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A strong convergence theorem for $H^1(\mathbb{T}^n)$

by

FENG DAI (Edmonton)

Abstract. Let $\mathbb{T}^n$ denote the usual n-torus and let $\widetilde{S}_u^\delta(f)$, $u > 0$, denote the Bochner–Riesz means of order $\delta > 0$ of the Fourier expansion of $f \in L^1(\mathbb{T}^n)$. The main result of this paper states that for $f \in H^1(\mathbb{T}^n)$ and the critical index $\alpha := (n-1)/2$,

$$\lim_{R\to\infty} \frac{1}{\log R} \int_0^R \frac{\|\widetilde{S}_u^\alpha(f) - f\|_{H^1(\mathbb{T}^n)}}{u+1}\, du = 0.$$

1. Introduction. In this introduction we describe the main results and their background with a minimum of definitions. We give the necessary details and appropriate definitions, as needed, in the next section.

Let Λ denote the unit lattice in the n-dimensional Euclidean space $\mathbb{R}^n$ having integral coordinates, and let $\mathbb{T}^n$ be the n-torus, identified with $\mathbb{R}^n/\Lambda$. By $H^p(\mathbb{T}^n)$, $0 < p \le 1$, we denote the usual Hardy spaces on $\mathbb{T}^n$. Let

$$f(x) \sim \sum_{k\in\Lambda} a_k(f) e^{2\pi i k\cdot x}$$

be the Fourier expansion of an integrable function on the fundamental cube

$$Q = \{(x_1,\dots,x_n) \in \mathbb{R}^n : -1/2 \le x_j < 1/2,\ j = 1,\dots,n\}.$$

For $u > 0$, we define the *Bochner–Riesz means* of order $\delta > -1$ of the Fourier expansion by

$$\widetilde{S}_u^\delta(f)(x) = \sum_{|k|<u} \left(1 - \frac{|k|^2}{u^2}\right)^\delta a_k(f) e^{2\pi i k\cdot x},$$

where $k = (k_1,\dots,k_n) \in \Lambda$ and $|k| = (k_1^2 + \dots + k_n^2)^{1/2}$. It is well known (see [STW] and [CF]) that for $\delta > \alpha := (n-1)/2$,

$$\sup_{u>0} \|\widetilde{S}_u^\delta(f)\|_{H^1(\mathbb{T}^n)} \le C\|f\|_{H^1(\mathbb{T}^n)}$$

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while for $\delta = (n-1)/2$,

$$\|\widetilde{S}_u^\delta\|_{(H^1(\mathbb{T}^n),L^1(\mathbb{T}^n))} \geq C\log(u+1).$$

The main purpose of this paper is to investigate the strong summability of the Bochner–Riesz means on $H^1(\mathbb{T}^n)$ at the critical index $\alpha = (n-1)/2$. We will always use the letter α for the critical index $(n-1)/2$ for the rest of the paper.

The background for the problem treated here is as follows. In 1983, B. Smith [Sm] proved that for every $f \in H^1(\mathbb{T})$,

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{1}{k}\,\|S_k(f)\|_{H^1(\mathbb{T})} \leq C\|f\|_{H^1(\mathbb{T})}, \tag{1.1}$$

where $S_k(f)$ denotes the usual kth partial sum of Fourier series. A new proof of this inequality was given by Belinskii [Be2] in 1996. However, the multidimensional generalization of this inequality seems to be more complicated. In fact, the two-dimensional result for rectangle partial sums with bounded ratio of sides was obtained by Weisz in [We] while the n-dimensional result for the cubic partial sums and a modified product $H^1(\mathbb{T}^n)$ space was obtained by Belinskii in [Be1].

It was Bochner [Bo] who first pointed out that when the dimension $n > 1$, summability at the critical index $(n-1)/2$ was the correct analogue of convergence, for phenomena near L^1. In this sense, versions of many of the results for S_k are known for S_u^α in the case of general n (see [SW, Ch. VII] and [St1]). Of related interest is the fact that an inequality similar to (1.1) was proved in [JLL] for the space $H^p(\mathbb{T}^n)$, $0 < p < 1$, with the Bochner–Riesz means with critical index $\delta = n/p - (n+1)/2$ instead of partial sums (see also [Lu, Theorem 4.3, p. 196]). Therefore, a problem that remained was what happens for functions in $H^1(\mathbb{T}^n)$, $n > 1$.

This paper is devoted to the proof of the following theorem, which gives an affirmative answer to a question raised by S. Z. Lu [Lu, p. 204].

THEOREM 1. *For $f \in H^1(\mathbb{T}^n)$ and $R > 0$,*

$$\frac{1}{\log(R+1)}\int_0^R \frac{\|\widetilde{S}_u^\alpha(f)\|_{H^1(\mathbb{T}^n)}}{u+1}\,du \leq C\|f\|_{H^1(\mathbb{T}^n)},$$

where C is a positive constant independent of f and R.

As a consequence, we have

COROLLARY 2. *For $f \in H^1(\mathbb{T}^n)$ and $R > 0$,*

$$\int_0^R \frac{\|\widetilde{S}_u^\alpha(f) - f\|_{H^1(\mathbb{T}^n)}}{u+1}\,du \approx \int_0^R \frac{E_u(f, H^1)}{u+1}\,du,$$

where

$$E_u(f,H^1)=\inf\Big\{\|f-g\|_{H^1(\mathbb{T}^n)}: g(x)=\sum_{|k|\le u} c_k e^{2\pi i k\cdot x},\ c_k\in\mathbb{C}\Big\},$$

and "≈" means that the ratio of both sides lies between two positive constants independent of f and R.

COROLLARY 3. *For $f\in H^1(\mathbb{T}^n)$ and $R>0$,*

$$\frac{1}{\log(R+1)}\int_0^R \frac{\|\widetilde{S}_u^\alpha(f)-f\|_{H^1(\mathbb{T}^n)}}{u+1}\,du \le C\omega\Big(f,\frac{1}{\log(R+1)}\Big)_{H^1(\mathbb{T}^n)},$$

where C is a positive constant independent of f and R, and $\omega(f,t)_{H^1(\mathbb{T}^n)}$ denotes the first-order modulus of smoothness of f on $H^1(\mathbb{T}^n)$.

We point out that in the one-dimensional case, Corollaries 2 and 3 for the partial sums of Fourier series are due to Belinskii [Be2] and the authors of [CJL], respectively.

The paper is organized as follows. Section 2 contains some basic definitions and notation. The proof of Theorem 1 is divided into two parts: the first part is given in Section 3, where we prove

$$\frac{1}{\log(R+1)}\int_0^R \frac{\|\widetilde{S}_u^\alpha(f)\|_{L^1(\mathbb{T}^n)}}{u+1}\,du \le C\|f\|_{H^1(\mathbb{T}^n)}, \tag{1.2}$$

while the second part is given in Section 4, where we show

$$\|\widetilde{S}_u^\alpha(f)\|_{H^1(\mathbb{T}^n)} \le C(\|f\|_{H^1(\mathbb{T}^n)}+\|\widetilde{S}_u^\alpha(f)\|_{L^1(\mathbb{T}^n)}).$$

This last inequality combined with (1.2) will prove Theorem 1. In the final Section 5, we prove Corollaries 2 and 3.

2. Basic definitions and notations. In this section we introduce some basic definitions and notations, most of which can be found in [SW] and [Lu].

Let $\mathcal{S}(\mathbb{T}^n)$ denote the space of test functions on $\mathbb{T}^n$ and $\mathcal{S}'(\mathbb{T}^n)$ be the dual of $\mathcal{S}(\mathbb{T}^n)$. The *Poisson kernel* on $\mathbb{T}^n$ is defined by

$$\widetilde{P}_t(x)=\sum_{k\in\Lambda} e^{-2\pi|k|t}e^{2\pi i k\cdot x},\qquad t>0,$$

where Λ is the unit lattice in $\mathbb{R}^n$, $k=(k_1,\ldots,k_n)$ and $|k|=(k_1^2+\cdots+k_n^2)^{1/2}$. For $f\in\mathcal{S}'(\mathbb{T}^n)$, we define

$$\widetilde{P}_+(f)(x)=\sup_{t>0}|f*\widetilde{P}_t(x)|.$$

Definition 2.1. The *Hardy space* $H^p(\mathbb{T}^n)$, $0 < p \le 1$, is the linear space of distributions $f \in \mathcal{S}'(\mathbb{T}^n)$ with $\|f\|_{H^p} \equiv \|\widetilde{P}_+(f)\|_{L^p} < \infty$.

We denote by $B(x,r)$ the ball

$$B(x,r) := \{y \in \mathbb{R}^n : |x-y| \le r\}$$

with center at $x \in \mathbb{R}^n$ and radius $r > 0$, and we write χ_E for the characteristic function of a measurable set $E \subset \mathbb{R}^n$. Let Q denote the *fundamental cube*

$$Q = \{(x_1, \dots, x_n) \in \mathbb{R}^n : -1/2 \le x_j < 1/2,\ j = 1, \dots, n\}.$$

We now turn to the "atomic" characterization of Hardy spaces.

Definition 2.2. Let $0 < p \le 1$. A function $a \in L^\infty(\mathbb{R}^n)$ is an $H^p(\mathbb{R}^n)$-*atom* with support $B(x,r)$, $x \in \mathbb{R}^n$, $r > 0$, if it satisfies

(i) $\operatorname{supp} a \subset B(x,r)$,
(ii) $\|a\|_\infty \le r^{-n/p}$,
(iii) $\int_{\mathbb{R}^n} a(x)P(x)\,dx = 0$ for all polynomials $P(x)$ of degree less than or equal to $[n(1/p-1)]$.

A function $a \in L^\infty(\mathbb{T}^n)$ is a *regular* $H^p(\mathbb{T}^n)$-*atom* having support $B(z,r)$, $z \in \mathbb{R}^n$, $r > 0$, if $a\chi_{z+Q}$ is an $H^p(\mathbb{R}^n)$-atom with support $B(z,r)$. An *exceptional* $H^1(\mathbb{T}^n)$-*atom* is a function $a \in L^\infty(\mathbb{T}^n)$ with $\|a\|_\infty \le 1$.

Lemma 2.1 ([F]). *Let $0 < p \le 1$. If $\{a_j\}_{j=0}^\infty$ is a sequence of exceptional or regular $H^p(\mathbb{T}^n)$-atoms, and $\{c_j\}_{j=0}^\infty$ is a sequence of complex numbers with*

$$\Big(\sum_{j=0}^\infty |c_j|^p\Big)^{1/p} < \infty,$$

then $\sum_{j=0}^\infty c_j a_j$ converges in $H^p(\mathbb{T}^n)$ and

$$\Big\|\sum_{j=0}^\infty c_j a_j\Big\|_{H^p} \le A\Big(\sum_{j=0}^\infty |c_j|^p\Big)^{1/p},$$

where $A > 0$ depends on p and n.

Conversely, if $f \in H^p(\mathbb{T}^n)$ then there exist a sequence of exceptional or regular $H^p(\mathbb{T}^n)$-atoms $\{a_j\}_{j=0}^\infty$ and a sequence of complex numbers $\{c_j\}_{j=0}^\infty$ such that

$$f = \sum_{j=0}^\infty c_j a_j \quad \text{and} \quad \Big(\sum_{j=0}^\infty |c_j|^p\Big)^{1/p} \le B\|f\|_{H^p},$$

where B depends on p and n.

The conclusion of Lemma 2.1 is often described as the "atomic" characterization of Hardy spaces.

Let m be a nonnegative integer, t be a positive real number, and let h be a vector in $\mathbb{R}^n$. For $f \in \mathcal{S}'(\mathbb{T}^n)$, we define

$$\Delta_h^m f(x,s) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (f * \widetilde{P}_s)(x+jh), \qquad x \in \mathbb{R}^n,\ s>0,$$

and

$$\omega^m(f,t)_{H^p(\mathbb{T}^n)} = \sup_{|h|\le t} \Big\| \sup_{s>0} |\Delta_h^m f(\cdot,s)| \Big\|_{L^p(\mathbb{T}^n)}.$$

$\omega^m(f,t)_{H^p(\mathbb{T}^n)}$ is called the mth *modulus of smoothness* of f on $H^p(\mathbb{T}^n)$.

3. Proof of Theorem 1: Part I. The main goal in this section is to prove

$$\frac{1}{\log(R+1)} \int_0^R \frac{\|\widetilde{S}_u^\alpha(f)\|_{L^1(\mathbb{T}^n)}}{u+1}\, du \le C\|f\|_{H^1(\mathbb{T}^n)}. \tag{3.1}$$

This same inequality with $L^1(\mathbb{T}^n)$-norm on the left-hand side replaced by $H^1(\mathbb{T}^n)$-norm will be shown in the next section.

Let

$$K_u^\alpha(x) := \sum_{|k|<u} \left(1 - \frac{|k|^2}{u^2}\right)^\alpha e^{2\pi i k\cdot x}.$$

Then we have

$$\widetilde{S}_u^\alpha(f)(x) = \int_Q f(x-y) K_u^\alpha(y)\, dy.$$

We also define

$$S_u(f)(x) = \pi^{(n-1)/2} \left(\frac{n+1}{2}\right) u^{1/2} \int_Q f(x-y)|y|^{-(n-1/2)} J_{n-1/2}(2\pi u|y|)\, dy,$$

where $J_\nu(t)$ denotes the Bessel function of order ν. Then by Lemma 2.1 and the following well known estimate of Stein (see [St2, Theorem 1], or [SW, p. 285]):

$$\sup_{u>0} \left\| K_u^\alpha(y) - \pi^{(n-1)/2}\left(\frac{n+1}{2}\right) u^{1/2} |y|^{-(n-1/2)} J_{n-1/2}(2\pi u|y|) \right\|_{L^1(Q)} \le A_n,$$

it will suffice to prove that for every $H^1(\mathbb{T}^n)$-atom a,

$$\frac{1}{\log(R+1)} \int_0^R \frac{\|S_u(a)\|_{L^1(Q)}}{u+1}\, du \le C. \tag{3.2}$$

For the proof of this last inequality, we claim that it is enough to prove (3.2) for every $H^1(\mathbb{R}^n)$-atom with support $B(z,r)$ for some $z\in[-1,1]^n$ and $r\in(0,0.001)$. To see this, first, we note that by the translation invariance of the operator $\widetilde{S}_u^\alpha$ and the fact that $\|\widetilde{S}_u^\alpha\|_{(L^2(\mathbb{T}^n),L^2(\mathbb{T}^n))}\le 1$, we may assume a is a regular $H^1(\mathbb{T}^n)$-atom with support $B(0,r)$ for some $r\in(0,0.001)$. Second, we note that by the definition, for every regular $H^1(\mathbb{T}^n)$-atom a with support $B(0,r)$, $r\in(0,0.001)$, $a\chi_{[-3/2,3/2]^n}$ can be expressed as a sum of 3^n $H^1(\mathbb{R}^n)$-atoms, each having a support $B(z,r)$ for some $z\in[-1,1]^n$. Since the definition of $S_u(a)(x)$ for $x\in Q$ involves only the values of a on $[-1,1]^n$, the claim follows.

For the rest of this section, the letter a will always denote an $H^1(\mathbb{R}^n)$-atom with support $B(z,r)$ for some $z\in[-1,1]^n$ and $r\in(0,0.001)$.

The proof of (3.2) for an $H^1(\mathbb{R}^n)$-atom a relies on the following

LEMMA 3.1. *With the above notation, we have*

(i) $$\int_0^\infty\Big[\int_{\{x\in Q\,:\,|x-z|\ge 5r\}}|S_u(a)(x)\,dx\Big]^2du\le C_n r^{-1}\log^2\frac{1}{r};$$

(ii) $$\int_{\{x\in Q\,:\,|x-z|\le 5r\}}|S_u(a)(x)|\,dx\le C_n;$$

(iii) $$\int_{\{x\in Q\,:\,|x-z|\ge 5r\}}|S_u(a)(x)|\,dx\le C_n\Big[(u+1)r\log\frac{1}{r}+1\Big].$$

For the moment we take this last lemma for granted and proceed with the proof of (3.2).

By Lemma 3.1(ii), it suffices to prove

$$\frac{1}{\log(R+1)}\int_0^R\frac{1}{u+1}\int_{\{x\in Q\,:\,|x-z|\ge 5r\}}|S_u(a)(x)|\,dx\,du\le C_n. \tag{3.3}$$

To prove (3.3), we consider the following two cases:

CASE 1: $r^{-1}\le R$. In this case, on one hand, by Lemma 3.1(iii),

$$\begin{aligned}\int_0^{r^{-1}}\frac{1}{u+1}&\int_{\{x\in Q\,:\,|x-z|\ge 5r\}}|S_u(a)(x)|\,dx\,du\\ &\le C_n\int_0^{r^{-1}}\frac{1}{u+1}\Big(1+(u+1)r\log\frac{1}{r}\Big)\,du\le C_n\log\frac{1}{r}\le C_n\log(R+1),\end{aligned}$$

but on the other hand, by Lemma 3.1(i) and Hölder's inequality,

$$\begin{aligned}\int_{r^{-1}}^{R} \frac{1}{u+1} \int_{\{x\in Q\,:\,|x-z|\ge 5r\}} |S_u(a)(x)|\,dx\,du \\ &\le \Big(\int_{r^{-1}}^{R} \frac{du}{(u+1)^2}\Big)^{1/2} \Big(\int_{r^{-1}}^{R} \Big[\int_{\{x\in Q\,:\,|x-z|\ge 5r\}} |S_u(a)(x)|\,dx\Big]^2 du\Big)^{1/2} \\ &\le Cr^{1/2}r^{-1/2}\log\frac{1}{r} \le C\log R.\end{aligned}$$

Case 2: $R < r^{-1}$. In this case, using Lemma 3.1(iii), we obtain

$$\begin{aligned}\int_{0}^{R} \frac{1}{u+1} \int_{\{x\in Q\,:\,|x-z|\ge 5r\}} |S_u(a)(x)|\,dx\,du \\ &\le C_n \int_0^R \frac{1}{u+1}\Big((u+1)r\log\frac{1}{r}+1\Big)\,du \\ &\le C_n Rr\log\frac{1}{r} + C_n\log(R+1) \le C_n\log(R+1).\end{aligned}$$

The last inequality follows since the function $\frac{\log x}{x}$ is decreasing over (e,∞).

Now combining the above two cases we obtain (3.3). This proves the first part of Theorem 1, assuming Lemma 3.1. ∎

For the proof of Lemma 3.1, we need the following

Lemma 3.2. *Let $x \in Q$ be such that $|x-z| \ge 5r$. For $t > 0$, put*

$$g_x(t) := t^{n-1} \int_{\mathbb{S}^{n-1}} a(x-ty)\chi_Q(ty)\,d\sigma(y),$$

where $d\sigma(y)$ denotes the usual Lebesgue measure on $\mathbb{S}^{n-1}$ normalized by $\sigma(\mathbb{S}^{n-1}) = 1$. Then

(i) $\operatorname{supp} g_x(\cdot) \subset [|x-z|-r, |x-z|+r]$;
(ii) $|g_x(t)| \le C_n r^{-1}$, *with $C_n > 0$ depending only on n.*

Proof. By the definition, we have

$$|g_x(t)| \le t^{n-1} \int_{\mathbb{S}^{n-1}} |a(x-ty)|\,d\sigma(y) = \int_{S(x,t)\cap B(z,r)} |a(y)|\,d\sigma_t(y), \tag{3.4}$$

where $S(x,t) = \{y \in \mathbb{R}^n : |x-y| = t\}$, and $d\sigma_t(y)$ denotes the usual Lebesgue measure on $S(x,t)$ normalized by $\sigma_t(S(x,t)) = t^{n-1}$. Since $S(x,t)\cap B(z,r) = \emptyset$ whenever $t \notin [|x-z|-r, |x-z|+r]$, (i) follows by (3.4).

To show (ii) we note that for $t \in [|x-z|-r, |x-z|+r]$,

$$\sigma_t(S(x,t)\cap B(z,r)) \le Cr^{n-1}.$$

Thus, by (3.4), it follows that

$$|g_x(t)| \le Cr^{n-1}r^{-n} = Cr^{-1},$$

which gives (ii). ■

Now we are in a position to prove Lemma 3.1.

Proof of Lemma 3.1. (i) Recall that the Plancherel theorem for the Fourier–Bessel transform (see, for instance, [GS, p. 656]) asserts that for any $\beta > -1/2$ and $f \in L^2((0,\infty), t^{2\beta+1}\,dt)$,

$$\int_0^\infty |f(t)|^2 t^{2\beta+1}\,dt = 4\pi^2 \int_0^\infty \Big|\int_0^\infty \frac{J_\beta(2\pi ts)}{(ts)^\beta} f(s)s^{2\beta+1}\,ds\Big|^2 t^{2\beta+1}\,dt. \tag{3.5}$$

This last formula will play an important role in our proof below.

Let $g_x(t)$ be as defined in Lemma 3.2, and write

$$\begin{aligned} S_u(a)(x) &= C_n u^{1/2} \int_{\mathbb{R}^n} a(x-y)\chi_Q(y)|y|^{-(n-1/2)} J_{n-1/2}(2\pi u|y|)\,dy \\ &= C_n u^{1/2} \int_0^\infty g_x(t) t^{-(n-1/2)} J_{n-1/2}(2\pi ut)\,dt. \end{aligned}$$

Then using Lemma 3.2 and (3.5) with $\beta = n - 1/2$ and $f(t) = g_x(t)t^{-2n}$, we deduce that for $|x-z| \ge 5r$,

$$\begin{aligned} \int_0^\infty |S_u(a)(x)|^2\,du &= C_n \int_0^\infty u\Big|\int_0^\infty g_x(t)t^{-(n-1/2)} J_{n-1/2}(2\pi ut)\,dt\Big|^2\,du \\ &= \frac{C_n}{4\pi^2} \int_{|x-z|-r}^{|x-z|+r} |g_x(t)|^2 t^{-2n}\,dt \\ &\le Cr^{-1}|x-z|^{-2n}. \end{aligned}$$

Noticing that $z \in [-1,1]^n$, we obtain, by Hölder's inequality,

$$\begin{aligned} &\int_0^\infty \Big[\int_{\{x\in Q\,:\,|x-z|\ge 5r\}} |S_u(a)(x)|\,dx\Big]^2 du \\ &\quad\le \int_0^\infty \Big[\int_{5r\le|x-z|\le 10} |x-z|^{-n}\,dx\Big]\Big[\int_{5r\le|x-z|\le 10} |x-z|^n |S_u(a)(x)|^2\,dx\Big]\,du \\ &\quad\le C_n r^{-1}\log\frac{1}{r} \int_{5r\le|x-z|\le 10} |x-z|^n|x-z|^{-2n}\,dx \le C_n r^{-1}\log^2\frac{1}{r}, \end{aligned}$$

which gives (i).

(ii) Since for $|x-z| \le 5r$,

$$x - Q \supseteq B(z,r),$$

it follows that

$$S_u(a)(x) = C_n u^{1/2} \int_Q a(x-y)|y|^{-(n-1/2)} J_{n-1/2}(2\pi u|y|)\,dy \tag{3.6}$$
$$= C_n u^{1/2} \int_{\mathbb{R}^n} a(x-y)|y|^{-(n-1/2)} J_{n-1/2}(2\pi u|y|)\,dy,$$
$$=: S_u^{(n-1)/2}(a)(x),$$

where

$$S_u^{(n-1)/2}(f)(x) = C_n u^{1/2} \int_{\mathbb{R}^n} f(x-y)|y|^{-(n-1/2)} J_{n-1/2}(2\pi u|y|)\,dy.$$

It is well known that (see [SW, Theorem 4.15, p. 171])

$$S_u^{(n-1)/2}(f)(x) = C_n \int_{|\xi|\le u} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\left(1-\frac{|\xi|^2}{u^2}\right)^{(n-1)/2} d\xi,$$

where

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i y\cdot\xi}\,dy,$$

and that (see [Du, Theorem 8.15, p. 169])

$$\sup_{u>0} \|S_u^{(n-1)/2}(f)\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)}. \tag{3.7}$$

Now using (3) and (3.7), we obtain, by Hölder's inequality,

$$\int_{\{x\in Q\,:\,|x-z|\le 5r\}} |S_u(a)(x)|\,dx$$
$$= \int_{\{x\in Q\,:\,|x-z|\le 5r\}} |S_u^{(n-1)/2}(a)(x)|\,dx$$
$$\le Cr^{n/2}\Big(\int_{\mathbb{R}^n} |S_u^{(n-1)/2}(a)(x)|^2\,dx\Big)^{1/2} \le Cr^{n/2}\|a\|_{L^2(\mathbb{R}^n)} \le C_n.$$

This gives (ii).

(iii) To show (iii), we consider the following two cases:

Case 1: $5r \le |x-z| \le 0.1$. In this case $x - Q \supset B(z,r)$, and hence

$$\int_Q a(x-y)\,dy = 0. \tag{3.8}$$

For simplicity, we put $t_0 = |x-z|$, $\varphi(t) = t^{-(n-1/2)} J_{n-1/2}(t)$, and

$$g_x(t) = t^{n-1} \int_{\mathbb{S}^{n-1}} a(x-ty)\chi_Q(ty)\,d\sigma(y).$$

Then, by Lemma 3.2(i) and (3.8), we have

$$S_u(a)(x) = C_n u^n \int_{t_0-r}^{t_0+r} g_x(t)[\varphi(2\pi ut) - \varphi(2\pi ut_0)]\,dt. \tag{3.9}$$

Since

$$|\varphi'(t)| = |t^{-(n-1/2)} J_{n+1/2}(t)| \le C \min\{t, t^{-n}\},$$

it follows that for $t_0 - r \le t \le t_0 + r$,

$$|\varphi(2\pi ut) - \varphi(2\pi ut_0)| \le Cur \min\{ut_0, (ut_0)^{-n}\}.$$

Hence, by (3.9) and Lemma 3.2(ii),

$$|S_u(a)(x)| \le C_n u^n \int_{t_0-r}^{t_0+r} |g_x(t)| ur \min\{ut_0, (ut_0)^{-n}\}\,dt \le C_n urt_0^{-n} \tag{3.10}$$

in the case when $5r \le t_0 = |x - z| \le 0.1$.

CASE 2: $|x - z| > 0.1$. In this case $B(x, 0.005) \cap B(z, r) = \emptyset$, so

$$\begin{aligned} |S_u(a)(x)| &= C_n u^{1/2} \Big| \int_{\{y\in Q\,:\,|y|\ge 0.005\}} a(x-y)|y|^{-(n-1/2)} J_{n-1/2}(2\pi u|y|)\,dy \Big| \\ &\le C_n u^{1/2} \int_{\{y\in x-B(z,r)\,:\,0.05\le|y|\le 10\}} r^{-n} \min\{u^{n-1/2}, u^{-1/2}\}\,dy \le C_n. \end{aligned} \tag{3.11}$$

Now putting the above two cases together, combining (3.10) with (3.11), we deduce

$$\begin{aligned} \int_{\{x\in Q\,:\,|x-z|\ge 5r\}} |S_u(a)(x)|\,dx &\le C(u+1)r \int_{5r\le|x-z|\le 0.1} \frac{1}{|x-z|^n}\,dx + C_n \\ &\le C_n\Big[(u+1)r\log\frac{1}{r} + 1\Big], \end{aligned}$$

which gives (iii).

This completes the proof of Lemma 3.1. ∎

4. Proof of Theorem 1: Part II. This section is devoted to the proof of the inequality

$$\|\widetilde{S}_u^\alpha(f)\|_{H^1(\mathbb{T}^n)} \le C(\|f\|_{H^1(\mathbb{T}^n)} + \|\widetilde{S}_u^\alpha(f)\|_{L^1(\mathbb{T}^n)}), \tag{4.1}$$

with $C > 0$ independent of f and u. This combined with (3.1) proved in the last section will complete the proof of Theorem 1.

The referee kindly pointed out to us that Theorem 1 is, in fact, a direct consequence of (3.1) proved in Part I because of the following fact: $\widetilde{S}_u^\alpha$ is translation invariant and the Hardy space $H^1(\mathbb{T}^n)$ can be characterized by a system of Riesz transforms (see, for instance, [Lu, Remark 6.1, p. 152]).

Our proof of (4.1) in this section is independent of this fact and may be of independent interest (see, for instance, [Da]).

For the proof of (4.1), we define

$$\sigma_u^\delta(f)(x) = \sum_{|k|<u} \left(1 - \frac{|k|}{u}\right)^\delta a_k(f) e^{2\pi i k\cdot x}, \qquad \delta > -1,\ u > 0,$$

and

$$\sigma_*^\delta(f)(x) = \sup_{u>0} |\sigma_u^\delta(f)(x)|.$$

We need the following lemmas.

LEMMA 4.1. *Let $0 < p \le 1$, $\delta > \delta(p) := n/p - (n+1)/2$ and $f \in \mathcal{S}'(\mathbb{T}^n)$. Then $f \in H^p(\mathbb{T}^n)$ if and only if $\sigma_*^\delta(f) \in L^p(\mathbb{T}^n)$. Moreover, if $f \in H^p(\mathbb{T}^n)$ then*

$$\|f\|_{H^p(\mathbb{T}^n)} \approx \|\sigma_*^\delta(f)\|_{L^p(\mathbb{T}^n)}.$$

LEMMA 4.2. *Let $\ell \ge 0$ be an integer and let m be an $\ell+1$ times differentiable function on $[0,\infty)$ such that $\lim_{u\to\infty} m(u) = 0$ and*

$$\int_0^\infty |m^{(\ell+1)}(u)| u^\ell \, du < \infty.$$

Define

$$T_m(f) := \sum_{k\in\Lambda} m(|k|) a_k(f) e^{2\pi i k\cdot x}.$$

Then for $f \in \mathcal{S}(\mathbb{T}^n)$,

$$T_m(f)(x) = \frac{(-1)^{\ell-1}}{\ell!} \int_0^\infty m^{(\ell+1)}(u) u^\ell \sigma_u^\ell(f)(x)\, du, \qquad x \in \mathbb{R}^n.$$

For the moment we take these last two lemmas for granted and proceed with the proof of (4.1). By Lemma 4.1, it is sufficient to prove that for $f \in \mathcal{S}(\mathbb{T}^n)$,

$$\sigma_*^{2n}(\widetilde{S}_u^\alpha(f))(x) \le C[\sigma_*^{[\alpha]+1}(f)(x) + |\widetilde{S}_u^\alpha(f)(x)|]. \tag{4.2}$$

To prove (4.2), we have to estimate $|\sigma_y^{2n}(\widetilde{S}_u^\alpha(f))(x)|$ for $y, u > 0$. We put $\ell = [\alpha] + 1$ and consider the following two cases:

CASE 1: $0 < y < u$. In this case we will prove

$$|\sigma_y^{2n}(\widetilde{S}_u^\alpha(f))(x)| \le C_n \sigma_*^\ell(f)(x). \tag{4.3}$$

To this end, we write

$$\sigma_y^{2n}(\widetilde{S}_u^\alpha(f))(x) = \sum_{|k|<y}\left(1-\frac{|k|}{y}\right)^{2n}\left(1-\frac{|k|^2}{u^2}\right)^\alpha a_k(f)e^{2\pi i k\cdot x}$$
$$= \sum_{k\in\Lambda} m(|k|)a_k(f)e^{2\pi ik\cdot x},$$

where

$$m(t) = \frac{(y-t)_+^{2n}}{y^{2n}}\frac{(u^2-t^2)^\alpha}{u^{2\alpha}}, \qquad g_+(x) = \max\{g(x),0\}. \tag{4.4}$$

We claim that for $0<t<y<u$,

$$|m^{(\ell+1)}(t)| \le C_n y^{-\ell-1}, \tag{4.5}$$

which, combined with Lemma 4.2, will imply that for $0<y<u$,

$$|\sigma_y^{2n}(\widetilde{S}_u^\alpha(f))(x)| = C_n\left|\int_0^y m^{(\ell+1)}(t)t^\ell \sigma_t^\ell(f)(x)\,dt\right| \le C_n\sigma_*^\ell(f)(x),$$

and hence will prove (4.3).

In fact, by (4.4), for $0<t<y$,

$$|m^{(\ell+1)}(t)| \le C \max_{i_1+i_2+i_3=\ell+1} \frac{(y-t)^{2n-i_1}}{y^{2n}}\,\frac{(u-t)^{\alpha-i_2}}{u^\alpha}\,\frac{(u+t)^{\alpha-i_3}}{u^\alpha}.$$

So, if $0<y<u/2$ then, clearly,

$$|m^{(\ell+1)}(t)| \le Cy^{-\ell-1}, \qquad 0<t<y;$$

if $u/2\le y\le u$ then for $0<t<y$,

$$|m^{(\ell+1)}(t)| \le Cy^{-2n-2\alpha}\max_{i_1+i_2+i_3=\ell+1}(u-t)^{2n+\alpha-i_1-i_2}u^{\alpha-i_3} \le Cy^{-\ell-1}.$$

Therefore, in either case, we have, for $0<t<y<u$,

$$|m^{(\ell+1)}(t)| \le Cy^{-\ell-1},$$

proving the claim.

Case 2: $y\ge u$. In this case we will prove

$$|\sigma_y^{2n}(\widetilde{S}_u^\alpha(f))(x)| \le C_n[\sigma_*^\ell(f)(x) + |\widetilde{S}_u^\alpha(f)(x)|], \tag{4.6}$$

which combined with (4.3) in Case 1 will complete the proof of (4.2).

We write

$$\sigma_y^{2n}(\widetilde{S}_u^\alpha(f))(x) = \sum_{|k|<u}\left(1-\frac{|k|}{y}\right)^{2n}\left(1-\frac{|k|^2}{u^2}\right)^\alpha a_k(f)e^{2\pi ik\cdot x} \tag{4.7}$$
$$=: T_m(f)(x) + \left(1-\frac{u}{y}\right)^{2n}\widetilde{S}_u^\alpha(f)(x),$$

where

$$T_m(f)(x) := \sum_{k\in\Lambda} m(|k|)a_k(f)e^{2\pi ik\cdot x},$$
$$m(t) = a(t)\left(1-\frac{t^2}{u^2}\right)_+^\alpha,$$
$$a(t) = \frac{1}{y^{2n}}\,[(y-t)^{2n}-(y-u)^{2n}].$$

For $0<t<u$, it is easy to verify

$$|a(t)| \le C(u-t)/y, \qquad \max_{1\le i\le \ell+1} y^i|a^{(i)}(t)| \le C.$$

Using these estimates, we have: if $\alpha=(n-1)/2$ is an integer then $\ell=\alpha+1$ and for $0<t<u$,

$$|m^{(\ell+1)}(t)| \le C \max_{\substack{i_1+i_2+i_3=\ell+1\\ i_2\le\alpha=\ell-1}} |a^{(i_1)}(t)|\,\frac{(u-t)^{\alpha-i_2}}{u^\alpha}\,\frac{(u+t)^{\alpha-i_3}}{u^\alpha} \le Cu^{-\ell-1};$$

if $\alpha=(n-1)/2$ is not an integer then $\ell=[\alpha]+1$ and for $0<t<u$,

$$\begin{aligned}|m^{(\ell+1)}(t)| &\le C|a(t)|\,\frac{(u-t)^{\alpha-\ell-1}}{u^\alpha} + C\,\frac{(u-t)^{\alpha-\ell}}{u^\alpha}\max_{i+j=1}|a^{(i)}(t)|\,\frac{(u+t)^{\alpha-j}}{u^\alpha}\\ &\quad + C\max_{\substack{i_1+i_2+i_3=\ell+1\\ i_2\le[\alpha]=\ell-1}}|a^{(i_1)}(t)|\,\frac{(u-t)^{\alpha-i_2}}{u^\alpha}\,\frac{(u+t)^{\alpha-i_3}}{u^\alpha}\\ &\le Cu^{-\ell-1} + C\,\frac{(u-t)^{\alpha-[\alpha]-1}}{u^{\alpha+1}}.\end{aligned}$$

In either case, we have

$$\int_0^u |m^{(\ell+1)}(t)|t^\ell\,dt \le C_n.$$

So, by Lemma 4.2,

$$|T_m(f)(x)| \le C\sigma_*^\ell(f)(x).$$

Now (4.6) follows by (4.7).

This completes the proof of Theorem 1, assuming the validity of Lemmas 4.1 and 4.2. ∎

So, it remains to prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. Using the transference theorem in [CF] and following the proof in [STW], one can easily verify that for $f\in H^p(\mathbb{T}^n)$ and $\delta>\delta(p):=n/p-(n+1)/2$,

$$\|\sigma_*^\delta(f)\|_{L^p(\mathbb{T}^n)} \le C\|f\|_{H^p(\mathbb{T}^n)}.$$

On the other hand, since

$$e^{-2\pi t|k|} = \frac{(2\pi t)^{1+\delta}}{\Gamma(1+\delta)} \int_0^\infty y^\delta e^{-2\pi t y}\left(1 - \frac{|k|}{y}\right)_+^\delta dy,$$

it follows that

$$\widetilde{P}_t(f)(x) = \frac{(2\pi t)^{1+\delta}}{\Gamma(1+\delta)} \int_0^\infty y^\delta e^{-2\pi t y}\sigma_y^\delta(f)(x)\, dy,$$

which implies

$$\widetilde{P}_+(f)(x) \le \sigma_*^\delta(f)(x)$$

and hence the inverse inequality

$$\|f\|_{H^p(\mathbb{T}^n)} \le \|\sigma_*^\delta(f)\|_{L^p(\mathbb{T}^n)}.$$

This completes the proof. ∎

Proof of Lemma 4.2. First, we note that under the assumptions of Lemma 4.2 the following is true:

$$\lim_{t\to\infty} m^{(i)}(t) = 0, \qquad i = 0, \dots, \ell,$$

and

$$\int_0^\infty |m^{(i+1)}(t)|t^i\, dt < \infty, \qquad i = 0, \dots, \ell.$$

In view of these last two facts, we obtain by integration by parts ℓ times

$$m(t) = \frac{(-1)^{\ell-1}}{\ell!} \int_0^\infty m^{(\ell+1)}(u)u^\ell \left(1 - \frac{t}{u}\right)_+^\ell du.$$

The identity

$$T_m(f)(x) = \frac{(-1)^{\ell-1}}{\ell!} \int_0^\infty m^{(\ell+1)}(u)u^\ell \sigma_u^\ell(f)(x)\, du, \qquad f \in \mathcal{S}(\mathbb{T}^n),$$

then follows. This completes the proof. ∎

5. Proof of Corollaries 2 and 3

Proof of Corollary 2. The lower estimate is obvious. For the proof of the upper estimate, we let η be a C^∞-function on $[0,\infty)$ such that $\eta(t) = 1$ for $0 \le t \le 1$, and $\eta(t) = 0$ for $t \ge 2$, and define for $u > 0$,

$$V_u(f)(x) = \sum_{k\in\Lambda} \eta\left(\frac{|k|}{u}\right) a_k(f) e^{2\pi i k\cdot x},$$

and for $u \le 0$,

$$V_u(f)(x) = a_0(f).$$

Then it is easy to show that

$$\|f - V_u(f)\|_{H^1} \le C E_u(f, H^1), \qquad u \ge 0.$$

For simplicity, we set

$$g_j = V_{2^{2^{j-2}}}(f), \qquad j \ge 2.$$

Without loss of generality, we may assume $R > 16$, $2^{2^m} \le R < 2^{2^{m+1}}$ with $m \ge 2$, and $\int_{\mathbb{T}^n} f(x)\,dx = 0$. Since

$$\int_0^{16} \frac{\|\widetilde{S}_u^\alpha(f) - f\|_{H^1}}{u+1}\,du \le C E_0(f, H^1),$$

it is sufficient to show

$$\int_{16}^{R} \frac{\|\widetilde{S}_u^\alpha(f) - f\|_{H^1}}{u+1}\,du \le C \int_0^R \frac{E_u(f, H^1)}{u+1}\,du. \tag{5.1}$$

We have

$$\begin{aligned}
\int_{16}^{R} \frac{\|\widetilde{S}_u^\alpha(f) - f\|_{H^1}}{u+1}\,du &\le \sum_{j=3}^{m+1} \int_{2^{2^{j-1}}}^{2^{2^j}} \frac{\|f - g_j\|_{H^1}}{u+1}\,du \\
&\quad + \sum_{j=3}^{m+1} \int_{2^{2^{j-1}}}^{2^{2^j}} \frac{1}{u+1}\,\|\widetilde{S}_u^\alpha(f - g_j)\|_{H^1}\,du \\
&\quad + \sum_{j=3}^{m+1} \int_{2^{2^{j-1}}}^{2^{2^j}} \frac{1}{u+1}\,\|\widetilde{S}_u^\alpha(g_j) - g_j\|_{H^1}\,du, \\
&=: I + J + L.
\end{aligned}$$

For the first sum, we have

$$\begin{aligned}
I &\le C \sum_{j=3}^{m+1} \left(\int_{2^{2^{j-1}}}^{2^{2^j}} \frac{E_{2^{2^{j-2}}}(f, H^1)}{u+1}\,du \right) \le C \sum_{j=3}^{m+1} \int_{2^{2^{j-3}}}^{2^{2^{j-2}}} \frac{E_u(f, H^1)}{u+1}\,du \\
&\le C \int_2^R \frac{E_u(f, H^1)}{u+1}\,du.
\end{aligned}$$

For the second sum, using Theorem 1, we have

$$J \le C \sum_{j=3}^{m+1} 2^j \|f - g_j\|_{H^1} \le C \sum_{j=3}^{m+1} \int_{2^{2^{j-3}}}^{2^{2^{j-2}}} \frac{E_u(f, H^1)}{u+1}\,du \le C \int_1^R \frac{E_u(f, H^1)}{u+1}\,du.$$

To estimate the third sum, we first claim that for $2^{2^{j-1}} \le u \le 2^{2^j}$ and $j \ge 3$,

$$\|\widetilde{S}_u^\alpha(g_j) - g_j\|_{H^1} \le C u^{-2} \|\Delta(g_j)\|_{H^1}, \tag{5.2}$$

where $\Delta = \sum_{j=1}^n (\partial/\partial x_j)^2$ denotes the Laplacian on $\mathbb{T}^n$. For the moment we take this last inequality for granted and proceed with the proof. Using Bernstein's inequality, we deduce that for $2^{2^{j-1}} \le u \le 2^{2^j}$,

$$u^{-2}\|\Delta(g_j)\|_{H^1} = u^{-2}\|\Delta(V_{2^{2^j-2}}(f))\|_{H^1} \le Cu^{-2}2^{2^{j-1}}\|V_{2^{2^j-2}}(f)\|_{H^1}$$
$$\le Cu^{-1}\|V_{2^{2^j-2}}(f)\|_{H^1}.$$

Since $a_0(f) = 0$, it follows that

$$u^{-1}\|V_{2^{2^j-2}}(f)\|_{H^1} \le u^{-1}\sum_{l=0}^{2^{j-2}} \|V_{2^{l-1}}(f) - V_{2^l}(f)\|_{H^1} \le Cu^{-1}\sum_{l=0}^{2^{j-2}} E_{2^{l-1}}(f, H^1),$$

where $E_{2^{-1}}(f, H^1) = E_0(f, H^1)$. Then from (5.2) we get

$$\|\widetilde{S}_u^\alpha(g_j) - g_j\|_{H^1} \le Cu^{-1}\sum_{l=0}^{2^{j-2}} E_{2^{l-1}}(f, H^1),$$

and hence

$$L \le C\sum_{j=3}^{m+1}\int_{2^{2^{j-1}}}^{2^{2^j}} \frac{du}{(u+1)^2}\sum_{l=0}^{2^{j-2}} E_{2^{l-1}}(f, H^1) \le C\sum_{j=3}^{m+1} 2^{-2^{j-1}}\sum_{l=0}^{2^{j-2}} E_{2^{l-1}}(f, H^1)$$
$$\le C\sum_{l=0}^{2^{m-1}} 2^{-2l}E_{2^{l-1}}(f, H^1) \le C\sum_{l=3}^{2^{m-1}}\int_{2^{l-2}}^{2^{l-1}} \frac{E_u(f, H^1)}{(u+1)^3}\,du + CE_0(f, H^1)$$
$$\le C\int_0^R \frac{E_u(f, H^1)}{(u+1)^3}\,du.$$

Putting the above together, we prove (5.1) and hence the desired upper estimate, assuming (5.2).

Now it remains to prove (5.2). To this end, let $\xi \in C^\infty(\mathbb{R})$ be such that $\xi(x) = 1$ for $0 \le |x| \le 1/2$ and $\xi(x) = 0$ for $|x| \ge 3/4$. For simplicity, we define

$$\mathbb{P}_u = \Big\{\sum_{|k|\le u} c_k e^{2\pi i k\cdot x} : c_k \in \mathbb{C},\ |k| \le u\Big\}.$$

Since

$$g_j = V_{2^{2^j-2}}(f) \in \mathbb{P}_{2\cdot 2^{2^j-2}},$$

it follows that for $j \ge 3$ and $2^{2^{j-1}} \le u \le 2^{2^j}$ we get $g_j \in \mathbb{P}_{u/2}$, and hence

$$\text{(5.3)}\qquad \widetilde{S}_u^\alpha(g_j) - g_j = \sum_{|k|<u}\bigg[\bigg(1 - \frac{|k|^2}{u^2}\bigg)^\alpha - 1\bigg]\xi\bigg(\frac{|k|}{u}\bigg)a_k(g_j)e^{2\pi i k\cdot x}$$
$$= u^{-2}\sum_{k\in\Lambda} m\bigg(\frac{|k|}{u}\bigg)a_k(\Delta(g_j))e^{2\pi i k\cdot x},$$

where

$$m(t) = \frac{(1-t^2)^\alpha - 1}{t^2}\xi(t).$$

We note that $m \in C^\infty[0,\infty)$ and $\operatorname{supp} m \subset [0,3/4]$. Hence,

$$\Big\| \sum_{k\in\Lambda} m\Big(\frac{|k|}{u}\Big) a_k(\Delta(g_j)) e^{2\pi i k\cdot x} \Big\|_{H^1} \le C_n \|\Delta(g_j)\|_{H^1},$$

and (5.2) then follows by (5.4).

This completes the proof of Corollary 2. ■

Proof of Corollary 3. By Corollary 2 and the Jackson inequality, we have

$$\begin{aligned}
&\frac{1}{\log(R+1)} \int_0^R \frac{\|\widetilde{S}_u^\alpha(f) - f\|_{H^1}}{u+1}\,du \\
&\quad \le \frac{C}{\log(R+1)} \int_0^R \frac{E_u(f,H^1)}{u+1}\,du \le \frac{C}{\log(R+1)} \int_0^R \frac{\omega(f,(u+1)^{-1})_{H^1}}{u+1}\,du \\
&\quad \le C\,\frac{\omega(f,1/\log(R+1))_{H^1}}{\log(R+1)} \int_0^{\log(R+1)} \frac{\log(R+1)}{(u+1)^2}\,du \\
&\qquad + C\,\frac{\omega(f,1/\log(R+1))_{H^1}}{\log(R+1)} \int_{\log(R+1)}^R \frac{1}{u+1}\,du \\
&\quad \le C\omega\Big(f, \frac{1}{\log(R+1)}\Big)_{H^1},
\end{aligned}$$

proving Corollary 3. ■

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References

[Be1] E. S. Belinskii, *Strong summability for the Marcinkiewicz means in the integral metric and related questions*, J. Austral. Math. Soc. Ser. A 65 (1998), 303–312.

[Be2] —, *Strong summability of Fourier series of the periodic functions from H^p* ($0 < p \le 1$), Constr. Approx. 12 (1996), 187–195.

[Bo] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. 40 (1936), 175–207.

[CF] D. N. Chen and D. S. Fan, *Multiplier transformations on H^p spaces*, Studia Math. 131 (1998), 189–204.

[CJL] G. L. Chen, Y. S. Jiang and S. Z. Lu, *Strong approximation of Riesz means at critical index on $H^p(T)$ $(0 < p \leq 1)$*, Approx. Theory Appl. 5 (1989), no. 2, 39–49.

[Da] F. Dai, *Strong convergence of spherical harmonic expansions on $H^1(\mathbb{S}^{n-1})$*, Constr. Approx. 22 (2005), 417–436.

[Du] J. Duoandikoetxea, *Fourier Analysis*, translated and revised from the 1995 Spanish original by D. Cruz-Uribe, Amer. Math. Soc., Providence, RI, 2001.

[F] D. S. Fan, *Hardy spaces on compact Lie groups*, Ph.D. thesis, Washington Univ., St. Louis, 1990.

[GS] J. Gosselin and K. Stempak, *A weak-type estimate for Fourier–Bessel multipliers*, Proc. Amer. Math. Soc. 106 (1989), 655–662.

[JLL] Y. S. Jiang, H. P. Liu and S. Z. Lu, *Some properties of elliptic Riesz means at critical index on $H^p(\mathbb{T}^n)$*, Approx. Theory Appl. 6 (1990), no. 2, 28–37.

[Lu] S. Z. Lu, *Four Lectures on Real H^p Spaces*, World Sci., Singapore, 1995.

[Sm] B. Smith, *A strong convergence theorem for $H^1(\mathbb{T}^n)$*, in: Lecture Notes in Math. 995, Springer, Berlin, 1983, 169–173.

[St1] E. M. Stein, *Localization and summability of multiple Fourier series*, Acta Math. 100 (1958), 93–147.

[St2] —, *On certain exponential sums arising in multiple Fourier series*, Ann. of Math. (2) 73 (1961), 87–109.

[STW] E. M. Stein, M. H. Taibleson and G. Weiss, *Weak type estimates for maximal operators on certain H^p classes*, Rend. Circ. Mat. Palermo (2) Suppl. 1 (1981), 81–97.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, NJ, 1971.

[We] F. Weisz, *Strong convergence theorems for two-parameter Walsh–Fourier and trigonometric–Fourier series*, Studia Math. 117 (1996), 173–194.

Department of Mathematical and Statistical Sciences
CAB 632, University of Alberta
Edmonton, Alberta, T6G 2G1, Canada
E-mail: dfeng@math.ualberta.ca

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