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The dual of the James tree space is asymptotically uniformly convex

by

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Abstract. The dual of the James tree space is asymptotically uniformly convex.

1. Introduction. In 1950, R. C. James [J1] constructed a Banach space which is now called the James space. This space, along with its many variants (such as the James tree space [J2]) and their duals and preduals, have been a rich source for further research and results (both positive ones and counterexamples), answering many questions, several of which date back to Banach [B, 1932]. See [FG] for a splendid survey of such spaces.

This paper's main result, Theorem 5, shows that the dual JT^* of the James tree space JT is asymptotically uniformly convex. (See Section 2 for definitions.)

Schachermayer [S, Theorem 4.1] showed that JT^* has the Kadec–Klee property. It follows from Theorem 5 of this paper that JT^* enjoys the *uniform* Kadec–Klee property. Of course, the same can be said about the (unique) predual JT_* of JT. In fact, Theorem 3 shows that the modulus of asymptotic convexity of JT_* is of power type 3.

Johnson, Lindenstrauss, Preiss, and Schechtman [JLPS] showed that an asymptotically uniformly convex space has the point of continuity property and asked whether an asymptotically uniformly convex space has the Radon–Nikodým property. It is well known that both JT_* and JT^* have the point of continuity property yet fail the Radon–Nikodým property. It follows from Theorem 5 of this paper that JT^* is an asymptotically uniformly convex (*dual*) space without the Radon–Nikodým property. Thus JT_* is a *separable* asymptotically uniformly convex space without the Radon–Nikodým property. To the best of the author's knowledge, these are the first known examples of asymptotically uniformly convex spaces without the Radon–Nikodým property.

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2. Definitions and notation. Throughout this paper $\mathfrak{X}$ denotes an arbitrary (infinite-dimensional real) Banach space. If $\mathfrak{X}$ is a Banach space, then $\mathfrak{X}^*$ is its dual space, $B(\mathfrak{X})$ is its (closed) unit ball, $S(\mathfrak{X})$ is its unit sphere, $\widehat{\imath} : \mathfrak{X} \to \mathfrak{X}^{**}$ is the natural point-evaluation isometric embedding, $\widehat{x} = \widehat{\imath}(x)$ and $\widehat{\mathfrak{X}} = \widehat{\imath}(\mathfrak{X})$. If Y is a subset of $\mathfrak{X}$, then $[Y]$ is the closed linear span of Y and

$$\mathfrak{N}(\mathfrak{X}) = \{[x_i^*]_{1\le i\le n}^{\top} : x_i^* \in \mathfrak{X}^* \text{ and } n \in \mathbb{N}\},$$

$$\mathcal{W}(\mathfrak{X}^*) = \{[x_i]_{1\le i\le n}^{\perp} : x_i \in \mathfrak{X} \text{ and } n \in \mathbb{N}\}.$$

Thus $\mathfrak{N}(\mathfrak{X})$ is the collection of (norm-closed) finite-codimensional subspaces of $\mathfrak{X}$ while $\mathcal{W}(\mathfrak{X}^*)$ is the collection of weak-star closed finite-codimensional subspaces of $\mathfrak{X}^*$. All notation and terminology, not otherwise explained, are as in [DU, LT1, LT2].

The *modulus of convexity* $\delta_{\mathfrak{X}} : [0,2] \to [0,1]$ of $\mathfrak{X}$ is

$$\delta_{\mathfrak{X}}(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S(\mathfrak{X}) \text{ and } \|x-y\| \ge \varepsilon\right\}$$

and $\mathfrak{X}$ is *uniformly convex* (UC) if and only if $\delta_{\mathfrak{X}}(\varepsilon) > 0$ for each $\varepsilon \in (0,2]$. The *modulus of asymptotic convexity* $\overline{\delta}_{\mathfrak{X}} : [0,1] \to [0,1]$ of $\mathfrak{X}$ is

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon) = \inf_{x\in S(\mathfrak{X})} \sup_{\mathcal{Y}\in\mathfrak{N}(\mathfrak{X})} \inf_{y\in S(\mathcal{Y})} [\|x+\varepsilon y\| - 1]$$

and $\mathfrak{X}$ is *asymptotically uniformly convex* (AUC) if and only if $\overline{\delta}_{\mathfrak{X}}(\varepsilon) > 0$ for each ε in $(0,1]$.

A space $\mathfrak{X}$ has the *Kadec–Klee* (KK) *property* provided the relative norm and weak topologies on $B(\mathfrak{X})$ coincide on $S(\mathfrak{X})$. A space $\mathfrak{X}$ has the *uniform Kadec–Klee* (UKK) *property* provided for each $\varepsilon > 0$ there exists $\delta > 0$ such that every ε-separated weakly convergent sequence $\{x_n\}$ in $B(\mathfrak{X})$ converges to an element of norm less than $1 - \delta$.

Related to the above geometric isometric properties are the following geometric isomorphic properties.

- $\mathfrak{X}$ has the *Radon–Nikodým property* (RNP) provided each bounded subset of $\mathfrak{X}$ has nonempty slices of arbitrarily small diameter.
- $\mathfrak{X}$ has the *point of continuity property* (PCP) provided each bounded subset of $\mathfrak{X}$ has nonempty relatively weakly open subsets of arbitrarily small diameter.
- $\mathfrak{X}$ has the *complete continuity property* (CCP) provided each bounded subset of $\mathfrak{X}$ is Bocce dentable.

Implications between these various properties are summarized in the diagram below.

$$\begin{array}{ccccccc} \text{UC} & \to & \text{AUC} & \to & \text{UKK} & \to & \text{KK} \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{RNP} & \to & \text{PCP} & \to & \text{CCP} & & \end{array}$$

Helpful notation is

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon) = \inf_{x \in S(\mathfrak{X})} \overline{\delta}_{\mathfrak{X}}(\varepsilon, x)$$

where

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon, x) = \sup_{\mathcal{Y} \in \mathfrak{N}(\mathfrak{X})} \inf_{y \in S(\mathcal{Y})} [\|x + \varepsilon y\| - 1].$$

Note that, for each $x \in S(\mathfrak{X})$,

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon, x) = \sup_{\mathcal{Y} \in \mathfrak{N}(\mathfrak{X})} \inf_{\substack{y \in \mathcal{Y} \\ \|y\| \geq \varepsilon}} [\|x + y\| - 1]$$

and so $\overline{\delta}_{\mathfrak{X}}(\varepsilon, x)$ is a nondecreasing function of ε. Thus $\overline{\delta}_{\mathfrak{X}}$ is a nondecreasing Lipschitz function with Lipschitz constant at most one. For any space $\mathfrak{X}$ and $\varepsilon \in [0, 1]$,

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon) \leq \varepsilon = \overline{\delta}_{\ell_1}(\varepsilon);$$

thus, ℓ_1 is, in some sense, the most asymptotically uniformly convex space.

Uniform convexity, the KK property, and the UKK property have been extensively studied (for example, see [DGZ, LT2]). Asymptotic uniform convexity has been examined explicitly in [JLPS, M] and implicitly in [GKL, KOS]. The RNP, PCP, and CCP have also been extensively studied (for example, see [DU, GGMS, G1, G2]).

The JT space is constructed on a (binary) *tree*

$$\mathcal{T} = \bigcup_{n=0}^{\infty} \Delta_n$$

where Δ_n is the nth level of the tree; thus,

$$\Delta_0 = \{\emptyset\} \quad \text{and} \quad \Delta_n = \{-1, +1\}^n$$

for each $n \in \mathbb{N}$. The *finite tree* $\mathcal{T}_N$ up through level $N \in \mathbb{N} \cup \{0\}$ is

$$\mathcal{T}_N = \bigcup_{n=0}^{N} \Delta_n.$$

The tree $\mathcal{T}$ is equipped with its natural (tree) ordering: if t_1 and t_2 are elements of $\mathcal{T}$, then $t_1 < t_2$ provided one of the following holds:

1. $t_1 = \emptyset$ and $t_2 \neq \emptyset$,
2. for some $n, m \in \mathbb{N}$,

$$t_1 = (\varepsilon_1^1, \varepsilon_2^1, \ldots, \varepsilon_n^1) \quad \text{and} \quad t_2 = (\varepsilon_1^2, \varepsilon_2^2, \ldots, \varepsilon_m^2)$$

with $n < m$ and $\varepsilon_i^1 = \varepsilon_i^2$ for each $1 \leq i \leq n$.

A (finite) *segment* of $\mathcal{T}$ is a linearly ordered subset $\{t_n, t_{n+1}, \ldots, t_{n+k}\}$ of $\mathcal{T}$ where $t_i \in \Delta_i$ for each $n \le i \le n+k$. A *branch* of $\mathcal{T}$ is a linearly ordered subset $\{t_0, t_1, t_2, \ldots\}$ of $\mathcal{T}$ where $t_i \in \Delta_i$ for each $i \in \mathbb{N} \cup \{0\}$.

The *James tree space* JT is the completion of the space of finitely supported functions $x : \mathcal{T} \to \mathbb{R}$ with respect to the norm

$$\|x\|_{JT} = \sup\Big\{\Big[\sum_{i=1}^{n}\Big|\sum_{t\in S_i} x_t\Big|^2\Big]^{1/2} : S_1, \ldots, S_n \text{ are disjoint segments of } \mathcal{T}\Big\}.$$

By lexicographically ordering $\mathcal{T}$, the sequence $\{\eta_t\}_{t\in\mathcal{T}}$ in JT, where

$$\eta_t(s) = \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{if } t \neq s, \end{cases}$$

forms a monotone boundedly complete (Schauder) basis of JT with biorthogonal functions $\{\eta_t^*\}_{t\in\mathcal{T}}$ in JT^*. Thus $\widehat{JT_*} = [\eta_t^*]_{t\in\mathcal{T}}$.

For $N, M \in \mathbb{N} \cup \{0\}$ with $N \le M$, the restriction maps from JT to JT given by

$$\pi_N(x) = \sum_{t\in\Delta_N} \eta_t^*(x)\eta_t,$$

$$\pi_{[N,M]}(x) = \sum_{t\in\bigcup_{i=N}^{M}\Delta_i} \eta_t^*(x)\eta_t,$$

$$\pi_{[N,\omega)}(x) = \sum_{t\in\bigcup_{i=N}^{\infty}\Delta_i} \eta_t^*(x)\eta_t$$

are each contractive projections (by the nature of the norm on JT); thus, so are their adjoints.

Let Γ be the set of all branches of $\mathcal{T}$. Then [LS, Theorem 1] the mapping $\pi_\infty : JT^* \to \ell_2(\Gamma)$ given by

$$\pi_\infty(x^*) = \{\lim_{t\in B} x^*(\eta_t)\}_{B\in\Gamma}$$

is an isometric quotient mapping with kernel $\widehat{JT_*}$. Also, for each $x^* \in JT^*$,

$$\|x^*\| = \lim_{N\to\infty} \|\pi_{[0,N]}^* x^*\|,$$

$$\|\pi_\infty x^*\| = \lim_{N\to\infty} \|\pi_{[N,\omega)}^* x^*\| = \lim_{N\to\infty} \|\pi_N^* x^*\|,$$

by the weak-star lower semicontinuity of the norm on JT^*.

To show that JT^* has the Kadec–Klee property, Schachermayer calculated the two quantitative bounds below.

FACT 1 [S, Lemma 3.8]. *Let $f_1 : (0,1) \to (0,\infty)$ be a continuous strictly increasing function satisfying $f_1(t) < 2^{-10}t^3$ for each $t \in (0,1)$. Let $N \in \mathbb{N}$ and $z^* \in JT^*$. If*

$$[1 - f_1(t)]\|z^*\| < \|\pi_{[0,N]}^* z^*\|$$

then

$$\|\pi^*_{[N,\omega)} z^*\| < \|\pi^*_N z^*\| + t\|z^*\|.$$

FACT 2 [S, Lemma 3.11]. *Let $f_2 : (0,1) \to (0,\infty)$ be a continuous strictly increasing function satisfying $f_2(t) < 2^{-26}t^5$ for each $t \in (0,1)$. Let $N \in \mathbb{N}$ and $\varepsilon_0 \in (0,1)$ and $\widetilde{x}^*, \widetilde{u}^* \in JT^*$. If*

$$\|\pi^*_{[N,\omega)}\widetilde{x}^*\| \le 1, \tag{2.1}$$

$$\|\pi^*_N\widetilde{x}^*\| > 1 - f_2(\varepsilon_0), \tag{2.2}$$

$$\|\pi_\infty\widetilde{x}^*\| > 1 - f_2(\varepsilon_0), \tag{2.3}$$

$$\|\pi^*_{[N,\omega)}(\widetilde{x}^* + \widetilde{u}^*)\| \le 1, \tag{2.4}$$

$$\|\pi^*_N\widetilde{u}^*\| < f_2(\varepsilon_0), \tag{2.5}$$

$$\|\pi_\infty\widetilde{u}^*\| < f_2(\varepsilon_0), \tag{2.6}$$

then

$$\|\pi^*_{[N,\omega)}\widetilde{u}^*\| < \varepsilon_0. \tag{2.7}$$

3. Results. Theorem 3 shows that the modulus of asymptotic convexity of JT_* is of power type 3. Its proof uses Fact 1.

THEOREM 3. *There exists a positive constant k so that*

$$\overline{\delta}_{JT_*}(\varepsilon) \ge k\varepsilon^3$$

for each $\varepsilon \in (0,1]$. Thus JT_* is asymptotically uniformly convex.

Proof. Fix $c \in (0, 2^{-10})$ and find k so that

$$0 < k(1+k)^2 \le c. \tag{1}$$

Fix $\varepsilon \in (0,1)$ and a finitely supported $x_* \in S(JT_*)$. It suffices to show that

$$\overline{\delta}_{JT_*}(\varepsilon, x_*) \ge k\varepsilon^3. \tag{2}$$

Find $N \in \mathbb{N}$ so that $\pi^*_{[0,N-1]}\widehat{x}_* = \widehat{x}_*$ and let $\mathcal{Y} = [\eta_t]^{\top}_{t\in\mathcal{T}_N}$. Fix $y_* \in S(\mathcal{Y})$. Assume that

$$\|x_* + \varepsilon y_*\| - 1 < k\varepsilon^3.$$

Then

$$\left[1 - \frac{k\varepsilon^3}{1+k\varepsilon^3}\right]\|\widehat{x}_* + \varepsilon\widehat{y}_*\| < 1 = \|\pi^*_{[0,N]}(\widehat{x}_* + \varepsilon\widehat{y}_*)\|.$$

Thus by Fact 1, with $f_1(t) = ct^3$,

$$\|\pi^*_{[N,\omega)}(\widehat{x}_* + \varepsilon\widehat{y}_*)\| < \|\pi^*_N(\widehat{x}_* + \varepsilon\widehat{y}_*)\| + f_1^{-1}\left(\frac{k\varepsilon^3}{1+k\varepsilon^3}\right)\|(\widehat{x}_* + \varepsilon\widehat{y}_*)\|$$

and so

$$\varepsilon < [1 + k\varepsilon^3]f_1^{-1}\left(\frac{k\varepsilon^3}{1+k\varepsilon^3}\right). \tag{3}$$

But inequality (3) is equivalent to

$$c^{1/3} < k^{1/3}(1 + k\varepsilon^3)^{2/3},$$

which contradicts (1). Thus $\|x_* + \varepsilon y_*\| - 1 \geq k\varepsilon^3$ and so (2) holds. ∎

A modification of the proof of Theorem 3 shows that, for each $\varepsilon \in (0,1)$, the $\overline{\delta}_{JT^*}(\varepsilon, x^*)$ stays uniformly bounded below from zero for $x^* \in S(JT^*)$ whose $\|\pi_\infty x^*\|$ is small. Recall that if $x_* \in JT_*$ then $\|\pi_\infty \widehat{x}_*\| = 0$.

LEMMA 4. *For each $\varepsilon \in (0,1)$ there exists $\eta = \eta(\varepsilon) > 0$ so that*

$$\inf_{\substack{x^* \in S(JT^*) \\ \|\pi_\infty x^*\| \leq \eta}} \sup_{\mathcal{Y} \in \mathcal{W}(JT^*)} \inf_{y^* \in S(\mathcal{Y})} [\|x^* + \varepsilon y^*\| - 1] > 0.$$

Proof. Fix $\varepsilon \in (0,1)$. With the notation of Fact 1, find $\delta, \eta_2 > 0$ so that

$$4\eta_2 + \frac{\delta}{1 - f_1(\delta)} < \varepsilon.$$

Fix $x^* \in S(JT^*)$ with $\|\pi_\infty x^*\| \equiv b \leq \eta_2$. It suffices to show that

$$\sup_{\mathcal{Y} \in \mathcal{W}(JT^*)} \inf_{y^* \in S(\mathcal{Y})} \|x^* + \varepsilon y^*\| \geq \frac{1}{1 - f_1(\delta)}. \tag{4}$$

Fix $\eta_1 \in (0,1)$. Find $N \in \mathbb{N}$ so that

$$1 - \eta_1 \leq \|\pi^*_{[0,N]} x^*\| \quad \text{and} \quad \|\pi^*_{[N,\omega)} x^*\| < b + \eta_2$$

and let $\mathcal{Y} = [\eta_t]^{\perp}_{t \in \mathcal{T}_N}$. Fix $y^* \in S(\mathcal{Y})$.

Assume that

$$\|x^* + \varepsilon y^*\| < \frac{1 - \eta_1}{1 - f_1(\delta)}.$$

Then

$$[1 - f_1(\delta)]\|x^* + \varepsilon y^*\| < \|\pi^*_{[0,N]} x^*\| = \|\pi^*_{[0,N]}(x^* + \varepsilon y^*)\|.$$

Thus by Fact 1,

$$\|\pi^*_{[N,\omega)}(x^* + \varepsilon y^*)\| < \|\pi^*_N(x^* + \varepsilon y^*)\| + \delta\|(x^* + \varepsilon y^*)\|$$

and so

$$\varepsilon - (b + \eta_2) < (b + \eta_2) + \frac{\delta}{1 - f_1(\delta)}.$$

But $b \leq \eta_2$ and so

$$\varepsilon < 4\eta_2 + \frac{\delta}{1 - f_1(\delta)}.$$

A contradiction, thus

$$\|x^* + \varepsilon y^*\| \geq \frac{1 - \eta_1}{1 - f_1(\delta)}.$$

Since $\eta_1 > 0$ was arbitrary, inequality (4) holds. ∎

Thus to show that JT^* is asymptotically uniformly convex, one just needs to examine $\overline{\delta}_{JT^*}(\varepsilon, x^*)$ for $x^* \in S(JT^*)$ whose $\|\pi_\infty x^*\|$ is not small. Fact 2 is used for this case.

THEOREM 5. *JT^* is asymptotically uniformly convex.*

Proof. Fix $\varepsilon \in (0,1)$ and let $\varepsilon_0 = \varepsilon/4$. Let $f_1 : (0,1) \to (0, 2^{-12})$ be given by $f_1(t) = 2^{-12}t^3$ and f_2 be a function satisfying the hypothesis in Fact 2. Find $\delta, \eta_2 > 0$ so that

$$4\eta_2 + \frac{\delta}{1 - f_1(\delta)} < \varepsilon.$$

Next find $\gamma_i > 0$ and $\tau > 1$ so that

$$\gamma_3 < \gamma_2 < 1/2, \tag{5}$$

$$\tau \le \frac{(1-\gamma_1)(1-\gamma_2)}{1 - f_2(\varepsilon_0)}, \tag{6}$$

$$\tau < \frac{1-\gamma_2}{\sqrt{1 - f_2^2(\varepsilon_0)}}, \tag{7}$$

$$\tau \le \frac{\eta_2^3\gamma_3^3}{2^{15}(1-\gamma_2)^3} - \gamma_4 + 1, \tag{8}$$

$$\frac{\tau - 1 + \gamma_4}{\tau} < f_1(1), \tag{9}$$

$$\tau \le \frac{1}{1 - f_1(\delta)}. \tag{10}$$

Fix $x^* \in S(JT^*)$. It suffices to show that

$$\sup_{\mathcal{Y} \in \mathfrak{N}(JT^*)} \inf_{y \in S(\mathcal{Y})} \|x^* + \varepsilon y^*\| \ge \tau. \tag{11}$$

Let $\|\pi_\infty x^*\| \equiv b$. If $b \le \eta_2$, then by the proof of Lemma 4 and (10), inequality (11) holds. So let $b > \eta_2$. Find $N \in \mathbb{N}$ so that

$$(1-\gamma_1)b < \|\pi_N^* x^*\| \le \|\pi_{[N,\omega)}^* x^*\| < b\left(\frac{1-\gamma_3}{1-\gamma_2}\right) < \frac{b}{1-\gamma_2}, \tag{12}$$

$$1 - \gamma_4 < \|\pi_{[0,N]}^* x^*\|. \tag{13}$$

Let $g_{x^*} \in JT^{**}$ be the functional given by

$$g_{x^*}(z^*) = \langle \pi_\infty z^*, \pi_\infty x^* \rangle_{H_2}$$

where the inner product is the natural inner product on $\ell_2(\Gamma)$. Let

$$\mathcal{Y} = [\eta_t]_{t \in \mathcal{T}_N}^{\perp} \cap [g_{x^*}]^{\top}$$

and fix $y^* \in S(\mathcal{Y})$.

Assume that

$$\|x^* + \varepsilon y^*\| < \tau. \tag{14}$$

It suffices to find a contradiction to (14). Towards this, let

$$\widetilde{x}^* = \frac{1-\gamma_2}{\tau b}\, x^* \quad \text{and} \quad \widetilde{y}^* = \frac{1-\gamma_2}{\tau b}\, y^*.$$

It suffices to show (keeping the same notation but with $\widetilde{u}^* = \varepsilon \widetilde{y}^*$) that conditions (2.1) through (2.6) of Fact 2 hold; for then condition (2.7) holds and so by (5),

$$\varepsilon_0 > \|\pi^*_{[N,\omega)} \varepsilon \widetilde{y}^*\| = \frac{1-\gamma_2}{\tau b}\, \varepsilon \geq \frac{\varepsilon}{4} = \varepsilon_0.$$

Condition (2.1) follows from (12) since

$$\|\pi^*_{[N,\omega)} \widetilde{x}^*\| \leq \frac{1-\gamma_2}{\tau b} \cdot \frac{b}{1-\gamma_2} \leq 1.$$

Condition (2.2) follows from (12) and (6) since

$$\|\pi^*_N \widetilde{x}^*\| > \frac{1-\gamma_2}{\tau b}\,(1-\gamma_1) b = \frac{(1-\gamma_1)(1-\gamma_2)}{\tau} \geq 1 - f_2(\varepsilon_0).$$

Towards condition (2.3), note that by (7),

$$\|\pi_\infty \widetilde{x}^*\| = \frac{1-\gamma_2}{\tau b}\, b = \frac{1-\gamma_2}{\tau} > \sqrt{1 - f_2^2(\varepsilon_0)} \tag{15}$$

and so

$$\|\pi_\infty \widetilde{x}^*\| > 1 - f_2(\varepsilon_0).$$

Towards condition (2.4), note that by (14) and (13),

$$\|x^* + \varepsilon y^*\| < \frac{\tau}{1-\gamma_4}\, \|\pi^*_{[0,N]}(x^* + \varepsilon y^*)\|.$$

Thus by Fact 1 and (9),

$$\begin{aligned}\|\pi^*_{[N,\omega)}(x^* + \varepsilon y^*)\| &< \|\pi^*_N (x^* + \varepsilon y^*)\| \\ &\qquad + f_1^{-1}\left(\frac{\tau - 1 + \gamma_4}{\tau}\right) \|(x^* + \varepsilon y^*)\| \\ &\leq b\,\frac{1-\gamma_3}{1-\gamma_2} + \tau 2^4 \left(\frac{\tau - 1 + \gamma_4}{\tau}\right)^{1/3}.\end{aligned}$$

Thus condition (2.4) holds provided

$$b\,\frac{1-\gamma_3}{1-\gamma_2} + \tau 2^4 \left(\frac{\tau - 1 + \gamma_4}{\tau}\right)^{1/3} \leq \frac{\tau b}{1-\gamma_2},$$

or equivalently

$$\tau^{2/3} (\tau - 1 + \gamma_4)^{1/3} \leq \frac{b(\tau - 1 + \gamma_3)}{2^4 (1-\gamma_2)}.$$

But by (8) and since $b > \eta_2$,

$$\tau^{2/3}(\tau - 1 + \gamma_4)^{1/3} \le 2(\tau - 1 + \gamma_4)^{1/3} \le \frac{2\eta_2\gamma_3}{2^5(1-\gamma_2)} \le \frac{b\gamma_3}{2^4(1-\gamma_2)} \le \frac{b(\tau - 1 + \gamma_3)}{2^4(1-\gamma_2)}.$$

Thus condition (2.4) holds.

Condition (2.5) follows from the fact that $y^* \in [\eta_t]^{\perp}_{t\in\mathcal{T}_N}$. Towards condition (2.6), since $y^* \in [g_{x^*}]^{\top}$, the vectors $\pi_\infty \widetilde{y}^*$ and $\pi_\infty \widetilde{x}^*$ are orthogonal in $\ell_2(\Gamma)$ and so

$$\|\pi_\infty \varepsilon \widetilde{y}^*\|^2 = \|\pi_\infty(\widetilde{x}^* + \varepsilon\widetilde{y}^*)\|^2 - \|\pi_\infty \widetilde{x}^*\|^2;$$

but $\pi_\infty = \pi_\infty \pi^*_{[N,\omega)}$ and so by condition (2.4) and (15),

$$\|\pi_\infty \varepsilon \widetilde{y}^*\|^2 \le \|\pi^*_{[N,\omega)}(\widetilde{x}^* + \varepsilon\widetilde{y}^*)\|^2 - \|\pi_\infty \widetilde{x}^*\|^2 < 1 - [1 - f_2^2(\varepsilon_0)] = f_2^2(\varepsilon_0).$$

Thus condition (2.6) holds. ∎

The proof in [JLPS] that an asymptotically uniformly convex space has the PCP shows that if $\overline{\delta}_{\mathfrak{X}}(\varepsilon) > 0$ for each $\varepsilon \in (0,1]$ then $\mathfrak{X}$ has the PCP. A bit more can be said.

Proposition 6. *If* $\overline{\delta}_{\mathfrak{X}}(1/2) > 0$ *then* $\mathfrak{X}$ *has the PCP.*

The proof of Proposition 6 uses the following (essentially known) lemma.

Lemma 7. *Let* $\mathfrak{X}$ *be a space without the PCP and* $0 < \varepsilon < 1$. *Then there is a closed subset* A *of* $\mathfrak{X}$ *so that*

(1) *each* (*nonempty*) *relatively weakly open subset of* A *has diameter larger than* $1 - \varepsilon$,

(2) $\sup\{\|a\| : a \in A\} = 1$.

Proof. Let $\mathfrak{X}$ fail the PCP and $0 < \varepsilon < 1$. By a standard argument (e.g., see [SSW, Prop. 4.10]), there is a closed subset $\widetilde{A}$ of $\mathfrak{X}$ of diameter one such that each (nonempty) relatively weakly open subset of $\widetilde{A}$ has diameter larger than $1-\varepsilon$. Without loss of generality $0 \in \widetilde{A}$ (just consider a translate of $\widetilde{A}$). Let

$$b = \sup\{\|x\| : x \in \widetilde{A}\} \quad \text{and} \quad A = \widetilde{A}/b.$$

Note that $0 < b \le 1$. If V is a (nonempty) relatively weakly open subset of A, then bV is a relatively weakly open subset of $\widetilde{A}$ and so

$$\operatorname{diam} V = \frac{1}{b}\operatorname{diam} bV > 1 - \varepsilon.$$

Thus A does the job. ∎

Proof of Proposition 6. Let $\mathfrak{X}$ be a Banach space without the PCP. Fix $t \in (0, 1/2)$ and $\delta \in (0, t)$. It suffices to show that $\overline{\delta}_{\mathfrak{X}}(t) \leq 2\delta$.

Find a subset A of $\mathfrak{X}$ which satisfies the conditions of Lemma 7 with $\varepsilon = 1 - 2t$ and find $a \in A$ so that

$$\left\| \frac{a}{\|a\|} - a \right\| < \delta.$$

Let $\mathcal{Y} \in \mathfrak{N}(\mathfrak{X})$. It suffices to show that

$$\inf_{\substack{y \in \mathcal{Y} \\ \|y\| \geq t}} \left[\left\| \frac{a}{\|a\|} + y \right\| - 1 \right] \leq 2\delta.$$

By condition (1) of Lemma 7 there exists $x \in A$ so that $\|x - a\| \geq t$ and $x - a$ is *almost* in $\mathcal{Y}$; thus, by a standard perturbation argument (e.g., see [GJ, Lemma 2]) there exists $y \in \mathcal{Y}$ so that

$$\|y\| \geq t \quad \text{and} \quad \|y - (x - a)\| < \delta.$$

Thus

$$\left\| \frac{a}{\|a\|} + y \right\| \leq \left\| \frac{a}{\|a\|} - a \right\| + \|y - x + a\| + \|x\| < 1 + 2\delta.$$

Thus $\overline{\delta}_{\mathfrak{X}}(1/2) = 0$. ∎

The observation below formalizes an essentially known fact, which to the best of the author's knowledge, has not appeared in print as such. Recall that the *modulus of asymptotic smoothness* $\overline{\varrho}_{\mathfrak{X}} : [0, 1] \to [0, 1]$ of $\mathfrak{X}$ is

$$\overline{\varrho}_{\mathfrak{X}}(\varepsilon) = \sup_{x \in S(\mathfrak{X})} \inf_{\mathcal{Y} \in \mathfrak{N}(\mathfrak{X})} \sup_{y \in S(\mathcal{Y})} [\|x + \varepsilon y\| - 1]$$

and $\mathfrak{X}$ is *asymptotically uniformly smooth* if and only if $\lim_{\varepsilon \to 0^+} \varrho_{\mathfrak{X}}(\varepsilon)/\varepsilon = 0$. Also, $L_p(\mathfrak{X})$ is the Lebesgue–Bochner space of strongly measurable $\mathfrak{X}$-valued functions defined on a separable nonatomic probability space, equipped with its usual norm.

Observation 8. *Let $1 < p < \infty$. For a Banach space $\mathfrak{X}$, the following are equivalent.*

(1) *$\mathfrak{X}$ is uniformly convexifiable.*
(2) *$L_p(\mathfrak{X})$ is uniformly convexifiable.*
(3) *$L_p(\mathfrak{X})$ is asymptotically uniformly convexifiable.*
(4) *$L_p(\mathfrak{X})$ admits an equivalent UKK norm.*
(5) *$L_p(\mathfrak{X})$ is asymptotically uniformly smoothable.*

Proof. Let $1 < p < \infty$ and $\mathfrak{X}$ be a Banach space.

That (1) through (4) are equivalent and that (2) implies (5) follows easily from the following known facts about a Banach space $\mathcal{Y}$:

(i) $\mathcal{Y}$ is uniformly convex if and only if $L_p(\mathcal{Y})$ is [Mc].

(ii) $\mathcal{Y}$ is uniformly convexifiable if and only if $L_p(\mathcal{Y})$ admits an equivalent UKK norm [DGK, Theorem 4].

(iii) $\mathcal{Y}$ is uniformly convexifiable if and only if $\mathcal{Y}$ is uniformly smoothable (cf. [DU, page 144]).

Towards showing that (5) implies (1), let $L_p(\mathfrak{X})$ be asymptotically uniformly smoothable and $\mathfrak{X}_0$ be a separable subspace of $\mathfrak{X}$. It suffices to show that $\mathfrak{X}_0$ is uniformly convexifiable (cf. [DGZ, Remark IV.4.4]).

It follows from [GKL, Proposition 2.6] that if $\mathcal{Y}$ is separable, then $\mathcal{Y}$ is asymptotically uniformly smooth if and only if $\mathcal{Y}^*$ has the UKK* property. Thus $[L_p(\mathfrak{X}_0)]^*$ admits an equivalent UKK* norm. But ℓ_1 cannot embed into $L_p(\mathfrak{X}_0)$ since $L_p(\mathfrak{X}_0)$ is asymptotically uniformly smoothable and so $[L_p(\mathfrak{X}_0)]^*$ is asymptotically weak* uniformly convexifiable and so is also asymptotically uniformly convexifiable. Thus $L_q(\mathfrak{X}_0^*)$ is asymptotically uniformly convexifiable where $1/p + 1/q = 1$. From (3)$\Rightarrow$(1) it follows that $\mathfrak{X}_0^*$ is uniformly convexifiable and hence so is $\mathfrak{X}_0$. ∎

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