of $\hat{R}_b(\mu,\lambda)$ and $\hat{R}_b(\lambda,\mu)$. Using (1.8), we compute $\hat{R}_b(t,\lambda)$ as follows:

$$\hat{R}_b(t,\lambda) = \int_0^t \frac{g_b(\sigma)}{(t-\sigma)^{\alpha-1}}\,d\sigma\,\frac{\hat{g}_b(\lambda)}{\lambda} + \Gamma(2-\alpha)\int_0^t \frac{g_b(\sigma)}{\lambda^{2-\alpha}}\,d\sigma\,\hat{g}_b(\lambda)$$
$$-\int_0^t g_b(\sigma)\widehat{(t+\tau-\sigma)^{1-\alpha}}(\lambda)\hat{g}_b(\lambda)\,d\sigma.$$

Denote by $(t+s)^{1-\alpha} * g_b(t)$ the convolution of the functions $(t+s)^{1-\alpha}$ and $g_b(t)$ at t. Taking the Laplace transform of $\hat{R}_b(t,\lambda)$ with respect to t, we have

$$\begin{aligned}
(3.6)\quad & \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}R_b(t,s)\,ds\,dt \\
&= \Gamma(2-\alpha)\mu^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\lambda} + \Gamma(2-\alpha)\lambda^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\mu} \\
&\quad - \int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}(t+s)^{1-\alpha} * g_b(t)\,dt\,ds\,\hat{g}_b(\lambda) \\
&= \Gamma(2-\alpha)\mu^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\lambda} + \Gamma(2-\alpha)\lambda^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\mu} \\
&\quad - \int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}(t+s)^{1-\alpha}\,dt\,ds\,\hat{g}_b(\mu)\hat{g}_b(\lambda) \\
&= \Gamma(2-\alpha)\mu^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\lambda} + \Gamma(2-\alpha)\lambda^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\mu} \\
&\quad - \int_0^\infty e^{(\mu-\lambda)s}\int_s^\infty e^{-\mu t}t^{1-\alpha}\,dt\,ds\,\hat{g}_b(\mu)\hat{g}_b(\lambda) \\
&= \Gamma(2-\alpha)\mu^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\lambda} + \Gamma(2-\alpha)\lambda^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\mu} \\
&\quad - \int_0^\infty e^{(\mu-\lambda)s}\Big(\int_0^\infty e^{-\mu t}t^{1-\alpha}\,dt - \int_0^s e^{-\mu t}t^{1-\alpha}\,dt\Big)\,ds\,\hat{g}_b(\mu)\hat{g}_b(\lambda) \\
&= \Gamma(2-\alpha)\mu^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\lambda} + \Gamma(2-\alpha)\lambda^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\mu} \\
&\quad - \int_0^\infty e^{(\mu-\lambda)s}\Big(\mu^{\alpha-2}\int_0^\infty e^{-\mu t}(\mu t)^{1-\alpha}\,d(\mu t) - \int_0^s e^{-\mu t}t^{1-\alpha}\,dt\Big)\,ds\hat{g}_b(\mu)\hat{g}_b(\lambda) \\
&= \Gamma(2-\alpha)\mu^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\lambda} + \Gamma(2-\alpha)\lambda^{\alpha-2}\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\mu}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\Gamma(2-\alpha)\mu^{\alpha-2}}{\lambda-\mu}\hat{g}_b(\mu)\hat{g}_b(\lambda)+\int_0^\infty e^{(\mu-\lambda)s}\int_0^s e^{-\mu t}t^{1-\alpha}\,dt\,ds\,\hat{g}_b(\mu)\hat{g}_b(\lambda)\\
&=\Gamma(2-\alpha)\mu^{\alpha-2}\,\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\lambda}+\Gamma(2-\alpha)\lambda^{\alpha-2}\,\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\mu}\\
&\quad-\frac{\Gamma(2-\alpha)\mu^{\alpha-2}}{\lambda-\mu}\hat{g}_b(\mu)\hat{g}_b(\lambda)+\int_0^\infty e^{-\lambda s}\int_0^s e^{\mu(s-t)}t^{1-\alpha}\,dt\,ds\,\hat{g}_b(\mu)\hat{g}_b(\lambda)\\
&=\Gamma(2-\alpha)\mu^{\alpha-2}\,\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\lambda}+\Gamma(2-\alpha)\lambda^{\alpha-2}\,\frac{\hat{g}_b(\mu)\hat{g}_b(\lambda)}{\mu}\\
&\quad-\frac{\Gamma(2-\alpha)\mu^{\alpha-2}}{\lambda-\mu}\hat{g}_b(\mu)\hat{g}_b(\lambda)+\frac{\Gamma(2-\alpha)\lambda^{\alpha-2}}{\lambda-\mu}\hat{g}_b(\mu)\hat{g}_b(\lambda).
\end{aligned}
$$

The equality (3.6) can be reduced to

$$
\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}R_b(t,s)\,ds\,dt=\frac{\Gamma(2-\alpha)(\lambda^\alpha-\mu^\alpha)}{\lambda\mu(\lambda-\mu)}\hat{g}_b(\mu)\hat{g}_b(\lambda). \tag{3.7}
$$

In the same way, it can be shown that

$$
\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}R_b(s,t)\,ds\,dt=\frac{\Gamma(2-\alpha)(\lambda^\alpha-\mu^\alpha)}{\lambda\mu(\lambda-\mu)}\hat{g}_b(\lambda)\hat{g}_b(\mu). \tag{3.8}
$$

The combination of (3.5), (3.7) and (3.8) implies

$$
\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}R_b(t,s)\,ds\,dt=\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}R_b(s,t)\,ds\,dt.
$$

Hence we easily deduce that

$$
\hat{g}_b(\mu)\hat{g}_b(\lambda)=\hat{g}_b(\lambda)\hat{g}_b(\mu).
$$

By the properties of the Laplace transform, it follows that

$$
g_b(t)g_b(s)=g_b(s)g_b(t),\qquad \forall t,s\ge 0.
$$

By the arbitrariness of b, we derive that

$$
T(t)T(s)=T(s)T(t),\qquad \forall t,s\ge 0. \ \blacksquare
$$

Remark 3.5. Observe that in the proof above, it is not the case that $R_b(t,s)=R_b(s,t)$ for all $t,s\ge 0$, though $R(t,s)=R(s,t)$ for all $t,s\ge 0$.

Theorem 3.6. *Assume that $\{T(t)\}_{t\ge 0}$ is a solution operator for system* (1.10). *Then it satisfies* (3.1) *and is therefore an α-order cosine function.*

Proof. Denote by $L(t,s)$ and $R(t,s)$ the left and right sides of (3.1), respectively. Obviously, we have only to prove that $L(t,s)=R(t,s)$ for all $t,s\ge 0$. For brevity, we set

$$
H(t,s)=T(t)J_s^\alpha T(s)-J_t^\alpha T(t)T(s),\quad K(t,s)=J_s^\alpha T(s)-J_t^\alpha T(t),\quad t,s\ge 0.
$$

Moreover, for sufficiently large $b > 0$, we denote by $g_b(t)$ the truncation of $T(t)$ at b, and by $R_b(t,s)$, $L_b(t,s)$, $H_b(t,s)$ and $K_b(t,s)$ the quantities resulting by replacing $T(t)$ with $g_b(t)$ in $R(t,s)$, $L(t,s)$, $H(t,s)$ and $K(t,s)$, respectively.

On the one hand, it follows from (3.7) that the Laplace transform of $R_b(t,s)$ with respect to t and s is given by

$$\hat{R}_b(\mu,\lambda) = \frac{\Gamma(2-\alpha)(\lambda^\alpha - \mu^\alpha)}{\lambda\mu(\lambda-\mu)}\hat{g}_b(\mu)\hat{g}_b(\lambda). \tag{3.9}$$

On the other hand, it can be shown that for all $t \geq 0$,

$$\begin{aligned}
\hat{L}_b(t,\lambda) &= \int_0^\infty e^{-\lambda s}\Bigg[\int_t^{t+s}\int_0^\sigma \frac{g_b(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma - \int_0^s\int_0^\sigma \frac{g_b(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\Bigg]\,ds\\
&= \int_t^\infty\int_0^\sigma g_b(\tau)\,d\tau \int_{\sigma-t}^\infty \frac{e^{-\lambda s}}{(t+s-\sigma)^{\alpha-1}}\,ds\,d\sigma\\
&\quad - \int_0^\infty\int_0^\sigma g_b(\tau)\,d\tau \int_\sigma^\infty \frac{e^{-\lambda s}}{(t+s-\sigma)^{\alpha-1}}\,ds\,d\sigma\\
&= \int_t^\infty e^{\lambda(t-\tau)}\int_0^\sigma g_b(\tau)\,d\tau \int_0^\infty e^{-\lambda r} r^{1-\alpha} dr\,d\sigma - \frac{e^{\lambda t}}{\lambda}\int_t^\infty e^{-\lambda s}s^{1-\alpha}\,ds\,\hat{g}_b(\lambda)\\
&= \lambda^{\alpha-2}\Gamma(2-\alpha)\int_t^\infty e^{\lambda(t-\sigma)}\int_0^\sigma g_b(\tau)\,d\tau\,d\sigma - \frac{e^{\lambda t}}{\lambda}\int_t^\infty e^{-\lambda s}s^{1-\alpha}\,ds\,\hat{g}_b(\lambda).
\end{aligned}$$

The Laplace transform of $\hat{L}_b(t,\lambda)$ with respect to t is

$$\begin{aligned}
\hat{L}_b(\mu,\lambda) &= \frac{\Gamma(2-\alpha)}{\lambda-\mu}\left(\lambda^{\alpha-2}\,\frac{\hat{g}_b(\mu)}{\mu} - \mu^{\alpha-2}\,\frac{\hat{g}_b(\lambda)}{\lambda}\right)\\
&= \frac{\Gamma(2-\alpha)}{\lambda\mu(\lambda-\mu)}\big(\lambda^{\alpha-1}\hat{g}_b(\mu) - \mu^{\alpha-1}\hat{g}_b(\lambda)\big).
\end{aligned} \tag{3.10}$$

We set

$$\begin{aligned}
P_b(t,s) &= \int_0^t\int_0^s \frac{H_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^t\int_0^s \frac{H_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma\\
&\quad - \int_0^t\int_0^s \frac{H_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma
\end{aligned}$$

and

$$\begin{aligned}
Q_b(t,s) &= \int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma\\
&\quad - \int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma.
\end{aligned} \tag{3.11}$$

By Chen and Li [CL], $H(t,s)=K(t,s)$ for any $t,s\ge 0$. Thus, for all $t,s\ge 0$,

$$\lim_{b\to\infty} P_b(t,s)=\lim_{b\to\infty} Q_b(t,s). \tag{3.12}$$

Now, we calculate the Laplace transforms of $P_b(t,s)$ and $Q_b(t,s)$ with respect to t and s, respectively. The Laplace transform of the first term of $P_b(t,s)$ is

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{H_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt\\
&\quad=\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{g_b(\sigma)J_\tau^\alpha g_b(\tau)-J_\sigma^\alpha g_b(\sigma)g_b(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt\\
&\quad=\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\bigg(\int_0^t \frac{g_b(\sigma)}{(t-\sigma)^{\alpha-1}}\,d\sigma\int_0^s J_\tau^\alpha g_b(\tau)\,d\tau\bigg)\,ds\,dt\\
&\qquad-\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\bigg(\int_0^t \frac{J_\sigma^\alpha g_b(\sigma)}{(t-\sigma)^{\alpha-1}}\,d\sigma\int_0^s g_b(\tau)\,d\tau\bigg)\,ds\,dt\\
&\quad=\Gamma(2-\alpha)\lambda^{-\alpha-1}\mu^{\alpha-2}\hat g_b(\mu)\hat g_b(\lambda)-\Gamma(2-\alpha)\mu^{-\alpha}\mu^{\alpha-2}\lambda^{-1}\hat g_b(\mu)\hat g_b(\lambda).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{H_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt\\
&\quad=\Gamma(2-\alpha)\mu^{-1}\lambda^{\alpha-2}\lambda^{-\alpha}\hat g_b(\mu)\hat g_b(\lambda)-\Gamma(2-\alpha)\mu^{-\alpha-1}\lambda^{\alpha-2}\hat g_b(\mu)\hat g_b(\lambda).
\end{aligned}$$

The Laplace transform of the last term of $P_b(t,s)$ with respect to t and s is

$$\begin{aligned}
&-\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{H_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt\\
&\quad=-\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{g_b(\sigma)J_\tau^\alpha g_b(\tau)-J_\sigma^\alpha g_b(\sigma)g_b(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt\\
&\quad=-\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{g_b(\sigma)J_\tau^\alpha g_b(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt\\
&\qquad+\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{J_\sigma^\alpha g_b(\sigma)g_b(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt\\
&\quad=-\frac{\Gamma(2-\alpha)\mu^{\alpha-2}\lambda^{-\alpha}}{\lambda-\mu}\hat g_b(\mu)\hat g_b(\lambda)+\frac{\Gamma(2-\alpha)\lambda^{\alpha-2}\lambda^{-\alpha}}{\lambda-\mu}\hat g_b(\mu)\hat g_b(\lambda)\\
&\qquad+\frac{\Gamma(2-\alpha)\mu^{\alpha-2}\mu^{-\alpha}}{\lambda-\mu}\hat g_b(\mu)\hat g_b(\lambda)-\frac{\Gamma(2-\alpha)\lambda^{\alpha-2}\mu^{-\alpha}}{\lambda-\mu}\hat g_b(\mu)\hat g_b(\lambda).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\hat{P}_b(\mu,\lambda) = &\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\bigg(\int_0^t\int_0^s \frac{H_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma \\ &+\int_0^t\int_0^s \frac{H_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma - \int_0^t\int_0^s \frac{H_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\bigg)\,ds\,dt \\ = &\frac{\Gamma(2-\alpha)(-\lambda^{-\alpha-1}\mu^{\alpha-1}+2\lambda^{-1}\mu^{-1}-\lambda^{\alpha-1}\mu^{-\alpha-1})}{\lambda-\mu}\hat{g}_b(\mu)\hat{g}_b(\lambda) \\ = &(\lambda^{-\alpha}-\mu^{-\alpha})\frac{\Gamma(2-\alpha)(\lambda^\alpha-\mu^\alpha)}{\lambda\mu(\lambda-\mu)}\hat{g}_b(\mu)\hat{g}_b(\lambda) = (\lambda^{-\alpha}-\mu^{-\alpha})\hat{R}_b(\mu,\lambda).\end{aligned}$$

By the properties of the Laplace transform, we obtain

$$P_b(t,s) = (J_s^\alpha - J_t^\alpha)R_b(t,s), \qquad \forall t,s\ge 0. \tag{3.13}$$

We now compute the Laplace transform of the first term of $Q_b(t,s)$ with respect to s and t:

$$\begin{aligned}&\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt \\ &\quad= \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{J_\tau^\alpha g_b(\tau)-J_\sigma^\alpha g_b(\sigma)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt \\ &\quad= \Gamma(2-\alpha)\mu^{\alpha-3}\lambda^{-\alpha-1}\hat{g}_b(\lambda) - \Gamma(2-\alpha)\mu^{\alpha-2}\mu^{-\alpha}\lambda^{-2}\hat{g}_b(\mu).\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt \\ &\quad= \Gamma(2-\alpha)\mu^{-2}\lambda^{\alpha-2}\lambda^{-\alpha}\hat{g}_b(\lambda) - \Gamma(2-\alpha)\lambda^{\alpha-3}\mu^{-\alpha-1}\hat{g}_b(\mu).\end{aligned}$$

The Laplace transform of the third term of $Q_b(t,s)$ with respect to s and t is

$$\begin{aligned}&-\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt \\ &\quad= -\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{J_\tau^\alpha g_b(\tau)-J_\sigma^\alpha g_b(\sigma)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\,ds\,dt \\ &\quad= -\int_0^\infty e^{-\mu t}\int_0^t\int_0^\infty e^{-\lambda s}\,\frac{1}{(t+s-\sigma)^{\alpha-1}}\,ds\,d\sigma\,dt\,\lambda^{-\alpha}\hat{g}_b(\lambda) \\ &\qquad+\lambda^{-1}\int_0^\infty e^{-\mu t}\int_0^t J_\sigma^\alpha g_b(\sigma)\int_0^\infty e^{-\lambda s}\,\frac{1}{(t+s-\sigma)^{\alpha-1}}\,ds\,d\sigma\,dt\end{aligned}$$

$$= -\int_0^\infty e^{-\mu t} e^{\lambda t} \int_0^t e^{-\lambda\sigma}\left(\Gamma(2-\alpha)\lambda^{\alpha-2} - \int_0^{t-\sigma} e^{-\lambda s}\frac{1}{s^{\alpha-1}}\,ds\right) d\sigma\,dt\,\lambda^{-\alpha}\hat{g}_b(\lambda)$$

$$+ \lambda^{-1}\mu^{-\alpha}\hat{g}_b(\mu)\int_0^\infty e^{-\mu t}\int_0^\infty \frac{e^{-\lambda s}}{(t+s)^{\alpha-1}}\,ds\,dt$$

$$= -\Gamma(2-\alpha)\mu^{-1}(\mu-\lambda)^{-1}\lambda^{-2}\hat{g}_b(\lambda) + \Gamma(2-\alpha)\mu^{-1}(\mu-\lambda)^{-1}\mu^{\alpha-2}\lambda^{-\alpha}\hat{g}_b(\lambda)$$

$$+ \lambda^{-1}\mu^{-\alpha}\hat{g}_b(\mu)\int_0^\infty e^{-\mu t}e^{\lambda t}\left(\Gamma(2-\alpha)\lambda^{\alpha-2} - \int_0^t \frac{e^{-\lambda s}}{s^{\alpha-1}}\,ds\right) dt$$

$$= -\Gamma(2-\alpha)\mu^{-1}(\mu-\lambda)^{-1}\lambda^{-2}\hat{g}_b(\lambda) + \Gamma(2-\alpha)\mu^{-1}(\mu-\lambda)^{-1}\mu^{\alpha-2}\lambda^{-\alpha}\hat{g}_b(\lambda)$$

$$+ \Gamma(2-\alpha)(\mu-\lambda)^{-1}\lambda^{-1}\lambda^{\alpha-2}\mu^{-\alpha}\hat{g}_b(\mu) - \Gamma(2-\alpha)(\mu-\lambda)^{-1}\mu^{\alpha-2}\lambda^{-1}\mu^{-\alpha}\hat{g}_b(\mu).$$

Therefore,

$$\begin{aligned}
\hat{Q}_b(\mu,\lambda) &= \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\left(\int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma\right.\\
&\quad\left. + \int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma - \int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\right) ds\,dt\\
&= \frac{\Gamma(2-\alpha)(\mu^{\alpha-2}\lambda^{-\alpha-1} - \mu^{-2}\lambda^{-1})}{\mu-\lambda}\hat{g}_b(\lambda)\\
&\quad + \frac{\Gamma(2-\alpha)(\lambda^{\alpha-2}\mu^{-1-\alpha} - \lambda^{-2}\mu^{-1})}{\mu-\lambda}\hat{g}_b(\mu)\\
&= (\lambda^{-\alpha} - \mu^{-\alpha})\hat{L}_b(\mu,\lambda).
\end{aligned}$$

By the properties of the Laplace transform, we obtain

$$Q_b(t,s) = (J_s^\alpha - J_t^\alpha)L_b(t,s), \qquad \forall t,s \ge 0. \tag{3.14}$$

The combination of (3.12)–(3.14) implies that

$$(J_s^\alpha - J_t^\alpha)L(t,s) = (J_s^\alpha - J_t^\alpha)R(t,s), \qquad \forall t,s \ge 0. \tag{3.15}$$

Therefore, $R(t,s) = L(t,s)$. ∎

4. Equivalence to a solution operator. In the previous section, it has been proved that a solution operator for the fractional order Cauchy problem with order $1 < \alpha < 2$ necessarily satisfies the equality (3.1). In this section we will further prove that (3.1) is also sufficient for a family of bounded linear operators to become a solution operator for a certain fractional abstract Cauchy problem, that is, a fractional cosine function is also a solution operator. To do this, we begin with the definition of the generator of an α-order cosine function.

DEFINITION 4.1. Let $\{T(t)\}_{t\geq 0}$ be an α-order cosine function on a Banach space X. Denote by $D(A)$ the set of all $x \in X$ such that the limit

$$\lim_{t\to 0^+} 2t^{-2} J_t^{2-\alpha}(T(t)x - x)$$

exists. Then the operator $A : D(A) \to X$ defined by

$$Ax = \lim_{t\to 0^+} 2t^{-2} J_t^{2-\alpha}(T(t)x - x)$$

is called the *generator* of $\{T(t)\}_{t\geq 0}$.

PROPOSITION 4.2. *Let $\{T(t)\}_{t\geq 0}$ be an α-order cosine function on X with generator A. Then:*

(a) *For all $x \in X$ and $t \geq 0$, we have $J_t^\alpha T(t)x \in D(A)$ and*

$$T(t)x = x + AJ_t^\alpha T(t)x.$$

(b) *$T(t)D(A) \subset D(A)$ and $T(t)Ax = AT(t)x$ for all $x \in D(A)$.*
(c) *For all $x \in D(A)$, we have $T(t)x = x + J_t^\alpha T(t)Ax$.*
(d) *A is equivalently defined by*

$$Ax = \Gamma(1+\alpha) \lim_{t\to 0^+} \frac{T(t)x - x}{t^\alpha}$$

and $D(A)$ consists precisely of those $x \in X$ such that the above limit exists.
(e) *A is closed and densely defined.*
(f) *A admits at most one α-order cosine function.*

Proof. (a) Let $x \in X$ and $t > 0$ be fixed. Define

$$H_t(r,s) = (g_t(r) - I)J_s^\alpha g_t(s)x \quad r,s \geq 0, \tag{4.1}$$

where $g_t(r)$ is the truncation of $T(r)$ at t. Obviously, for $0 < r \leq t$,

$$H_t(r,t) = (T(r) - I)J_t^\alpha T(t)x. \tag{4.2}$$

Taking the Laplace transform with respect to r and s successively on both sides of (4.1), we derive

$$\hat{H}_t(\mu,\lambda) = \lambda^{-\alpha}\hat{g}_t(\mu)\hat{g}_t(\lambda)x - \lambda^{-\alpha}\mu^{-1}\hat{g}_t(\lambda)x. \tag{4.3}$$

The combination of (3.9), (3.10) and (4.3) implies that

$$\begin{aligned}\hat{H}_t(\mu,\lambda) &= \mu^{-\alpha}\hat{g}_t(\mu)\hat{g}_t(\lambda)x - \mu^{-\alpha}\lambda^{-1}\hat{g}_t(\lambda)x \\ &\quad + \frac{\lambda^{1-\alpha}\mu^{1-\alpha}(\lambda-\mu)}{\Gamma(2-\alpha)}(\hat{L}_t(\mu,\lambda) - \hat{R}_t(\mu,\lambda))x.\end{aligned}$$

Observe that $L_t(r,0) = L_t(0,s) = R_t(r,0) = R_t(0,s) = 0$. By the properties of the Laplace transform, it follows from (1.6) that

$$H_t(r,s) = (g_t(s)-I)J_r^\alpha g_t(r)x \\ + \frac{J_r^{\alpha-1}({}^C D_s^{2-\alpha}) - ({}^C D_r^{2-\alpha})J_s^{\alpha-1}(L_t(r,s)-R_t(r,s))x}{\Gamma(2-\alpha)}.$$

By the definition of g_t, it follows that $L_t(r,s) = R_t(r,s)$ for all $r,s \le t$, so

$$H_t(r,s) = (T(s)-I)J_r^\alpha T(r)x, \quad \forall r,s \le t.$$

In particular,

$$H_t(r,t) = (T(t)-I)J_r^\alpha T(r)x, \quad \forall 0 < r \le t. \tag{4.4}$$

Combining (4.2) and (4.4), we derive that

$$\begin{aligned}\lim_{r\to 0^+} & 2r^{-2}J_r^{2-\alpha}(T(r)-I)J_t^\alpha T(t)x \\ &= \lim_{r\to 0^+} 2r^{-2}(T(t)-I)J_r^2 T(r)x \\ &= (T(t)-I)\lim_{r\to 0^+} 2\int_0^r (1-\sigma/r)T(\sigma)x\, d(\sigma/r) \\ &= (T(t)-I)\lim_{r\to 0^+} 2\int_0^1 (1-\sigma)T(r\sigma)x\, d\sigma = T(t)x - x.\end{aligned}$$

This implies that $J_t^\alpha T(t)x \in D(A)$ and $AJ_t^\alpha T(t)x = T(t)x - x$.

(b) and (c) follow directly from Proposition 3.4 and (a).

(d) Denote by D the set of those $x \in X$ such that $\lim_{t\to 0^+} t^{-\alpha}(T(t)x-x)$ exists. Then, by [PL2, (3.12) and (3.13)], we obtain $D(A) \subset D$. Conversely, let $x \in D$. The existence of the above limit implies that

$$\begin{aligned}\lim_{t\to 0^+} 2t^{-2}J_t^{2-\alpha}(T(t)x-x) &= \lim_{t\to 0^+} \frac{2}{\Gamma(2-\alpha)t^2}\int_0^t (t-\sigma)^{1-\alpha}(T(\sigma)x-x)\, d\sigma \\ &= \lim_{t\to 0^+} \frac{2}{\Gamma(2-\alpha)}\int_0^1 (1-\sigma)^{1-\alpha}\sigma^\alpha \frac{T(t\sigma)x-x}{(t\sigma)^\alpha}\, d\sigma \\ &= \frac{2B(2-\alpha,\alpha+1)}{\Gamma(2-\alpha)}\lim_{t\to 0^+}\frac{T(t)x-x}{t^\alpha} \\ &= \Gamma(\alpha+1)\lim_{t\to 0^+}\frac{T(t)x-x}{t^\alpha}\end{aligned}$$

where the Beta function satisfies $B(2-\alpha,\alpha+1) = \Gamma(2-\alpha)\Gamma(1+\alpha)/\Gamma(3)$. Hence, $x \in D(A)$, which implies that $D \subset D(A)$. Moreover,

$$Ax = \Gamma(1+\alpha)\lim_{t\to 0^+}\frac{T(t)x-x}{t^\alpha}.$$

(e) The properties that A is closed and densely defined are obtained by the same procedure as in [PL2, Proposition 4(d)].

(f) Assume that both $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ are α-order cosine functions. Then, by (c), for all $x \in D(A)$, we have

$$S(t)x = x + J_t^\alpha S(t)Ax.$$

Observe that $D(A)$ is dense in X. The uniqueness is proved directly by setting $v = T(\cdot)x$ in the proof of [Pr, Proposition 1.1]. ∎

By Proposition 4.2 and Definition 4.1, it follows that any α-order cosine function is indeed a solution operator for the associated Volterra equation. Hence, the combination of Theorem 3.6 and Proposition 4.2 implies that $\{T(t)\}_{t\geq 0}$ is an α-order cosine function if and only if it is a solution operator for the associated Volterra equation. Therefore, equation (1.10) is well-posed if and only if A generates an α-order cosine function. Here the well-posedness is defined as follows:

Definition 4.3 ([B]). (i) A function $u \in C([0,\infty), X)$ is called a *strong solution* of (1.10) if $u(t) \in D(A)$ for all $t \geq 0$, and the mapping

$$t \mapsto \int_0^t (t-\sigma)^{-\alpha}(u(\sigma) - u(0))\, d\sigma$$

is continuously differentiable and such that (1.2) holds on $[0,\infty)$.

(ii) The problem (1.10) is said to be *well-posed* if for any $x \in D(A)$ there exists a unique strong solution $u(t,x)$, and $D(A) \ni x_n \to 0$ implies that $u(t,x_n) \to 0$ as $n \to \infty$ in X, uniformly on any compact subinterval of $[0,\infty)$.

Remark 4.4. Lizama and Poblete [LP] recently studied the relations between (a,k)-regularized resolvents and the functional equation

$$(4.5)\qquad T(t)(a*T)(s) - (a*T)(t)T(s) = k(t)(a*T)(s) - k(s)(a*T)(t), \qquad t,s \geq 0,$$

where the convolution $a*T$ is given by $(a*T)(t) = \int_0^t a(\theta)T(t-\theta)\, d\theta$, $t \geq 0$. In fact, a solution operator for (1.10) is just a $(t^{\alpha-1}/\Gamma(\alpha), 1)$-regularized resolvent for (1.10), and it is an α-times resolvent family (see [CL]). By [LP, Theorems 3.1 and 4.1] and [CL, Theorem 3.4], $\{T(t)\}_{t\geq 0}$ is a $(t^{\alpha-1}/\Gamma(\alpha), 1)$-regularized resolvent if and only if $T(0) = I$ and (4.5) holds for $a = t^{\alpha-1}/\Gamma(\alpha)$ and $k = 1$. This means that the functional equation (4.5) is equivalent to (3.1). However, (4.5) cannot be written as a functional equation in terms of the sum of the time variables, $s+t$. This is rather important in concrete applications of the algebraic functional equation, and makes our functional equation (3.1) especially valuable.

Theorem 4.5. *Assume that a linear operator A generates an α-order cosine function $\{T(t)\}_{t\geq 0}$ on Banach space X. Then the abstract fractional*

Cauchy problem

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t), & t > 0, \\ u(0) = x, \quad u'(0) = y, \end{cases} \tag{4.6}$$

has a unique solution given by $u(t) = T(t)x + \int_0^t T(s)y\,ds$.

Proof. We have only to prove that if A generates an α-order cosine function $\{T(t)\}_{t\ge 0}$, then A generates an α-order sine function defined by $G(t) = \int_0^t T(s)\,ds$.

By Theorem 3.6, $T(t)$ satisfies

$$T(t)x = x + A\int_0^t (t-\sigma)^{\alpha-1} T(\sigma)x\,d\sigma, \qquad \forall x \in X,\ t \ge 0. \tag{4.7}$$

Observe that A is closed. Integrating both sides of (4.7), we obtain

$$\int_0^t T(s)x = x + A\int_0^t (t-\sigma)^{\alpha-1} \int_0^\sigma T(s)\,ds\,x\,d\sigma, \qquad \forall x \in X,\ t \ge 0.$$

This means that $G(t)$ is an α-order sine function. Therefore, (4.6) has a unique solution given by $u(t) = T(t)x + \int_0^t T(s)y\,ds$. ∎

To close this paper, we mention that due to their applications in physics, chemistry, materials science and engineering, fractional linear systems have received increasing interest in the last few decades, and the related theory has gained a rapid development (see, e.g., Anh and Leonenko [AL], Eidelman and Kochubei [EK], Hilfer [H], Kilbas et al. [KST], Niu and Xie [NX], Podlubny [P]). The notion of fractional cosine function might be viewed as an extension of fractional semigroup of order $0 < \alpha < 1$ to $1 < \alpha < 2$. On the other hand, it is proved in [B] that, for all $\alpha > 2$, the generator A of the solution operator corresponding to (1.2) is a bounded linear operator, and therefore the associated solution operator can be defined by $E_\alpha(At^\alpha)$, which is similar to the situation of C_0-semigroups. Hence, for every $\alpha > 0$, the algebraic description that is similar to semigroup and cosine function corresponding to (1.2) is clear.

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