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Differentiation of Banach-space-valued additive processes

by

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Dedicated to Professor Kôzô Yabuta on his sixtieth birthday

Abstract. Let X be a Banach space and (Ω, Σ, μ) be a σ-finite measure space. Let L be a Banach space of X-valued strongly measurable functions on (Ω, Σ, μ). We consider a strongly continuous d-dimensional semigroup $T = \{T(u) : u = (u_1, \dots, u_d),\ u_i > 0,\ 1 \le i \le d\}$ of linear contractions on L. We assume that each $T(u)$ has, in a sense, a contraction majorant and that the strong limit $T(0) = \text{strong-lim}_{u\to 0} T(u)$ exists. Then we prove, under some suitable norm conditions on the Banach space L, that a differentiation theorem holds for d-dimensional bounded processes in L which are additive with respect to the semigroup T. This generalizes a differentiation theorem obtained previously by the author under the assumption that L is an X-valued L_p-space, with $1 \le p < \infty$.

1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space and (Ω, Σ, μ) be a σ-finite measure space. Let $(L, \|\cdot\|_L)$ denote a Banach space of X-valued strongly measurable functions on (Ω, Σ, μ) under pointwise operations. Two functions f and g in L are not distinguished provided that $f(\omega) = g(\omega)$ for almost all $\omega \in \Omega$. Thus all statements and relations are assumed to hold modulo sets of measure zero. In this paper we will also assume that the norm $\|\cdot\|_L$ of L has the following properties:

(I) If $f, g \in L$ and $\|f(\omega)\| \le \|g(\omega)\|$ for almost all $\omega \in \Omega$, then $\|f\|_L \le \|g\|_L$.

(II) If g is an X-valued strongly measurable function on Ω and if there exists an $f \in L$ such that $\|g(\omega)\| \le \|f(\omega)\|$ for almost all $\omega \in \Omega$, then $g \in L$.

(III) If $E_n \in \Sigma$, $E_n \supset E_{n+1}$ for each $n \ge 1$, and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, then for any $f \in L$ we have $\lim_{n\to\infty} \|\chi_{E_n} f\|_L = 0$, where χ_{E_n} denotes the characteristic function of E_n.

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(IV) If f and g are in L, $\|f(\omega)\| \le \|g(\omega)\|$ for almost all $\omega \in \Omega$, and $\|f\|_L = \|g\|_L$, then $\|f(\omega)\| = \|g(\omega)\|$ for almost all $\omega \in \Omega$.

(V) If $(f_n,\ n \ge 1)$ is a sequence of functions in L such that $\|f_n(\omega)\| \le \|f_{n+1}(\omega)\|$ for almost all $\omega \in \Omega$, for each $n \ge 1$, and also such that $\sup_{n\ge 1} \|f_n\|_L < \infty$, then there exists an $f \in L$ such that $\|f_n(\omega)\| \le \|f(\omega)\|$ for almost all $\omega \in \Omega$, for every $n \ge 1$.

It is worth noting that in addition to the usual X-valued L_p-spaces, with $1 \le p < \infty$, there are many interesting X-valued function spaces with properties (I) to (V). Examples are some (X-valued) Lorentz spaces and Orlicz spaces, etc. (see, for example, [8] and [9]). By simple examples we observe that properties (III), (IV) and (V) are independent of each other.

Fix $x_1 \in X$ with $\|x_1\| = 1$. We denote by $L(\mathbb{R})$ the set of all real-valued measurable functions $\widetilde{f}$ on (Ω, Σ, μ) such that the function

$$f(\omega) = \widetilde{f}(\omega)x_1 \tag{1}$$

is in L. Define

$$\|\widetilde{f}\|_{L(\mathbb{R})} = \|f\|_L \quad \text{for } \widetilde{f} \in L(\mathbb{R}). \tag{2}$$

It follows that $(L(\mathbb{R}), \|\cdot\|_{L(\mathbb{R})})$ becomes a Banach space. In an obvious manner $(L(\mathbb{R}), \|\cdot\|_{L(\mathbb{R})})$ can be regarded as a closed subspace of $(L, \|\cdot\|_L)$.

We call a positive linear operator P defined on $L(\mathbb{R})$ a *majorant* of a linear operator U defined on L if

$$\|Uf(\omega)\| \le [P\|f(\cdot)\|](\omega) \tag{3}$$

for almost all $\omega \in \Omega$, for every $f \in L$. We call U a *contraction* if the operator norm $\|U\|$ of U is less than or equal to one.

For an integer d, with $d \ge 1$, we put $\mathbf{P}_d = \{u = (u_1, \dots, u_d) : u_i > 0,\ 1 \le i \le d\}$ and $\mathbb{R}_d^+ = \{u = (u_1, \dots, u_d) : u_i \ge 0,\ 1 \le i \le d\}$. Further, $\mathcal{I}_d$ denotes the class of all bounded intervals in $\mathbf{P}_d$, and λ_d is the d-dimensional Lebesgue measure. We will consider a strongly continuous d-dimensional semigroup $T = \{T(u) : u \in \mathbf{P}_d\}$ of linear contractions on L. This means that T satisfies:

(a) $\|T(u)\| \le 1$ for $u \in \mathbf{P}_d$,
(b) $T(u + v) = T(u)T(v)$ for $u, v \in \mathbf{P}_d$, and
(c) $\lim_{u\to v} \|T(u)f - T(v)f\|_L = 0$ for $v \in \mathbf{P}_d$ and $f \in L$.

It follows that for each $f \in L$ the L-valued function $u \mapsto T(u)f$ is Bochner integrable over every $I \in \mathcal{I}_d$.

By a (d-dimensional) *process* F in L we mean a set function $F : \mathcal{I}_d \to L$. It is *bounded* if

$$K(F) := \sup\left\{ \frac{\|F(I)\|_L}{\lambda_d(I)} : I \in \mathcal{I}_d,\ \lambda_d(I) > 0 \right\} < \infty, \tag{4}$$

and *additive* (with respect to T) if it satisfies the following conditions:

(i) $T(u)F(I) = F(u+I)$ for all $u \in \mathbf{P}_d$ and $I \in \mathcal{I}_d$.

(ii) If $I_1, \ldots, I_k \in \mathcal{I}_d$ are pairwise disjoint and $I = \bigcup_{i=1}^k I_i \in \mathcal{I}_d$, then $F(I) = \sum_{i=1}^k F(I_i)$.

Thus if $F(I) = \int_I T(u) f\,du$ for all $I \in \mathcal{I}_d$, where f is a fixed function in L, then $F(I)$ defines a bounded additive process in L.

In this paper we study the almost everywhere convergence of the averages

$$\alpha^{-d} F((0,\alpha]^d) \tag{5}$$

as α approaches zero. But this is meaningless when the averages denote equivalence classes and not actual functions and α ranges through all positive numbers. Therefore in this paper we let α range through a countable dense subset $\mathbf{D}$ of the positive numbers. It may be assumed that $\mathbf{D}$ includes all positive rational numbers. We use the following notations:

$$\operatorname*{q\text{-}lim}_{\alpha\to 0} \quad \text{and} \quad \operatorname*{q\text{-}lim\,sup}_{\alpha\to 0}, \tag{6}$$

which mean that these limits are taken as α tends to zero through the set $\mathbf{D}$ (cf. [2], [3]).

We are now in a position to state our differentiation theorem:

Theorem. *Assume that each $T(u)$, $u \in \mathbf{P}_d$, has a contraction majorant $P(u)$ defined on $L(\mathbb{R})$ and that the strong limit $T(0) = \text{strong-lim}_{u\to 0} T(u)$ exists. Then for each $f \in L$, we have*

$$T(0)f(\omega) = \operatorname*{q\text{-}lim}_{\alpha\to 0} \alpha^{-d}\Big(\int\limits_{(0,\alpha]^d} T(u) f\,du\Big)(\omega) \tag{7}$$

for almost all $\omega \in \Omega$. Further, if X is assumed to be a reflexive Banach space, then to each bounded additive process $F : \mathcal{I}_d \to L$ there corresponds a function $f \in L$, with $T(0)f = f$, for which

$$f(\omega) = \operatorname*{q\text{-}lim}_{\alpha\to 0} \alpha^{-d} F((0,\alpha]^d)(\omega) \tag{8}$$

for almost all $\omega \in \Omega$.

We remark that in [11] such differentiation theorems have been examined within the framework of X-valued L_p-spaces, with $1 \le p < \infty$. Since the existence of the strong limit $T(0) = \text{strong-lim}_{u\to 0} T(u)$ was essentially assumed there, the above theorem may be considered a generalization of the main result of [11]. It is also interesting to remark that in [11] a brief discussion was presented about the condition on the existence of a contraction majorant $P(u)$ for each $T(u)$.

The idea of the proof is as follows. First we show, as in [11], that there exists a strongly *measurable* subsemigroup $\{\tau(u) : u \in \mathbf{P}_d\}$ of positive linear contractions defined on $L(\mathbb{R})$ which dominates the semigroup $T = \{T(u) : u \in \mathbf{P}_d\}$ in the sense that for each $u \in \mathbf{P}_d$ and $f \in L$,

$$\|T(u)f(\omega)\| \le [\tau(u)\|f(\cdot)\|](\omega) \tag{9}$$

for almost all $\omega \in \Omega$. By using this, we then combine a reduction method on the dimension d of the semigroup T and the process F, due to Emilion [6] (see also [2], [5] and [13]), with the recent results of [12] in order to adapt the arguments of [11] to the present situation. This is an outline. In Section 2 we develop these in detail and provide some necessary lemmas. The proof of the theorem is given in Section 3, and in Section 4 we first construct an example showing that the theorem fails to hold when property (IV) is not assumed, and then we remark that in spite of this example the theorem holds without assuming property (IV) if the strong limit operator $T(0)$ of the theorem satisfies $\|T(0)f\|_L \ne 0$ whenever $\|f\|_L \ne 0$. Hence, in particular, if $T(0) = I$ (the identity operator), then the theorem holds without property (IV).

2. Preliminaries and lemmas. The next four lemmas clarify properties of the Banach space $(L, \|\cdot\|_L)$. Since proofs can be found in [12], we omit them.

LEMMA 1. *If $(f_n,\ n \ge 1)$ is a sequence of functions in L such that $\sum_{n=1}^{\infty} \|f_n\|_L < \infty$, then $\sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty$ for almost all $\omega \in \Omega$, and the function $f(\omega) = \sum_{n=1}^{\infty} f_n(\omega)$ is in L and satisfies*

$$\lim_{k\to\infty} \Big\|f - \sum_{n=1}^{k} f_n\Big\|_L = 0.$$

LEMMA 2. *Let $(f_n,\ n \ge 1)$ be a sequence of functions in L. If*

$$\lim_{n\to\infty} \|f - f_n\|_L = 0 \quad \text{for some } f \in L,$$

then there exists a subsequence $(f_{n'})$ of (f_n) such that $\lim_{n'\to\infty} f_{n'}(\omega) = f(\omega)$ for almost all $\omega \in \Omega$.

LEMMA 3. *Let $(f_n,\ n \ge 1)$ be a sequence of functions in L. If*

$$\lim_{n\to\infty} f_n(\omega) = 0 \quad \text{for almost all } \omega \in \Omega$$

and if there exists an $f \in L$ such that $\|f_n(\omega)\| \le \|f(\omega)\|$ for almost all $\omega \in \Omega$, for every $n \ge 1$, then $\lim_{n\to\infty} \|f_n\|_L = 0$.

LEMMA 4. *There exists a real-valued measurable function w on Ω, with $w(\omega) > 0$ on Ω, such that*

$$\int \|f(\omega)\| w(\omega)\, d\mu < \infty$$

for all $f \in L$.

Hereafter we may and do assume $d = 2^m$ with $m \geq 1$, where d is the dimension of the semigroup $T = \{T(u)\}$. This is done because if $2^{m-1} \leq d < 2^m$, then by setting $\widehat{T}(u,v) = T(u)$ for $(u,v) \in \mathbf{P}_{2^m}$, where $u \in \mathbf{P}_d$, we have a 2^m-dimensional semigroup $\widehat{T} = \{\widehat{T}(\widehat{u}) : \widehat{u} \in \mathbf{P}_{2^m}\}$, and if $R : \mathcal{I}_d \to L$ is a bounded additive process in L with respect to $T = \{T(u) : u \in \mathbf{P}_d\}$, then by setting

$$\widehat{F}(I \times I') = \lambda_{2^m - d}(I')F(I) \quad \text{for } I \times I' \in \mathcal{I}_{2^m},$$

where $I \in \mathcal{I}_d$, we have a 2^m-dimensional bounded process $\widehat{F} : \mathcal{I}_{2^m} \to L$ which is additive with respect to $\widehat{T}$. By an obvious argument it suffices to prove the theorem for $\widehat{T}$ and $\widehat{F}$ instead of T and F, respectively.

LEMMA 5 (cf. Lemma 1 of [11]). *Assume that each $T(u)$, $u \in \mathbf{P}_d$, has a contraction majorant $P(u)$ defined on $L(\mathbb{R})$. Then there exists a positive linear contraction $\tau(u)$ on $L(\mathbb{R})$, called the* linear modulus *of $T(u)$, such that*

(i) $\|T(u)f(\omega)\| \leq [\tau(u)\|f(\cdot)\|](\omega) \leq [P(u)\|f(\cdot)\|](\omega)$ *for almost all $\omega \in \Omega$, for every $f \in L$,*

(ii) $\tau(u)\widetilde{g} = \operatorname{ess\,sup}\{\sum_{i=1}^k \|[T(u)f_i](\cdot)\| : f_i \in L, \sum_{i=1}^k \|f_i(\omega)\| \leq \widetilde{g}(\omega)$ *on $\Omega\}$ for all $\widetilde{g} \in L(\mathbb{R})^+$, where $L(\mathbb{R})^+ = \{\widetilde{h} \in L(\mathbb{R}) : \widetilde{h}(\omega) \geq 0$ on $\Omega\}$,*

(iii) $\tau(s+t) \leq \tau(s)\tau(t)$ *for $s,t \in \mathbf{P}_d$,*

(iv) $\lim_{t\to u} \|[\tau(t)\widetilde{g} - \tau(u)\widetilde{g}]^-\|_{L(\mathbb{R})} = 0$ *for every $u \in \mathbf{P}_d$ and $\widetilde{g} \in L(\mathbb{R})^+$.*

Proof. If $\widetilde{g} \in L(\mathbb{R})^+$ and $u \in \mathbf{P}_d$, then define a nonnegative measurable function $\tau(u)\widetilde{g}$ on Ω by the relation

$$\tau(u)\widetilde{g} = \operatorname{ess\,sup}\Big\{ \sum_{i=1}^k \|[T(u)f_i](\cdot)\| : \\ f_i \in L,\ \sum_{i=1}^k \|f_i(\omega)\| \leq \widetilde{g}(\omega) \text{ on } \Omega,\ 1 \leq k < \infty\Big\}.$$

Since

$$\sum_{i=1}^k \|[T(u)f_i](\omega)\| \leq \sum_{i=1}^k [P(u)\|f_i(\cdot)\|](\omega) \leq P(u)\widetilde{g}(\omega)$$

for almost all $\omega \in \Omega$, it follows that $\tau(u)\widetilde{g}(\omega) \leq P(u)\widetilde{g}(\omega)$ for almost all $\omega \in \Omega$. Hence from properties (I) and (II) we get $\tau(u)\widetilde{g} \in L(\mathbb{R})^+$ and $\|\tau(u)\widetilde{g}\|_{L(\mathbb{R})} \leq \|P(u)\widetilde{g}\|_{L(\mathbb{R})} \leq \|\widetilde{g}\|_{L(\mathbb{R})}$. Thus (i) follows. Since $\tau(u)$ is linear on $L(\mathbb{R})^+$, it uniquely extends to a linear contraction on $L(\mathbb{R})$, and (iii) follows from the semigroup property $T(s+t) = T(s)T(t)$.

To prove (iv), let $\widetilde{g} \in L(\mathbb{R})^+$ and $\varepsilon > 0$. Then we can choose functions g_j in L, $1 \le j \le n$, so that each g_j has the form

$$g_j = \sum_{i=1}^{k} f_{ij} \quad \text{with} \quad \sum_{i=1}^{k} \|f_{ij}(\omega)\| \le \widetilde{g}(\omega) \quad \text{on } \Omega,$$

where $f_{ij} \in L$ for each i with $1 \le i \le k$, and so that

$$\Big\| [\tau(u)\widetilde{g}](\cdot) - \max_{1\le j\le n} \sum_{i=1}^{k} \|[T(u)f_{ij}](\cdot)\| \Big\|_{L(\mathbb{R})} < \varepsilon.$$

Since the strong continuity of $T = \{T(u)\}$ and properties (I) and (II) imply

$$\lim_{t\to u} \big\| \|[T(t)f_{ij}](\cdot)\| - \|[T(u)f_{ij}](\cdot)\| \big\|_{L(\mathbb{R})} = 0,$$

and since $\varepsilon > 0$ was arbitrary, it follows from the definition of $\tau(u)\widetilde{g}$ that

$$\lim_{t\to u} \|[\tau(t)\widetilde{g} - \tau(u)\widetilde{g}]^-\|_{L(\mathbb{R})} = 0,$$

whence (iv) follows, and the proof is complete.

Lemma 6. *For the proof of the theorem we may assume that L is separable.*

Proof. First we notice that $1 \in L(\mathbb{R})$ can be assumed without loss of generality. In fact, since μ is σ-finite, we can apply Lemma 1 to take an $h \in L$ with $\{\omega : h(\omega) \neq 0\} \supset \{\omega : f(\omega) \neq 0\}$ for every $f \in L$. Here obviously we may assume without loss of generality that $\{\omega : h(\omega) \neq 0\} = \Omega$. Then, by defining

$$L(h) = \left\{ \frac{f}{\|h(\cdot)\|} : f \in L \right\} \quad \text{and} \quad \left\| \frac{f}{\|h(\cdot)\|} \right\|_{L(h)} = \|f\|_L,$$

$(L(h), \|\cdot\|_{L(h)})$ becomes a Banach space which is isometrically isomorphic to $(L, \|\cdot\|_L)$ via the mapping $f/\|h(\cdot)\| \mapsto f$. Hence it follows that we may consider $(L(h), \|\cdot\|_{L(h)})$ instead of $(L, \|\cdot\|_L)$ for the proof of the theorem. Thus $1 \in L(\mathbb{R})$ can be assumed from the beginning.

Let $F : \mathcal{I}_d \to L$ be a bounded additive process with respect to the semigroup $T = \{T(u) : u \in \mathbf{P}_d\}$. It follows from the boundedness of F that the set $\{F(I) : I \in \mathcal{I}_d\}$ is separable in L, and thus by the strong continuity of the semigroup $T = \{T(u)\}$ the set $\{T(u)F(I) : u \in \mathbf{P}_d,\ I \in \mathcal{I}_d\}$ is also separable in L. Since $T(u)F(I)$ is a μ-essentially separably valued function for every $u \in \mathbf{P}_d$ and $I \in \mathcal{I}_d$, we then apply Lemma 2 to infer that there exists a separable Banach subspace X_1 of X for which

$$T(u)F(I)(\omega) \in X_1$$

for almost all $\omega \in \Omega$, for every $u \in \mathbf{P}_d$ and $I \in \mathcal{I}_d$. Further, there exists a separable σ-subalgebra Σ_1 of Σ such that $T(u)F(I)$ becomes a strongly measurable function with respect to (Ω, Σ_1) for every $u \in \mathbf{P}_d$ and $I \in \mathcal{I}_d$. Thus the linear manifold M_1 in L defined by

$$M_1 = \{g \in L : g \text{ is } X_1\text{-valued and strongly measurable with respect to } (\Omega, \Sigma_1)\}$$

includes the set $\{T(u)F(I) : u \in \mathbf{P}_d,\ I \in \mathcal{I}_d\}$, and Lemma 2 implies that M_1 is a closed subset of L. To see that it is separable, let g be a function in M_1. By a standard argument there exists a sequence $(g_n,\ n \geq 1)$ of X_1-valued simple functions, strongly measurable with respect to (Ω, Σ_1), such that $\|g_n(\omega)\| \leq \|g_{n+1}(\omega)\|$, $\|g_n(\omega) - g(\omega)\| \leq 2\|g(\omega)\|$ and $\lim_{n\to\infty} \|g_n(\omega) - g(\omega)\| = 0$ for all $n \geq 1$ and $\omega \in \Omega$. By Lemmas 2 and 3 we then have

$$\lim_{n\to\infty} \|g_n - g\|_L = 0.$$

On the other hand, since $1 \in L(\mathbb{R})$, Σ_1 is separable, and μ is σ-finite, it follows from Lemma 3 together with Theorem 13.D of [7] that the set of all X_1-valued simple functions in L that are strongly measurable with respect to (Ω, Σ_1) is separable in L. Hence it follows that M_1 is separable.

Since the union $M_1 \cup \{T(u)g : u \in \mathbf{P}_d,\ g \in M_1\}$ is separable, we can continue this argument to obtain a separable Banach subspace X_2 of X with $X_1 \subset X_2$, and a separable σ-subalgebra Σ_2 of Σ with $\Sigma_1 \subset \Sigma_2$. Then the linear manifold M_2 in L defined by

$$M_2 = \{g \in L : g \text{ is } X_2\text{-valued and strongly measurable with respect to } (\Omega, \Sigma_2)\}$$

becomes a separable Banach subspace of L such that $M_1 \subset M_2$.

By repeating this process we obtain an infinite sequence $((X_n, \Sigma_n, M_n), n \geq 1)$. Finally, define

$$X_\infty = \text{the closed linear subspace of } X \text{ generated by } \bigcup_{n=1}^{\infty} X_n,$$

$$\Sigma_\infty = \text{the } \sigma\text{-subalgebra of } \Sigma \text{ generated by } \bigcup_{n=1}^{\infty} \Sigma_n, \text{ and}$$

$$M_\infty = \{g \in L : g \text{ is } X_\infty\text{-valued and strongly measurable with respect to } (\Omega, \Sigma_\infty)\}.$$

Clearly, M_∞ is a separable Banach subspace of L such that $\{F(I) : I \in \mathcal{I}_d\} \subset M_\infty$. Further, using an approximation argument and Lemma 3 together with Theorem 13.D of [7], we observe that $T(u)M_\infty \subset M_\infty$ for every $u \in \mathbf{P}_d$. Hence we may consider M_∞ instead of L for the proof of the theorem, since M_∞ inherits properties (I) to (V) from L. This completes the proof of Lemma 6.

From now on L will be assumed to be separable. Thus $L(\mathbb{R})$ is also separable.

Following [6] and [11], we will call a set function $F^0 : \mathcal{I}_d \to L(\mathbb{R})^+$ an *almost additive process* in $L(\mathbb{R})^+$ with respect to the subsemigroup $\{\tau(u) : u \in \mathbf{P}_d\}$ of Lemma 5 if it is bounded and satisfies the following conditions:

(i) $\tau(u)F^0(I)(\omega) \geq F(u+I)(\omega)$ for almost all $\omega \in \Omega$, for every $u \in \mathbf{P}_d$ and $I \in \mathcal{I}_d$.

(ii) If $I_1, \ldots, I_k \in \mathcal{I}_d$ are pairwise disjoint and $I = \bigcup_{i=1}^k I_i \in \mathcal{I}_d$, then $F^0(I) = \sum_{i=1}^k F^0(I_i)$.

EXAMPLES. (a) If $f \in L$ is given, then, since $L(\mathbb{R})$ is separable, Lemma 5(iv) implies that the function $u \mapsto \tau(u)\|f(\cdot)\|$ from $\mathbf{P}_d$ to $L(\mathbb{R})^+$ is strongly Lebesgue measurable, and thus it is Bochner integrable over every $I \in \mathcal{I}_d$. If we set

$$F^0(I) = \int_I \tau(u)\|f(\cdot)\|\,du$$

for $I \in \mathcal{I}_d$, then $F^0(I)$ defines an almost additive process in $L(\mathbb{R})^+$ with respect to $\{\tau(u) : u \in \mathbf{P}_d\}$ by Lemma 5(iii). It is clear that

$$\Big\|\Big(\int_I T(u)f\,du\Big)(\omega)\Big\| \leq F^0(I)(\omega)$$

for almost all $\omega \in \Omega$, for every $I \in \mathcal{I}_d$.

(b) Let $F : \mathcal{I}_d \to L$ be a bounded additive process in L with respect to $T = \{T(u) : u \in \mathbf{P}_d\}$. For an $I \in \mathcal{I}_d$, let $\mathcal{P}(I)$ denote the class of all finite partitions of I into pairwise disjoint intervals in $\mathbf{P}_d$, and define

$$F^0(I) = \operatorname{ess\,sup}\Big\{\sum_{i=1}^k \|F(I_i)(\cdot)\| : \{I_1, \ldots, I_k\} \in \mathcal{P}(I)\Big\}.$$

Since

$$\Big\|\sum_{i=1}^k \|F(I_i)(\cdot)\|\Big\|_{L(\mathbb{R})} \leq \sum_{i=1}^k \|F(I_i)\|_L \leq K(F)\lambda_d(I) < \infty,$$

it follows from properties (V) and (II) that $F^0(I)$ is a function in $L(\mathbb{R})^+$. Thus, by Lemmas 3 and 5(i), $F^0(I)$ defines an almost additive process in $L(\mathbb{R})^+$ with respect to $\{\tau(u) : u \in \mathbf{P}_d\}$. Clearly, we have $K(F^0) = K(F) < \infty$ and $\|F(I)(\omega)\| \leq F^0(I)(\omega)$ for almost all $\omega \in \Omega$, for every $I \in \mathcal{I}_d$.

Since the function $u \mapsto \tau(u)\widetilde{g}$ from $\mathbf{P}_d$ to $L(\mathbb{R})$ is strongly Lebesgue measurable for every $\widetilde{g} \in L(\mathbb{R})$, we can apply the same proof of Lemma 2 of [11] to obtain the following lemma. We omit the details.

LEMMA 7 (cf. Lemma 2 of [11]). *Let $d = 2^m$ with $m \geq 1$. Then there exists a constant C_d, depending only on d, and a strongly continuous one-*

dimensional subsemigroup $T^m = \{T^m(t) : t > 0\}$ *of positive linear contractions defined on* $L(\mathbb{R})$ *such that to each d-dimensional almost additive process* F^0 *in* $L(\mathbb{R})^+$ *with respect to the subsemigroup* $\{\tau(u) : u \in \mathbf{P}_d\}$ *there corresponds a one-dimensional almost additive process* F^m *in* $L(\mathbb{R})^+$, *with respect to the subsemigroup* $T^m = \{T^m(t) : t > 0\}$, *such that*

$$\alpha^{-d} F^0((0,\alpha]^d)(\omega) \le C_d \widetilde{\alpha}^{-1} F^m((0,\widetilde{\alpha}])(\omega)$$

for almost all $\omega \in \Omega$, *for every* $\alpha > 0$, *where* $\log \widetilde{\alpha} = 2^{-m} \log \alpha$. *In particular, if* $F^0(I) = \int_I \tau(u)\|f(\cdot)\|\,du$ *for all* $I \in \mathcal{I}_d$, *where* $f \in L$, *then we have* $F^m((a,b]) = \int_a^b T^m(t)\|f(\cdot)\|\,dt$ *for every* $a, b \in \mathbb{R}_1^+$ *with* $a < b$.

Next, for $t > 0$ and $\widetilde{g} \in L(\mathbb{R})^+$, define

$$U(t)\widetilde{g} = \operatorname{ess\,sup} \Big\{ T^m(t_1)T^m(t_2)\ldots T^m(t_k)\widetilde{g} : t_i > 0,\ \sum_{i=1}^k t_i = t \Big\}.$$

Since $0 \le T^m(s+t) \le T^m(s)T^m(t)$ (by the subsemigroup property) and $\|T^m(t)\| \le 1$ for $s, t > 0$, it follows from properties (V), (II), (I) and Lemma 3 that $U(t)\widetilde{g}$ is a function in $L(\mathbb{R})^+$ such that

$$\|U(t)\widetilde{g}\|_{L(\mathbb{R})} \le \|\widetilde{g}\|_{L(\mathbb{R})}.$$

Since $U(t)$ is linear on $L(\mathbb{R})^+$, it can be uniquely extended to a linear contraction on $L(\mathbb{R})$ in an obvious manner. Clearly, the construction implies $U(s+t) = U(s)U(t)$ for $s, t > 0$. That is, $U = \{U(t) : t > 0\}$ becomes a semigroup of operators.

LEMMA 8. $U = \{U(t) : t > 0\}$ *is a strongly continuous semigroup of positive linear contractions on* $L(\mathbb{R})$.

Proof. It only remains to prove the strong continuity of the semigroup U. To do so, let $t > 0$ and $\widetilde{g} \in L(\mathbb{R})^+$. For an $\varepsilon > 0$, take $t_1, \ldots, t_k > 0$ with $t = \sum_{i=1}^k t_i$ so that

$$\|U(t)\widetilde{g} - T^m(t_1)T^m(t_2)\ldots T^m(t_k)\widetilde{g}\|_{L(\mathbb{R})} < \varepsilon.$$

Since $T^m = \{T^m(t)\}$ is strongly continuous on $(0,\infty)$ by Lemma 7, it follows that

$$\|T^m(s_1)T^m(s_2)\ldots T^m(s_k)\widetilde{g} - T^m(t_1)T^m(t_2)\ldots T^m(t_k)\widetilde{g}\|_{L(\mathbb{R})} \to 0$$

as s_i approaches t_i for each i with $1 \le i \le k$. Hence there exists a $\delta > 0$ so that if $|s_i - t_i| < \delta$ for $1 \le i \le k$, then

$$\|U(t)\widetilde{g} - T^m(s_1)T^m(s_2)\ldots T^m(s_k)\widetilde{g}\|_{L(\mathbb{R})} < \varepsilon.$$

By the fact that $U(s_1 + \ldots + s_k)\widetilde{g}(\omega) \ge T^m(s_1)T^m(s_2)\ldots T^m(s_k)\widetilde{g}(\omega)$ for almost all $\omega \in \Omega$, we now get

$$\|(U(s)\widetilde{g} - U(t)\widetilde{g})^-\|_{L(\mathbb{R})} < \varepsilon$$

for $s = s_1 + \ldots + s_k$, and hence

$$\lim_{s \to t} \|(U(s)\widetilde{g} - U(t)\widetilde{g})^-\|_{L(\mathbb{R})} = 0,$$

which implies that the function $t \mapsto U(t)\widetilde{g}$ is strongly measurable, since $L(\mathbb{R})$ is separable. Hence the semigroup $U = \{U(t)\}$ is strongly continuous on the interval $(0, \infty)$ by Lemma VIII.1.3 of [5]. This completes the proof.

Lemma 9. *For every $f \in L$ there exists an X-valued function $(u, \omega) \mapsto T(u)(f, \omega)$ defined on $\mathbf{P}_d \times \Omega$, strongly measurable with respect to the usual product σ-algebra $\mathcal{M}_d \times \Sigma$, where $\mathcal{M}_d$ is the σ-algebra of all Lebesgue measurable subsets of $\mathbf{P}_d$, such that, for each fixed $u \in \mathbf{P}_d$, $T(u)(f, \cdot)$ is, as a function on Ω, a representative of the element $T(u)f \in L$.*

Proof. Since the strong limit $T(0) = \lim_{u \to 0} T(u)$ exists and $\|T(u)\| \le 1$ for all $u \in \mathbf{P}_d$ by hypotheses, the semigroup $T = \{T(u) : u \in \mathbf{P}_d\}$ can be extended uniquely to a strongly continuous semigroup on $\mathbb{R}_d^+$ by an easy approximation argument. We will denote this extended semigroup by $T = \{T(u) : u \in \mathbb{R}_d^+\}$. (Incidentally, we note that each $T(u)$ with $u \in \mathbb{R}_d^+$ has the linear modulus $\tau(u)$ defined on $L(\mathbb{R})$. A proof can be found in the proof of Lemma 1 of [11].)

For $n \ge 1$, define a step functin $F_n : \mathbb{R}_d^+ \to L$ by

$$F_n(u) = T\left(\frac{i_1}{n!}, \ldots, \frac{i_d}{n!}\right) f \quad \text{for } u = (u_1, \ldots, u_d) \in \mathbb{R}_d^+,$$

where $i_1, \ldots, i_d$ are nonnegative integers such that

$$\frac{i_l}{n!} \le u_l < \frac{i_l + 1}{n!} \quad \text{for } 1 \le l \le d.$$

Since the contraction semigroup $\{T(u) : u \in \mathbb{R}_d^+\}$ is strongly continuous on $\mathbb{R}_d^+$, there exists a subsequence $(F_{n(k)},\ k \ge 1)$ of (F_n) such that

$$\sum_{k=1}^{\infty} \|F_{n(k)}(u) - T(u)f\|_L < \infty$$

for all $u \in \mathbb{R}_d^+$. Then we apply Lemma 1 to infer that, for each fixed $u \in \mathbb{R}_d^+$,

$$\sum_{k=1}^{\infty} \|(F_{n(k)}(u))(\omega) - (T(u)f)(\omega)\| < \infty$$

for almost all $\omega \in \Omega$. Hence the function $(u, \omega) \mapsto T(u)(f, \omega)$ on $\mathbf{P}_d \times \Omega$ defined by the relation

$$T(u)(f, \omega) = \begin{cases} \lim_{k \to \infty} (F_{n(k)}(u))(\omega) & \text{if the limit exists,} \\ 0 & \text{otherwise,} \end{cases}$$

is as required, and the proof is complete.

LEMMA 10. *Let $f \in L$. If $a, b \in \mathbb{R}_1^+$ with $a < b$, define a function $F(\omega)$ on Ω by the relation*

$$F(\omega) = \begin{cases} \int_{(a,b]^d} T(u)(f,\omega)\,du & \text{if the integral exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $F(\cdot)$ is a representative of the element $\int_{(a,b]^d} T(u)f\,du \in L$.

Proof. First we notice that

$$\int_{(a,b]^d} \|T(u)(f,\omega)\|\,du < \infty$$

for almost all $\omega \in \Omega$. In fact, there exists a strictly positive measurable function w on Ω such that

$$\int_{\Omega} \|f(\omega)\| w(\omega)\,d\mu < \infty$$

for all $f \in L$, by Lemma 4. Thus we can find a constant $\widetilde{K} > 0$ so that

$$\int_{\Omega} \|f(\omega)\| w(\omega)\,d\mu(\omega) \le \widetilde{K} \|f\|_L \tag{10}$$

for all $f \in L$ (cf. the proof of Lemma 4 of [12]). Then by Fubini's theorem

$$\begin{aligned} \int_{\Omega} \int_{(a,b]^d} \|T(u)(f,\omega)\| w(\omega)\,du\,d\mu(\omega) &= \int_{(a,b]^d} \int_{\Omega} \|T(u)(f,\omega)\| w(\omega)\,d\mu(\omega)\,du \\ &\le \int_{(a,b]^d} \widetilde{K} \|f\|_L\,du < \infty. \end{aligned}$$

Since $w(\omega) > 0$ on Ω, this shows that $\int_{(a,b]^d} \|T(u)(f,\omega)\|\,du < \infty$ for almost all $\omega \in \Omega$.

Now, by the strong continuity of $T = \{T(u)\}$ on $\mathbb{R}_d^+$, there exists a sequence $(\widehat{F}_n,\ n \ge 1)$ of step functions $\widehat{F}_n : (a,b]^d \to L$ such that

$$\|\widehat{F}_n(u)\|_L \le \|f\|_L \quad \text{and} \quad \lim_{n\to\infty} \|\widehat{F}_n(u) - T(u)f\|_L = 0$$

for all $u \in (a,b]^d$. Thus by Lebesgue's convergence theorem we have

$$\int_{(a,b]^d} T(u)f\,du = \text{strong-}\lim_{n\to\infty} \int_{(a,b]^d} \widehat{F}_n(u)\,du. \tag{11}$$

Here we may assume without loss of generality that

$$\sum_{n=1}^{\infty} \int_{(a,b]^d} \|\widehat{F}_n(u) - T(u)f\|_L\,du < \infty.$$

Then by (10) and Fubini's theorem

$$\infty > \sum_{n=1}^{\infty} \int_{(a,b]^d} \int_{\Omega} \|(\widehat{F}_n(u))(\omega) - T(u)(f,\omega)\| w(\omega)\, d\mu(\omega)\, du$$
$$= \sum_{n=1}^{\infty} \int_{\Omega} w(\omega) \Big(\int_{(a,b]^d} \|(\widehat{F}_n(u))(\omega) - T(u)(f,\omega)\|\, du \Big) d\mu(\omega),$$

whence the inequality

$$\sum_{n=1}^{\infty} \int_{(a,b]^d} \|(\widehat{F}_n(u))(\omega) - T(u)(f,\omega)\|\, du < \infty$$

must hold for almost all $\omega \in \Omega$, since $w(\omega) > 0$ on Ω. Consequently,

$$\lim_{n\to\infty} \int_{(a,b]^d} (\widehat{F}_n(u))(\omega)\, du = \int_{(a,b]^d} T(u)(f,\omega)\, du \tag{12}$$

for almost all $\omega \in \Omega$. On the other hand, since $\widehat{F}_n : (a,b] \to L$ is a step function, it is clear that the function $\omega \mapsto \int_{(a,b]^d} (\widehat{F}_n(u))(\omega)\, du$ is a representative of the element $\int_{(a,b]^d} \widehat{F}_n(u)\, du \in L$, for each $n \geq 1$. Hence the lemma follows from (11) and (12) together with Lemma 2.

Lemma 11. *Let $f \in L$ and $\beta > 0$. Then the function $f_\beta = \int_{(0,\beta]^d} T(u) f\, du$ satisfies*

$$f_\beta(\omega) = \operatorname*{q\text{-}lim}_{\alpha\to 0} \alpha^{-d} \Big(\int_{(0,\alpha]^d} T(u) f_\beta\, du \Big)(\omega)$$

for almost all $\omega \in \Omega$.

Proof. By Fubini's theorem, the X-valued function $F_\beta(u,\omega)$ on $\mathbf{P}_d \times \Omega$ defined by

$$F_\beta(u,\omega) = \begin{cases} \int_{u+(0,\beta]^d} T(t)(f,\omega)\, dt & \text{if the integral exists,} \\ 0 & \text{otherwise,} \end{cases}$$

is strongly measurable with respect to $\mathcal{M}_d \times \Sigma$. By Lemmas 9 and 10 we have

$$F_\beta(u,\omega) = T(u)(f_\beta,\omega)$$

for almost all $(u,\omega) \in \mathbf{P}_d \times \Omega$ with respect to the product measure of λ_d and μ. Thus

$$\operatorname*{q\text{-}lim}_{\alpha\to 0} \alpha^{-d} \Big(\int_{(0,\alpha]^d} T(u) f_\beta\, du \Big)(\omega)$$
$$= \operatorname*{q\text{-}lim}_{\alpha\to 0} \alpha^{-d} \int_{(0,\alpha]^d} T(u)(f_\beta,\omega)\, du \qquad \text{(by Lemma 10)}$$
$$= \operatorname*{q\text{-}lim}_{\alpha\to 0} \alpha^{-d} \int_{(0,\alpha]^d} F_\beta(u,\omega)\, du = \int_{(0,\beta]^d} T(u)(f,\omega)\, du$$

for almost all $\omega \in \Omega$, where the last equality follows from the obvious fact that

$$\lim_{\substack{u\to 0\\ u\in \mathbf{P}_d}} F_\beta(u,\omega) = \int_{(0,\beta]^d} T(t)(f,\omega)\,dt$$

for almost all $\omega \in \Omega$. Since the function $\omega \mapsto \int_{(0,\beta]^d} T(u)(f,\omega)\,du$ is a representative of f_β by Lemma 10, this completes the proof.

3. Proof of the Theorem

Part I: *Proof of the first half.* If $f \in L$ then, since

$$T(0) = \text{strong-}\lim_{u\to 0} T(u),$$

we find

$$\lim_{\beta\to 0} \Big\| T(0)f - \beta^{-d} \int_{(0,\beta]^d} T(u)f\,du \Big\|_L = 0$$

and

$$T(u)(f - T(0)f) = 0 \quad \text{for } u \in \mathbf{P}_d.$$

Hence by Lemma 11 there exists a sequence $(f_n,\ n \ge 1)$ of functions in L so that $\|T(0)f - f_n\|_L < 2^{-n}$, $T(0)f_n = f_n$, and

$$\text{q-}\lim_{\alpha\to 0} \alpha^{-d}\Big(\int_{(0,\alpha]^d} T(u)f_n\,du\Big)(\omega) = f_n(\omega) \tag{13}$$

for almost all $\omega \in \Omega$, for every $n \ge 1$. We now define a nonnegative measurable function $\widetilde{G}(\omega)$ on Ω by the relation

$$\widetilde{G}(\omega) = \text{q-}\limsup_{\alpha\to 0} \Big\| \alpha^{-d}\Big(\int_{(0,\alpha]^d} T(u)f\,du\Big)(\omega) - T(0)f(\omega)\Big\|.$$

It then suffices to prove that $\widetilde{G}(\omega) = 0$ for almost all $\omega \in \Omega$. To do so, we use the equation

$$\begin{aligned}\alpha^{-d} &\int_{(0,\alpha]^d} T(u)f\,du - T(0)f\\ &= \alpha^{-d}\int_{(0,\alpha]^d} T(u)(T(0)f - f_n)\,du + \alpha^{-d}\int_{(0,\alpha]^d} T(u)f_n\,du\\ &\quad - T(0)f_n + T(0)(f_n - f).\end{aligned}$$

From (13) we obtain

$$\begin{aligned}\widetilde{G}(\omega) &\le \text{q-}\limsup_{\alpha\to 0} \Big\| \alpha^{-d}\Big(\int_{(0,\alpha]^d} T(u)(T(0)f - f_n)\,du\Big)(\omega)\Big\|\\ &\quad + \|T(0)(f_n - f)(\omega)\|\\ &= \mathbf{I}_n(\omega) + \mathbf{II}_n(\omega)\end{aligned}$$

for almost all $\omega \in \Omega$. Since

$$\sum_{n=1}^{\infty} \|T(0)(f_n - f)\|_L = \sum_{n=1}^{\infty} \|f_n - T(0)f\|_L < \sum_{n=1}^{\infty} 2^{-n} < \infty,$$

it follows from Lemma 1 that

$$\lim_{n\to\infty} \mathbf{II}_n(\omega) = \lim_{n\to\infty} \|T(0)(f_n - f)(\omega)\| = 0$$

for almost all $\omega \in \Omega$.

To estimate $\mathbf{I}_n(\omega)$, we notice that

$$\Big\|\Big(\int\limits_{(0,\alpha]^d} T(u)(T(0)f - f_n)\,du\Big)(\omega)\Big\| \le \Big(\int\limits_{(0,\alpha]^d} \tau(u)(\|T(0)f(\cdot) - f_n(\cdot)\|)\,du\Big)(\omega)$$

for almost all $\omega \in \Omega$. Hence by Lemma 7,

$$\Big\|\alpha^{-d}\Big(\int\limits_{(0,\alpha]^d} T(u)(T(0)f - f_n)\,du\Big)(\omega)\Big\| \le C_d \widetilde{\alpha}^{-1}\Big(\int\limits_0^{\widetilde{\alpha}} T^m(t)(\|T(0)f(\cdot) - f_n(\cdot)\|)\,dt\Big)(\omega)$$

for almost all $\omega \in \Omega$, where $\log \widetilde{\alpha} = 2^{-m} \log \alpha$. Since $0 \le T^m(t) \le U(t)$ for $t > 0$ by the construction of the semigroup $U = \{U(t)\}$, it follows that

$$\mathbf{I}_n(\omega) \le C_d \operatorname*{q\text{-}lim\,sup}_{\widetilde{\alpha}\to 0} \widetilde{\alpha}^{-1}\Big(\int\limits_0^{\widetilde{\alpha}} U(t)(\|T(0)f(\cdot) - f_n(\cdot)\|)\,dt\Big)(\omega) \tag{14}$$

for almost all $\omega \in \Omega$.

Here we need two decompositions $\Omega = P + N$ and $\Omega = C + D$ of Ω (with respect to $U = \{U(t)\}$) mentioned in [12]. We recall that $\Omega = P + N$ is the measurable decomposition such that

(a) if $\widetilde{f} \in L(\mathbb{R})$ and $\{\omega : \widetilde{f}(\omega) \neq 0\} \subset N$, then $\|U(t)\widetilde{f}\|_{L(\mathbb{R})} = 0$ for all $t > 0$,

(b) if $0 \neq \widetilde{f} \in L(\mathbb{R})^+$ and $\mu(\{\omega : \widetilde{f}(\omega) \neq 0\} \cap P) > 0$, then $\|U(t)\widetilde{f}\|_{L(\mathbb{R})} > 0$ for some $t > 0$.

Next, $\Omega = C + D$ is the measurable decomposition such that

(c) $U(t)\widetilde{f}(\omega) = 0$ on D for every $t > 0$ and $\widetilde{f} \in L(\mathbb{R})$,

(d) $C = \bigcup_{n=1}^{\infty} \{\omega : U(1/n)\widetilde{h}(\omega) > 0\}$ for some $\widetilde{h} \in L(\mathbb{R})^+$.

Since $(L, \|\cdot\|_L)$ has property (IV), it follows from Proposition 2 of [12] that

$$C \subset P. \tag{15}$$

On the other hand, from (c) and Lemma 2 we see that

$$\Big(\int_0^{\widetilde{\alpha}} U(t)(\|T(0)f(\cdot) - f_n(\cdot)\|)\,dt\Big)(\omega) = 0 \quad \text{on } D.$$

Therefore it follows from the Theorem of [12] and (15) that for each $n \geq 1$ the limit

$$\widetilde{h}_n(\omega) := \operatorname*{q\text{-}lim}_{\widetilde{\alpha}\to 0} \widetilde{\alpha}^{-1}\Big(\int_0^{\widetilde{\alpha}} U(t)(\|T(0)f(\cdot) - f_n(\cdot)\|)\,dt\Big)(\omega) \tag{16}$$

exists and is finite for almost all $\omega \in \Omega$.

Now, for an integer $m \geq 1$, define a nonnegative measurable function $\widetilde{h}_{n,m}$ on Ω by

$$\widetilde{h}_{n,m}(\omega) = \inf_{k\geq m} k\Big(\int_0^{1/k} U(t)(\|T(0)f(\cdot) - f_n(\cdot)\|)\,dy\Big)(\omega).$$

Clearly, $\widetilde{h}_{n,m} \in L(\mathbb{R})^+$, $0 \leq \widetilde{h}_{n,1}(\omega) \leq \widetilde{h}_{n,2}(\omega) \leq \ldots \uparrow \widetilde{h}_n(\omega)$ for almost all $\omega \in \Omega$, and

$$\|\widetilde{h}_{n,m}\|_{L(\mathbb{R})} \leq \|T(0)f - f_n\|_L < 2^{-n} \quad \text{for every } m \geq 1$$

by properties (I) and (II). Thus from properties (V), (II) and Lemma 3 we see that $\widetilde{h}_n$ is a function in $L(\mathbb{R})^+$ such that

$$\|\widetilde{h}_n\|_{L(\mathbb{R})} = \lim_{m\to\infty} \|\widetilde{h}_{n,m}\|_{L(\mathbb{R})} \leq \|T(0)f - f_n\|_L < 2^{-n}.$$

Since $\sum_{n=1}^{\infty} \|\widetilde{h}_n\|_{L(\mathbb{R})} < \infty$, it follows from Lemma 1 that $\sum_{n=1}^{\infty} \widetilde{h}_n(\omega) < \infty$ for almost all $\omega \in \Omega$, which yields

$$\lim_{n\to\infty} \mathbf{I}_n(\omega) = 0$$

for almost all $\omega \in \Omega$, because $0 \leq \mathbf{I}_n(\omega) \leq C_d\widetilde{h}_n(\omega)$ for almost all $\omega \in \Omega$ by (14) and (16). Hence we have proved that $\widetilde{G}(\omega) = 0$ for almost all $\omega \in \Omega$, and this establishes the first half of the theorem.

Part II: *Proof of the second half.* For this purpose, X will be assumed below to be reflexive. Let $F : \mathcal{I}_d \to L$ be a bounded additive process with respect to the semigroup $T = \{T(u) : u \in \mathbf{P}_d\}$, where $d = 2^m$ with $m \geq 1$. By Example (b) there exists an almost additive process $F^0 : \mathcal{I}_d \to L(\mathbb{R})^+$ with respect to the subsemigroup $\{\tau(u) : u \in \mathbf{P}_d\}$ such that for every $I \in \mathcal{I}_d$,

$$\|F(I)(\omega)\| \leq F^0(I)(\omega) \tag{17}$$

for almost all $\omega \in \Omega$. By Lemma 7 there exists a one-dimensional strongly continuous subsemigroup $T^m = \{T^m(t) : t > 0\}$ and a one-dimensional almost additive process $F^m : \mathcal{I}_1 \to L(\mathbb{R})^+$ with respect to $T^m = \{T^m(t)\}$ such that for every $\alpha > 0$,

$$\alpha^{-d} F^0((0,\alpha]^d)(\omega) \le C_d \widetilde{\alpha}^{-1} F^m((0,\widetilde{\alpha}])(\omega) \tag{18}$$

for almost all $\omega \in \Omega$, where $\log\widetilde{\alpha} = 2^{-m}\log\alpha$. Thus, taking into account property (V) and Lemma 3, we can apply the same proof of Lemma 3 of [11] to obtain a bounded one-dimensional process $\widetilde{G} : \mathcal{I}_1 \to L(\mathbb{R})^+$, additive with respect to the semigroup $U = \{U(t) : t > 0\}$ of Lemma 8, such that for every $\alpha > 0$,

$$F^m((0,\alpha])(\omega) \le \widetilde{G}((0,\alpha])(\omega) \tag{19}$$

for almost all $\omega \in \Omega$.

Since

$$\begin{cases} \widetilde{G}((0,\alpha]) = \widetilde{G}((0,\varepsilon]) + U(\varepsilon)\widetilde{G}((0,\alpha-\varepsilon]), \\ \|\widetilde{G}((0,\varepsilon])\|_{L(\mathbb{R})} \to 0 \quad \text{as } \varepsilon \to 0, \quad \text{and} \\ U(\varepsilon)\widetilde{G}((0,\alpha-\varepsilon])(\omega) = 0 \quad \text{on } D \quad \text{(by (c))}, \end{cases}$$

it follows from Lemma 2 that for every $\alpha > 0$,

$$\widetilde{G}((0,\alpha])(\omega) = 0 \quad \text{on } D. \tag{20}$$

Hence from (15) and (c) we see that it may be assumed without loss of generality that

$$\Omega = C = P \tag{21}$$

(In fact, if $f \in L$ is such that $f(\omega) = 0$ on D then, since $(U(t)\|f(\cdot)\|)(\omega) = 0$ on D for every $t > 0$, it follows from the construction of $U = \{U(t)\}$ that $(T^m(t)\|f(\cdot)\|)(\omega) = 0$ on D for every $t > 0$, which in turn implies that $T(u)f(\omega) = 0$ on D for every $u \in \mathbf{P}_d$, by the costruction of $T^m = \{T^m(t) : t > 0\}$ (see the proof of Lemma 2 of [11]). From this and (15), without loss of generality, we may assume (21) for the proof of the theorem.)

Then, by Theorem of [12], the limit

$$\widetilde{g}(\omega) := \operatorname*{q\text{-}lim}_{\widetilde{\alpha}\to 0} \widetilde{\alpha}^{-1}\widetilde{G}((0,\widetilde{\alpha}])(\omega) \tag{22}$$

exists and is finite for almost all $\omega \in \Omega = C = P$, and since $\widetilde{G}$ is a bounded process in $L(\mathbb{R})^+$, it follows from properties (I), (II) and (V) that the limit function $\widetilde{g}$ is in $L(\mathbb{R})^+$.

To complete the proof we use Lemma 8 of [12] as follows. By this lemma we can find a positive number β and a sequence $(v_n,\ n \ge 1)$ of nonnegative measurable functions on Ω so that

(i) $0 \le v_1(\omega) \le v_2(\omega) \le \ldots$ for almost all $\omega \in \Omega$,
(ii) $\lim_{n\to\infty} v_n(\omega) > 0$ for almost all $\omega \in \Omega = P = C$,

(iii) for each $\widetilde{f} \in L(\mathbb{R})^+$, $t > 0$ and $n \geq 1$ we have

$$\int_\Omega (U(t)\widetilde{f}) \cdot v_n \, d\mu \leq e^{\beta t} \int_\Omega \widetilde{f} \cdot v_n \, d\mu < \infty. \tag{23}$$

For $n \geq 1$, we let

$$P_n = \{\omega : v_n(\omega) > 0\}. \tag{24}$$

Clearly, $P_1 \subset P_2 \subset \ldots$ and $\bigcup_{n=1}^\infty P_n = \Omega$. If $f \in L$ is such that $f(\omega) = 0$ on P_n, then (23) implies that $(U(t)\|f(\cdot)\|)(\omega) = 0$ on P_n for every $t > 0$. Thus it follows, as before, that $T(u)f(\omega) = 0$ on P_n for every $u \in \mathbf{P}_d$. Hence if we let

$$T_n(u)f = (T(u)f)\chi_{P_n} \tag{25}$$

for $u \in \mathbf{P}_d$ and $f \in L_n$, where L_n is a subspace of L defined by

$$L_n = \{f \in L : f(\omega) = 0 \text{ on } \Omega \setminus P_n\},$$

then $T_n = \{T_n(u) : u \in \mathbf{P}_d\}$ becomes a strongly continuous d-dimensional semigroup of linear contractions on L_n for which the strong limit $T_n(0) = \text{strong-lim}_{u \to 0} T_n(u)$ exists. It is clear that

$$T_n(0)f = (T(0)f)\chi_{P_n}$$

for every $f \in L_n$. Next let

$$F_n(I) = F(I)\chi_{P_n} \tag{26}$$

for $I \in \mathcal{I}_d$. Then $F_n : \mathcal{I}_d \to L_n$ becomes a bounded process in L_n which is additive with respect to the semigroup $T_n = \{T_n(u) : u \in \mathbf{P}_d\}$. Moreover, if $U_n = \{U_n(t) : t > 0\}$ denotes the one-dimensional semigroup of positive linear contractions on $L_n(\mathbb{R})$, where

$$L_n(\mathbb{R}) := \{\widetilde{f} \in L(\mathbb{R}) : \widetilde{f}(\omega) = 0 \text{ on } \Omega \setminus P_n\},$$

induced from the semigroup $T_n = \{T_n(u) : u \in \mathbf{P}_d\}$ through Lemmas 7 and 8, then

$$U_n(t)f = (U(t)f)\chi_{P_n} \tag{27}$$

for every $f \in L_n(\mathbb{R})$ and $t > 0$.

Taking into account (27), we will limit ourselves to the semigroup $T_n = \{T_n(u) : u \in \mathbf{P}_d\}$ for the moment; hence we may and do assume for a while that

$$P_n = P = \Omega = C. \tag{28}$$

Thus $U_n(t) = U(t)$ holds for $t > 0$; and by the right-hand side inequality of (23) there exists a constant $K > 0$ such that

$$\int_\Omega \|f(\omega)\| v_n(\omega) \, d\mu \leq K \|f\|_L \tag{29}$$

for all $f \in L$ (cf. Lemma 4 of [12]). It follows that

$$L \subset L_1(v_n\, d\mu; X) := L_1((\Omega, \Sigma, v_n\, d\mu); X),$$

and in particular

$$L(\mathbb{R}) \subset L_1(v_n\, d\mu; \mathbb{R}) := L_1((\Omega, \Sigma, v_n\, d\mu); \mathbb{R}). \tag{30}$$

Since $L(\mathbb{R})$ is dense in $L_1(v_n\, d\mu; \mathbb{R})$, (23) and (29) show that $U = \{U(t)\}$ can be regarded as a strongly continuous semigroup of positive linear operators on $L_1(v_n\, d\mu; \mathbb{R})$ such that

$$\|U(t)\|_{L_1(v_n\, d\mu; \mathbb{R})} \le e^{\beta t} \quad \text{for } t > 0,$$

and thus $\{e^{-\beta t}U(t) : t > 0\}$ becomes a strongly continuous contraction semigroup on $L_1(v_n\, d\mu; \mathbb{R})$. Since $C = \Omega$ by (28), it follows from Akcoglu and Chacon [1] that the strong limit

$$U(0) = \operatorname*{strong-lim}_{t \to 0} e^{-\beta t} U(t) \tag{31}$$

exists in $L_1(v_n d\mu; \mathbb{R})$.

Let $\widetilde{g} \in L(\mathbb{R})^+$ be the function in (22). Then, by putting

$$\widetilde{g}_k = k\widetilde{G}((0, k^{-1}]) \quad \text{for } k \ge 1, \tag{32}$$

we have

$$0 \le U(0)\widetilde{g}(\omega) \le \liminf_{n \to \infty} U(0)\widetilde{g}_k(\omega) = \liminf_{k \to \infty} \widetilde{g}_k(\omega) = \widetilde{g}(\omega) \tag{33}$$

for almost all $\omega \in \Omega$. On the other hand, since $C = \Omega$ by hypothesis, it follows from (d) of the decomposition $\Omega = C + D$ that there exists a strictly positive function $\widetilde{f}$ in $L_1(v_n\, d\mu; \mathbb{R})$ such that $U(0)\widetilde{f} = \widetilde{f}$. Since $U(0)$ is a positive linear contraction on $L_1(v_n\, d\mu; \mathbb{R})$, we see that

$$U(0)^* 1 = 1 \in L_\infty(v_n\, d\mu; \mathbb{R}) = L_1(v_n\, d\mu; \mathbb{R})^*,$$

so that

$$\int_\Omega [U(0)\widetilde{g}] v_n\, d\mu = \int_\Omega \widetilde{g}[U(0)^* 1] v_n\, d\mu = \int_\Omega \widetilde{g} v_n\, d\mu,$$

and by (33),

$$U(0)\widetilde{g} = \widetilde{g} \in L(\mathbb{R})^+ \subset L_1(v_n\, d\mu; \mathbb{R})^+. \tag{34}$$

Next, define

$$\widetilde{h}_k = \widetilde{g} \wedge \widetilde{k}\widetilde{G}((0, \widetilde{k}^{-1}]) \quad \text{for } k \ge 1, \tag{35}$$

where $\log \widetilde{k} = 2^{-m} \log k$. Since $\lim_{k\to\infty} \widetilde{g}_k(\omega) = \widetilde{g}(\omega) = \lim_{k\to\infty} \widetilde{h}_k(\omega)$ for almost all $\omega \in \Omega$ by (22), it follows from Lemma 3 that

$$\lim_{k \to \infty} \|\widetilde{g} - \widetilde{h}_k\|_{L(\mathbb{R})} = 0. \tag{36}$$

On the other hand, since the functions f_k on Ω defined by

$$f_k(\omega) = \begin{cases} k^d F((0,k^{-1}]^d)(\omega) & \text{if } k^d\|F((0,k^{-1}]^d)(\omega)\| \le C_d\widetilde{h}_k(\omega), \\ C_d\widetilde{h}_k(\omega)\,\mathrm{sgn}[F((0,k^{-1}]^d)(\omega)] & \text{otherwise}, \end{cases}$$

where $\mathrm{sgn}\, x = x/\|x\|$ if $0 \ne x \in X$, and $\mathrm{sgn}\, 0 = 0 \in X$, satisfy

$$\|f_k(\omega)\| \le C_d\widetilde{h}_k(\omega) \le C_d\widetilde{g}(\omega) \quad \text{for } \omega \in \Omega, \tag{37}$$

and since X is a reflexive Banach space by hypothesis, it follows that the set $\{f_k : k \ge 1\}$ is weakly sequentially compact in $L_1(v_n\, d\mu; X)$ (cf. Theorem IV.2.1 of [4]). So, if necessary, taking a subsequence of (f_k), we may assume without loss of generality that the weak limit function

$$f_\infty = \text{weak-}\lim_{k\to\infty} f_k \tag{38}$$

exists in $L_1(v_n\, d\mu; X)$. Here from (37) we see that $\|f_\infty(\omega)\| \le C_d\widetilde{g}(\omega)$ on Ω, so that by property (II),

$$f_\infty \in L \quad (\subset L_1(v_n\, d\mu; X)). \tag{39}$$

Now, we prove that for each fixed $\alpha > 0$ the mapping

$$g \mapsto \int_{(0,\alpha]^d} T(u)g\, du \tag{40}$$

from L into itself can be uniquely extended to a bounded linear operator from $L_1(v_n\, d\mu; X)$ into itself. For this purpose, let $g \in L$. Since

$$\Big\|\alpha^{-d}\Big(\int_{(0,\alpha]^d} T(u)g\, du\Big)(\omega)\Big\| \le C_d\widetilde{\alpha}^{-1}\Big(\int_0^{\widetilde{\alpha}} U(t)\|g(\cdot)\|dt\Big)(\omega)$$

for almost all $\omega \in \Omega$, where $\log\widetilde{\alpha} = 2^{-m}\log\alpha$, we have

$$\begin{aligned}
&\int_\Omega \alpha^{-d}\Big\|\Big(\int_{(0,\alpha]^d} T(u)g\, du\Big)(\omega)\Big\| v_n(\omega)\, d\mu(\omega) \\
&\quad \le C_d\widetilde{\alpha}^{-1}\int_\Omega \Big[\Big(\int_0^{\widetilde{\alpha}} U(t)\|g(\cdot)\|\, dt\Big)(\omega)v_n(\omega)\Big] d\mu(\omega) \\
&\quad = C_d\widetilde{\alpha}^{-1}\int_0^{\widetilde{\alpha}}\Big[\int_\Omega (U(t)\|g(\cdot)\|)(\omega)v_n(\omega)\, d\mu(\omega)\Big] dt \qquad \text{(by (29))} \\
&\quad \le C_d\widetilde{\alpha}^{-1}\int_0^{\widetilde{\alpha}}\Big[e^{\beta t}\int_\Omega \|g(\omega)\|\cdot v_n(\omega)\, d\mu(\omega)\Big] dt \qquad \text{(by (23))} \\
&\quad \le C_d e^{\beta\widetilde{\alpha}}\int_\Omega \|g(\omega)\| v_n(\omega)\, d\mu(\omega),
\end{aligned}$$

and thus

$$\left\|\alpha^{-d}\int_{(0,\alpha]^d} T(u)g\,du\right\|_{L_1(v_n\,d\mu;X)} \le C_d e^{\beta\widetilde{\alpha}}\|g\|_{L_1(v_n\,d\mu;X)} \tag{41}$$

for every $g \in L$. Since L is dense in $L_1(v_n\,d\mu;X)$, this establishes the desired conclusion.

We will apply this result to the weak limit function f_∞ of (38) as follows. Since $f_\infty \in L$ by (39), for every $\alpha > 0$ we have

$$\begin{aligned}(42)\quad F((0,\alpha]^d) &- \int_{(0,\alpha]^d} T(u)f_\infty\,du \\ &= \text{strong-}\lim_{k\to\infty}\int_{(0,\alpha]^d} T(u)[k^dF((0,k^{-1}]^d)]\,du \\ &\quad - \int_{(0,\alpha]^d} T(u)f_\infty\,du \quad \text{in } L \qquad \text{(cf. the proof of Lemma 3.2 of [1])} \\ &= \text{weak-}\lim_{k\to\infty}\int_{(0,\alpha]^d} T(u)[k^dF((0,k^{-1}]^d)]\,du \\ &\quad - \text{weak-}\lim_{k\to\infty}\int_{(0,\alpha]^d} T(u)f_k\,du\end{aligned}$$

in $L_1(v_n d\mu;X)$ by (29) and (38).

On the other hand, from the definition of f_k we have

$$\begin{aligned}&\|k^dF((0,k^{-1}]^d)(\omega) - f_k(\omega)\| \\ &\qquad = \max\{0, [k^d\|F((0,k^{-1}]^d)(\omega)\| - C_d\widetilde{h}_k(\omega)]\} \\ &\qquad \le C_d\widetilde{k}\widetilde{G}((0,\widetilde{k}^{-1}])(\omega) - C_d\widetilde{h}_k(\omega) \qquad \text{(by (17)–(19))}\end{aligned}$$

for almost all $\omega \in \Omega$, where $\log\widetilde{k} = 2^{-m}\log k$. Thus by Lemma 7 and the construction of $U = \{U(t)\}$ we have

$$\begin{aligned}(43)\quad \alpha^{-d}&\left\|\left(\int_{(0,\alpha]^d} T(u)[k^dF((0,k^{-1}]^d) - f_k]\,du\right)(\omega)\right\| \\ &\le C_d\widetilde{\alpha}^{-1}\left(\int_0^{\widetilde{\alpha}} T^m(t)[C_d\widetilde{k}\widetilde{G}((0,\widetilde{k}^{-1}]) - C_d\widetilde{h}_k]\,dt\right)(\omega) \\ &\le C_d^2\widetilde{\alpha}^{-1}\left(\int_0^{\widetilde{\alpha}} U(t)[\widetilde{k}\widetilde{G}((0,\widetilde{k}^{-1}]) - \widetilde{h}_k]\,dt\right)(\omega)\end{aligned}$$

for almost all $\omega \in \Omega$. Next we use the following relations:

$$(44)\quad \widetilde{\alpha}^{-1}\widetilde{G}((0,\widetilde{\alpha}]) - \widetilde{\alpha}^{-1}\int_0^{\widetilde{\alpha}} U(t)\widetilde{g}\,dt$$

$$= \widetilde{\alpha}^{-1}\widetilde{G}((0,\widetilde{\alpha}]) - \text{strong-}\lim_{k\to\infty} \widetilde{\alpha}^{-1}\int_0^{\widetilde{\alpha}} U(t)\widetilde{h}_k\,dt \quad \text{in } L(\mathbb{R}) \qquad (\text{by } (36))$$

$$= \text{strong-}\lim_{k\to\infty} \widetilde{\alpha}^{-1}\int_0^{\widetilde{\alpha}} U(t)[\widetilde{k}\widetilde{G}((0,\widetilde{k}^{-1}]) - \widetilde{h}_k]\,dt$$

in $L_1(v_n d\mu;\mathbb{R})$ by Lemma 3.2 of [2], (30) and (29).

It follows from (42)–(44), together with Lemma 5 of [11], that

$$(45)\quad \alpha^{-d}\Big\|F((0,\alpha]^d)(\omega) - \Big(\int_{(0,\alpha]^d} T(u)f_\infty\,du\Big)(\omega)\Big\|$$

$$\le C_d^2\widetilde{\alpha}^{-1}\Big[\widetilde{G}((0,\widetilde{\alpha}])(\omega) - \Big(\int_0^{\widetilde{\alpha}} U(t)\widetilde{g}\,dt\Big)(\omega)\Big]$$

for almost all $\omega \in \Omega$. Therefore

$$\text{q-}\limsup_{\alpha\to 0}\Big\|\alpha^{-d}F((0,\alpha]^d)(\omega) - \alpha^{-d}\Big(\int_{(0,\alpha]^d} T(u)f_\infty\,du\Big)(\omega)\Big\|$$

$$\le C_d^2\,\text{q-}\limsup_{\widetilde{\alpha}\to 0}\widetilde{\alpha}^{-1}\Big[\widetilde{G}((0,\widetilde{\alpha}])(\omega) - \Big(\int_0^{\widetilde{\alpha}} U(t)\widetilde{g}\,dt\Big)(\omega)\Big]$$

$$= C_d^2[\widetilde{g}(\omega) - U(0)\widetilde{g}(\omega)] = 0 \qquad (\text{by } (34))$$

for almost all $\omega \in \Omega$, and thus

$$\text{q-}\lim_{\alpha\to 0}\alpha^{-d}F((0,\alpha]^d)(\omega) = \text{q-}\lim_{\alpha\to 0}\alpha^{-d}\Big(\int_{(0,\alpha]^d} T(u)f_\infty\,du\Big)(\omega) = T(0)f_\infty(\omega)$$

for almost all $\omega \in \Omega$ by (7).

Since (28) was assumed in the above argument, and since $\Omega = C = P$ by (21), it is now easy to see that the limit

$$f(\omega) := \text{q-}\lim_{\alpha\to 0}\alpha^{-d}F((0,\alpha]^d)(\omega)$$

exists for almost all $\omega \in \bigcup_{n=1}^\infty P_n = P = C = \Omega$. Obviously, the limit function f is in L, and since the functions

$$h_n(\omega) := f(\omega)\chi_{P_n}(\omega)$$

satisfy $T(0)h_n(\omega) = h_n(\omega) = f(\omega)$ for almost all $\omega \in P_n$, it follows from Lemma 2 that

$$T(0)f(\omega) = \lim_{n\to\infty} T(0)h_n(\omega) = f(\omega)$$

for almost all $\omega \in \Omega$. This establishes the second half of the theorem.

4. Concluding remarks

(A) In order to see that property (IV) is necessary for the Theorem, we give the following example: By Theorem 2 of [10], for $\varepsilon > 0$, there exists a strongly continuous one-dimensional semigroup $T = \{T(t) : t \geq 0\}$ of positive linear operators on $L_1(\mu;\mathbb{R})$, where (Ω, Σ, μ) is a finite measure space, such that

$$T(t)1 = 1 \quad \text{and} \quad \|T(t)\|_{L_1(\mu;\mathbb{R})} = 1 + \varepsilon \tag{46}$$

for all $t \geq 0$, and also such that for some $h \in L_1(\mu;\mathbb{R})^+$ the limit

$$\operatorname*{q-lim}_{\alpha \to 0} \alpha^{-1}\Big(\int_0^\alpha T(t)h\,dt\Big)(\omega) \tag{47}$$

fails to exist for almost all $\omega \in N$, where $\Omega = P + N$ is the decomposition of Ω (with respect to $T = \{T(t)\}$) in Section 3 and $\mu(N) > 0$.

As a set, we let $L = L_1(\mu;\mathbb{R})$; we define the norm $\|f\|_L$ of $f \in L$ by

$$\|f\|_L = \sup\{\|(T(t)|f(\cdot)|)\|_1 \vee \|f\|_1 : t \geq 0\}.$$

By (46), $(L, \|\cdot\|_L)$ becomes a Banach space which is equivalent to $(L_1(\mu;\mathbb{R}), \|\cdot\|_1)$ via the identity mapping. Thus it is clear that $(L, \|\cdot\|_L)$ has properties (I), (II), (III) and (V). To see that $(L, \|\cdot\|_L)$ fails (IV), let $f = \chi_P$ and $g = \chi_P + \delta\chi_N$, where $\delta > 0$ is a constant. By the construction of the semigroup $T = \{T(t)\}$ (see the proof of Theorem 2 of [10]), if δ is sufficiently small, then we have

$$\|T(t)f\|_1 = (1+\varepsilon)\|\chi_P\|_1 = \|T(t)g\|_1$$

for all $t \geq 0$, whence $\|f\|_L = \|g\|_L$ follows. Thus property (IV) fails to hold for $(L, \|\cdot\|_L)$.

On the other hand, it is obvious that $\|T(t)f\|_L \leq \|f\|_L$ for every $t \geq 0$ and $f \in L$, and so $T = \{T(t) : t \geq 0\}$ becomes a strongly continuous semigroup of positive linear contractions on $(L, \|\cdot\|_L)$. Since the Bochner integral $\int_0^\alpha T(t)h\,dt$ with respect to $(L, \|\cdot\|_L)$ is the same as the one with respect to $(L_1(\mu;\mathbb{R}), \|\cdot\|_1)$, it follows that the theorem fails to hold in this case.

(B) In spite of the above example, if the strong limit operator $T(0)$ of the Theorem satisfies

$$\|T(0)f\|_L \neq 0 \quad \text{whenever} \quad \|f\|_L \neq 0, \tag{48}$$

then the Theorem holds without assuming (IV).

To see this we note that property (IV) was used just once for all in the above proof of the Theorem, to deduce that $C \subset P$. Thus if $P = \Omega$ is known, then the theorem must hold without property (IV). We now prove that (48) implies $P = \Omega$. To do so, assume the contrary: $\mu(N) > 0$. Then there exists an $\widetilde{f} \in L(\mathbb{R})^+$ such that $\|\widetilde{f}\|_{L(\mathbb{R})} \neq 0$ and $\{\omega : \widetilde{f}(\omega) > 0\} \subset N$. By (a) of Section 3 we have $\|U(t)\widetilde{f}\|_{L(\mathbb{R})} = 0$ for every $t > 0$, which implies, as in Section 3, that $\|T(u)f\|_L = 0$ for every $u \in \mathbf{P}_d$, where we let $f(\omega) = \widetilde{f}(\omega)x_1$ for $\omega \in \Omega$. Thus we have $T(0)f = 0$. But this contradicts (48), because $\|f\|_L \neq 0$. Hence $P = \Omega$ must follow.

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