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Sums of commuting operators with maximal regularity

by

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Abstract. Let Y be a Banach space and let $S \subset L_p$ be a subspace of an L_p space, for some $p \in (1,\infty)$. We consider two operators B and C acting on S and Y respectively and satisfying the so-called maximal regularity property. Let $\mathcal{B}$ and $\mathcal{C}$ be their natural extensions to $S(Y) \subset L_p(Y)$. We investigate conditions that imply that $\mathcal{B}+\mathcal{C}$ is closed and has the maximal regularity property. Extending theorems of Lamberton and Weis, we show in particular that this holds if Y is a UMD Banach lattice and e^{-tB} is a positive contraction on L_p for any $t \ge 0$.

1. Introduction. Let X be a Banach space. Given any $p \in (1,\infty)$ we consider the vector-valued L_p space $L_p(\mathbb{R};X)$ and we let $\mathcal{A}_X$ be the derivation operator on $L_p(\mathbb{R};X)$, defined on its natural domain $W^{1,p}(\mathbb{R};X)$. Let $-B$ be the generator of a bounded analytic semigroup on X, with domain $D(B)$. We denote by $\mathcal{B}$ the operator on $L_p(\mathbb{R};X)$ defined by $D(\mathcal{B}) = L_p(\mathbb{R};D(B))$ and $\mathcal{B}u(t) = B(u(t))$ for all u in $D(\mathcal{B})$ and t in $\mathbb{R}$. By definition we say that B has the *maximal regularity* property (MR_∞ for short) if there exists a constant $K > 0$ such that

$$\forall u \in D(\mathcal{A}_X) \cap D(\mathcal{B}), \qquad \|\mathcal{A}_X u\|_p \le K \|\mathcal{A}_X u + \mathcal{B}u\|_p. \tag{1.1}$$

This property implies that for any $T > 0$ and any $f \in L_p(0,T;X)$, the Cauchy problem

$$(\mathrm{CP})_T \quad \begin{cases} u'(t) + Bu(t) = f(t), & t \in (0,T), \\ u(0) = 0, \end{cases} \tag{1.2}$$

admits a (necessarily unique) solution $u \in W_0^{1,p}(0,T;X) \cap L_p(0,T;D(B))$. It follows e.g. from [4] or [8] that the maximal regularity property MR_∞ for B does not depend on $p \in (1,\infty)$. In 1964, de Simon [33] showed that if X is a Hilbert space then MR_∞ is satisfied by every negative generator of a bounded analytic semigroup. Then in 1987, Dore and Venni [12] showed that B satisfies MR_∞ if X is a UMD Banach space and if B has bounded imaginary powers, with an estimate $\|B^{is}\| \le K e^{\theta|s|}$ for some $\theta \in (0,\pi/2)$. Very recently, Kalton and Lancien [15] showed that the latter result does not hold

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true if we remove the assumption on imaginary powers. They proved that if X is a separable Banach lattice and if every negative generator of a bounded analytic semigroup on X satisfies MR_∞, then X is isomorphic to a Hilbert space. We refer to [4], [8], [11], and [21] for some background on MR_∞ and its variants, and to [31] for general information on UMD Banach spaces.

We are interested in the following general problem. Let $-B$ and $-C$ be the generators of two commuting bounded analytic semigroups on X, and assume that B and C satisfy MR_∞. Which additional conditions ensure that the sum $B+C$ is closed and in that case, does it satisfy MR_∞ as well? Our motivation for this problem lies in results from [13], [30], and [18] which show that in many natural situations, sufficient conditions for MR_∞, such as having bounded imaginary powers or a bounded H^∞ functional calculus, are preserved by taking sums of commuting operators. In this paper, we shall obtain positive results in cases when X is a tensor product of two Banach spaces and the operators B and C act on one of the components of the tensor product. A typical situation is that of vector-valued L_p spaces. Let Y be a Banach space, let (Ω,μ) be a measure space, and let $X = L_p(\Omega;Y)$ for some $p \in (1,\infty)$. Let B and C be negative generators of bounded analytic semigroups on $L_p(\Omega, d\mu)$ and Y respectively. It is not hard to see that $B\otimes I_Y$ and $I_{L_p}\otimes C$ admit closures $\mathcal{B}$ and $\mathcal{C}$ on X. In Section 4 (Theorem 4.3), we will show that if e^{-tB} is contractively regular for any $t \geq 0$ (see below for a definition), if Y is a UMD Banach lattice, and if C satisfies MR_∞, then $\mathcal{B}+\mathcal{C}$ is closed and satisfies MR_∞. This result extends a well known result of Lamberton [17] (see also [8, Section 5]), corresponding to the case when $C = 0$, and complements some recent work of Weis ([34], [35]). In Section 5 (Theorem 5.2), we will show that if Y is a Hilbert space, if C admits a bounded H^∞ functional calculus, and if B satisfies MR_∞, then $\mathcal{B}+\mathcal{C}$ is closed and satisfies MR_∞. This result complements [18, Theorem 1.4].

We will work in the more general context of the so-called vector-valued SL_p spaces, and will establish a general result (Theorem 4.1) from which the two theorems presented above will be deduced. For any closed subspace $S \subset L_p(\Omega, d\mu)$, called an SL_p space, and any Banach space Y, we will consider the Banach space $X = S(Y)$, defined as the closure of $S\otimes Y$ in $L_p(\Omega;Y)$, and consider operators B and C acting on S and Y respectively. Theorem 4.1 will provide a general sufficient condition ensuring that the sum $\mathcal{B}+\mathcal{C}$ of the extensions of B and C to X is closed and satisfies MR_∞. Its proof requires several preparatory results of independent interest which are established in the next two sections.

All Banach spaces considered here, including Banach lattices, are complex. Given a Banach space X, we denote by $B(X)$ the Banach algebra of all bounded linear operators on X.

2. A domination principle for contractively regular semigroups. Let $1 \le p \le \infty$, let (Ω, μ) be a measure space, and let S be a closed subspace of $L_p(\Omega, d\mu)$. It is plain that for any Banach spaces Y_1, Y_2 and any bounded operator $b : Y_1 \to Y_2$, the tensor product mapping $I_S \otimes b$ extends to a bounded operator from $S(Y_1)$ into $S(Y_2)$, with

$$\|I_S \otimes b : S(Y_1) \to S(Y_2)\| = \|b\|. \tag{2.1}$$

In particular, given a Banach space Y, the tensorization by I_S yields an isometric embedding

$$B(Y) \subset B(S(Y)). \tag{2.2}$$

The tensorization of a bounded operator on S by I_Y requires some special assumptions. We say that a bounded operator $T : S \to S$ is *regular* if there exists a constant $K > 0$ such that

$$\forall n \in \mathbb{N}^* \ \forall x_1, \ldots, x_n \in S, \qquad \|\sup_{1\le i\le n} |Tx_i|\|_p \le K\|\sup_{1\le i\le n} |x_i|\|_p. \tag{2.3}$$

We denote by $\|T\|_{\mathrm{r}}$ the smallest constant K which satisfies (2.3). Clearly $\|\ \|_{\mathrm{r}}$ is a norm on the vector space of regular operators on S. When $\|T\|_{\mathrm{r}} \le 1$ we say that T is *contractively regular* and by extension a *contractively regular semigroup* $(T_t)_{t\ge 0}$ on S is a c_0-semigroup such that for all $t \ge 0$, T_t is contractively regular. We refer to [29] for some information on regular operators on SL_p spaces. This notion extends the well known one of regular operators on L_p spaces. We recall that any bounded operator on $L_1(\Omega, d\mu)$ or on $L_\infty(\Omega, d\mu)$ is regular and that if $p \in (1, \infty)$, a bounded operator $T : L_p(\Omega, d\mu) \to L_p(\Omega, d\mu)$ is regular if and only if T is a linear combination of positive operators on $L_p(\Omega, d\mu)$ (see e.g. [24] or [32]). In particular we mention that a positive operator T on $L_p(\Omega, d\mu)$ satisfies $\|T\|_{\mathrm{r}} = \|T\|$. More generally, T is contractively regular if and only if there exists a positive contraction $\widehat{T} : L_p(\Omega, d\mu) \to L_p(\Omega, d\mu)$ such that $|T(f)| \le \widehat{T}(|f|)$ for every $f \in L_p(\Omega, d\mu)$. The following reformulation of regularity will be useful.

LEMMA 2.1. *Let $T : S \to S$ be a bounded operator. Then T is regular if and only if for any Banach space Y, the tensor product $T \otimes I_Y$ extends to a bounded operator on $S(Y)$. Furthermore, we have*

$$\|T \otimes I_Y : S(Y) \to S(Y)\| \le \|T\|_{\mathrm{r}}. \tag{2.4}$$

Proof. Note that (2.3) means that $\|T \otimes I_{\ell_n^\infty} : S(\ell_n^\infty) \to S(\ell_n^\infty)\| \le K$ for any integer $n \ge 1$. Assume that T is regular and let Y be a finite-dimensional Banach space. For any $\varepsilon > 0$, there exist an integer $n \ge 1$, a subspace $E \subset \ell_n^\infty$, and an isomorphism $b : Y \to E$ such that $\|b\| \cdot \|b^{-1}\| \le 1 + \varepsilon$. Using (2.1) twice, we obtain $\|T \otimes I_Y\| \le \|T\|_{\mathrm{r}}(1 + \varepsilon)$. Since ε is arbitrary, we obtain (2.4) for any finite-dimensional Y. The inequality for arbitrary Y follows at once because $S \otimes Y$ is dense in $S(Y)$ by definition. Conversely, the boundedness of $T \otimes I_{c_0}$ on $S(c_0)$ implies (2.3). ∎

If $T : S \subset L_p(\Omega, d\mu) \to S$ is regular and $b : Y \to Y$ is bounded, then $T \otimes b = (T \otimes I_Y)(I_S \otimes b)$ extends to a bounded operator on $S(Y)$ that we denote by $T \overline{\otimes} b$. By (2.1) and (2.4), we have

$$\|T \overline{\otimes} b : S(Y) \to S(Y)\| \leq \|T\|_{\mathrm{r}} \|b\|.$$

If B (resp. C) is a closed operator on S (resp. Y), then the tensor product $B \otimes I_Y$ (resp. $I_S \otimes C$), defined on $D(B) \otimes Y$ (resp. $S \otimes D(C)$), is closable on $S(Y)$ (see e.g. [19, Lemma 1]). These closures will be denoted by $\mathcal{B}$ and $\mathcal{C}$.

If $(T_t)_{t \geq 0}$ is a contractively regular semigroup on S, then $(T_t \overline{\otimes} I_Y)_{t \geq 0}$ is obviously a contraction c_0-semigroup on $S(Y)$. It is easy to check that if $-B$ is the generator of $(T_t)_{t \geq 0}$, then $-\mathcal{B}$ is the generator of $(T_t \overline{\otimes} I_Y)_{t \geq 0}$. Note that similarly, if $(V_t)_{t \geq 0}$ is a bounded c_0-semigroup on Y with generator $-C$, then $(I_S \overline{\otimes} V_t)_{t \geq 0}$ is a bounded c_0-semigroup on $S(Y)$ with generator $-\mathcal{C}$. It should be noticed that if $(V_t)_{t \geq 0}$ extends to a bounded analytic semigroup on Y, then the same property holds for $(I_S \overline{\otimes} V_t)_{t \geq 0}$ on $S(Y)$.

We now wish to establish a domination principle for contractively regular semigroups on S which will extend a famous inequality of Coifman–Weiss [7, Corollary 4.17]. Our result is also clearly related to [5, Theorem 5.6], and actually extends it. We start with the discrete counterpart of this principle. Let us denote by σ the shift operator on $\ell_p(\mathbb{Z})$ defined by

$$\forall (x_n)_{n \in \mathbb{Z}} \in \ell_p, \qquad \sigma[(x_n)_{n \in \mathbb{Z}}] = (x_{n-1})_{n \in \mathbb{Z}}.$$

LEMMA 2.2. *Let S be a closed subspace of $L_p(\Omega, d\mu)$ for some $p \in [1, \infty)$. Let T be a contractively regular operator on S. Let Y be a Banach space. Then for any sequence $b \in \ell_1(\mathbb{N}; B(Y))$ we have*

$$\Big\| \sum_{k \geq 0} T^k \overline{\otimes} b(k) \Big\|_{B(S(Y))} \leq \Big\| \sum_{k \geq 0} \sigma^k \overline{\otimes} b(k) \Big\|_{B(\ell_p(\mathbb{Z}; Y))}. \tag{2.5}$$

Proof. Regard $T : S \to S \subset L_p(\Omega, d\mu)$ as having values in $L_p(\Omega, d\mu)$. Since T is contractively regular, it admits an extension $\widetilde{T} : L_p(\Omega, d\mu) \to L_p(\Omega, d\mu)$ such that $\|\widetilde{T}\|_{\mathrm{r}} = \|T\|_{\mathrm{r}}$. This extension property of regular operators is due to Pisier [29, Theorem 3]. Then for any sequence $b \in \ell_1(\mathbb{N}; B(Y))$ we have

$$\Big\| \sum_{k \geq 0} T^k \overline{\otimes} b(k) \Big\|_{B(S(Y))} \leq \Big\| \sum_{k \geq 0} \widetilde{T}^k \overline{\otimes} b(k) \Big\|_{B(L_p(\Omega; Y))}. \tag{2.6}$$

We can now apply Akcoglu's dilation theorem [1] and its generalizations ([6], [26]), which ensure that there exist a measure space (Ω', μ'), two contractively regular operators

$$J : L_p(\Omega, d\mu) \to L_p(\Omega', d\mu') \quad \text{and} \quad P : L_p(\Omega', d\mu') \to L_p(\Omega, d\mu),$$

and an invertible isometric operator $U : L_p(\Omega', d\mu') \to L_p(\Omega', d\mu')$ such that both U and U^{-1} are contractively regular, and

$$\forall n \in \mathbb{N}, \qquad \widetilde{T}^n = PU^n J. \tag{2.7}$$

Note by Lemma 2.1 that $\|U\|_{\mathrm{r}} \le 1$ and $\|U^{-1}\|_{\mathrm{r}} \le 1$ imply that $\|U^n \overline{\otimes} I_Y\| = 1$ for any $n \in \mathbb{Z}$. The Coifman–Weiss transference principle [7] (in fact, a vector-valued version of it) can therefore be applied to the sequence $(U^n \overline{\otimes} I_Y)_{n\in\mathbb{Z}}$ and we find that for any sequence $b \in \ell_1(\mathbb{Z}; B(Y))$,

$$\Big\|\sum_{k\in\mathbb{Z}} U^k \overline{\otimes} b(k)\Big\|_{B(L_p(\Omega';Y))} \le \Big\|\sum_{k\in\mathbb{Z}} \sigma^k \overline{\otimes} b(k)\Big\|_{B(\ell_p(\mathbb{Z};Y))}. \tag{2.8}$$

Assume that b is supported by $\mathbb{N}$. From (2.7), we deduce

$$\sum_{k\ge 0} \widetilde{T}^k \otimes b(k) = (P \otimes I_Y)\Big(\sum_{k\ge 0} U^k \otimes b(k)\Big)(J \otimes I_Y) \tag{2.9}$$

on $L_p(\Omega, d\mu) \otimes Y$. Since P and J are contractively regular, Lemma 2.1 implies that

$$\|P \overline{\otimes} I_Y\|_{B(L_p(\Omega';Y),L_p(\Omega;Y))} \le 1 \quad \text{and} \quad \|J \overline{\otimes} I_Y\|_{B(L_p(\Omega;Y),L_p(\Omega';Y))} \le 1.$$

Therefore (2.6), (2.8) and (2.9) give the desired inequality (2.5). ∎

We shall denote by $(U_t)_{t\ge 0}$ the translation semigroup on $L_p(\mathbb{R})$ defined for any f in $L_p(\mathbb{R})$ by $U_t(f)(s) = f(s-t)$, $s \in \mathbb{R}$. Note that it is obviously a contractively regular semigroup. The following result is a generalization of [5, Theorem 5.6], which we recover when $S = L_p(\Omega, d\mu)$, the T_t's are positive contractions, and b is scalar-valued.

Theorem 2.3. *Let Y be a Banach space and S be a closed subspace of $L_p(\Omega, d\mu)$ for some $p \in [1,\infty)$. Let $(T_t)_{t\ge 0}$ be a contractively regular semigroup on S. Then for any $b \in L_1(\mathbb{R}_+; B(Y))$ we have*

$$\Big\|\int_0^\infty T_t \overline{\otimes} b(t)\,dt\Big\|_{B(S(Y))} \le \Big\|\int_0^\infty U_t \overline{\otimes} b(t)\,dt\Big\|_{B(L_p(\mathbb{R};Y))}. \tag{2.10}$$

Proof. We shall only outline the proof. Indeed we follow a well known discretization principle introduced in [7], and whose details appear e.g. in [5, Appendix]. First note that compactly supported functions in $L_1(\mathbb{R}_+; B(Y))$ are dense in $L_1(\mathbb{R}_+; B(Y))$, hence we may assume that the support of b is compact. Under this assumption, there exist sequences $b_N = (b_N(k))_{k\ge 0} \in \ell_1(\mathbb{N}; B(Y))$ such that for all $x \in S(Y)$,

$$\int_0^\infty (T_t \overline{\otimes} b(t))(x)\,dt = \lim_{N\to\infty} \sum_{k\ge 0} (T_{1/N}^k \overline{\otimes} b_N(k))(x) \tag{2.11}$$

and for any $N \ge 1$,

$$(2.12)\qquad \Big\|\sum_{k\geq 0}\sigma^k \,\overline{\otimes}\, b_N(k)\Big\|_{B(\ell_p(\mathbb{Z};Y))} \leq \Big\|\int\limits_0^\infty U_t \,\overline{\otimes}\, b(t)\,dt\Big\|_{B(L_p(\mathbb{R};Y))}.$$

Since $\|T_{1/N}\|_{\mathrm{r}} \leq 1$, we can apply Lemma 2.2 to obtain

$$(2.13)\quad \Big\|\sum_{k\geq 0}(T_{1/N}^k \,\overline{\otimes}\, b_N(k))(x)\Big\|_{S(Y)} \leq \Big\|\sum_{k\geq 0}\sigma^k \,\overline{\otimes}\, b_N(k)\Big\|_{B(\ell_p(\mathbb{Z};Y))}\|x\|_{S(Y)}.$$

The estimate (2.10) follows from (2.11)–(2.13). ■

3. Generalized H^∞ functional calculus for generators of contractively regular semigroups. H^∞ functional calculus for generators of bounded semigroups, or more generally for sectorial operators, was introduced by McIntosh [23] on Hilbert spaces and then developed on general Banach spaces in [10]. Its deep connections with maximal regularity are well known; see e.g. [21] for a survey. Here we shall especially use the so-called generalized H^∞ functional calculus introduced in [2]. This approach was already exploited in [18], [19], and [22]. We briefly recall the relevant definitions and refer to the papers quoted above for complements.

For any $\omega \in (0,\pi)$, let Σ_ω be the set of all $z \in \mathbb{C}^*$ such that $|\mathrm{Arg}(z)| < \omega$. Given a linear operator A on a Banach space X, we denote by $D(A)$, $R(A)$, and $N(A)$ the domain, range and kernel of A respectively. We denote by $\sigma(A)$ the spectrum of A and we let $\varrho(A)$ be the resolvent set of A. For any $\lambda \in \varrho(A)$, we denote by $R(\lambda, A) = (\lambda I_X - A)^{-1} \in B(X)$ the corresponding resolvent operator. We say that A is *sectorial* of type $\omega \in (0,\pi)$ if A is closed, densely defined, with the property that $\sigma(A) \subset \overline{\Sigma_\omega}$ and

$$\forall \theta \in (\omega,\pi)\ \exists C > 0\ \forall z \in (\overline{\Sigma_\theta})^{\mathrm{c}}, \qquad \|zR(z,A)\| \leq C.$$

We recall that the negative generator of a bounded c_0-semigroup is sectorial of type $\pi/2$ and that an operator $-A$ is the generator of a bounded analytic semigroup if and only if A is sectorial of type strictly less than $\pi/2$.

Given a sectorial operator A of type $\omega \in (0,\pi)$ we define its commutant by

$$E_A = \{T \in B(X) : \forall \lambda \in \varrho(A),\ TR(\lambda,A) = R(\lambda,A)T\}.$$

For any $\theta \in (\omega,\pi)$, we let $H^\infty(\Sigma_\theta; E_A)$ be the space of all bounded analytic functions $F : \Sigma_\theta \to E_A$. This is a Banach algebra for the norm

$$\|F\|_{H^\infty(\Sigma_\theta;E_A)} = \sup\{\|F(z)\|_{B(X)} : z \in \Sigma_\theta\}.$$

We then define the (non-closed) subalgebra

$$H_0^\infty(\Sigma_\theta;E_A) = \Big\{F \in H^\infty(\Sigma_\theta;E_A) : \text{there are } s, C > 0 \text{ such that}$$
$$\|F(z)\|_{B(X)} \leq C\frac{|z|^s}{(1+|z|)^{2s}} \text{ for } z \in \Sigma_\theta\Big\}.$$

Let $\omega < \omega' < \theta < \pi$, and let $\Gamma_{\omega'}$ be the path defined by

$$\Gamma_{\omega'}(t) = \begin{cases} -te^{i\omega'}, & t \in \mathbb{R}_-, \\ te^{-i\omega'}, & t \in \mathbb{R}_+. \end{cases}$$

Then for any function $F \in H_0^\infty(\Sigma_\theta; E_A)$ we set

$$u_A(F) = \frac{1}{2\pi i} \int\limits_{\Gamma_{\omega'}} F(\lambda) R(\lambda, A)\, d\lambda. \tag{3.1}$$

Since A is sectorial and $F \in H_0^\infty(\Sigma_\theta; E_A)$, $u_A(F)$ is well defined and belongs to $B(X)$. Furthermore, the definition (3.1) does not depend on the choice of $\omega' \in (\omega, \theta)$ and the mapping $u_A : H_0^\infty(\Sigma_\theta; E_A) \to B(X)$ is an algebra homomorphism. Note that u_A is not bounded in general. If we moreover assume that $N(A) = \{0\}$ and $R(A)$ is dense in X, then for any $F \in H^\infty(\Sigma_\theta; E_A)$ we may define a possibly unbounded operator $u_A(F)$ as follows. We let φ be the scalar-valued function defined by $\varphi(z) = z/(1+z)^2$. Then for $F \in H^\infty(\Sigma_\theta; E_A)$, the product function $F\varphi$ belongs to $H_0^\infty(\Sigma_\theta; E_A)$ and we set

$$u_A(F) = \varphi(A)^{-1} u_A(F\varphi),$$

with domain equal to

$$D(u_A(F)) = \{x \in X : u_A(F\varphi)(x) \in D(A) \cap R(A)\}.$$

The point here is that the range of $\varphi(A)$ is equal to $D(A) \cap R(A)$ and that the latter space is dense in X. Consequently, the operator $u_A(F)$ is a closed and densely defined operator, with $D(A) \cap R(A) \subset D(u_A(F))$. Note that $u_A(F)$ is unbounded in general. If F is scalar-valued (i.e. with values in $\mathrm{Span}\{I_X\}$), then the operator $u_A(F)$ is simply denoted by $F(A)$.

Let Y be a Banach space. Let $p \in [1, \infty)$, let $S \subset L_p(\Omega, d\mu)$ and let $-B$ be the generator of a contractively regular semigroup $(T_t)_{t \geq 0}$ on S. We consider the Banach space $X = S(Y)$. Recall that we denote by $\mathcal{B}$ the negative generator of $(T_t \mathbin{\overline{\otimes}} I_Y)_{t \geq 0}$ on X. This operator is then sectorial of type $\pi/2$. Via the isometric embedding (2.2), we may consider $B(Y)$ as a (closed) subalgebra of the commutant $E_{\mathcal{B}}$; hence for any $\theta > \pi/2$, we may regard $H_0^\infty(\Sigma_\theta; B(Y))$ as a subalgebra of $H_0^\infty(\Sigma_\theta; E_{\mathcal{B}})$, which allows us to define $u_{\mathcal{B}}(F)$ for any $F \in H_0^\infty(\Sigma_\theta; B(Y))$. Likewise we may regard $B(Y)$ as a subspace of $B(L_p(\mathbb{R}; Y))$, which is actually included in the commutant algebra $E_{\mathcal{A}_Y}$ of the derivation operator $\mathcal{A}_Y$, and we will therefore consider operators $u_{\mathcal{A}_Y}(F)$ for $F \in H_0^\infty(\Sigma_\theta; B(Y))$. Note that $\mathcal{A}_Y$ is 1-1 with a dense range.

Theorem 3.1. *Let Y be a Banach space. Let $S \subset L_p(\Omega, d\mu)$ for some $p \in [1, \infty)$, and let $-B$ be the generator of a contractively regular semigroup on S.*

(i) *For any* $\theta \in (\pi/2, \pi)$ *and any* $F \in H_0^\infty(\Sigma_\theta; B(Y))$,

$$\|u_{\mathcal{B}}(F)\|_{B(S(Y))} \le \|u_{\mathcal{A}_Y}(F)\|_{B(L_p(\mathbb{R};Y))}. \tag{3.2}$$

(ii) *Assume that* B *is* 1-1 *with dense range, and let* $F \in H^\infty(\Sigma_\theta; B(Y))$ *for some* $\theta \in (\pi/2, \pi)$. *If* $u_{\mathcal{A}_Y}(F)$ *is bounded on* $L_p(\mathbb{R}; Y)$, *then* $u_{\mathcal{B}}(F)$ *is bounded on* $S(Y)$.

Proof. Let $\theta \in (\pi/2, \pi)$ and let $F \in H_0^\infty(\Sigma_\theta; B(Y))$. We choose $\omega' \in (\pi/2, \theta)$. By the definition of $H_0^\infty(\Sigma_\theta; B(Y))$, the integral $\int_{\Gamma_{\omega'}} \|F(\lambda)\| \left|\frac{d\lambda}{\lambda}\right|$ is finite and hence

$$\int_{\Gamma_{\omega'}} \int_0^\infty \|F(\lambda)\| \cdot |e^{\lambda t}| \, |d\lambda| \, dt < \infty.$$

By Fubini's Theorem, we may therefore define $b \in L_1(\mathbb{R}_+; B(Y))$ by letting

$$b(t) = -\frac{1}{2\pi i} \int_{\Gamma_{\omega'}} F(\lambda) e^{\lambda t} \, d\lambda$$

and for any $f \in S$ and any $y \in Y$ we have

$$\begin{aligned}
\int_0^\infty (T_t \otimes b(t))(f \otimes y) \, dt &= -\frac{1}{2\pi i} \int_0^\infty \int_{\Gamma_{\omega'}} T_t(f) \otimes F(\lambda)(y) e^{\lambda t} \, d\lambda \, dt \\
&= \frac{1}{2\pi i} \int_{\Gamma_{\omega'}} \Big(-\int_0^\infty T_t(f) e^{\lambda t} \, dt\Big) \otimes F(\lambda)(y) \, d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_{\omega'}} (R(\lambda, B) \otimes F(\lambda))(f \otimes y) \, d\lambda.
\end{aligned}$$

Applying formula (3.1), we deduce that for any $x \in S \otimes Y$,

$$u_{\mathcal{B}}(F)(x) = \int_0^\infty (T_t \otimes b(t))(x) \, dt.$$

Similarly,

$$u_{\mathcal{A}_Y}(F)(x) = \int_0^\infty (U_t \otimes b(t))(x) \, dt.$$

Indeed, $\mathcal{A}_Y$ is the negative generator of $(U_t \overline{\otimes} I_Y)_{t \ge 0}$ on $L_p(\mathbb{R}; Y)$. The inequality (3.2) therefore follows from (2.10).

Let us show (ii). We assume that B (hence $\mathcal{B}$) is 1-1 with dense range. We let $\theta \in (\pi/2, \pi)$ and $F \in H^\infty(\Sigma_\theta; B(Y))$ and we assume that $u_{\mathcal{A}_Y}(F)$ is bounded. We introduce the sequence $(\varphi_n)_{n \ge 1}$ of rational functions defined by

$$\varphi_n(z) = \frac{n^2 z}{(n+z)(1+nz)}.$$

Let $x \in X = S(Y)$. Each $F\varphi_n$ belongs to $H_0^\infty(\Sigma_\theta; B(Y))$, hence by (i) we have

$$\|u_{\mathcal{B}}(F\varphi_n)(x)\|_{S(Y)} \le \|u_{\mathcal{A}_Y}(F\varphi_n)\|_{B(L_p(\mathbb{R};Y))}\|x\|_{S(Y)}.$$

For any $n \ge 1$, $\varphi_n(\mathcal{B})(x)$ belongs to $R(\mathcal{B}) \cap D(\mathcal{B})$, hence to $D(u_{\mathcal{B}}(F))$, and since $u_{\mathcal{B}}$ is a homomorphism on $H_0^\infty(\Sigma_\theta; B(Y))$, we see that $u_{\mathcal{B}}(F\varphi_n)(x) = u_{\mathcal{B}}(F)[\varphi_n(\mathcal{B})(x)]$. Similarly, $u_{\mathcal{A}_Y}(F\varphi_n) = u_{\mathcal{A}_Y}(F)\varphi_n(\mathcal{A}_Y)$. Consequently,

$$\|u_{\mathcal{B}}(F)[\varphi_n(\mathcal{B})(x)]\|_{S(Y)} \le \|u_{\mathcal{A}_Y}(F)\|_{B(L_p(\mathbb{R};Y))}\|\varphi_n(\mathcal{A}_Y)\|_{B(L_p(\mathbb{R};Y))}\|x\|_{S(Y)}. \tag{3.3}$$

By the sectoriality of $\mathcal{B}$, the sequence $\varphi_n(\mathcal{B})$ strongly converges to the identity on X (see e.g. [21]). Again the sectoriality of $\mathcal{A}_Y$ implies that the sequence $(\varphi_n(\mathcal{A}_Y))_{n\ge 0}$ is bounded. Hence the boundedness of $u_{\mathcal{B}}(F)$ follows from (3.3). ∎

We shall deduce two corollaries from Theorem 3.1. If A is a sectorial operator of type $\omega \in (0,\pi)$ on a Banach space X, and if $\theta \in (\omega,\pi)$, we say that A admits a *bounded $H^\infty(\Sigma_\theta)$ functional calculus* if there exists a constant $K > 0$ such that

$$\forall F \in H_0^\infty(\Sigma_\theta), \qquad \|F(A)\|_{B(X)} \le K\|F\|_{H^\infty(\Sigma_\theta)}.$$

We recall (see [23], [10]) that if A is 1-1 with dense range, then this is equivalent to the property that $F(A)$ is a bounded operator for any $F \in H^\infty(\Sigma_\theta)$. It was proved in [9] and [14] that negative generators of contractively regular semigroups on L_p spaces $(1 < p < \infty)$ admit a bounded H^∞ functional calculus. We provide a generalization to subspaces of L_p spaces.

Corollary 3.2. *Let $p \in (1,\infty)$, let $S \subset L_p(\Omega, d\mu)$ be an SL_p space, and let $-B$ be the generator of a contractively regular semigroup on S. Then for any $\theta > \pi/2$, B admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof. We fix $\theta > \pi/2$ and apply Theorem 3.1 with $Y = \mathbb{C}$. For any $F \in H_0^\infty(\Sigma_\theta)$, we have $\|F(B)\| \le \|F(A)\|$, where $A = \mathcal{A}_{\mathbb{C}}$ is the derivation on $L_p(\mathbb{R})$. This operator admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus (see [9]), hence the result follows at once. ∎

Remark 3.3. If Y is UMD, the operator $\mathcal{A}_Y$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, hence for any B as in Corollary 3.2 and any UMD Banach space Y, the operator $\mathcal{B}$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $S(Y)$.

Corollary 3.4. *Let $p \in (1,\infty)$ and let $-B$ be the generator of a contractively regular semigroup on some SL_p space $S \subset L_p(\Omega, d\mu)$. Let C be an operator on a Banach space Y which satisfies* MR_∞. *Then the operator $\mathcal{B} + \mathcal{C} : D(\mathcal{B}) \cap D(\mathcal{C}) \to S(Y)$ is closed, and there exists a constant $K > 0$*

such that

$$(3.4)\qquad \forall u\in D(\mathcal{B})\cap D(\mathcal{C}),\qquad \|\mathcal{C}u\|_{S(Y)}\leq K\|(\mathcal{B}+\mathcal{C})u\|_{S(Y)}.$$

Proof. Here we denote by $\mathcal{C}$ the closure of $I_S\otimes C$ on $S(Y)$ and to avoid confusion, we denote by $\mathcal{C}_1$ the closure of $I_{L_p(\mathbb{R})}\otimes C$ on $L_p(\mathbb{R};Y)$. Since C is sectorial of type $\omega<\pi/2$, the function $F_C : z\mapsto z(z+C)^{-1}$ belongs to $H^\infty(\Sigma_\theta;B(Y))$ for some $\theta>\pi/2$. We assume that C satisfies MR_∞, hence applying (1.1), there is a constant $K>0$ such that

$$\forall v\in D(\mathcal{A}_Y)\cap D(\mathcal{C}_1),\qquad \|\mathcal{A}_Y v\|_{L_p(\mathbb{R};Y)}\leq K\|(\mathcal{A}_Y+\mathcal{C}_1)v\|_{L_p(\mathbb{R};Y)}.$$

By [18, Proposition 2.6], this implies that $u_{\mathcal{A}_Y}(F_C)$ is bounded. Assume for simplicity that B is 1-1 with dense range. Then Theorem 3.1(ii) implies that $u_{\mathcal{B}}(F_C)$ is bounded, hence again by [18, Proposition 2.6], we see that $\mathcal{B}+\mathcal{C}$ is closed and that (3.4) holds. When B is not 1-1 with dense range we can consider the operators $B+\varepsilon I_S$ for $\varepsilon>0$, which are invertible and are negative generators of contractively regular semigroups. Then it is easy to check that they satisfy (3.4) with a constant K not depending on $\varepsilon>0$. We conclude by letting ε tend to 0. ∎

REMARK 3.5. Let $S\subset L_p(\Omega,d\mu)$, Y, B, and C be as in Corollary 3.4. Then let $(T_t)_{t\geq0}$ and $(V_t)_{t\geq0}$ be the semigroups generated by $-B$ and $-C$ on S and Y respectively. It follows from [25, A-I, 3.7] that $\mathcal{B}+\mathcal{C}$ is the negative generator of the semigroup $(T_t\overline{\otimes}V_t)_{t\geq0}$. We may derive two simple properties from this fact. First, if $\mathcal{B}$ is sectorial of type $<\pi/2$, then the operator $\mathcal{B}+\mathcal{C}$ is sectorial of type $<\pi/2$ as well. Second, assume that $Y\subset L_p(\Omega',\mu')$ is also an SL_p space. Then using Fubini, we may regard $S(Y)\subset L_p(\Omega\times\Omega',\mu\otimes\mu')$ as an SL_p space in an obvious way. Assume moreover that $(V_t)_{t\geq0}$ is contractively regular. Using the identity $S(Y)(\ell_n^\infty)=S(Y(\ell_n^\infty))=Y(S(\ell_n^\infty))$ for any $n\geq1$, it is easy to check that $(T_t\overline{\otimes}V_t)_{t\geq0}$ is also contractively regular. Thus the assumption that $-B$ and $-C$ generate contractively regular semigroups implies that the same is true for $-(\mathcal{B}+\mathcal{C})$.

4. A generalization of theorems of Lamberton and Weis on maximal regularity. For operators B and C as in Corollary 3.4, the next general result gives a sufficient condition under which the sum $\mathcal{B}+\mathcal{C}$ satisfies MR_∞.

THEOREM 4.1. *Let $p\in(1,\infty)$, let S be a closed subspace of some $L_p(\Omega,d\mu)$ and let Y be a Banach space. Let $-B$ be the generator of a contractively regular semigroup on S and assume that $\mathcal{B}$ is sectorial of type strictly less than $\pi/2$ and satisfies* MR_∞ *on $S(Y)$. Then for any operator C on Y satisfying* MR_∞, *the sum $\mathcal{B}+\mathcal{C}$ is closed and satisfies* MR_∞ *on $S(Y)$.*

Proof. We let $X = S(Y)$. We know from Corollary 3.4 and Remark 3.5 that $\mathcal{C} + \mathcal{B}$ is closed and sectorial of type strictly less than $\pi/2$ on X. Let A be the derivation operator on $L_p(\mathbb{R})$, and let $\Delta = D(A) \otimes D(B) \otimes D(C) \subset L_p(\mathbb{R}) \otimes S \otimes Y \subset L_p(\mathbb{R}; X)$. Using the bounded net of mappings $nR(-n, A) \overline{\otimes}\, n'R(-n', B) \overline{\otimes}\, n''R(-n'', C) : L_p(\mathbb{R}; X) \to \Delta \subset L_p(\mathbb{R}; X)$, for $n, n', n'' \geq 1$, it is not hard to see that $\mathcal{B} + \mathcal{C}$ satisfies MR_∞ provided that there exists a constant $K > 0$ such that for any $u \in \Delta$,

$$\|(A \otimes I_S \otimes I_Y)u\| \leq K\|(A \otimes I_S \otimes I_Y + I_{L_p} \otimes B \otimes I_Y + I_{L_p} \otimes I_S \otimes C)u\|. \tag{4.1}$$

Let $\mathcal{B}_0$ be the closure of $I_{L_p} \otimes B$ on $L_p(\mathbb{R}; S)$. Our assumption that $\mathcal{B}$ satisfies MR_∞ implies that B satisfies MR_∞, hence by Corollary 3.4 and Remark 3.5, $\mathcal{B}_0 + \mathcal{A}_S$ is the negative generator of a contractively regular semigroup on $L_p(\mathbb{R}; S)$. Using the identification $L_p(\mathbb{R}; S)(Y) = L_p(\mathbb{R}; X)$ and applying Corollary 3.4, we deduce that there is a constant $K_1 > 0$ such that for any $u \in \Delta$,

$$\|(A \otimes I_S \otimes I_Y + I_{L_p} \otimes B \otimes I_Y)u\| \leq K_1\|(A \otimes I_S \otimes I_Y + I_{L_p} \otimes B \otimes I_Y + I_{L_p} \otimes I_S \otimes C)u\|.$$

We assumed that $\mathcal{B}$ satisfies MR_∞ on X, hence we have an estimate $\|(A \otimes I_S \otimes I_Y)u\| \leq K_2\|(A \otimes I_S \otimes I_Y + I_{L_p} \otimes B \otimes I_Y)u\|$ on Δ, whence (4.1) with $K = K_1K_2$. ∎

REMARK 4.2. Let S, Y, B, C, $(T_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ be as in Theorem 4.1 and Remark 3.5. Then our Theorem 4.1 says that if $(T_t \overline{\otimes} I_Y)_{t\geq 0}$ extends to a bounded analytic semigroup on $S(Y)$ whose negative generator satisfies MR_∞, and if that of $(I_S \overline{\otimes} V_t)_{t\geq 0}$ satisfies MR_∞, then the negative generator of the product semigroup $(T_t \overline{\overline{\otimes}} V_t)_{t\geq 0}$ satisfies MR_∞ as well.

We now turn to the special case when $S = L_p(\Omega, d\mu)$, with $p \in (1, \infty)$. Let $(T_t)_{t\geq 0}$ be a bounded analytic semigroup on $L_p(\Omega, d\mu)$. It was proved by Weis [35, Section 4] that if in addition, $(T_t)_{t\geq 0}$ is a contractively regular semigroup, then its negative generator satisfies MR_∞. (In fact Weis only stated this result in the case when the T_t's are positive contractions but his proof works as well in the general case.) Recall that Lamberton [17] had obtained the same conclusion under the assumption that for any $t \geq 0$, T_t extends to contractions from $L_1(\Omega, d\mu)$ into itself and from $L_\infty(\Omega, d\mu)$ into itself. It should be noticed that Weis's theorem contains Lamberton's as a special case. Indeed, using interpolation (see [3]), it is easy to see that if a linear operator $T : L_p(\Omega, d\mu) \to L_p(\Omega, d\mu)$ is both contractive on $L_1(\Omega, d\mu)$ and on $L_\infty(\Omega, d\mu)$, then $\|T\|_{\mathrm{r}} \leq 1$. It was observed in [8] that Lamberton's Theorem may be extended to $L_p(\Omega; Y)$, provided that Y is any UMD Banach lattice. Here is an extension of these results.

Theorem 4.3. *Let Y be a UMD Banach lattice, let (Ω, μ) be a measure space, and let $p \in (1, \infty)$. Let $(T_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ be two bounded analytic semigroups on $L_p(\Omega, d\mu)$ and Y respectively. Assume that $\|T_t\|_r \leq 1$ for any $t \geq 0$.*

(i) *$(T_t \overline{\otimes} I_Y)_{t\geq 0}$ extends to a bounded analytic semigroup on $L_p(\Omega; Y)$ whose negative generator satisfies* MR_∞.

(ii) *If the negative generator of $(V_t)_{t\geq 0}$ satisfies MR_∞, then the negative generator of $(T_t \overline{\otimes} V_t)_{t\geq 0}$ satisfies MR_∞ as well on $L_p(\Omega; Y)$.*

Proof. Clearly (ii) follows from (i), Theorem 4.1, and Remark 4.2 hence we only need to prove (i). We shall use complex interpolation, for which we refer to [3]. Improving an earlier result of Pisier [27], Rubio de Francia [31, Part IIIc] showed the following extrapolation result. Given a UMD Banach lattice Y, there exist a Hilbert space H_0 and a UMD Banach space Y_0 such that $Y = [H_0, Y_0]_\alpha$ for some $\alpha \in (0, 1)$. We then have

$$L_p(\Omega; Y) = [L_p(\Omega; H_0), L_p(\Omega; Y_0)]_\alpha$$

by [3, Theorem 5.1.2]. We let B be the negative generator of $(T_t)_{t\geq 0}$. Then we denote by $\mathcal{B}_0$, $\mathcal{B}_\alpha$ and $\mathcal{B}_1$ the negative generators of $(T_t \overline{\otimes} I_{H_0})_{t\geq 0}$, $(T_t \overline{\otimes} I_Y)_{t\geq 0}$, and $(T_t \overline{\otimes} I_{Y_0})_{t\geq 0}$ respectively.

Assume for a while that these operators are invertible, so that we may consider their imaginary powers. Our goal is to show that

$$\exists K > 0\ \exists \theta < \pi/2\ \forall s \in \mathbb{R}, \qquad \|\mathcal{B}_\alpha^{is}\| \leq K e^{\theta|s|}. \tag{4.2}$$

Once it is proved, we can conclude as follows. By [30, Theorem 2], this estimate shows that $-\mathcal{B}_\alpha$ generates a bounded analytic semigroup on $L_p(\Omega; Y)$. Furthermore Y is a UMD Banach space, hence $L_p(\Omega; Y)$ is UMD as well and so by [12], (4.2) ensures that $\mathcal{B}_\alpha$ satisfies MR_∞.

We now proceed to the proof of (4.2). It follows from [16, Corollary 5.2] (and its proof) that B admits a bounded $H^\infty(\Sigma_{\theta_0})$ functional calculus for some $\theta_0 < \pi/2$. In particular, there is a constant $K_0 > 0$ such that $\|B^{is}\| \leq K_0 e^{\theta_0|s|}$ for any $s \in \mathbb{R}$. Moreover the space H_0 is a Hilbert space, hence for any $T \in B(L_p(\Omega))$, the operator $T \otimes I_{H_0}$ extends to a bounded operator of norm equal to $\|T\|$ on $L_p(\Omega; H_0)$. Since $\mathcal{B}_0^{is}$ is the closure of $B^{is} \otimes I_{H_0}$ for any $s \in \mathbb{R}$, we obtain

$$\exists K_0 > 0\ \exists \theta_0 < \pi/2\ \forall s \in \mathbb{R}, \qquad \|\mathcal{B}_0^{is}\| \leq K_0 e^{\theta_0|s|}. \tag{4.3}$$

On the other hand, since Y_0 is UMD the operator $\mathcal{B}_1$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \pi/2$. Indeed, this is implicit in [5]; see also Remark 3.3. In particular,

$$\forall \theta_1 > \pi/2\ \exists K_1 > 0\ \forall s \in \mathbb{R}, \qquad \|\mathcal{B}_1^{is}\| \leq K_1 e^{\theta_1|s|}. \tag{4.4}$$

We then choose θ_1 such that

$$\theta = (1-\alpha)\theta_0 + \alpha\theta_1 < \pi/2. \tag{4.5}$$

By construction, the imaginary powers $\mathcal{B}_0^{is}$, $\mathcal{B}_\alpha^{is}$ and $\mathcal{B}_1^{is}$ are compatible, hence by interpolation,

$$\forall s \in \mathbb{R}, \quad \|\mathcal{B}_\alpha^{is}\| \le \|\mathcal{B}_0^{is}\|^{1-\alpha}\|\mathcal{B}_1^{is}\|^{\alpha}.$$

The estimate (4.2) now follows from (4.3)–(4.5), with $K = K_0^{1-\alpha}K_1^{\alpha}$.

The general case can be deduced as follows. For any $\varepsilon > 0$, replace $(T_t)_{t\ge 0}$ by $(e^{-\varepsilon t}T_t)_{t\ge 0}$. Then $\mathcal{B}_0$, $\mathcal{B}_\alpha$ and $\mathcal{B}_1$ are replaced by $\mathcal{B}_0 + \varepsilon I$, $\mathcal{B}_\alpha + \varepsilon I$ and $\mathcal{B}_1 + \varepsilon I$. These operators are invertible, hence the preceding reasoning can be applied to them. The point is that the constants K_0 and K_1 appearing in (4.3) and (4.4) can be chosen to be independent of $\varepsilon > 0$. Indeed, this follows from the boundedness of the H^∞ functional calculi of $\mathcal{B}_0$ and $\mathcal{B}_1$. Consequently, (4.2) is now replaced by

$$\exists K > 0\ \exists \theta < \pi/2\ \forall \varepsilon > 0\ \forall s \in \mathbb{R}, \qquad \|(\mathcal{B}_\alpha + \varepsilon I)^{is}\| \le K e^{\theta|s|}.$$

Applying [12], we obtain an estimate $\|\mathcal{A}_X u\| \le K'\|\mathcal{A}_X u + \overline{I_{L_p} \otimes \mathcal{B}_\alpha} u + \varepsilon u\|$ for some constant K' only depending on K, θ, p and Y. In particular K' does not depend on $\varepsilon > 0$, hence we finally get the desired inequality. ■

REMARK 4.4. It is clear from the above proof that Theorem 4.3 remains true if Y is any UMD Banach space with the property that $Y = [H, Z]_\alpha$ for some space H isomorphic to a quotient of a subspace of an L_p space (this includes Hilbert spaces), some UMD Banach space Z, and some $\alpha \in (0,1)$. This holds in particular if Y is the Schatten p-class, for $p \in (1,\infty)$, or more generally a non-commutative L_p space for $p \in (1,\infty)$. We do not know if Theorem 4.3 is true for any UMD Banach space Y.

5. Maximal regularity on Hilbert-space-valued L_p spaces. Let H be a Hilbert space, let $p \in (1,\infty)$, and let (Ω,μ) be a measure space. We let B and C be two sectorial operators of type strictly less than $\pi/2$ on H and $L_p(\Omega, d\mu)$ respectively, and denote as usual by $\mathcal{B}$ and $\mathcal{C}$ their extensions to $L_p(\Omega; H)$. We look for conditions under which the sum $\mathcal{B} + \mathcal{C}$ on $L_p(\Omega; H)$ is closed and satisfies MR_∞. It was proved in [18] that this holds true if we assume that C admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $L_p(\Omega, d\mu)$ for some $\theta < \pi/2$. In Theorem 5.2, we prove that the same result holds if the assumption of bounded H^∞ functional calculus is assigned to B (and C satisfies MR_∞).

We fix an orthonormal basis $(e_i)_{i\in I}$ on H, for some index set I. Let $(g_i)_{i\in I}$ be a family of complex independent Gaussian normal variables on a probability space (Ω', μ'). Then we let $S \subset L_p(\Omega', \mu')$ be the closed linear span of $\{g_i : i \in I\}$. For any finitely supported family of complex numbers $(t_i)_{i\in I}$, we have $\|\sum_i t_i g_i\|_p = \alpha_p\big(\sum_i |t_i|^2\big)^{1/2}$, where α_p is the L_p norm of any g_i. Thus the mapping $e_i \mapsto \alpha_p^{-1} g_i$ induces an isometric identification $H = S$, whence

$$L_p(\Omega; H) = S(L_p(\Omega, d\mu)). \tag{5.1}$$

LEMMA 5.1. *Any bounded operator $T : S \to S$ is automatically regular, with $\|T\| = \|T\|_{\mathrm{r}}$.*

Proof. When I is finite, this follows from [28, Proposition 3.7] and Lemma 2.1. The general case follows by a simple approximation argument. ∎

THEOREM 5.2. *Assume that C satisfies* MR_∞ *on $L_p(\Omega, d\mu)$ with $p \in (1,\infty)$. Let B be the negative generator of a bounded analytic semigroup on H, which admits a bounded H^∞ functional calculus. Then $\mathcal{B}+\mathcal{C}$ is closed and satisfies* MR_∞ *on $L_p(\Omega; H)$.*

Proof. The operator B satisfies MR_∞ (see [33]), hence using the identification $L_p(\mathbb{R}; L_p(\Omega; H)) = L_p(\Omega; L_p(\mathbb{R}; H))$, we see that $\mathcal{B}$ satisfies MR_∞. Let $(T_t)_{t\geq 0}$ be generated by $-B$ on H. Since B admits a bounded H^∞ functional calculus, it follows from [20, Theorem 4.3] that there exists an invertible operator R on H such that $(RT_tR^{-1})_{t\geq 0}$ is a contraction semigroup. We let $\mathcal{R} = I_{L_p(\Omega,d\mu)} \overline{\otimes} R \in B(L_p(\Omega; H))$. Then $\mathcal{R}\mathcal{B}\mathcal{R}^{-1}$ clearly satisfies MR_∞. Let us identify H with $S \subset L_p(\Omega', \mu')$ as explained above. Then $(RT_tR^{-1})_{t\geq 0}$ is contractively regular thanks to Lemma 5.1. It therefore follows from Theorem 4.1 and (5.1) that $\mathcal{R}\mathcal{B}\mathcal{R}^{-1} + \mathcal{C}$ is closed and satisfies MR_∞ on $S(L_p(\Omega, d\mu))$, hence on $L_p(\Omega; H)$. Since

$$\mathcal{R}\mathcal{B}\mathcal{R}^{-1} + \mathcal{C} = \mathcal{R}(\mathcal{B} + \mathcal{C})\mathcal{R}^{-1},$$

the result follows at once. ∎

REMARK 5.3. We wish to mention a result which essentially follows from [34] and was indicated to us by Nigel Kalton (in June 2000). Let X be a Banach space and let $-B$ and $-C$ be the generators of two commuting bounded analytic semigroups $(T_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ on X. Recall from [25, A-I, 3.7] that the product semigroup $(T_tV_t)_{t\geq 0}$ is bounded analytic and that its generator is $-(\overline{B+C})$. Kalton's observation is that if B and C satisfy MR_∞ and if X is UMD, then $\overline{B+C}$ satisfies MR_∞. Indeed, since X is UMD, it follows from [34, Theorem 4.2] that there exists $\theta > 0$ such that the two sets $\{e^{-zB} : z \in \Sigma_\theta\}$ and $\{e^{-zC} : z \in \Sigma_\theta\}$ are R-bounded. Then the "product set" $\{e^{-zB}e^{-zC} : z \in \Sigma_\theta\}$ is R-bounded as well. Hence applying [34, Theorem 4.2] again, we deduce that $\overline{B+C}$ satisfies MR_∞. This yields an alternate route to prove the second half of either Theorem 4.3 or Theorem 5.2. We also refer to [16] for recent developments.

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