The Bloch space for the minimal ball

by

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Abstract. We introduce the Bloch space for the minimal ball and we prove that this space can be identified with the dual of a certain analytic space which is strongly related to the Bergman theory on the minimal ball.

1. Introduction and main result. Let $\mathbb{B}_*$ be the domain in $\mathbb{C}^n, n \geq 2$, defined by

$$\mathbb{B}_* := \{z \in \mathbb{C}^n : |z|^2 + |z \bullet z| < 1\},$$

where $z \bullet w := \sum_{j=1}^n z_j w_j$ for z and w in $\mathbb{C}^n$. This is the unit ball with respect to the norm

$$N_*(z) := \sqrt{|z|^2 + |z \bullet z|}, \quad z \in \mathbb{C}^n.$$

The norm $N := N_*/\sqrt{2}$ was introduced by Hahn and Pflug [HP], and was shown to be the smallest norm in $\mathbb{C}^n$ that extends the euclidian norm in $\mathbb{R}^n$ under certain restrictions. The automorphism group of $\mathbb{B}_*$ is compact and its identity component is $\mathrm{Aut}_{\mathcal{O}}^0(\mathbb{B}_*) = S^1 \cdot \mathrm{SO}(n, \mathbb{R})$, where the S^1-action is diagonal and the $\mathrm{SO}(n, \mathbb{R})$-action is the matrix multiplication (see [K] or [OY]). The ball $\mathbb{B}_*$ is a nonhomogeneous domain with singular boundary consisting of all its boundary points z that satisfy $z \bullet z = 0$. The regular part of the boundary of $\mathbb{B}_*$ consists of strictly pseudoconvex points. The Bergman theory on the minimal ball, developed in [MY], showed the importance of this singularity. $\mathbb{B}_*$ was also used to construct counter-examples to the Lu Qi-Keng conjecture [PY]. This makes the analysis on the minimal ball very interesting. Furthermore E. H. Youssfi recently proved [Y] that the methods used in [MY] can be extended to a more general class of domains in $\mathbb{C}^n$ containing the minimal ball, the unit ball and the complex

ellipsoids. The study of $\mathbb{B}_*$ seems to be a good way to understand a much wider case.

Let v be the normalized Lebesgue measure on $\mathbb{B}_*$. For $p \geq 1$ let $L^p(\mathbb{B}_*, |z \bullet z|^{(p-2)/2}dv)$ be the Banach space of all functions on $\mathbb{B}_*$ that are L^p-integrable with respect to the measure $|z \bullet z|^{(p-2)/2}dv(z)$. We denote by $\mathcal{A}^p(\mathbb{B}_*)$ the space of all holomorphic functions on $\mathbb{B}_*$ which are in the space $L^p(\mathbb{B}_*, |z \bullet z|^{(p-2)/2}dv)$. The spaces $\mathcal{A}^p(\mathbb{B}_*)$ appear naturally in Bergman theory associated to the minimal ball (see [MY]). It is known that these spaces furnished with the norm of $L^p(\mathbb{B}_*, |z \bullet z|^{(p-2)/2}dv)$ are Banach spaces (see Lemma 4.1 of [MY]).

In [M] we proved that for $p > 1$ the dual space of $\mathcal{A}^p(\mathbb{B}_*)$ can be identified with $\mathcal{A}^q(\mathbb{B}_*)$ where $1/p + 1/q = 1$. Here we are interested in the case $p = 1$. For the unit ball it is well known that the dual space of the Bergman space of order 1 is the Bloch space (see for instance [A], [Z2] for the one-dimensional case and [Ch], [Tim1], [Tim2] for generalizations). There are also various results for other domains (see for instance [B], [Z1], [Tem] for bounded symmetric domains, and [Co], [L], [KM] for strictly pseudoconvex domains). But these results are not applicable in the case of $\mathbb{B}_*$.

Usually the definition of the Bloch space depends on the gradient growth. But Timoney showed that we can also characterize it using the radial derivative (see Theorem 4.10 of [Tim1]). In the case of $\mathbb{B}_*$ we will use the following definition.

Let $\mathcal{B}(\mathbb{B}_*)$ be the space of all holomorphic functions f on $\mathbb{B}_*$ such that

$$\|f\|_{\mathcal{B}(\mathbb{B}_*)} = |f(0)| + \sup_{\mathbb{B}_*} \frac{|Rf(z)|}{|z|}|z \bullet z|^{1/2}(1 - N_*^2(z)) < \infty,$$

where

$$Rf(z) = \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}(z) \quad \text{for all } f \in \text{Hol}(\mathbb{B}_*) \text{ and } z \in \mathbb{B}_*.$$

Rf is called the *radial derivative* of f. We easily see that $\|\cdot\|_{\mathcal{B}(\mathbb{B}_*)}$ defines a norm on $\mathcal{B}(\mathbb{B}_*)$. We will prove that $\mathcal{B}(\mathbb{B}_*)$ is a Banach space and that, for all $p \geq 1$, $\mathcal{B}(\mathbb{B}_*) \subset \mathcal{A}^p(\mathbb{B}_*)$ (see Corollary 4.2).

Our main result is the following

THEOREM 1.1. *The dual of the space $\mathcal{A}^1(\mathbb{B}_*)$ can be identified with $\mathcal{B}(\mathbb{B}_*)$. More precisely, there is a bounded bilinear complex form $\Lambda_{\mathbb{B}_*}$ on $\mathcal{A}^1(\mathbb{B}_*) \times \mathcal{B}(\mathbb{B}_*)$ such that every bounded linear functional on $\mathcal{A}^1(\mathbb{B}_*)$ is of the form*

$$f \mapsto \Lambda_{\mathbb{B}_*}(f, g)$$

for some unique $g \in \mathcal{B}(\mathbb{B}_*)$. Furthermore the norm of the linear functional on $\mathcal{A}^1(\mathbb{B}_*)$ is equivalent to the norm of g in $\mathcal{B}(\mathbb{B}_*)$, i.e. there is a positive

constant τ such that

$$\tau \|g\|_{\mathcal{B}(\mathbb{B}_*)} \leq \|A_{\mathbb{B}_*}(\cdot, g)\| \leq \tau^{-1} \|g\|_{\mathcal{B}(\mathbb{B}_*)}$$

for all $g \in \mathcal{B}(\mathbb{B}_*)$.

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2. Preliminaries

NOTATION 2.1. In this paper we denote by $\mathrm{Hol}(X)$ the set of all holomorphic functions on X where X is a complex manifold.

The domain $\mathbb{B}_*$ is strongly related to the hypersurface $\mathbb{M}$ of the unit ball in $\mathbb{C}^{n+1}$ defined by

$$\mathbb{M} = \{z \in \mathbb{C}^{n+1} \setminus \{0\} : z \bullet z = 0, \ |z| < 1\},$$

where $z \bullet w := \sum_{j=1}^{n} z_j w_j$ for z and w in $\mathbb{C}^{n+1}$. Let $\mathrm{Pr} : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be defined by $\mathrm{Pr}(z_1, \ldots, z_n, z_{n+1}) = (z_1, \ldots, z_n)$ and $F = \mathrm{Pr}_{|\mathbb{M}}$. Then $F : \mathbb{M} \to \mathbb{B}_* \setminus \{0\}$ is a proper holomorphic mapping of degree 2. We denote by W the branching locus of F. The image $F(W)$ is an analytic subset of $\mathbb{B}_* \setminus \{0\}$. We set $V := F(W) \cup \{0\}$. The local inverses ϕ and ψ of F are given for $z \in \mathbb{B}_* \setminus V$ by

$$\phi(z) = (z, i\sqrt{z \bullet z}), \qquad \psi(z) = (z, -i\sqrt{z \bullet z}).$$

Set

$$\mathbb{H} = \{z \in \mathbb{C}^{n+1} \setminus \{0\} : z \bullet z = 0\}.$$

It was proved in [OPY] that there is a unique (up to a multiplicative constant) $\mathrm{SO}(n+1, \mathbb{C})$-invariant holomorphic form α on $\mathbb{H}$. The restriction of this form to $\mathbb{H} \cap (\mathbb{C} \setminus \{0\})^{n+1}$ is given by

$$\alpha(z) = \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_{n+1}.$$

DEFINITION 2.2. 1. $L^1(\mathbb{M})$ is the space of measurable complex functions on $\mathbb{M}$ such that

$$\|f\|_{L^1(\mathbb{M})} = \int_{\mathbb{M}} |f(z)| \frac{\alpha(z) \wedge \overline{\alpha(z)}}{C} < \infty \quad \text{where} \quad C := (-1)^{n(n+1)/2}(2i)^n.$$

2. $\mathcal{A}^1(\mathbb{M})$ is the subspace of all holomorphic functions in $L^1(\mathbb{M})$.

All along this note we will use the following important operator.

DEFINITION 2.3. Let $f : \mathbb{B}_* \to \mathbb{C}$ be a measurable function. We define Tf by

$$(Tf)(z) := \frac{1}{2(n+1)^2} z_{n+1} (f \circ F)(z)$$

for all $z = (z_1, \ldots, z_{n+1}) \in \mathbb{M}$.

In [MY] (Lemma 4.1) we proved the following.

LEMMA 2.4. *The operator T is an isometry from $L^1(\mathbb{B}_*, dv(z)/|z \bullet z|^{1/2})$ into $L^1(\mathbb{M})$. More precisely, we have*

$$\int_{\mathbb{M}} |(Tf)(z)| \frac{\alpha(z) \wedge \overline{\alpha(z)}}{C} = \int_{\mathbb{B}_*} |f(w)| \cdot |w \bullet w|^{-1/2} \, dv(w).$$

In addition, the image $\mathcal{E}^1(\mathbb{M})$ of $\mathcal{A}^1(\mathbb{B}_*)$ under T is a closed proper subspace of $\mathcal{A}^1(\mathbb{M})$ and T is a unitary operator from $\mathcal{A}^1(\mathbb{B}_*)$ onto $\mathcal{E}^1(\mathbb{M})$. In particular, $\mathcal{A}^1(\mathbb{B}_*)$ is a Banach space.

REMARK 2.5. In fact we have

$$\mathcal{E}^1(\mathbb{M}) = L^1(\mathbb{M}) \cap T(\mathrm{Hol}(\mathbb{B}_*)).$$

3. An intermediate result. In order to prove the main theorem on $\mathbb{B}_*$ we will establish an intermediate result on $\mathbb{M}$. More precisely we will find the dual space of $\mathcal{E}^1(\mathbb{M})$ (see Theorem 3.14). For this purpose we must first introduce some new analytic spaces.

3.1. *Other analytic spaces on $\mathbb{M}$*

DEFINITION 3.1. Let f be a holomorphic function on $\mathbb{M}$ and $z \in \mathbb{M}$.

1. As usual f_z denotes the *slice function* defined by

$$f_z(\tau) = f(\tau z)$$

for all $\tau \in \mathbb{C}$ such that $0 < |\tau| < 1/|z|$.

2. We set

$$Rf(z) = \frac{\partial f_z}{\partial \tau}\bigg|_{\tau=1}.$$

Rf is called the *radial derivative* of f.

LEMMA 3.2. *Let $f \in \mathrm{Hol}(\mathbb{M})$.*

1. *f can be uniquely extended to the complex hypersurface $\mathbb{M} \cup \{0\}$ so that we can define $f(0)$.*

2. *We have*

$$f(z) = f(0) + \int_0^1 \frac{Rf(tz)}{t} \, dt.$$

Proof. Let $\mathbb{B}_{n+1}$ be the unit ball in $\mathbb{C}^{n+1}$. By the proof of Lemma 3.1 of [MY] there is a function $g \in \mathrm{Hol}(\mathbb{B}_{n+1})$ such that $g_{|\mathbb{M}} = f$.

1. Thus $\lim_{z \to 0, \, z \in \mathbb{M}} f(z)$ exists and we can define $f(0)$.

2. For all $z \in \mathbb{B}_{n+1}$ we can write $g(z) = g(0) + \int_0^1 \frac{\partial g_z}{\partial t}(t) \, dt$. But by Definition 3.1 it is also clear that for all $z \in \mathbb{M}$,

$$\frac{\partial g_z}{\partial t}(t) = \frac{\partial f_z}{\partial t}(t) = \frac{Rf(tz)}{t}. \quad \blacksquare$$

DEFINITION 3.3. $\mathcal{B}(\mathbb{M})$ is the space of all holomorphic functions such that

$$\|f\|_{\mathcal{B}(\mathbb{M})} = |f(0)| + \sup_{z \in \mathbb{M}} \frac{|Rf(z)|}{|z|}(1 - |z|^2) < \infty.$$

From Lemma 3.2 it follows that $\|f\|_{\mathcal{B}(\mathbb{M})}$ is a norm on $\mathcal{B}(\mathbb{M})$.

LEMMA 3.4. *Let* $f \in \mathcal{B}(\mathbb{M})$.

1. *We have*

$$\sup_{z \in \mathbb{M}} |f(z)|(1 - |z|^2) \le 2\|f\|_{\mathcal{B}(\mathbb{M})}.$$

2. *For all* $p > 0$, $f \in L^p(\mathbb{M})$. *The inclusion of* $\mathcal{B}(\mathbb{M})$ *in* $L^p(\mathbb{M})$ *is continuous.*

Proof. 1. By Lemma 3.2 we have

$$|f(z)| \le |f(0)| + \int_0^1 \left| \frac{Rf(tz)}{t} \right| dt$$

$$\le \|f\|_{\mathcal{B}(\mathbb{M})} + \|f\|_{\mathcal{B}(\mathbb{M})} \int_0^1 \frac{t|z|}{t(1 - (t|z|)^2)} \, dt$$

$$\le \|f\|_{\mathcal{B}(\mathbb{M})} + \|f\|_{\mathcal{B}(\mathbb{M})} \int_0^{|z|} \frac{du}{1 - t^2} \, dt$$

$$\le 2\|f\|_{\mathcal{B}(\mathbb{M})}(1 - \ln(1 - |z|^2)).$$

But it is clear that

$$1 - \ln(1 - |z|^2) \le \frac{1}{1 - |z|^2}$$

on $\mathbb{M}$, which completes the proof of assertion 1.

2. Now we fix $p' > p$. Then there is a positive constant $M_{p'}$ such that

$$1 - \ln(1 - |z|^2) \le \frac{M_{p'}}{(1 - |z|^2)^{1/p'}}$$

on $\mathbb{M}$. But integration in polar coordinates (see Lemma 2.1 of [MY]) proves that the function $M_{p'}/(1 - |z|^2)^{1/p'}$ is in $L^p(\mathbb{M})$ since $p/p' < 1$. This leads to assertion 2. $\blacksquare$

DEFINITION 3.5.
$$\mathcal{E}(\mathbb{M}) = \mathcal{B}(\mathbb{M}) \cap T(\mathrm{Hol}(\mathbb{B}_*)).$$

LEMMA 3.6. *Furnished with the Bloch norm, $\mathcal{B}(\mathbb{M})$ and $\mathcal{E}(\mathbb{M})$ are Banach spaces.*

Proof. The fact that $\mathcal{B}(\mathbb{M})$ is a Banach space is a classical consequence of Lemma 3.4. Then it suffices to show that $\mathcal{E}(\mathbb{M})$ is a closed subset of $\mathcal{B}(\mathbb{M})$. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{E}(\mathbb{M})$ tend to f in $\mathcal{B}(\mathbb{M})$. Again by Lemma 3.4 we know that $f_k \to f$ in $L^1(\mathbb{M})$. But by Remark 2.5, $f_k \in \mathcal{E}^1(\mathbb{M})$, and by Lemma 2.4, $\mathcal{E}^1(\mathbb{M})$ is a closed subset of $L^1(\mathbb{M})$ so that $f \in \mathcal{E}^1(\mathbb{M})$. In particular $f \in T(\mathrm{Hol}(\mathbb{B}_*))$. ∎

We will use the following space.

DEFINITION 3.7.
$$\mathcal{E}^\infty(\mathbb{M}) = L^\infty(\mathbb{M}) \cap T(\mathrm{Hol}(\mathbb{B}_*)).$$

LEMMA 3.8. *$\mathcal{E}^\infty(\mathbb{M})$ is dense in $\mathcal{E}^1(\mathbb{M})$.*

Proof. By usual methods we can show that if $f \in L^1(\mathbb{M})$ then $f_r \to f$ in $L^1(\mathbb{M})$ as $r \to 1^-$. But if $f \in \mathcal{E}^1(\mathbb{M})$ then $f_r \in \mathcal{E}^\infty(\mathbb{M})$. ∎

3.2. *The dual space of $\mathcal{E}^1(\mathbb{M})$*

NOTATION 3.9. 1. $\widetilde{P}$ denotes the orthogonal projection from $L^2(\mathbb{M})$ onto $\mathcal{E}^2(\mathbb{M})$ (which is a closed subspace of $L^2(\mathbb{M})$ by Lemma 4.1 of [MY]).

2. $K_{\mathbb{B}_*}$ denotes the Bergman kernel of $\mathbb{B}_*$ (see [OPY] or [MY]).

We recall the following (see Lemma 2 of [M]).

LEMMA 3.10. *The projection $\widetilde{P}$ is an integral operator induced by the kernel*
$$\widetilde{K}(z,w) = \frac{1}{2(n+1)^2} z_{n+1}\overline{w}_{n+1} K_{\mathbb{B}_*}(F(z), F(w))$$

for all z and w in $\mathbb{M}$.

LEMMA 3.11. *The projection $\widetilde{P}$ is a well defined bounded operator from $L^\infty(\mathbb{M})$ into $\mathcal{E}(\mathbb{M})$.*

Proof. In this proof M will denote a constant depending only on n which may differ at each appearance. Let $f \in L^\infty(\mathbb{M})$. Then $\widetilde{P}f \in \mathcal{E}^2(\mathbb{M})$ so that $\widetilde{P}f \in T(\mathrm{Hol}(\mathbb{M}))$. So we must only prove that $\widetilde{P}f \in \mathcal{B}(\mathbb{M})$. We have

$$\widetilde{P}f(z) = \int_{\mathbb{M}} f(w)\widetilde{K}(z,w)\, \frac{\alpha(w) \wedge \overline{\alpha(w)}}{C}.$$

Thus

$$R(\widetilde{P}f)(z) = \int_{\mathbb{M}} f(w) R(z \mapsto \widetilde{K}(z,w))\, \frac{\alpha(w) \wedge \overline{\alpha(w)}}{C}.$$

Using Lemma 4.2 of [MY] we find that, for all z and w in $\mathbb{M}$,

$$\widetilde{K}(z, w) = M[K_{\mathbb{M}}(z, w) - K_{\mathbb{M}}(z, A(w))]$$

where A is the transformation of $\mathbb{C}^{n+1}$ defined by

$$A(z_1, \ldots, z_n, z_{n+1}) = (z_1, \ldots, z_n, -z_{n+1})$$

and $K_{\mathbb{M}}$ is the Bergman kernel of $\mathbb{M}$ with respect to the volume form $\alpha(w) \wedge \overline{\alpha(w)}/C$ (see Theorem 3.2 of [MY]). Thus

$$R(z \mapsto \widetilde{K}(z, w)) = M[R(z \mapsto K_{\mathbb{M}}(z, w)) - R(z \mapsto K_{\mathbb{M}}(z, A(w)))].$$

By Theorem 3.2 of [MY] and a little computation we obtain

$$R(z \mapsto K_{\mathbb{M}}(z, w)) = M \frac{(z \bullet \overline{w})[1 + z \bullet \overline{w}]}{(1 - z \bullet \overline{w})^{n+2}}.$$

Hence

$$|R(z \mapsto K_{\mathbb{M}}(z, w))| \leq M \frac{|z|}{|1 - z \bullet \overline{w}|^{n+2}}.$$

Then Lemma 5.1 of [MY] gives

$$|R(\widetilde{P}f)(z)| \leq M \frac{|z| \cdot \|f\|_{\infty}}{1 - |z|^2}.$$

Since $\widetilde{P}f(0) = 0$, this exactly says that

$$\|\widetilde{P}f\|_{\mathcal{B}(\mathbb{M})} \leq M \|f\|_{\infty}. \quad \blacksquare$$

DEFINITION 3.12. 1. For $f \in \mathrm{Hol}(\mathbb{M})$ and $z \in \mathbb{M}$ we put

$$Qf(z) = (1 - |z|^2)(nf(z) + Rf(z)).$$

2. For all functions f and g measurable in $\mathbb{M}$ we set

$$\langle f, g \rangle_{\mathbb{M}} = \int_{\mathbb{M}} f(w)\overline{g(w)} \, \frac{\alpha(w) \wedge \overline{\alpha(w)}}{C}$$

provided the above integral is defined.

Then we have the following

LEMMA 3.13. 1. *Q is a well defined bounded operator from $\mathcal{B}(\mathbb{M})$ into $L^{\infty}(\mathbb{M})$.*

2. *For all $h \in \mathcal{A}(\mathbb{M}) := L^{\infty}(\mathbb{M}) \cap \mathrm{Hol}(\mathbb{M})$ and $g \in \mathcal{B}(\mathbb{M})$ we have*

$$\langle h, g \rangle_{\mathbb{M}} = \langle h, Qg \rangle_{\mathbb{M}}.$$

Proof. 1. Let $f \in \mathcal{B}(\mathbb{M})$ and $z \in \mathbb{M}$. We have

$$|Qf(z)| \leq n(1 - |z|^2)|f(z)| + (1 - |z|^2)|Rf(z)|$$

and by Lemma 3.4 we obtain

$$|Qf(z)| \leq (2n + 1)\|f\|_{\mathcal{B}(\mathbb{M})}.$$

2. Firstly we note that by Lemma 3.4 and by the first assertion the above scalar products are well defined. Then recall that by the proof of Theorem 3.2 of [MY] we know that if $h \in \mathcal{A}(\mathbb{M})$ then

$$h = \sum_{k=0}^{\infty} p_k$$

in $L^2(\mathbb{M})$ where $p_k \in \mathcal{P}_k(\mathbb{M})$, the set of homogeneous polynomials of degree k on $\mathbb{M}$. So we need only verify the formula for $h = p_k$ and $g \in \mathcal{B}(\mathbb{M})$. But we also know that the sum is orthogonal in $L^2(\mathbb{M})$ and that $g \in L^2(\mathbb{M})$. Thus it suffices to prove that

$$\langle p_k, q_k \rangle_{\mathbb{M}} = \langle p_k, Q q_k \rangle_{\mathbb{M}}$$

for all p_k and q_k in $\mathcal{P}_k(\mathbb{M})$. But $R q_k = k q_k$ so that

$$Q(q_k)(z) = (n + k)(1 - |z|^2) q_k(z).$$

Using integration in polar coordinates (see Lemma 2.1 of [MY]) we can prove the previous equality and the result follows. ∎

Now we can prove the intermediate theorem on $\mathbb{M}$.

THEOREM 3.14. *The dual of the space $\mathcal{E}^1(\mathbb{M})$ can be identified with $\mathcal{E}(\mathbb{M})$. More precisely, there is a bounded bilinear complex form $\Lambda_{\mathbb{M}}$ on $\mathcal{E}^1(\mathbb{M}) \times \mathcal{E}(\mathbb{M})$ such that every bounded linear functional on $\mathcal{E}^1(\mathbb{M})$ is of the form $\Lambda_{\mathbb{M}}(\cdot, g)$ for some unique $g \in \mathcal{B}(\mathbb{B}_*)$. $\Lambda_{\mathbb{M}}$ is defined for all $h \in \mathcal{E}^{\infty}(\mathbb{M})$ and $g \in \mathcal{E}(\mathbb{M})$ by*

$$\Lambda_{\mathbb{M}}(h, g) = \langle h, g \rangle_{\mathbb{M}}.$$

Furthermore the norms of g and $\Lambda_{\mathbb{M}}(\cdot, g)$ are equivalent.

Proof. Firstly by Lemma 3.13 we can write, for all $h \in \mathcal{E}^{\infty}(\mathbb{M})$ and $g \in \mathcal{E}(\mathbb{M})$,

$$|\langle h, g \rangle_{\mathbb{M}}| = |\langle h, Q g \rangle_{\mathbb{M}}| \leq \|h\|_1 \|Q g\|_{\infty} \leq \|h\|_1 \|Q\| \cdot \|g\|_{\mathcal{B}(\mathbb{M})}.$$

So $\Lambda_{\mathbb{M}}$ is bounded on $\mathcal{E}^{\infty}(\mathbb{M}) \times \mathcal{E}(\mathbb{M})$. By Lemma 3.8, $\Lambda_{\mathbb{M}}$ can be uniquely extended to a bounded bilinear form on $\mathcal{E}^1(\mathbb{M}) \times \mathcal{E}(\mathbb{M})$. Lemma 3 of [M] shows that the g as in the statement is unique.

Now let L be a bounded linear functional on $\mathcal{E}^1(\mathbb{M})$. By the Hahn–Banach theorem, L can be extended to a bounded linear functional $\widetilde{L}$ on $L^1(\mathbb{M})$ satisfying $\|L\| = \|\widetilde{L}\|$ and by the Riesz representation theorem we can find a function $f \in L^{\infty}(\mathbb{M})$ such that

$$\widetilde{L}(h) = \int_{\mathbb{M}} h(z) \overline{f(z)} \frac{\alpha(z) \wedge \overline{\alpha(z)}}{C}$$

for all $h \in L^1(\mathbb{M})$ and $\|\widetilde{L}\| = \|f\|_\infty$. By Lemma 3.11 the function $g = \widetilde{P}(f)$ belongs to $\mathcal{E}(\mathbb{M})$ and

$$\|g\|_{\mathcal{E}(\mathbb{M})} \le \|\widetilde{P}\| \cdot \|f\|_\infty = \|\widetilde{P}\| \cdot \|L\|.$$

Then for all $h \in \mathcal{E}^\infty(\mathbb{M})$ we have

$$\int_\mathbb{M} h(z)\overline{\overline{g(z)}} \frac{\alpha(z) \wedge \overline{\alpha(z)}}{C} = \int_\mathbb{M} h(z)\overline{\widetilde{P}(f)} \frac{\alpha(z) \wedge \overline{\alpha(z)}}{C}$$

$$= \int_\mathbb{M} h(z)\overline{\left(\int_\mathbb{M} \widetilde{K}(z,w)f(w)\frac{\alpha(w) \wedge \overline{\alpha(w)}}{C} \right)} \frac{\alpha(z) \wedge \overline{\alpha(z)}}{C}$$

$$= \int_\mathbb{M} \overline{f(w)}\left(\int_\mathbb{M} \widetilde{K}(w,z)h(z)\frac{\alpha(z) \wedge \overline{\alpha(z)}}{C} \right) \frac{\alpha(w) \wedge \overline{\alpha(w)}}{C}$$

$$= \langle h, f \rangle = \widetilde{L}(h) = L(h).$$

Note that the proof of Lemma 4 of [MY] enables us to use Fubini's theorem. The equivalence of the norms is clear. ∎

4. Proof of the main theorem. To prove the main theorem we need the next lemma and its corollary.

LEMMA 4.1. *Let $g \in \text{Hol}(\mathbb{M})$. For all $z \in \mathbb{M}$ set $\widetilde{T}g(z) = z_{n+1}g(z)$ and define*

$$\widetilde{S}(g) = \sup_{z \in \mathbb{M}} \frac{|R(\widetilde{T}g)(z)|}{|z|}(1 - |z|^2),$$

$$S(g) = |g(0)| + \sup_{z \in \mathbb{M}} \frac{|R(g)(z)|}{|z|}(1 - |z|^2)|z_{n+1}|.$$

Then there is a constant γ independent of g such that

$$\gamma^{-1}\widetilde{S}(g) \le S(g) \le \gamma\widetilde{S}(g).$$

Proof. First of all note that

(1) $$R(\widetilde{T}g)(z) = z_{n+1}[g(z) + Rg(z)] \quad \text{for all } z \in \mathbb{M}.$$

Then we recall that $g = \sum_{k=0}^\infty p_k$ for some $p_k \in \mathcal{P}_k$ as in the proof of Lemma 3.13. Thus for all $z \in \mathbb{M}$ we can write

$$Rg(z) = \sum_{k=0}^\infty k p_k(z), \quad \widetilde{T}g(z) = \sum_{k=0}^\infty z_{n+1}p_k(z)$$

and so

$$z_{n+1}Rg(z) = \sum_{k=0}^\infty z_{n+1}k p_k(z).$$

Now suppose that

(2) $$\widetilde{S}(g) = \sup_{z \in \mathbb{M}} |R(\widetilde{T}g)(z)|(1 - |z|^2) < \infty.$$

By (1) we can write

(3) $$S(g) - |g(0)| \leq \widetilde{S}(g) + \sup_{z \in \mathbb{M}} \frac{|\widetilde{T}g(z)|}{|z|}(1 - |z|^2).$$

So it suffices to study the last supremum. Let $z \in \partial\mathbb{M}$ where

$$\partial\mathbb{M} = \{z \in \mathbb{C}^{n+1} : z \bullet z = 0, \ |z| = 1\}.$$

The slice function $(R(\widetilde{T}g))_z$ is holomorphic on $\mathbb{D}$, the unit disc in $\mathbb{C}$. For all $\zeta \in \mathbb{D}$, we have

$$(R(\widetilde{T}g))_z(\zeta) = \sum_{k=0}^{\infty} (k+1)[z_{n+1}p_k(z)]\zeta^{k+1} = \zeta \sum_{k=0}^{\infty} (k+1)a_k(z)\zeta^k$$

with $a_k(z) = z_{n+1}p_k(z)$. But by (2) we have

$$\frac{|(R(\widetilde{T}g))_z(\zeta)|}{|\zeta|} \leq \frac{\widetilde{S}(g)}{1 - |\zeta|^2}.$$

Thus for all $k \in \mathbb{N}$ we get

$$|(k+1)a_k(z)| \leq \widetilde{S}(g)e(k+1).$$

Therefore we have, for all $z \in \partial\mathbb{M}$ and $k \in \mathbb{N}$,

$$|a_k(z)| \leq \widetilde{S}(g)e.$$

Note that, in particular, $|a_0(z)| = |z_{n+1}g(0)| \leq \widetilde{S}(g)e$ for all $z \in \partial\mathbb{M}$. For $z = (1/\sqrt{2}, 0, \ldots, 0, i/\sqrt{2})$ this gives

(4) $$|g(0)| \leq \sqrt{2}\,\widetilde{S}(g)e.$$

We also have

$$\frac{(\widetilde{T}g)_z(\zeta)}{\zeta} = \sum_{k=0}^{\infty} a_k\zeta^k$$

for $z \in \partial\mathbb{M}$ and $\zeta \in \mathbb{D}$. Hence

$$\left|\frac{(\widetilde{T}g)_z(\zeta)}{\zeta}\right| \leq \widetilde{S}(g)e\sum_{k=0}^{\infty}|\zeta|^k \leq \frac{2\widetilde{S}(g)e}{1 - |\zeta|^2}.$$

So we get

$$\sup_{z \in \mathbb{M}} \frac{|\widetilde{T}g(z)|}{|z|}(1 - |z|^2) \leq 2\widetilde{S}(g)e.$$

This combined with (3) and (4) shows that there exists $\gamma > 0$ such that $S(g) \leq \gamma\widetilde{S}(g)$.

We use the same method to establish the other inequality. ∎

COROLLARY 4.2. *T is a linear isomorphism from $\mathcal{B}(\mathbb{B}_*)$ onto $\mathcal{E}(\mathbb{M})$ and there is a constant δ such that for all $f \in \mathcal{B}(\mathbb{B}_*)$,*

$$\delta^{-1}\|f\|_{\mathcal{B}(\mathbb{B}_*)} \leq \|Tf\|_{\mathcal{E}(\mathbb{M})} \leq \delta\|f\|_{\mathcal{B}(\mathbb{B}_*)}.$$

In particular $\mathcal{B}(\mathbb{B}_*)$ is a Banach space and, for all $p \geq 1$, $\mathcal{B}(\mathbb{B}_*) \subset \mathcal{A}^p(\mathbb{B}_*)$.

Proof. In fact the radial derivative of holomorphic functions on $\mathbb{B}_*$ can be defined using slice functions as in Definition 3.1 for holomorphic functions on $\mathbb{M}$. This fact, the previous lemma and the equivalence of the norms $|\cdot|$ and N_* lead to the desired inequalities. This implies that T is a linear isomorphism from $\mathcal{B}(\mathbb{B}_*)$ onto $\mathcal{E}(\mathbb{M})$ and that $\mathcal{B}(\mathbb{B}_*)$ is a Banach space. Then the inclusion $\mathcal{B}(\mathbb{B}_*) \subset \mathcal{A}^p(\mathbb{B}_*)$ follows from Lemma 4.1 of [MY] and Lemma 3.4. ∎

Now we can prove the main result. Consider the bilinear form $\Lambda_{\mathbb{B}_*}$ defined on $\mathcal{A}^1(\mathbb{B}_*) \times \mathcal{B}(\mathbb{B}_*)$ by

$$(f, g) \mapsto \Lambda_{\mathbb{B}_*}(f, g) = \Lambda_{\mathbb{M}}(Tf, Tg).$$

This mapping is well defined and bounded by Lemma 2.4, the previous corollary and Theorem 3.14. Now let L be a bounded linear functional on $\mathcal{A}^1(\mathbb{B}_*)$. Then $L \circ T^{-1}$ is a bounded linear functional on $\mathcal{E}^1(\mathbb{M})$. By Theorem 3.14 there is a function $\widetilde{g} \in \mathcal{E}(\mathbb{M})$ such that $L \circ T^{-1}(\widetilde{f}) = \Lambda_{\mathbb{M}}(\widetilde{f}, \widetilde{g})$ for all $\widetilde{f} \in \mathcal{E}^1(\mathbb{M})$. But $\widetilde{g} = Tg$ for some $g \in \mathcal{B}(\mathbb{B}_*)$. Then for all $f \in \mathcal{A}^1(\mathbb{B}_*)$ we have

$$L(f) = L \circ T^{-1}(Tf) = \Lambda_{\mathbb{M}}(Tf, Tg) = \Lambda_{\mathbb{B}_*}(f, g).$$

Therefore $L = \Lambda_{\mathbb{B}_*}(\cdot, g)$. We also have the uniqueness and the equivalence of the norms. The proof is complete.

PROBLEM 4.3. Theorem 4.10 of [Tim1] shows that the Bloch space of the unit ball can also be characterized by the norm

$$|f(0)| + \sup_{z \in \mathbb{B}} |\nabla f(z)|(1 - |z|^2)$$

where $\nabla f(z)$ is the complex gradient of the holomorphic function f. In the case of $\mathbb{B}_*$ it does not seem to be easy to prove the analogue of Theorem 4.10 of [Tim1]. Thus it would be very interesting to find a similar characterization of $\mathcal{B}(\mathbb{B}_*)$—if it is possible—in terms of the gradient, the norm N_* and the weight $|z \bullet z|^{1/2}$.

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