Monotone extenders for bounded $c$-valued functions

by

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Abstract. Let $c$ be the Banach space consisting of all convergent sequences of reals with the sup-norm, $C_\infty(A,c)$ the set of all bounded continuous functions $f : A \to c$, and $C_A(X,c)$ the set of all functions $f : X \to c$ which are continuous at each point of $A \subset X$. We show that a Tikhonov subspace $A$ of a topological space $X$ is strong Choquet in $X$ if there exists a monotone extender $u : C_\infty(A,c) \to C_A(X,c)$. This shows that the monotone extension property for bounded $c$-valued functions can fail in GO-spaces, which provides a negative answer to a question posed by I. Banakh, T. Banakh and K. Yamazaki.

In this paper, vector spaces mean real vector spaces. Let $X$ and $Y$ be topological spaces. Then $C(X,Y)$ stands for the set of all continuous functions $f : X \to Y$. If $Y$ is a topological vector space, the set of all bounded continuous functions $f : X \to Y$ is denoted by $C_\infty(X,Y)$, where $f : X \to Y$ is bounded if $f(X)$ is a bounded subset of $Y$, that is, for each 0-neighborhood $U$ of $Y$ there exists $r \in \mathbb{R}$ such that $f(X) \subset rU$. For $A \subset X$, a map $u : C(A,Y) \to C(X,Y)$ is called an extender if $u(f)|A = f$ for each $f \in C(A,Y)$. For a topological vector space $Y$, an extender $u : C(A,Y) \to C(X,Y)$ is said to be a conv-extender (resp. conv-extender) if $u(f)(X)$ is a subset of the convex hull (resp. the closed convex hull) of $f(A)$ for each $f \in C(A,Y)$. For a topological space $Y$ with a partial order structure $\leq$, an extender $u : C(A,Y) \to C(X,Y)$ (or $u : C_\infty(A,Y) \to C(X,Y)$) is said to be monotone if $u(f) \leq u(g)$ for each $f,g \in C(A,Y)$ (or $f,g \in C_\infty(A,Y)$) with $f \leq g$. A vector space $Y$ with a partial order structure $\leq$ is called an ordered vector space if the following axioms are satisfied:

$$(O)_1 x \leq y \text{ implies } x + z \leq y + z \text{ for all } x,y,z \in Y;$$

$$(O)_2 x \leq y \text{ implies } \lambda x \leq \lambda y \text{ for all } x,y \in Y \text{ and all } \lambda > 0.$$
closed. Note that, for an ordered topological vector space $Y$, each linear \textit{conv}-extender $u : C(A,Y) \to C(X,Y)$ is monotone. As usual, $C(X)$ and $C_\infty(X)$ stand for $C(X,\mathbb{R})$ and $C_\infty(X,\mathbb{R})$, respectively.

Dugundji’s extension theorem \cite{6} states that for a metric space $X$, a closed subset $A$ of $X$ and a locally convex topological vector space $Y$, there exists a linear \textit{conv}-extender $u : C(A,Y) \to C(X,Y)$; this is an improvement of an earlier result by K. Borsuk \cite{4} that for a closed separable subset of a metric space $X$, there exists a norm-one linear extender $u : C_\infty(A) \to C_\infty(X)$. Now it is known that Dugundji’s extension theorem holds in some classes of generalized metric spaces $X$ (C. J. R. Borges \cite{3}, I. S. Stares \cite{11}), but does not hold for all GO-spaces $X$. Indeed, for the Michael line $\mathbb{R}_\mathbb{Q}$, R. W. Heath and D. J. Lutzer \cite{9} show that there exists no linear \textit{conv}-extender $u : C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$; E. K. van Douwen \cite{5} extends it by showing that there is no monotone extender $u : C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$ (see also I. S. Stares and J. E. Vaughan \cite{12}). For related results on Dugundji extenders and retracts in GO-spaces, see G. Gruenhage, Y. Hattori and H. Ohta \cite{8}.

On extenders for bounded functions, R. W. Heath and D. J. Lutzer \cite{9} establish that for a closed subset $A$ of a GO-space $X$, there exists a linear \textit{conv}-extender $u : C_\infty(A) \to C_\infty(X)$; van Douwen’s result \cite{5} shows that “\textit{conv}-extender” in the above cannot be strengthened to “\textit{conv}-extender”. For normed-space-valued functions, I. Banakh, T. Banakh and K. Yamazaki \cite[Theorem 4.1]{1} prove that a normed space $Y$ is reflexive if and only if for every closed subset $A$ of a GO-space $X$, there exists a linear \textit{conv}-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$. From these viewpoints, a natural further question arises: let $Y$ be a non-reflexive normed space which is an ordered topological vector space; does there exist, for every closed subset $A$ of every GO-space $X$, a monotone extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$? The answer is “yes” for $Y = l_1$ \cite[Theorem 9.1]{1}, “no” for $Y = c_0$ \cite[Corollary 6.3]{1}), and for $Y = c$ the following is asked in \cite[Question 6.4]{1}: \textit{Is there a monotone extender $u : C_\infty(\mathbb{Q},c) \to C(\mathbb{R}_\mathbb{Q},c)$?} In this paper, we give a negative answer to this question (Corollary 4). In fact, we show that any subset $A$ which is not strong Choquet in $X$ fails to possess such monotone extenders (Corollary 3).

Let us recall some terminology. The symbol $c_0$ (resp. $c$) stands for the Banach space consisting of all sequences of reals that converge to 0 (resp. of all convergent sequences of reals) with the sup-norm and a partial order structure $\leq$, where for $x = (x_n)_{n \in \omega}$ and $y = (y_n)_{n \in \omega} \in c_0$ (or $c$, $x \leq y$ if $x_n \leq y_n$ for each $n \in \omega$. Recall from \cite{10} and \cite{2} that a Hausdorff space $X$ is a \textit{generalized ordered space} (= GO-space) if $X$ has a linear order structure and has a base of the topology consisting of order-convex sets. The \textit{Michael line} $\mathbb{R}_\mathbb{Q}$ is the set $\mathbb{R}$ endowed with the topology $\{ U \cup V : U \in \tau, V \subset \mathbb{R} \setminus \mathbb{Q} \}$, where
\( \tau \) is the usual topology of \( \mathbb{R} \) and \( \mathbb{Q} \) is the set of all rational numbers. (\[7\])

The Michael line \( \mathbb{R}_\omega \) is a typical example of a GO-space.

As in [1], the relative strong Choquet game \( G_r(A,X) \) is played by two players, I and II, for a subset \( A \) of a topological space \( X \). Player I starts the game selecting a point \( a_0 \in A \) and a neighborhood \( U_0 \) of \( a_0 \) in \( X \). Player II responds with a neighborhood \( V_0 \subset U_0 \) of \( a_0 \) in \( X \). At the next inning player I selects a point \( a_n \in V_{n-1} \cap A \) and a neighborhood \( U_n \subset V_{n-1} \) of \( a_n \) in \( X \), while player II responds with a neighborhood \( V_n \subset U_n \) of \( a_n \) in \( X \). Thus players construct a sequence \( \{a_n\}_{n \in \omega} \) of points of \( A \) and sequences \( \{U_n\}_{n \in \omega} \) and \( \{V_n\}_{n \in \omega} \) of open sets of \( X \) such that \( a_n \in V_n \subset U_n \subset V_{n-1} \) for all \( n \in \mathbb{N} \). Player I declares the winner in the game \( G_r(A,X) \) if \( \emptyset \neq \bigcap_{n \in \omega} U_n \subset X \setminus A \). Otherwise, player II wins. If player II has a winning strategy in the game \( G_r(A,X) \), then the set \( A \) is said to be strong Choquet in \( X \). Note that \( \mathbb{Q} \) is not strong Choquet in \( \mathbb{R}_\omega \) ([1 Corollary 3.7]). Another important example of a subset which is not strong Choquet in the whole space is given in [12] (see [1, Remark 3.8]).

Let \( Y \) be a topological space with a partial order structure. Then a continuous function \( \gamma : [0, \infty) \rightarrow Y \) is an \( \omega \)-increasing ray (resp. \( \omega \)-decreasing ray) if \( \gamma(n) \leq \gamma(t) \) (resp. \( \gamma(n) \geq \gamma(t) \)) for any integer \( n \in \omega \) and any real \( t \geq n \) ([1]). In order to improve [1 Theorem 6.1], we introduce key notions which are modifications of (almost) upper boundedness of \( \gamma(\omega) \) for an \( \omega \)-increasing ray \( \gamma \) appearing in [1]. For \( Y_0 \subset Y \), we say \( Y_0 \) has the \( \omega \)-decreasing intersection property in \( Y \) if for each increasing sequence \( \{y_n\}_{n \in \omega} \) in \( Y_0 \) and each decreasing sequence \( \{z_n\}_{n \in \omega} \) in \( Y_0 \) with \( y_n \leq z_n \) for each \( n \in \omega \), \( \bigcap_{n \in \omega} \{y \in Y : y_n \leq y \leq z_n\} \neq \emptyset \). We also say \( Y_0 \) has the almost \( \omega \)-decreasing intersection property in \( Y \) if for each \( \omega \)-increasing ray \( \gamma_1 : [0, \infty) \rightarrow Y_0 \), each \( \omega \)-decreasing ray \( \gamma_2 : [0, \infty) \rightarrow Y_0 \) with \( \gamma_1(r_1) \leq \gamma_2(r_2) \) for each \( r_1, r_2 \in [0, \infty) \), and each family \( \{G_{\alpha,n}^i\}_{n \in \omega} \) of \( G_\delta \)-sets of \( Y \) with \( \gamma_i(n) \in G_{\alpha,n}^i \), \( n \in \omega \), \( i = 1,2 \), it follows that \( \bigcap_{n \in \omega} \bigcup_{b_1 \in G_{\alpha,n}^1, b_2 \in G_{\alpha,n}^2} \{y \in Y : b_1 \leq y \leq b_2\} \neq \emptyset \).

For \( A \subset X \), \( C_A(X,Y) \) denotes the set of all functions \( f : X \rightarrow Y \) which are continuous at each point of \( A \).

**Theorem 1.** Let \( A \) be a Tikhonov subset of a topological space \( X \), \( Y \) a topological space with a partial order structure and \( Y_0 \subset Y \). If there exists a monotone extender \( u : C(A,Y_0) \rightarrow C_A(X,Y) \), then either \( A \) is strong Choquet in \( X \) or else \( Y_0 \) has the almost \( \omega \)-decreasing intersection property in \( Y \).

**Proof.** Let \( u : C(A,Y_0) \rightarrow C_A(X,Y) \) be a monotone extender. Assume \( Y_0 \) does not have the almost \( \omega \)-decreasing intersection property in \( Y \). Namely, there exist \( \omega \)-increasing ray \( \gamma_1 : [0, \infty) \rightarrow Y_0 \), an \( \omega \)-decreasing ray \( \gamma_2 : [0, \infty) \rightarrow Y_0 \) with \( \gamma_1(r_1) \leq \gamma_2(r_2) \) for each \( r_1, r_2 \in [0, \infty) \), and a
family \( \{G^i_n\}_{n \in \omega} \) of \( G^i \)-sets of \( Y \) with \( \gamma_i(n) \in G^i_n, n \in \omega, i = 1, 2 \), such that
\[
\bigcap_{n \in \omega} \bigcup_{b_1 \in G^1_n, b_2 \in G^2_n} \{y \in Y : b_1 \leq y \leq b_2\} = \emptyset.
\]
Set \( y_n = \gamma_1(n) \) and \( z_n = \gamma_2(n) \) for each \( n \in \omega \). For each \( n \in \omega \), take decreasing sequences \( (O_m(y_n))_{m \geq n} \) and \( (O_m(z_n))_{m \geq n} \) of open neighborhoods of \( y_n \) and \( z_n \), respectively, such that \( \bigcap_{m \geq n} O_m(y_n) \subset G^1_n \) and \( \bigcap_{m \geq n} O_m(z_n) \subset G^2_n \).

Now we describe a winning strategy of player II in the game \( G \). For \( a_0 \in A \) and a neighborhood \( U_0 \) of \( a_0 \) in \( X \), player II takes a neighborhood \( V_0 \) of \( a_0 \) in \( X \) with \( V_0 \subset U_0 \) and sets a function \( \lambda_0 \equiv 0 \) on \( A \). In the \( n \)th inning \((n > 0)\), for \( a_n \in V_{n-1} \cap A \) and a neighborhood \( U_n \subset V_{n-1} \) of \( a_n \), player II takes a continuous function \( \lambda_n : A \to [0,1] \) and a neighborhood \( V_n \) of \( a_n \) in \( X \) such that
\[
V_n \subset U_n, \quad a_n \in V_n \cap A \subset \lambda^{-1}_n(1) \subset \lambda^{-1}_n((0,1]) \subset U_n,
\]
for each \( k \leq n \). Indeed, since \( A \) is Tikhonov, take a continuous function \( \lambda_n : A \to [0,1] \) such that \( a_n \in \text{int}_A \lambda^{-1}_n(1) \subset \lambda^{-1}_n((0,1]) \subset U_n \). For each \( i \) with \( 1 \leq i < n \), it follows from \( a_n \in U_n \cap A \subset V_i \cap A \subset \lambda^{-1}_i(1) \) that \( \lambda_i(a_n) = 1 \).

Hence, \( u(\gamma_1 \circ \sum_{i=0}^{k} \lambda_i)(a_n) = \gamma_1(k) = y_k \) and \( u(\gamma_2 \circ \sum_{i=0}^{k} \lambda_i)(a_n) = \gamma_2(k) = z_k \) for each \( k \leq n \). Since \( u(\gamma_1 \circ \sum_{i=0}^{k} \lambda_i) \) and \( u(\gamma_2 \circ \sum_{i=0}^{k} \lambda_i) \) are in \( C_A(X,Y) \), choose a neighborhood \( V_n \) of \( a_n \) in \( X \) such that \( u(\gamma_1 \circ \sum_{i=0}^{k} \lambda_i)(V_n) \subset O_n(y_k) \) and \( u(\gamma_2 \circ \sum_{i=0}^{k} \lambda_i)(V_n) \subset O_n(z_k) \) for each \( k \leq n \) and \( V_n \cap A \subset \lambda^{-1}_n(1) \).

Then the condition \( \emptyset \neq \bigcap_{n \in \omega} U_n \subset X \setminus A \) fails. To show this, assume on the contrary that there exists \( c \in \bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} V_n \subset X \setminus A \). By (3),
\[
u(\gamma_2 \circ \sum_{i=0}^{k} \lambda_i)(c) \in \bigcap_{n \geq k} O_n(z_k) \subset G^2_k,
\]
for each \( k \in \omega \). Define a continuous function \( s : A \to [0,\infty) \) by \( s(a) = \sum_{i \in \omega} \lambda_i(a) \) for each \( a \in A \); this is possible by (2) and the fact that \( \bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} V_n \subset X \setminus A \). Then we show \( (\gamma_1 \circ \sum_{i=0}^{k} \lambda_i)(a) \leq (\gamma_1 \circ s)(a) \) for each \( a \in A \). Indeed, if \( \sum_{i=0}^{k} \lambda_i(a) \) is an integer, this follows from \( \gamma_1 \) being \( \omega \)-increasing. If \( \sum_{i=0}^{k} \lambda_i(a) \) is not an integer, the fact that \( \lambda^{-1}_n((0,1]) \subset \lambda^{-1}_n(1) \) for each \( n \in \mathbb{N} \) implies that \( \sum_{i=0}^{k} \lambda_i(a) = s(a) \). Similarly, \( \gamma_1 \circ \sum_{i=0}^{k} \lambda_i \leq \gamma_1 \circ s \leq \gamma_2 \circ s \leq \gamma_2 \circ \sum_{i=0}^{k} \lambda_i \) for each \( k \in \omega \). Since \( \gamma_1 \circ \sum_{i=0}^{k} \lambda_i, \gamma_1 \circ s, \) and \( \gamma_2 \circ \sum_{i=0}^{k} \lambda_i \) are in \( C(A,Y_0) \) and \( u \) is monotone, it follows that
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u(γ₁ ∘ ∑ₖᵢ₌₀ λᵢ) ≤ u(γ₁ ∘ s) ≤ u(γ₂ ∘ ∑ₖᵢ₌₀ λᵢ) for each k ∈ ω. By (4),

u(γ₁ ∘ s)(c) ∈ ∩ₙ∈ω ∪ₜ₁∈G₁ₙ,ₜ₂∈G₂ₙ {y ∈ Y : b₁ ≤ y ≤ b₂}, a contradiction
to [1]. Hence A is strong Choquet in X. This completes the proof. ■

Lemma 2. Let Y be a topological vector space and an ordered vector
space, and Y₀ a convex subset of Y. If Y₀ has countable pseudo-character
in Y, the following conditions are equivalent:

(1) Y₀ has the almost ω-decreasing intersection property in Y;
(2) Y₀ has the ω-decreasing intersection property in Y.

Proof. (2)⇒(1) is obvious. To show (1)⇒(2), let \{yₙ\}ₙ∈ω be an increas-
ing sequence in Y₀ and \{zₙ\}ₙ∈ω a decreasing sequence in Y₀ with yₙ ≤ zₙ for
each n ∈ ω. Since Y₀ has countable pseudo-character in Y, we set G₁ₙ = \{yₙ\}
and G₂ₙ = \{zₙ\} for each n ∈ ω. Define γ₁, γ₂ : [0, ∞) → Y₀ by γ₁(r) =
(nᵣ + 1 − r)yₙᵣ + (r − nᵣ)yₙᵣ₊₁ and γ₂(r) = (nᵣ + 1 − r)zₙᵣ + (r − nᵣ)zₙᵣ₊₁,
where nᵣ ∈ ω with r ∈ [nᵣ, nᵣ + 1]. Then γ₁ and γ₂ are Y₀-valued continu-
ous, because Y₀ is convex and \{[n, n + 1] : n ∈ ω\} is a locally finite closed
collection in [0, ∞). It is easy to see that γ₁ is an ω-increasing ray and γ₂
is an ω-decreasing ray. To show γ₁(r) ≤ γ₂(s) for each r, s ∈ [0, ∞), fix
r, s ∈ [0, ∞) arbitrarily. Set n₀ = max\{nᵣ + 1, nₛ + 1\}. Then it follows from
(O)₁ and (O)₂ that

γ₁(r) = (nᵣ + 1 − r)yₙᵣ + (r − nᵣ)yₙᵣ₊₁ ≤ (nᵣ + 1 − r)yₙᵣ₊₁ + (r − nᵣ)yₙᵣ₊₁
      = yₙᵣ₊₁ ≤ yₙ₀ ≤ zₙ₀ ≤ zₙₛ₊₁ = (nₛ + 1 − s)zₙₛ₊₁ + (s − nₛ)zₙₛ₊₁
      ≤ (nₛ + 1 − s)zₙₛ + (s − nₛ)zₙₛ₊₁ = γ₂(s).

Since γ₁(n) = yₙ and γ₂(n) = zₙ for each n ∈ ω, it follows from (1) that
∩ₙ∈ω\{y ∈ Y : yₙ ≤ y ≤ zₙ\} ≠ ∅. ■

Corollary 3. A Tikhonov subspace A of a topological space X is strong
Choquet in X if there exists a monotone extender u : C∞(A, c) → CA(X, c).

Proof. Let Y = c and Y₀ = \{y ∈ c : 0 ≤ y ≤ 1\}, where 0 = (0, 0, . . .),
1 = (1, 1, . . .) ∈ c. Then Y₀ is convex and bounded in Y. Define yₙ, zₙ ∈ c,
n ∈ ω, by yₙ(m) = 1 if m = 2j + 1, j ≤ n − 1, j ∈ ω; yₙ(m) = 0
otherwise; zₙ(m) = 0 if m = 2j, j ≤ n − 1, j ∈ ω; zₙ(m) = 1 otherwise.
Then \{yₙ\}ₙ∈ω ⊂ Y₀ is increasing, \{zₙ\}ₙ∈ω ⊂ Y₀ is decreasing with yₙ ≤ zₙ
for each n ∈ ω, and ∩ₙ∈ω\{y ∈ c : yₙ ≤ y ≤ zₙ\} = ∅. By Lemma 2,
Y₀ does not have the almost ω-decreasing intersection property in c. Since
C(A, Y₀) ⊂ C∞(A, c), Theorem 1 completes the proof of Corollary 3. ■

Corollary 4. There is no monotone extender u : C∞(Q, c) → C(Rₜ, c).

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