STUDIA MATHEMATICA 213 (1) (2012)

Projectively invariant Hilbert–Schmidt kernels and convolution type operators

by

JAESEONG HEO (Seoul)

Abstract. We consider positive definite kernels which are invariant under a multiplier and an action of a semigroup with involution, and construct the associated projective isometric representation on a Hilbert C^*-module. We introduce the notion of C^*-valued Hilbert–Schmidt kernels associated with two sequences and construct the corresponding reproducing Hilbert C^*-module. We also discuss projective invariance of Hilbert–Schmidt kernels. We prove that the range of a convolution type operator associated with a Hilbert–Schmidt kernel coincides with the reproducing Hilbert C^*-module associated with its convolution kernel. We show that the integral operator associated with a Hilbert–Schmidt kernel is Hilbert–Schmidt. Finally, we discuss a relation between an integral type operator and convolution type operator.

1. Introduction. The theory of reproducing kernels is fundamental and applicable widely in various areas of mathematics [1, 3, 7, 11, 15]. In particular, reproducing kernel Hilbert spaces have played an important role in operator theory and applications [1, 7, 11]. For example, if a Hilbert space of functions has a reproducing kernel, then the kernel is characterized as the unique solution of an extremal problem [15, 10]. One of the advantages of reproducing kernel Hilbert spaces is that the norm in such spaces is easily computed only on the linear span of the kernels, which is a dense set but not the whole space in general. Schoenberg [16] introduced functions which are positive definite on the m-dimensional sphere. A class of positive definite kernels on a closed compact Riemannian manifold provides some method for solving uniquely a generalized Hermite interpolation problem [12]. Some kernels in [12] can be regarded as an example of Hilbert–Schmidt kernels which we are considering in this paper.

Let $\mathcal{H}$ be a reproducing kernel Hilbert space consisting of functions on a set S. The Riesz representation theorem says that for each element $s \in S$, there is a unique vector $\phi_s \in \mathcal{H}$ such that $f(s) = (f, \phi_s)$ for all $f \in \mathcal{H}$.

2010 *Mathematics Subject Classification*: Primary 46E22, 46C50; Secondary 46L08, 46N55.
Key words and phrases: C^*-valued positive definite kernel, C^*-valued Hilbert–Schmidt kernel, convolution type operator, convolution kernel, reproducing kernel Hilbert C^*-module.

DOI: 10.4064/sm213-1-5

The function κ defined by $\kappa(s,t) = \phi_s(t)$ is called the *reproducing kernel* for $\mathcal{H}$. Moreover, if $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis for H, then the kernel κ is expressed by $\kappa(s,t) = \sum_{n\in\mathbb{N}} \phi_n(s)\overline{\phi_n(t)}$. In this paper, we consider C^*-valued kernels associated with a sequence of positive numbers as a quantization of such kernels. We will call such a kernel a *C^*-valued Hilbert–Schmidt kernel*. We discuss positive definite (Hilbert–Schmidt) kernels which are projectively invariant under a multiplier and an action of a semigroup with involution.

The contents of the sections are as follows. In Section 2, we discuss positive definite von Neumann algebra-valued kernels which are projectively invariant under an action of a semigroup with involution. We construct a projective isometric representation on a Hilbert W^*-module associated to a unitary multiplier with a 2-cocycle property. In the third section, we construct a concrete reproducing Hilbert C^*-module associated with a C^*-valued Hilbert–Schmidt kernel and discuss projective invariance of Hilbert–Schmidt kernels. Some example is given for a countably generated Hilbert C^*-module. In the fourth section, we discuss C^*-valued kernels given by convolution of C^*-valued kernels. We prove that the range of a convolution type operator associated with a C^*-valued kernel is contained in the reproducing Hilbert C^*-module associated with the kernel. The range of a Hilbert–Schmidt kernel coincides with the reproducing Hilbert C^*-module associated with its convolution kernel. In the last section, we show that the integral operator associated with a Hilbert–Schmidt kernel is again Hilbert–Schmidt. Finally, we discuss an inner product of convolution type operators in the space of square summable sequences on a countable discrete group.

2. Hilbert C^*-modules and positive definite kernels. Let $\mathcal{A}$ be a C^*-algebra. A right $\mathcal{A}$-module X is called a (*right*) *pre-Hilbert $\mathcal{A}$-module* if there is an $\mathcal{A}$-valued mapping $\langle\cdot,\cdot\rangle_X : X \times X \to \mathcal{A}$ which is linear in the second variable and has the following properties:

(i) $\langle x,x\rangle_X \geq 0$, and equality holds only if $x = 0$;
(ii) $\langle x,y\rangle_X = \langle y,x\rangle_X^*$;
(iii) $\langle x,y\cdot b\rangle_X = \langle x,y\rangle_X b$.

If, in addition, X is complete with respect to the norm $\|x\| = \|\langle x,x\rangle_X\|^{1/2}$, then X is called a (*right*) *Hilbert $\mathcal{A}$-module*.

Let X and Y be Hilbert $\mathcal{A}$-modules. We denote by $\mathcal{L}_{\mathcal{A}}(X,Y)$ the set of all right $\mathcal{A}$-module maps $T : X \to Y$ for which there is an operator $T^* : Y \to X$, called the *adjoint* of T, such that

$$\langle Tx,y\rangle_Y = \langle x,T^*y\rangle_X \quad \text{for } x \in X \text{ and } y \in Y.$$

It follows from the uniform boundedness theorem that each operator T in $\mathcal{L}_{\mathcal{A}}(X,Y)$ is bounded. We write $\mathcal{L}_{\mathcal{A}}(X)$ for $\mathcal{L}_{\mathcal{A}}(X,X)$, which becomes a

C^*-algebra with the operator norm. For $y \in Y$ and $x \in X$, the operator $\theta_{y,x} : X \to Y$ defined by $\theta_{y,x}(z) = y\langle x, z\rangle_X$ is called a *rank one operator*. We denote by $\mathcal{K}_{\mathcal{A}}(X, Y)$ the closed linear span of all rank one operators and call elements in $\mathcal{K}_{\mathcal{A}}(X, Y)$ *compact operators*.

By a *representation* of a C^*-algebra $\mathcal{B}$ on a Hilbert $\mathcal{A}$-module X, we mean a $*$-homomorphism π from $\mathcal{B}$ into $\mathcal{L}_{\mathcal{A}}(X)$.

Let X' be the set of all bounded $\mathcal{A}$-module maps of X into $\mathcal{A}$. We call a Hilbert $\mathcal{A}$-module X *self-dual* if $X \simeq X'$, that is, every bounded $\mathcal{A}$-module map $f : X \to \mathcal{A}$ is of the form $\langle x_f, \cdot\rangle$ for an element $x_f \in X$. One pleasant property of self-dual Hilbert $\mathcal{A}$-modules is that every bounded module map between two such modules has an adjoint. Note that self-dual modules have some properties in common with both Hilbert spaces and von Neumann algebras [13]. For detailed information on Hilbert C^*-modules, we refer to [9].

Example 2.1. The following are typical examples of Hilbert C^*-modules.

(1) Hilbert $\mathbb{C}$-modules are Hilbert spaces over $\mathbb{C}$ with scalar multiplication and inner product which is linear in the second variable.
(2) Every C^*-algebra $\mathcal{A}$ itself becomes a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b\rangle = a^*b$ and the usual multiplication in $\mathcal{A}$.
(3) Let $E = (E^0, E^1, r, s)$ be a directed graph where E^0 is the set of vertices, E^1 is the set of edges, r is the range map and s is the source map. We denote by $\mathcal{A}$ the C^*-algebra $C_0(E^0)$ of continuous functions $f : E^0 \to \mathbb{C}$ vanishing at infinity. We denote by $C_c(E^1)$ the space of continuous functions $x : E^1 \to \mathbb{C}$ with finite support. On the space $C_c(E^1)$ we define multiplication and inner product by
$$(x \cdot f)(e) = x(e)x(s(e)) \quad \text{and} \quad \langle x, y\rangle(v) = \sum_{\{e \in E^1 : s(e) = v\}} \overline{x(e)}y(e).$$
Then we can obtain a Hilbert $\mathcal{A}$-module X by completing the space $C_c(E^1)$.
(4) Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ which is linear in the second variable. The algebraic tensor product $\mathcal{A} \otimes \mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle$ defined on elementary tensors as follows:
$$\langle \xi \otimes a, \eta \otimes b\rangle = \langle \xi, \eta\rangle_{\mathcal{H}} a^*b.$$
Then the norm closure of $\mathcal{A} \otimes \mathcal{H}$ becomes a Hilbert $\mathcal{A}$-module. In particular, if $L^2(\Omega, \mu)$ is a Hilbert space of square-integrable functions on Ω where (Ω, μ) is a measure space and if $\mathcal{A}$ is a C^*-algebra, then we denote by $L^2_{\mathcal{A}}(\Omega, \mu)$ the norm closure of $\mathcal{A} \otimes L^2(\Omega, \mu)$. If $\mathcal{H}$ is the Hilbert space $l^2(\mathbb{N})$ of square-integrable sequences, we will denote by $l^2(\mathcal{A})$ the norm closure of $\mathcal{A} \otimes l^2(\mathbb{N})$.

Throughout this paper, Ω, $\mathcal{A}$ and X denote a non-empty set, a C^*-algebra and a Hilbert $\mathcal{A}$-module, respectively, unless specified otherwise.

In numerical analysis, functions of the form $\phi = \sum_{j=1}^{n} \lambda_j \Phi(\cdot, w_j)$ are useful for approximation, where $\{w_1, \ldots, w_n\}$ is a data set and Φ is a kernel function, when dealing with data dependent spaces of functions of many variables. With this motivation we start with a short review of C^*-valued kernels.

A kernel $\kappa : \Omega \times \Omega \to \mathcal{A}$ is *positive definite* if for any $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathcal{A}$ and $w_1, \ldots, w_n \in \Omega$, the sum $\sum_{i,j=1}^{n} a_i^* \kappa(w_i, w_j) a_j$ is positive in $\mathcal{A}$. Let X be the self-dual Hilbert $\mathcal{A}$-module generated by $\mathcal{A}$-valued bounded functions on a topological space Ω. Assume that each evaluation map $\mathrm{ev}_w : \psi \mapsto \psi(w) \in \mathcal{A}$ is continuous, that is, $\|\mathrm{ev}_w(\psi)\|_{\mathcal{A}} \le L_w \|\psi\|_X$ for some constant $L_w > 0$. Then, for each $\mathcal{A}$-valued function ψ and for any $w \in \Omega$ there exists an element $\phi_w \in X$ such that $\psi(w) = \langle \phi_w, \psi \rangle_X$. An $\mathcal{A}$-valued kernel $\kappa : \Omega \times \Omega \to \mathcal{A}$ defined by $\kappa(w, v) = \langle \phi_w, \phi_v \rangle_X \in \mathcal{A}$ is called a *reproducing kernel* on Ω. In this case, we denote by $\kappa(\cdot, v)$ the function $\phi_v \in X$.

The theorem below says that any C^*-valued positive definite kernel κ on a non-empty set Ω can be a reproducing kernel of a Hilbert C^*-module. It has been proved in [2, Proposition 3.1.3] and independently in [8, Theorem 3.2]. Moreover, a number of interesting results concerning the structure of type I product systems of Hilbert modules are given in [2].

THEOREM 2.2. *Let Ω be a non-empty set and let $\mathcal{A}$ be a C^*-algebra. If a kernel $\kappa : \Omega \times \Omega \to \mathcal{A}$ is positive definite, then there exists a Hilbert $\mathcal{A}$-module X_κ of $\mathcal{A}$-valued functions on Ω such that κ is the reproducing kernel of X_κ.*

We call the Hilbert C^*-module X_κ in Theorem 2.2 *a reproducing Hilbert $\mathcal{A}$-module associated with κ*. We can see from the construction of X_κ that the $\mathcal{A}$-submodule generated by the set $\{\kappa(\cdot, w) : w \in \Omega\}$ is dense in X_κ.

In the remainder of this section, we consider von Neumann algebra valued kernels projectively invariant under semigroup actions. Constantinescu and Gheondea [4] studied scalar-valued Hermitian kernels which are projectively invariant under an action of a semigroup with involution.

Let $\mathcal{S}$ be a unital semigroup with an involution J, that is, $J(J(s)) = s$ and $J(st) = J(t)J(s)$ for every $s, t \in \mathcal{S}$. We denote by θ an action of $\mathcal{S}$ on Ω, which means that

$$\theta(s, \theta(t, v)) = \theta(st, v) \quad \text{and} \quad \theta(e, v) = v$$

where e is the unit element of $\mathcal{S}$. Let $\mathcal{M}$ be a von Neumann algebra with center $\mathcal{Z}(\mathcal{M})$ and let $\alpha : \mathcal{S} \times \Omega \to \mathcal{U}(\mathcal{Z}(\mathcal{M}))$ be a map satisfying

$$\alpha(st, v)\alpha(st, w)^* = \alpha(s, \theta(t, v))\alpha(s, \theta(t, w))^* \alpha(t, v)\alpha(t, w)^* \tag{2.1}$$

where $\mathcal{U}(\mathcal{Z}(\mathcal{M}))$ is the set of unitaries in the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$. Then we see that

$$\sigma(s,t) = \alpha(s,\theta(t,v))^*\alpha(t,v)^*\alpha(st,v) \tag{2.2}$$

does not depend on v. Moreover, σ has the 2-cocycle property:

$$\sigma(r,s)\sigma(rs,t) = \sigma(r,st)\sigma(s,t) \quad \text{for all } r,s,t \in \mathcal{S}. \tag{2.3}$$

By assuming that $\alpha(sJ(s),v) = 1$ for all $s \in \mathcal{S}$ and $v \in \Omega$, we get the equality

$$\sigma(s,e) = \sigma(e,s) = 1 \quad \text{for any } s \in \mathcal{S}. \tag{2.4}$$

With the notation and the assumption as before, we introduce the following definition. See [6, 4] for scalar-valued versions.

Definition 2.3. Let a unital semigroup $\mathcal{S}$ with an involution J act on a set Ω by θ.

(i) A *$\mathcal{U}(\mathcal{M})$-multiplier on $\mathcal{S}$* is a function $\sigma : \mathcal{S} \times \mathcal{S} \to \mathcal{U}(\mathcal{Z}(\mathcal{M}))$ satisfying (2.3) and (2.4).

(ii) A *projective isometric σ-representation* of $\mathcal{S}$ is a map $W : \mathcal{S} \to \mathcal{M}$, $s \mapsto W_s$, having the following properties:

  (a) W_s is an isometry for each $s \in \mathcal{S}$.

  (b) $W_{st} = \sigma(s,t)W_sW_t$ for all $s,t \in \mathcal{S}$.

(iii) A Hermitian kernel $\kappa : \Omega \times \Omega \to \mathcal{M}$ is *projectively invariant* if

$$\kappa(v,\theta(s,w)) = \alpha(s,w)\alpha(s,\theta(J(s),v))^*\kappa(\theta(J(s),v),w) \tag{2.5}$$

for all $s \in \mathcal{S}$ and $v,w \in \Omega$.

Let $\mathcal{S}$ be a unital semigroup with involution J and let θ be an action of $\mathcal{S}$ on a set Ω such that $\theta(J(s)s,v) = v$ for all $s \in \mathcal{S}$ and $v \in \Omega$. Suppose that $\alpha : \mathcal{S} \times \Omega \to \mathcal{U}(\mathcal{Z}(\mathcal{M}))$ is a map satisfying (2.1) and $\alpha(J(s)s,v) = 1$ for all $s \in \mathcal{S}$, $v \in \Omega$, and let σ be given by (2.2).

Theorem 2.4. *Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{S}$ be a unital semigroup with involution J. Assume that θ and α are as above. If a positive definite kernel $\kappa : \Omega \times \Omega \to \mathcal{M}$ is projectively invariant, then there exists a Hilbert $\mathcal{M}$-module Y_κ and a projective isometric σ-representation $W : \mathcal{S} \to \mathcal{L}_{\mathcal{M}}(Y_\kappa)$ such that*

$$\kappa(v,w) = \langle \varphi_v, \varphi_w \rangle \quad \textit{and} \quad \varphi_{\theta(s,v)} = \alpha(s,v)W_s\varphi_v, \tag{2.6}$$

where each φ_v is an $\mathcal{M}$-valued map given by $\varphi_v(w) = \kappa(v,w)$.

Proof. By Theorem 2.2, we can construct a Hilbert $\mathcal{M}$-module Y_κ and a map $\varphi : \Omega \to \mathcal{M}$ such that $\kappa(v,w) = \langle \varphi_v, \varphi_w \rangle$ for all $v,w \in \Omega$. It follows from (2.1) and $\alpha(J(s)s,v) = 1$ ($s \in \mathcal{S}$) that

$$\beta(s) := \alpha(s,v)\alpha(J(s),\theta(s,v))$$

does not depend on v. Let $\Omega_\varphi = \{\varphi_v : v \in \Omega\}$. For any $s \in \mathcal{S}$, we define a map $W_s : \Omega_\varphi \to \mathcal{M}$ by

$$W_s\varphi_v = \alpha(s,v)^*\varphi_{\theta(s,v)}$$

and extend linearly. Then W_s is adjointable and its adjoint is given by $W_s^* = \beta(s)W_{J(s)}$ for $s \in \mathcal{S}$.

For any $s \in \mathcal{S}$ and $v, w \in \Omega$, we have

$$\begin{aligned}\langle W_s\varphi_v, W_s\varphi_w\rangle &= \langle \alpha(s,v)^*\varphi_{\theta(s,v)}, \alpha(s,w)^*\varphi_{\theta(s,w)}\rangle \\ &= \alpha(s,v)\langle \varphi_{\theta(s,v)}, \varphi_{\theta(s,w)}\rangle\alpha(s,w)^* \\ &= \alpha(s,v)\kappa(\theta(s,v),\theta(s,w))\alpha(s,w)^* = \langle \varphi_v, \varphi_w\rangle,\end{aligned}$$

which implies that W_s is an isometry. Moreover, for any $s, t \in \mathcal{S}$ we obtain

$$\begin{aligned}W_sW_t\varphi_v &= \alpha(t,v)^*W_s\alpha(s,\theta(t,v))^*\varphi_{\theta(s,\theta(t,v))} \\ &= \alpha(t,v)^*W_s\alpha(s,\theta(t,v))^*\varphi_{\theta(st,v)} \\ &= \alpha(t,v)^*W_s\alpha(s,\theta(t,v))^*\alpha(st,v)W_{st}\varphi_v = \sigma(s,t)W_{st}\varphi_v\end{aligned}$$

where the third equality follows from the definition of W. This completes the proof. ∎

We assume that $\alpha(s,\theta(J(s),v))\alpha(J(s),v) = 1$ for all $s \in \mathcal{S}$ and $v \in \Omega$. Then we deduce from (2.1) that $\alpha(sJ(s)) = 1$ for any s. Moreover, if κ is projectively invariant, then $W_s^* = W_{J(s)}$ for any $s \in \mathcal{S}$ where W is as in Theorem 2.4.

3. C^*-valued Hilbert–Schmidt kernels. To study the well-posedness of a generalized Hermite interpolation problem, Narcowich [12] discussed kernels $\kappa \in L^2(M^m \times M^m)$ having an eigenfunction expansion of the form

$$\kappa(p,q) = \sum_{j=1}^{\infty} a_jF_j(p)\bar{F}_j(q) \quad \text{with} \quad \sum_{j=1}^{\infty} |a_j|^2 < \infty \tag{3.1}$$

where M^m is a closed, compact, connected, orientable, m-dimensional C^∞ Riemannian manifold and $\{F_j\}$ is the set of eigenfunctions corresponding to eigenvalues of the Laplace–Beltrami operator. The positivity of all coefficients of the kernel κ in (3.1) implies the positive definiteness of the kernel [12]. For example, the heat kernel κ_t for a manifold M^m is of such form:

$$\kappa_t(p,q) = \sum_{j=1}^{\infty} e^{-\lambda_j t}F_j(p)\bar{F}_j(q).$$

Hence the heat kernel κ_t in $C^\infty(M^m \times M^m)$ is a positive definite kernel on M^m.

We now define a Hilbert–Schmidt kernel. Let $\mathcal{A}$ be a unital C^*-algebra and let $\{r_n : n \in \mathbb{N}\}$ be a sequence of positive real numbers. Consider a sequence of $\mathcal{A}$-valued functions $\phi_n : \Omega \to \mathcal{A}$ ($n \in \mathbb{N}$) with the following properties:

(a) For all $w \in \Omega$, $\sum_{n\in\mathbb{N}} r_n\|\phi_n(w)\|^2 < \infty$.
(b) Any finite subset of $\{\phi_n : n \in \mathbb{N}\}$ is $\mathcal{A}$-linearly independent over Ω, that is, for any finite subset $\Lambda \subset \mathbb{N}$ and given sequence $\{a_n\}_{n\in\Lambda}$ in $\mathcal{A}$, the equality $\sum_{n\in\Lambda} \phi_n(w)a_n = 0$ for all $w \in \Omega$ implies $a_n = 0$ for all $n \in \Lambda$.

Definition 3.1. An $\mathcal{A}$-valued kernel κ on Ω given by

$$\kappa(v,w) = \sum_{n\in\mathbb{N}} r_n\phi_n(v)^*\phi_n(w) \qquad (v,w \in \Omega) \tag{3.2}$$

is called the *$\mathcal{A}$-valued Hilbert–Schmidt kernel associated with $\{r_n\}$ and $\{\phi_n\}$*.

Condition (a) implies that the kernel κ is absolutely summable, so that it exists for all $v, w \in \Omega$ and it is obvious that κ is positive definite. We consider the space

$$\mathfrak{X} = \Big\{\sum_{n=1}^{k} \phi_n(\cdot)a_n : a_n \in \mathcal{A},\ k \in \mathbb{N}\Big\} \tag{3.3}$$

of $\mathcal{A}$-valued functions on Ω. By condition (b) on linear independence, all finite combinations of ϕ_n's have unique coefficients in $\mathcal{A}$. Hence we can define an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathfrak{X}}$ on $\mathfrak{X}$ as follows:

$$\Big\langle \sum_{n=1}^{k} \phi_n(\cdot)a_n, \sum_{n=1}^{l} \phi_n(\cdot)b_n \Big\rangle_{\mathfrak{X}} = \sum_{n=1}^{\min\{k,l\}} \frac{a_n^*b_n}{r_n}.$$

It immediately follows from the definition that

$$\langle \phi_n(\cdot), \phi_m(\cdot)\rangle_{\mathfrak{X}} = \frac{\delta_{nm}}{r_n} 1_{\mathcal{A}},$$

so that $\mathfrak{X}$ becomes a pre-Hilbert $\mathcal{A}$-module with the $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathfrak{X}}$. Furthermore, we have the norm

$$\Big\|\sum_{n=1}^{k} \phi_n(\cdot)a_n\Big\|_{\mathfrak{X}} = \Big\|\sum_{n=1}^{k} \frac{a_n^*a_n}{r_n}\Big\|^{1/2}.$$

We see that $(\sqrt{r_n}\phi_n(w))_{n\in\mathbb{N}}$ is (absolutely) Bochner square-integrable for each $w \in \Omega$. Note that this X is in fact isomorphic to the Hilbert C^*-module $l^2(\mathcal{A})$ defined in Example 2.1. Moreover, $l^2(\mathcal{A})$ is self-dual if and only if $\mathcal{A}$ is finite-dimensional [5]. The following theorem says that any Hilbert–Schmidt kernel has a reproducing Hilbert $\mathcal{A}$-module which is spanned by a set of the form (3.3). We denote by X the completion of $\mathfrak{X}$ with respect to the norm $\|\cdot\|_{\mathfrak{X}}$.

THEOREM 3.2. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\{r_n\}$ and $\{\phi_n\}$ be sequences satisfying conditions* (a) *and* (b). *If κ is the $\mathcal{A}$-valued Hilbert–Schmidt kernel associated with $\{r_n\}$ and $\{\phi_n\}$, then X is the reproducing Hilbert $\mathcal{A}$-module associated with κ.*

Proof. Under condition (a) that $\sum_{n\in\mathbb{N}} r_n\|\phi_n(w)\|^2 < \infty$ for all $w \in \Omega$, it follows from the Cauchy–Schwarz inequality that the infinite sum defining

$$\kappa(v,w) = \sum_{n\in\mathbb{N}} r_n\phi_n(v)^*\phi_n(w)$$

converges for all $(v,w) \in \Omega \times \Omega$. Thus, the Hilbert–Schmidt kernel is well-defined. From the above construction, we see that X is the set of all series of the form $\sum_{n\in\mathbb{N}} \phi_n(\cdot)a_n$ where (a_n) ranges over all sequences with $\sum_{n\in\mathbb{N}} r_n^{-1}\|a_n\|^2 < \infty$. By the $\mathcal{A}$-linear independence of $\{\phi_n\}$, the sequence can be regarded as the function $\sum_{n\in\mathbb{N}} \phi_n(\cdot)a_n \in X$. Thus, the function $\kappa(\cdot,w)$ has a representation $\sum_{n\in\mathbb{N}} \phi_n(\cdot)^*a_n$ where $a_n = r_n\phi_n(w)$ since

$$\sum_{n\in\mathbb{N}} \frac{\|a_n\|^2}{r_n} = \sum_{n\in\mathbb{N}} r_n\|\phi_n(w)\|^2 < \infty.$$

For any element $\sum_{n=1}^{k} \phi_n(\cdot)a_n \in \mathfrak{X}$ and $w \in \Omega$, we obtain

$$\Big\langle \kappa(\cdot,v), \sum_{n=1}^{k} \phi_n(\cdot)^*a_n \Big\rangle_{\mathfrak{X}} = \sum_{n=1}^{k} \frac{r_n\phi_n(w)^*a_n}{r_n} = \sum_{n=1}^{k} \phi_n(v)^*a_n.$$

Hence, we have the equation $\langle \kappa(\cdot,v), f\rangle_{\mathfrak{X}} = f(v)$ for every $f \in \mathfrak{X}$ and $v \in \Omega$, so that any ϕ in X can be regarded as an $\mathcal{A}$-valued function on Ω by $\langle \kappa(\cdot,v), \phi\rangle_{\mathfrak{X}} = \phi(v)$ for all $v \in \Omega$. For any v,w in Ω, we have

$$\langle \kappa(\cdot,v), \kappa(\cdot,w)\rangle_{\mathfrak{X}} = \sum_{n\in\mathbb{N}} r_n\phi_n(v)^*\phi_n(w) = \kappa(v,w),$$

where the second equality follows from the joint continuity of the inner product. Therefore, X is a reproducing Hilbert $\mathcal{A}$-module associated with the reproducing kernel κ. ∎

A Hilbert $\mathcal{A}$-module X is called *countably generated* if there is a countable set $\{x_n\}_{n=1}^{\infty}$ in X such that the linear span of $\{x_na : a \in \mathcal{A},\ n = 1,2,\ldots\}$ is dense in X. When $\mathcal{A}$ is unital, a set $\{x_n\}_{n\in\mathbb{I}}$ in X is called *orthonormal* if $\langle x_n, x_m\rangle = \delta_{nm}1_{\mathcal{A}}$. A set $\{x_n\}_{n\in\mathbb{I}}$ in X is called a *basis* of X if

(i) $\|x_n\| = 1$ for all $n \in \mathbb{I}$,
(ii) finite sums of the form $\sum_n x_na_n$ are dense in X,

(iii) an $\mathcal{A}$-linear combination $\sum_{n\in\mathbb{J}} x_n a_n$ with $\mathbb{J} \subseteq \mathbb{I}$ is equal to 0 if and only if every summand $x_n a_n$ is equal to 0 for $n \in \mathbb{J}$.

EXAMPLE 3.3. Let X be a countably generated Hilbert $\mathcal{A}$-module with an orthonormal basis $\{x_n : n \in \mathbb{N}\}$ and let $\{r_n : n \in \mathbb{N}\}$ be a sequence of positive real numbers. Suppose that $\sum_n r_n \|\langle x_n, x\rangle\|^2 < \infty$ for all $x \in X$. For each $n \in \mathbb{N}$, let $\phi_n : X \to \mathcal{A}$ be the $\mathcal{A}$-valued map given by $\phi_n(y) = \langle x_n, y\rangle$. Now we define an $\mathcal{A}$-valued kernel $\kappa : X \times X \to \mathcal{A}$ by

$$\kappa(x, y) = \sum_{n=1}^{\infty} r_n \langle x_n, x\rangle^* \langle x_n, y\rangle \qquad (x, y \in X).$$

It follows from the Cauchy–Schwarz inequality that $\sum r_n \langle x_n, x\rangle^* \langle x_n, y\rangle$ is absolutely summable, so that the kernel κ is well-defined. The orthogonality of $\{x_n\}$ implies that $\{\phi_n\}$ satisfies condition (b) in the definition of an $\mathcal{A}$-valued Hilbert–Schmidt kernel.

By taking $\Omega = X$ in Definition 3.1, we can see that κ is an $\mathcal{A}$-valued Hilbert–Schmidt kernel associated with the sequences $\{r_n\}$ and $\{\phi_n\}$. Let $\mathfrak{X}_\kappa = \big\{\sum_{n=1}^{k} \langle x_n, \cdot\rangle a_n : a_n \in \mathcal{A},\ k \in \mathbb{N}\big\}$. If we define an $\mathcal{A}$-valued inner product on $\mathfrak{X}_\kappa$ by

$$\Big\langle \sum_{n=1}^{k} \langle x_n, \cdot\rangle a_n, \sum_{n=1}^{l} \langle x_n, \cdot\rangle b_n \Big\rangle_{\mathfrak{X}_\kappa} = \sum_{n=1}^{\min\{k,l\}} \frac{a_n^* b_n}{r_n},$$

then $\mathfrak{X}_\kappa$ is a pre-Hilbert $\mathcal{A}$-module. By Theorem 3.2, the reproducing Hilbert $\mathcal{A}$-module X_κ is the completion of the space $\mathfrak{X}_\kappa$ with respect to the norm induced by the above inner product.

REMARK. Let κ be the $\mathcal{A}$-valued Hilbert–Schmidt kernel associated with sequences $\{r_n\}$ and $\{\phi_n\}$ and let X_κ be its reproducing Hilbert $\mathcal{A}$-module. The set $\{\kappa(\cdot, w) : w \in \Omega\}$ is $\mathcal{A}$-linearly independent over Ω if and only if for any pairwise distinct $w_1, \ldots, w_N \in \Omega$ and $a_1, \ldots, a_N \in \mathcal{A}$, the equation $\sum_{i=1}^{N} a_i \phi(w_i) = 0$ for all $\phi \in X_\kappa$ implies that $a_i = 0$ for all $i = 1, \ldots, N$. Indeed, for any finite $\Lambda \subset \mathbb{N}$, the equation $\sum_{n\in\Lambda} \kappa(\cdot, w_n) a_n = 0$ is equivalent to the fact that $\langle \sum_{n\in\Lambda} \kappa(\cdot, w_n) a_n, \phi\rangle = \sum_{n\in\Lambda} a_n^* \phi(w_n) = 0$.

If for any finite subset $\{w_1, \ldots, w_n\}$ in Ω, there are elements $\psi_1, \ldots, \psi_n$ in X_κ such that $\psi_i(w_j) = \delta_{ij} 1_{\mathcal{A}}$, then any finite subset of $\{\kappa(\cdot, w) : w \in \Omega\}$ is $\mathcal{A}$-linearly independent over Ω. Furthermore, the functions in X_κ separate the points. Indeed, take $N = 1$ and $a_1 = 1_{\mathcal{A}}$, $a_2 = -1_{\mathcal{A}}$. We see that for any two distinct $t_1, t_2 \in \Omega$ there must exist a $\phi \in X_\kappa$ so that $\phi(t_1) \neq \phi(t_2)$.

Let $\mathcal{M}$ be a von Neumann algebra with center $\mathcal{Z}(\mathcal{M})$ and let S, θ and α be as in Definition 2.3. A map $\phi : \Omega \to \mathcal{M}$ is called *projectively invariant under θ and α* if

$$\phi(\theta(s, v)) = \alpha(s, v)\phi(v) \quad \text{for all } s \in \mathcal{S} \text{ and } v \in \Omega. \tag{3.4}$$

In the following proposition, we use the same notation as in Section 2.

PROPOSITION 3.4. *Let $\mathcal{M}$ be a von Neumann algebra with center $\mathcal{Z}(\mathcal{M})$ and let S, θ and α be as in Definition* 2.3. *Suppose that*

$$\alpha(s, \theta(J(s), v))\alpha(J(s), v) = 1$$

for all $s \in \mathcal{S}$ and $v \in \Omega$. If each $\phi_n : \Omega \to \mathcal{M}$ is projectively invariant under θ and α, then the Hilbert–Schmidt kernel $k : \Omega \times \Omega \to \mathcal{M}$ given by (3.2) *is also projectively invariant.*

Proof. Assume that each $\phi_n : \Omega \to \mathcal{M}$ is projectively invariant under θ and α. Let $s \in \mathcal{S}$ and $v, w \in \Omega$. By the definition of a Hilbert–Schmidt kernel, we have

$$\kappa(v, \theta(s, w)) = \alpha(s, w) \sum_n r_n \phi_n(v)^* \phi_n(w),$$
$$\kappa(\theta(J(s), v), w) = \alpha(J(s), v)^* \sum_n r_n \phi_n(v)^* \phi_n(w).$$

It follows from $\alpha(s, \theta(J(s), v))\alpha(J(s), v) = 1$ that

$$\kappa(v, \theta(s, w)) = \alpha(s, w)\alpha(s, \theta(J(s), v))^* \kappa(\theta(J(s), v), w),$$

which implies that κ is projectively invariant. ■

4. C^*-valued convolution type operators. In this section, we denote by (Ω, μ) a measure space with a positive measure μ. Let $L^2(\Omega, \mu)$ be the Hilbert space of square-integrable functions on Ω and let $\mathcal{A}$ be a unital C^*-algebra. Recall that $L^2_{\mathcal{A}}(\Omega, \mu)$ is the norm closure of $\mathcal{A} \otimes L^2(\Omega, \mu)$ with the $\mathcal{A}$-valued inner product defined in Example 2.1. Let $\mathfrak{S}(\Omega, \mathcal{A})$ be the set of $\mathcal{A}$-valued integrable simple functions on Ω. Then $\mathfrak{S}(\Omega, \mathcal{A})$ may be identified with the subspace $\mathcal{A} \otimes \mathfrak{S}(\Omega)$ of $\mathcal{A} \otimes L^2(\Omega, \mu)$ where $\mathfrak{S}(\Omega)$ is the set of integrable simple functions on Ω and $\mathcal{A} \otimes \mathfrak{S}(\Omega)$ is the algebraic tensor product. It follows from the density of $\mathfrak{S}(\Omega)$ in $L^2(\Omega, \mu)$ that $\mathfrak{S}(\Omega, \mathcal{A})$ is dense in $L^2_{\mathcal{A}}(\Omega, \mu)$. For any $f \in \mathfrak{S}(\Omega, \mathcal{A})$, we have

$$\|f\|_2^2 = \Big\| \int_\Omega f(w)^* f(w)\, d\mu(w) \Big\| \le \int_\Omega \|f(w)\|^2\, d\mu(w)$$

where the integral $\int_\Omega f(w)^* f(w)\, d\mu(w)$ is in the sense of Bochner. Hence, the set $L^2(\Omega, \mathcal{A})$ of square-integrable functions is contained in $L^2_{\mathcal{A}}(\Omega, \mu)$. The Hilbert $\mathcal{A}$-module $L^2_{\mathcal{A}}(\Omega, \mu)$ can be obtained by completing $L^2(\Omega, \mathcal{A})$ with respect to the norm given by $\|f\|_2$. The following lemma says that there is an easy way to embed a reproducing Hilbert $\mathcal{A}$-module into a space $L^2_{\mathcal{A}}(\Omega, \mu)$.

LEMMA 4.1. *Let $\kappa : \Omega \times \Omega \to \mathcal{A}$ be a positive definite kernel such that $\kappa(\cdot, \omega)$ is measurable for any $\omega \in \Omega$. If X_κ is the reproducing Hilbert $\mathcal{A}$-module associated with κ such that*

$$L(\kappa)^2 := \int_{\Omega} \|\kappa(w,w)\| \, d\mu(w) < \infty,$$

then there is a continuous linear inclusion map ι of X_κ into $L^2_{\mathcal{A}}(\Omega,\mu)$ with norm $\le L(\kappa)$.

Proof. Let ϕ be any element in X_κ. Since X_κ is a reproducing Hilbert $\mathcal{A}$-module, we find that for any $w \in \Omega$,

$$\begin{aligned}\|\phi(w)\|_{\mathcal{A}} &= \|\langle \kappa(\cdot,w), \phi(\cdot)\rangle_{X_\kappa}\| \\ &\le \|\phi(\cdot)\|_{X_\kappa} \|\kappa(\cdot,w)\|_{X_\kappa} = \|\phi(\cdot)\|_{X_\kappa} \|\kappa(w,w)\|^{1/2}\end{aligned}$$

where the inequality follows from the Cauchy–Schwarz inequality. If we denote by $\|\cdot\|_2$ the norm on $L^2_{\mathcal{A}}(\Omega,\mu)$, then

$$\|\phi\|_2^2 \le \int_{\Omega} \|\phi(w)^*\phi(w)\| \, d\mu \le \int_{\Omega} \|\phi(\cdot)\|^2_{X_\kappa} \|\kappa(w,w)\| \, d\mu.$$

This means that the embedding $\iota : X_\kappa \hookrightarrow L^2(\Omega, \mathcal{A})$ has a norm $\le L(\kappa)$. ∎

Let κ be as in Lemma 4.1. For any $\phi \in L^2_{\mathcal{A}}(\Omega,\mu)$, we define a map $\mathfrak{C}_\kappa(\phi) : \Omega \to \mathcal{A}$ by

$$\mathfrak{C}_\kappa(\phi)(w) = \int_{\Omega} \kappa(v,w)^* \phi(v) \, d\mu(v) \qquad (w \in \Omega).$$

We say that $\mathfrak{C}_\kappa$ is the *convolution type operator associated with* κ, briefly a *convolution type operator*.

REMARK. Let $\{r_n\}$ and $\{\phi_n\}$ be sequences satisfying conditions (a) and (b) given before Definition 3.1.

(1) Suppose that $\kappa(\cdot, w) = \sum_{n=0}^{\infty} r_n \phi_n(\cdot)^* \phi_n(w)$ is in $L^2_{\mathcal{A}}(\Omega,\mu)$ for each $w \in \Omega$ where the sum also converges in the $L^2_{\mathcal{A}}(\Omega,\mu)$ metric. Then

$$\mathfrak{C}_\kappa(\phi)(w) = \sum_{n=0}^{\infty} \phi_n(w)^* \Big(\int_{\Omega} r_n \phi_n(v) \phi(v) \, d\mu(v) \Big).$$

We can see that $\mathfrak{C}_\kappa(\phi)$ is in X_κ if and only if

$$\sum_{n=0}^{\infty} \frac{\| \int_{\Omega} r_n \phi_n(v)\phi(v) \, d\mu(v)\|^2}{r_n} = \sum_{n=0}^{\infty} r_n \Big\| \int_{\Omega} \phi_n(v)\phi(v) \, d\mu(v) \Big\|^2 < \infty.$$

(2) Let each map ϕ_n be projectively invariant under θ and α and let $\phi \in L^2_{\mathcal{M}}(\Omega,\mu)$. If $\kappa : \Omega \times \Omega \to \mathcal{M}$ is the Hilbert–Schmidt kernel given by (3.2), then $\mathfrak{C}_\kappa(\phi)$ is projectively invariant under θ and α^* where $\alpha^*(s,v) = \alpha(s,v)^*$.

Let $F(X_\kappa)$ be the linear span of all point evaluation $\mathcal{A}$-valued functions on X_κ; here, a point evaluation $\mathcal{A}$-valued function $\delta_w : X_\kappa \to \mathcal{A}$ is given by $\delta_w(\phi) = \phi(w)$ $(w \in \Omega,\ \phi \in X_\kappa)$. Then each element ϕ in X_κ satisfies the inequality

$$\|\chi(\phi)\| \le L_\phi \|\chi\| \quad \text{for all } \chi \in F(X_\kappa) \tag{4.1}$$

for some constant L_ϕ. If X_κ is self-dual, then X_κ is the space of functions ϕ with a constant L_ϕ satisfying (4.1). Note that not every bounded $\mathcal{A}$-module map between Hilbert $\mathcal{A}$-modules has a bounded adjoint [13].

The following theorem says that the convolution type operator $\mathfrak{C}_\kappa$ is the adjoint operator of the embedding ι of X_κ into $L^2_{\mathcal{A}}(\Omega, \mu)$ in Lemma 4.1.

Theorem 4.2. *Let κ and $L(\kappa)$ be as in Lemma 4.1. If the reproducing Hilbert $\mathcal{A}$-module X_κ is self-dual, then the convolution type operator $\mathfrak{C}_\kappa$ maps $L^2_{\mathcal{A}}(\Omega, \mu)$ into X_κ. Moreover, the norm of $\mathfrak{C}_\kappa$ is at most $L(\kappa)$ and the equation*

$$\langle \phi, \iota(\psi)\rangle_2 = \langle \mathfrak{C}_\kappa(\phi), \psi\rangle_{X_\kappa}$$

holds for any $\phi \in L^2_{\mathcal{A}}(\Omega, \mu)$ and $\psi \in X_\kappa$ where $\langle \cdot, \cdot\rangle_2$ and $\langle \cdot, \cdot\rangle_{X_\kappa}$ are the inner products on $L^2_{\mathcal{A}}(\Omega, \mu)$ and X_κ, respectively.

Proof. Let $\chi = \sum_{i=1}^n \delta_{w_i}(\cdot) a_i$ be any element in $F(X_\kappa)$ with finite support. Then we have

$$\begin{aligned}
\|\chi(\mathfrak{C}_\kappa(\phi))\| &= \Big|\Big(\sum_{i=1}^n \delta_{w_i} a_i\Big)[\mathfrak{C}_\kappa(\phi)]\Big| = \Big|\sum_{i=1}^n a_i^*[\mathfrak{C}_\kappa(\phi)](w_i)\Big| \\
&= \Big|\int_\Omega \Big[\sum_{i=1}^n a_i^* \kappa(w, w_i)^* \phi(w)\Big]\, d\mu(w)\Big| \\
&\le \|\phi\|_2 \Big\|\sum_{i=1}^n \kappa(\cdot, w_i) a_i\Big\|_2 \le L(\kappa)\|\phi\|_2\|\chi\| = L_\phi\|\chi\|
\end{aligned}$$

where $L_\phi = L(\kappa)\cdot\|\phi\|_2$ and the last inequality follows from the self-duality of X_κ. This inequality implies that $\mathfrak{C}_\kappa(\phi)$ belongs to X_κ for all $\phi \in L^2_{\mathcal{A}}(\Omega, \mu)$.

Let w_0 be any fixed element in Ω and let ψ be an element of X_κ given by $\psi = \kappa(\cdot, w_0)a_0$. For any element ϕ in $L^2_{\mathcal{A}}(\Omega, \mu)$, we have

$$\begin{aligned}
\langle \iota(\psi), \phi\rangle_2 &= \int_\Omega a_0^* \kappa(w, w_0)^* \phi(w)\, d\mu(w) \\
&= a_0^* \mathfrak{C}_\kappa(\phi)(w_0) = a_0^* \langle \kappa(\cdot, w_0), \mathfrak{C}_\kappa(\phi)\rangle_{X_\kappa} \\
&= \langle \kappa(\cdot, w_0)a_0, \mathfrak{C}_\kappa(\phi)\rangle_{X_\kappa} = \langle \psi, \mathfrak{C}_\kappa(\phi)\rangle_{X_\kappa}.
\end{aligned}$$

Since every element in X_κ can be approximated by elements of the form $\sum_{i=1}^n \kappa(\cdot, w_i)a_i$, it follows by continuity that $\langle \iota(\psi), \phi\rangle_2 = \langle \psi, \mathfrak{C}_\kappa(\phi)\rangle_{X_\kappa}$ for all $\psi \in X_\kappa$. ∎

We now consider a kernel on a countable discrete set. Let $S = \{s_1, s_2, \ldots\}$ be a countable discrete set with counting measure ν. If e_n is the function on S defined by $e_n(s_m) = \delta_{nm}1_{\mathcal{A}}$, then each function ϕ in $L^2_{\mathcal{A}}(S,\nu)$ can be expressed as the Fourier expansion

$$\phi(\cdot) = \sum_{n=1}^{\infty} e_n(\cdot)\langle e_n, \phi\rangle_2$$

where $\langle e_n(\cdot), \phi\rangle_2 = \sum_{m=1}^{\infty} e_n(s_m)^*\phi(s_m) = \phi(s_n)$. If $\{r_n : n \in \mathbb{N}\}$ is a sequence of positive real numbers and if m is a positive integer, then we have $\sum_{n=1}^{\infty} r_n\|e_n(s_m)\| = r_m$. It is clear that any finite subset of $\{e_n\}$ is $\mathcal{A}$-linearly independent over S.

Let κ be the $\mathcal{A}$-valued Hilbert–Schmidt kernel associated with the sequences $\{r_n\}$ and $\{e_n\}$, that is, $\kappa(s,t) = \sum_{n=1}^{\infty} r_n e_n(s)^* e_n(t)$ where e_n's are as above. By Theorem 3.2, the corresponding reproducing Hilbert $\mathcal{A}$-module X is the completion of the space

$$\mathfrak{X} = \Big\{ \sum_{n=1}^{k} e_n(\cdot)a_n : a_n \in \mathcal{A},\ k \in \mathbb{N} \Big\}$$

with respect to the norm induced by the $\mathcal{A}$-valued inner product

$$\langle \phi, \psi\rangle_X = \sum_{n=1}^{\infty} \frac{\langle e_n, \phi\rangle_2^* \langle e_n, \psi\rangle_2}{r_n} = \sum_{n=1}^{\infty} \frac{\langle \phi, e_n\rangle_2 \langle e_n, \psi\rangle_2}{r_n}.$$

Proposition 4.3. *Let S, κ, $\{e_n\}$ and $\{r_n\}$ be as above. Suppose that the sequence $\{r_n\}$ is summable and that the corresponding reproducing Hilbert $\mathcal{A}$-module X_κ is self-dual.*

(i) *For all $\phi \in L^2_{\mathcal{A}}(S,\nu)$, $\mathfrak{C}_\kappa(\phi) \in X_\kappa$.*
(ii) *The convolution type operator $\mathfrak{C}_\kappa : L^2_{\mathcal{A}}(S,\nu) \to X_\kappa$ is compact.*

Proof. (i) For each $s \in S$ we have

$$\begin{aligned}[\mathfrak{C}_\kappa(\phi)](s) &= \sum_{n=1}^{\infty} \kappa(s_n, s)^*\phi(s_n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} r_n e_m(s)^* e_m(s_n)\phi(s_n) \\ &= \sum_{n=1}^{\infty} r_n e_n(s)^*\phi(s_n).\end{aligned}$$

Thus, $[\mathfrak{C}_\kappa(\phi)](\cdot) = \sum_{n\in\mathbb{N}} r_n e_n(\cdot)^*\phi(s_n)$. By letting $a_n = r_n\phi(s_n)$, we obtain

$$\sum_{n\in\mathbb{N}} \frac{\|a_n\|^2}{r_n} = \sum_{n\in\mathbb{N}} r_n\|\phi(s_n)\|^2 < \infty$$

since $\{r_n\}$ is summable. It follows from Theorem 4.2 that $\mathfrak{C}_\kappa(\phi) \in X_\kappa$ for all $\phi \in L^2_{\mathcal{A}}(S,\nu)$.

(ii) Let $\kappa_N(s,t) = \sum_{n=1}^N r_n e_n(s)^* e_n(t)$. We define an operator $\mathfrak{C}_N$ on $L^2_{\mathcal{A}}(S,\nu)$ by

$$[\mathfrak{C}_N(\phi)](s) = \sum_{n=1}^{\infty} \kappa_N(s_n, s)^* \phi(s_n) \quad (\phi \in L^2_{\mathcal{A}}(S,\nu)).$$

For any $\phi \in L^2_{\mathcal{A}}(S,\nu)$, we have

$$\begin{aligned}[\mathfrak{C}_N(\phi)](\cdot) &= \sum_{n=1}^{N} r_n e_n(\cdot)^* \phi(s_n) \\ &= \sum_{n=1}^{N} e_n \langle r_n e_n(\cdot), \phi(\cdot)\rangle_2 = \sum_{n=1}^{N} \theta_{e_n, r_n e_n(\cdot)}(\phi).\end{aligned}$$

Since we can regard $r_n e_n(\cdot)$ as an element in $L^2_{\mathcal{A}}(S,\nu)$, $\mathfrak{C}_N$ is a finite rank operator. For the summable sequence $\{r_n\}$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \|\kappa(s_n,s_m)\|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Big\| \sum_{l\in\mathbb{N}} r_l e_l(s_n)^* e_l(s_m)\Big\|^2 = \sum_{n=1}^{\infty} r_n^2. \tag{4.2}$$

Since a summable sequence of positive numbers is square-summable, it follows from (4.2) that the convolution type operator $\mathfrak{C}_\kappa : L^2_{\mathcal{A}}(S,\nu) \to X_\kappa$ is compact. ∎

For any element ϕ in $L^2_{\mathcal{A}}(S,\nu)$, we have

$$[\mathfrak{C}_\kappa(\phi)](\cdot) = \sum_{n=1}^{\infty} r_n e_n(\cdot)^* \langle e_n, \phi\rangle_2,$$

which implies that the convolution type operator $\mathfrak{C}_\kappa$ is multiplication of each Fourier coefficient by r_n. That is, $\mathfrak{C}_\kappa$ can be regarded as the diagonal operator with diagonal $\mathrm{diag}(r_1, r_2, \ldots)$ with respect to the basis $\{e_n\}$, so that its compactness is immediate. The operator norm of the convolution type operator $\mathfrak{C}_\kappa$ is $\sup_n r_n$. Moreover, we see that the range of the convolution type operator $\mathfrak{C}_\kappa$ is

$$\begin{aligned}\mathrm{Im}(\mathfrak{C}_\kappa) &= \Big\{ \sum_{n=1}^{\infty} r_n e_n(\cdot)^* \langle e_n, \phi\rangle_2 : \sum_n \|\langle e_n, \phi\rangle_2\|^2 < \infty,\ \phi \in L^2_{\mathcal{A}}(S,\nu)\Big\} \\ &= \Big\{ \phi \in L^2_{\mathcal{A}}(S,\nu) : \sum_n \frac{\|\langle e_n,\phi\rangle_2\|^2}{r_n^2} < \infty \Big\}.\end{aligned}$$

Let κ_1 and κ_2 be $\mathcal{A}$-valued kernels on a discrete countable set $S = \{s_1, s_2, \ldots\}$. We define the *convolution kernel* $\kappa_1 * \kappa_2$ *of* κ_1 *and* κ_2 by

$$(\kappa_1 * \kappa_2)(s,t) = \sum_{n=1}^{\infty} \kappa_1(s, s_n)^* \kappa_2(t, s_n), \quad (s, t \in S).$$

In particular, if $\kappa_1 = \kappa_2 = \kappa$, then $K = \kappa * \kappa$ is called the *convolution kernel of* κ.

Let S, $\{e_n\}$ and $\{r_n\}$ be as in Proposition 4.3. If κ is the $\mathcal{A}$-valued Hilbert–Schmidt kernel associated with the sequences $\{r_n\}$ and $\{e_n\}$, then we have

$$\begin{aligned} K(s,t) &= \sum_{n=1}^{\infty} \Big[\sum_{m=1}^{\infty} r_m e_m(s)^* e_m(s_n) \Big]^* \Big[\sum_{l=1}^{\infty} r_l e_l(t)^* e_l(s_n) \Big] \\ &= \sum_{n=1}^{\infty} r_n^2 e_n(s) e_n(t)^*. \end{aligned}$$

It follows from Theorem 3.2 that the reproducing Hilbert $\mathcal{A}$-module X_K associated with K is the completion of the space of $\mathcal{A}$-valued functions on S with finite supports with respect to the norm induced by the inner product

$$\Big\langle \sum_{n=1}^{k} e_n(\cdot) a_n, \sum_{n=1}^{l} e_n(\cdot) b_n \Big\rangle = \sum_{n=1}^{\min\{k,l\}} \frac{a_n^* b_n}{r_n^2}.$$

This observation yields the following result.

THEOREM 4.4. *Let S, $\{e_n\}$ and $\{r_n\}$ be as in Proposition* 4.3. *If κ is an $\mathcal{A}$-valued Hilbert–Schmidt kernel associated with $\{r_n\}$ and $\{e_n\}$, then the reproducing Hilbert $\mathcal{A}$-module associated with the convolution kernel K coincides with the range of the convolution type operator $\mathfrak{C}_\kappa$.*

Let S and κ be as in Theorem 4.4. Then for any s, t in S we have

$$\begin{aligned} \mathfrak{C}_\kappa(\kappa(\cdot, s))(t) &= \sum_{n=1}^{\infty} \Big[\sum_{m=1}^{\infty} r_m e_m(s_n)^* e_m(s) \Big]^* \Big[\sum_{l=1}^{\infty} r_l e_l(s_n)^* e_l(t) \Big] \\ &= \sum_{n=1}^{\infty} r_n^2 e_n(s)^* e_n(t) = K(s,t). \end{aligned}$$

This is the reason why we call $\mathfrak{C}_\kappa$ a convolution type operator.

Let G be a transformation group acting on the right on a set T. Then there is a canonical action π of G on the space of $\mathcal{A}$-valued functions given by the formula

$$(\pi(g)\phi)(t) = \phi(tg) \quad (g \in G,\ t \in T)$$

where $\phi : T \to \mathcal{A}$ is an $\mathcal{A}$-valued function. In many cases, the domain T of functions admits a transformation group G such that the Hilbert $\mathcal{A}$-module X of $\mathcal{A}$-valued functions on T is invariant under G, that is, for every $\phi, \psi \in X$,

$$\phi \circ \pi(g) \in X \quad \text{and} \quad \langle \pi(g) \circ \phi, \pi(g) \circ \psi \rangle_X = \langle \phi, \psi \rangle_X.$$

If X is a self-dual Hilbert $\mathcal{A}$-module of $\mathcal{A}$-valued functions on T with a unitary representation π of G on X, that is, $\pi(g)\phi \in X$ and $\|\pi(g)\phi\| = \|\phi\|$ for all $g \in G$ and $\phi \in X$, then X is called a (G, π)-*Hilbert* $\mathcal{A}$-*module* [8].

PROPOSITION 4.5 ([8]). *Let G be a transformation group acting on the right on a set T and let X be a (G, π)-Hilbert $\mathcal{A}$-module of $\mathcal{A}$-valued functions on T. Then κ is invariant under G in the sense that $\kappa(sg, tg) = \kappa(s, t)$ for all $g \in G$ and $s, t \in T$.*

REMARK. One can read off some invariance properties inherited by reproducing kernels κ from their Hilbert $\mathcal{A}$-modules of $\mathcal{A}$-valued functions on a set T:

(1) Let $\mathcal{A}$ be a C^*-algebra and let X be a Hilbert $\mathcal{A}$-module. The invariance on the unit sphere in X under all unitary transformations leads to the function $\kappa(x, y) = \phi(\langle x, y\rangle_X)$ for a function $\phi : \mathcal{A}_1 \to \mathcal{A}$ where A_1 is the unit ball of a C^*-algebra $\mathcal{A}$. We may regard X as a space of $\mathcal{A}$-valued functions on X defined by $x(y) = \langle x, y\rangle$ for $x, y \in X$.

(2) Let T be the homogeneous space $L\backslash G$ of right cosets of L in G where G is a connected semisimple Lie group with finite center and L is a maximal compact subgroup of G. For any s, t in T, set

$$\kappa(s, t) = \phi(xy^{-1})$$

where $x \in Ls$, $y \in Lt$ and ϕ is an $\mathcal{A}$-valued function on G. If ϕ is invariant under translation by L on both sides, then κ is well-defined. Moreover, if ϕ is positive definite, then κ is a positive definite kernel which is invariant under right translations.

5. Convolution type operators and integral type operators. Throughout this section, Γ denotes a countable discrete group, unless specified otherwise. Given a complex-valued kernel $\kappa : \Gamma \times \Gamma \to \mathbb{C}$, we define an operator $\mathcal{I}_\kappa$ by

$$\mathcal{I}_\kappa(\phi)(s) = \sum_{t\in\Gamma} \kappa(s, t)\phi(t) \quad \text{for } \phi \in l^2(\Gamma), \tag{5.1}$$

which is called the *integral type operator associated with* κ.

Let $\{f_n\}$ be an orthonormal basis for $l^2(\Gamma)$ and $\{r_n\}$ a positive summable sequence. If $\kappa : \Gamma \times \Gamma \to \mathbb{C}$ is the Hilbert–Schmidt kernel associated with $\{r_n\}$ and $\{f_n\}$, then $\kappa(s, \cdot)$ can be regarded as an element in $l^2(\Gamma)$ for each $s \in \Gamma$. Indeed,

$$\begin{aligned}\|\kappa(s, \cdot)\|_2^2 &\le \sum_{t\in\Gamma} \Big(\sum_{n\in\mathbb{N}} r_n|f_n(s)|^2\Big)\Big(\sum_{n\in\mathbb{N}} r_n|f_n(t)|^2\Big) \\ &= \Big(\sum_{n\in\mathbb{N}} r_n|f_n(s)|^2\Big)\Big(\sum_{n\in\mathbb{N}} r_n\Big) < \infty.\end{aligned}$$

We have $\sum_{s,t\in\Gamma}|\kappa(s,t)|^2 \le \big(\sum_{n\in\mathbb{N}} r_n\big)^2 < \infty$, so that κ can be regarded as a function in $l^2(\Gamma\times\Gamma)$. Furthermore, if each f_n is projectively invariant under θ and α, then $\mathcal{I}_\kappa(\phi)$ is also projectively invariant for any $\phi\in l^2(\Gamma)$.

PROPOSITION 5.1. *The integral operator $\mathcal{I}_\kappa$ associated with a Hilbert–Schmidt kernel κ is a Hilbert–Schmidt operator on $l^2(\Gamma)$ and the map $\kappa\mapsto\mathcal{I}_\kappa$ is an isometry. Moreover, $\mathcal{I}_\kappa$ is self-adjoint.*

Proof. It follows from Proposition 3.4.16 in [14] that $\mathcal{I}_\kappa$ is a Hilbert–Schmidt operator on $l^2(\Gamma)$ and that the map $\kappa\mapsto\mathcal{I}_\kappa$ is an isometry. Since κ is positive definite, it is conjugate symmetric, that is, $\kappa(s,t)=\overline{\kappa(t,s)}$ for all $s,t\in\Gamma$. The self-adjointness of $\mathcal{I}_\kappa$ also follows from Proposition 3.4.16 in [14]. ∎

In the following theorem, we see that an integral operator associated with a positive definite kernel induces an embedding of an l^2-space into the reproducing Hilbert space.

THEOREM 5.2. *If $\kappa:\Gamma\times\Gamma\to\mathbb{C}$ is a positive definite kernel such that $\sum_{s\in\Gamma}\kappa(s,s)<\infty$, then the integral operator $\mathcal{I}_\kappa$ maps $l^2(\Gamma)$ into the reproducing Hilbert space $\mathcal{H}_\kappa$. Moreover,*

$$\langle\phi,\psi\rangle_2=\langle\mathcal{I}_{\overline{\kappa}}(\phi),\psi\rangle_\kappa \quad \text{for all } \phi\in l^2(\Gamma) \text{ and } \psi\in\mathcal{H}_\kappa,$$
$$\langle\phi,\psi\rangle_2=\langle\mathcal{I}_{\kappa}(\overline{\psi}),\phi\rangle_\kappa \quad \text{for all } \phi\in\mathcal{H}_\kappa \text{ and } \psi\in l^2(\Gamma).$$

Proof. The proof of the first part is the same as that of Theorem 4.2. The proof of the equations is similar to that of Theorem 4.2, so that we only give a sketch. Let $\psi=\kappa(s,\cdot)$ be in $\mathcal{H}_\kappa$ for some $s\in\Gamma$. For any ϕ in $l^2(G)$, we have

$$\begin{aligned}\langle\phi,\psi\rangle_2&=\sum_{t\in\Gamma}\phi(t)\overline{\kappa(s,t)}=\mathcal{I}_{\overline{\kappa}}(\phi)(s)\\&=\langle\mathcal{I}_{\overline{\kappa}}(\phi),\kappa(s,\cdot)\rangle_\kappa=\langle\mathcal{I}_{\overline{\kappa}}(\phi),\psi\rangle_\kappa.\end{aligned}$$

Since any $\psi\in\mathcal{H}_\kappa$ can be approximated by linear combinations of $\kappa(s,\cdot)$'s, the first equation holds by continuity. To prove the second, we need only consider elements of the form $\phi=\kappa(s,\cdot)$ in $\mathcal{H}_\kappa$. For any ψ in $l^2(\Gamma)$ we have

$$\begin{aligned}\langle\phi,\psi\rangle_2&=\sum_{t\in\Gamma}\kappa(s,t)\overline{\psi(t)}=\mathcal{I}_{\kappa}(\overline{\psi})(s)\\&=\langle\mathcal{I}_{\kappa}(\overline{\psi}),\kappa(s,\cdot)\rangle_\kappa=\langle\mathcal{I}_{\kappa}(\overline{\psi}),\phi\rangle_\kappa,\end{aligned}$$

which completes the proof. ∎

Suppose that κ is a complex-valued positive definite kernel satisfying the assumption as in Lemma 4.1, that is, $\sum_{s\in\Gamma}\kappa(s,s)<\infty$. We can easily see that $\mathcal{I}_\kappa(\overline{\kappa(t,\cdot)})(s)=K(s,t)$ for all $s,t\in\Gamma$ where K is the convolution kernel of κ. In particular, if κ is given by $\kappa(s,t) = \sum_n r_n \overline{f_n(s)} f_n(t)$ where $\{f_n\}$ is an orthonormal basis for $l^2(\Gamma)$, we get the equality

$$\mathfrak{C}_\kappa(\kappa(s,\cdot))(t) = \sum_n r_n^2 \overline{f_n(s)} f_n(t) \qquad (s,t \in \Gamma).$$

To compute l^2-inner products of elements in the range of $\mathfrak{C}_\kappa$, let ϕ, ψ be in $l^2(\Gamma)$. We have

$$\begin{aligned}\langle \mathfrak{C}_\kappa(\phi), \mathfrak{C}_\kappa(\psi)\rangle_{l^2(\Gamma)} &= \sum_u \Big(\sum_t \overline{\kappa(u,t)}\phi(t)\Big)\Big(\sum_s \kappa(u,s)\overline{\psi(s)}\Big)\\ &= \sum_{s,t} K(s,t)\phi(t)\overline{\psi(s)}.\end{aligned}$$

In particular, if $\phi = \kappa(t,\cdot)$ and $\psi = \kappa(s,\cdot)$ for some $s,t \in \Gamma$ then

$$\begin{aligned}\langle \mathfrak{C}_\kappa(\kappa(t,\cdot)), \mathfrak{C}_\kappa(\kappa(s,\cdot))\rangle_{l^2(\Gamma)} &= \sum_u \Big(\sum_v \overline{\kappa(u,v)}\kappa(t,v)\Big)\Big(\sum_w \kappa(u,w)\overline{\kappa(s,w)}\Big)\\ &= \sum_u \overline{K(s,u)}K(t,u) = \mathfrak{C}_K(K(t,\cdot))(s).\end{aligned}$$

Thus, we can regard the l^2-inner product in the range of $\mathfrak{C}_\kappa$ as the evaluation of the image of $\mathfrak{C}_K$ associated with the convolution kernel K.

Acknowledgements. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012-0002569).

References

[1] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337–404.

[2] S. Barreto, B. V. Bhat, V. Liebscher and M. Skeide, *Type I product systems of Hilbert modules*, J. Funct. Anal. 212 (2004), 121–181.

[3] C. Berg and G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Ergeb. Math. Grenzgeb. 87, Springer, New York, 1975.

[4] T. Constantinescu and A. Gheondea, *Representations of Hermitian kernels by means of Kreĭn spaces II. Invariant kernels*, Comm. Math. Phys. 216 (2001), 409–430.

[5] M. Frank, *Self-duality and C^*-reflexivity of Hilbert C^*-moduli*, Z. Anal. Anwend. 9 (1990), 165–176.

[6] J. Heo, *Hilbert C^*-modules and projective representations associated with multipliers*, J. Math. Anal. Appl. 331 (2007), 499–505.

[7] J. Heo, *Stationary stochastic processes in a group system*, J. Math. Phys. 48 (2007), no. 10, 103502.

[8] J. Heo, *Reproducing kernel Hilbert C^*-modules and kernels associated with cocycles*, J. Math. Phys. 49 (2008), no. 10, 103507.

[9] E. Lance, *Hilbert C^*-modules. A Toolkit for Operator Algebraists*, London Math. Soc. Lecture Note Ser. 210, Cambridge Univ. Press, Cambridge, 1995.

[10] J.-L. Lions, *Remarks on reproducing kernels of some function spaces*, in: Function Spaces, Interpolation Theory and Related Topics (Lund, 2000), de Gruyter, Berlin, 2002, 49–59.

[11] J. Mercer, *Functions of positive and negative type and their connection with the theory of integral equations*, Philos. Trans. Roy. Soc. London Ser. A 209 (1909), 415–446.

[12] F. Narcowich, *Generalized Hermite interpolation and positive definite kernels on a Riemannian manifold*, J. Math. Anal. Appl. 190 (1995), 165–193.

[13] W. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. 182 (1973), 443–468.

[14] G. Pedersen, *Analysis Now*, Springer, New York, 1989.

[15] R. Schaback, *A unified theory of radial basis functions. Native Hilbert spaces for radial basis functions II*, J. Comput. Appl. Math. 121 (2000), 165–177.

[16] I. Schoenberg, *Positive definite functions on spheres*, Duke Math. J. 9 (1942), 96–108.

Jaeseong Heo
Department of Mathematics
Research Institute for Natural Sciences
Hanyang University
Seoul 133-791, Korea
E-mail: hjs@hanyang.ac.kr

Received May 22, 2012
Revised version December 6, 2012 (7522)