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Another fixed point theorem for nonexpansive potential operators

by

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Abstract. We prove the following result: Let X be a real Hilbert space and let $J : X \to \mathbb{R}$ be a C^1 functional with a nonexpansive derivative. Then, for each $r > 0$, the following alternative holds: either J' has a fixed point with norm less than r, or

$$\sup_{\|x\|=r} J(x) = \sup_{\|u\|_{L^2([0,1],X)}=r} \int_0^1 J(u(t))\,dt.$$

Here and in what follows, $(X, \langle\cdot,\cdot\rangle)$ is a real Hilbert space and $T : X \to X$ is a nonexpansive potential operator. That is, we have

$$\|T(x) - T(y)\| \le \|x - y\|$$

for all $x, y \in X$ and there exists a C^1 functional $J : X \to \mathbb{R}$, with $J(0) = 0$, such that $J' = T$. It is easy to see that

$$J(x) = \int_0^1 \langle T(sx), x\rangle\,ds$$

for all $x \in X$.

A challenging problem is to decide whether T has a fixed point, and if yes, to give an explicit bound for its norm. The aim of this very short note is to give a further contribution along this direction, besides the ones we recently gave in [4]. We refer to the monograph [1] for a thorough treatment of nonexpansive mappings.

To simplify the statement of our main result, let us introduce some notation. $(\Omega, \mathcal{F}, \mu)$ is a measure space, with $\mu(\Omega) = 1$, and $L^2(\Omega, X)$ is the usual space of all μ-strongly measurable functions $u : \Omega \to X$ such that

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$\int_\Omega \|u(t)\|^2\,d\mu < +\infty$, with the norm

$$\|u\|_{L^2} = \Big(\int_\Omega \|u(t)\|^2\,d\mu\Big)^{1/2}.$$

For each $r > 0$, we also put

$$\begin{aligned} B_r &= \{x \in X : \|x\| \le r\},\\ S_r &= \{x \in X : \|x\| = r\},\\ \tilde{S}_r &= \{u \in L^2(\Omega, X) : \|u\|_{L^2} = r\} \end{aligned}$$

and

$$\tilde{J}(u) = \int_\Omega J(u(t))\,d\mu$$

for all $u \in L^2(\Omega, X)$. Clearly, we have

$$\sup_{S_r} J \le \sup_{\tilde{S}_r} \tilde{J} \tag{1}$$

for all $r > 0$. Here is our main result:

Theorem 1. *With the above notation, for each $r > 0$, at least one of the following assertions holds:*

(a) *The operator T has a fixed point lying in* $\operatorname{int}(B_r)$.
(b) $\sup_{S_r} J = \sup_{\tilde{S}_r} \tilde{J}$.

Proof. Assume that there is no fixed point of T lying in $\operatorname{int}(B_r)$. Then we have to prove (b). So, in view of (1), we have to show that

$$\sup_{\tilde{S}_r} \tilde{J} \le \sup_{S_r} J. \tag{2}$$

For each $x, y \in X$, since T is nonexpansive, we have

$$\begin{aligned} \langle x - T(x) - (y - T(y)), x - y\rangle &= \|x-y\|^2 - \langle T(x) - T(y), x - y\rangle\\ &\ge \|x-y\|^2 - \|T(x) - T(y)\|\,\|x-y\| \ge 0. \end{aligned}$$

This shows that the derivative of the functional $x \mapsto \frac{1}{2}\|x\|^2 - J(x)$ is monotone. As a consequence, the functional $x \mapsto J(x) - \frac{1}{2}\|x\|^2$ is concave (besides being continuous). So, there exists $\hat{x} \in B_r$ such that

$$J(x) - \frac{1}{2}\|x\|^2 \le J(\hat{x}) - \frac{1}{2}\|\hat{x}\|^2$$

for all $x \in B_r$. Clearly, $\hat{x} \in S_r$, since otherwise $\hat{x}$ would be a fixed point of T lying in $\operatorname{int}(B_r)$. Now, fix $u \in \tilde{S}_r$. Observe that

$$\Big\|\int_\Omega u(t)\,d\mu\Big\| \le \|u\|_{L^2} = r, \tag{3}$$

where $\int_{\Omega} u(t)\,d\mu$ denotes the Bochner integral of u. Moreover, thanks to the Jensen inequality and to (3), we have

$$\int_{\Omega} J(u(t))\,d\mu - \frac{1}{2}\int_{\Omega} \|u(t)\|^2\,d\mu \le J\Big(\int_{\Omega} u(t)\,d\mu\Big) - \frac{1}{2}\Big\|\int_{\Omega} u(t)\,d\mu\Big\|^2 \le J(\hat{x}) - \frac{1}{2}\|\hat{x}\|^2.$$

Therefore

$$\int_{\Omega} J(u(t))\,d\mu \le J(\hat{x}) = \sup_{S_r} J,$$

from which (2) follows. ■

REMARK 1. If $\Phi : X \to X$ is a generic nonexpansive mapping having no fixed points lying in $\operatorname{int}(B_r)$ (for some $r > 0$), then it is well-known that there exist $x_0 \in S_r$ and $\lambda \ge 1$ such that

$$\Phi(x_0) = \lambda x_0.$$

In this connection, see [3] and Theorem 16.2 of [1]. In our current case, we can show that x_0 coincides with the unique global minimum point of the restriction of the functional $x \mapsto \frac{1}{2}\|x\|^2 - J(x)$ to B_r. Indeed, for some $r > 0$, assume that (a) does not hold. Accordingly, there exist $x_0 \in S_r$ and $\lambda \ge 1$ such that

$$T(x_0) = \lambda x_0.$$

The fact that $\lambda \ge 1$ implies that the functional $x \mapsto \lambda\|x\|^2/2 - J(x)$ is convex. Consequently, x_0 turns out to be one of its global minimum points in X. On the other hand, again because (a) does not hold, the set of all global minimum points of the restriction of the functional $x \mapsto \frac{1}{2}\|x\|^2 - J(x)$ to B_r is a nonempty and convex subset of S_r, and hence it reduces to a singleton, say $\{\hat{x}\}$. Now, if $\lambda = 1$, we are done. So, assume that $\lambda > 1$. In this case, we still have $x_0 = \hat{x}$, since otherwise, by Proposition 2.2 of [4], we would have $\|x_0\| < \|\hat{x}\|$, contrary to the fact that $x_0, \hat{x} \in S_r$.

We now point out two particular consequences of Theorem 1.

THEOREM 2. *Assume that there exists $r > 0$ such that*

$$\sup_{S_r} J < r^2 \sup_{\|x\|>r} \frac{J(x)}{\|x\|^2}. \tag{4}$$

Then the operator T has a fixed point lying in $\operatorname{int}(B_r)$.

Proof. Assume that $\Omega = [0,1]$ with the Lebesgue measure. In view of (4), there exists $\hat{x} \in X$ with $\|\hat{x}\| > r$ such that

$$\sup_{S_r} J < r^2 \frac{J(\hat{x})}{\|\hat{x}\|^2}. \tag{5}$$

Fix any measurable set $A \subset [0,1]$ so that

$$\text{meas}(A) = \frac{r^2}{\|\hat{x}\|^2} \tag{6}$$

and define the function $v : [0,1] \to X$ by

$$v(t) = \begin{cases} \hat{x} & \text{if } t \in A, \\ 0 & \text{if } t \in [0,1] \setminus A. \end{cases}$$

In view of (6), we have

$$\int_0^1 \|v(t)\|^2 \, dt = r^2.$$

Then from (5) and (6) it follows that

$$\sup_{S_r} J < \text{meas}(A) J(\hat{x}) = \tilde{J}(v) \le \sup_{\tilde{S}_r} \tilde{J}.$$

So, condition (b) of Theorem 1 does not hold. Therefore, (a) holds, as claimed. ∎

If $f : X \to \mathbb{R}$ is a convex continuous function, we denote by $\partial f(x_0)$ the subdifferential of f at x_0, i.e.,

$$\partial f(x_0) = \Big\{ z \in X : \inf_{x \in X} (f(x) - \langle z, x \rangle) \ge f(x_0) - \langle z, x_0 \rangle \Big\}.$$

Recall that $\partial f(x_0)$ is nonempty and closed, and that $0 \in \partial f(x_0)$ if and only if x_0 is a global minimum point of f.

THEOREM 3. *Let $f : X \to \mathbb{R}$ be a convex continuous function such that* 0 *is not a global minimum point of f. Then, for each $r \in]0, \text{dist}(0, \partial f(0))]$,*

$$\sup_{x \in S_r} \inf_{y \in X} \left(f(y) + \frac{1}{2}\|x - y\|^2 \right) = \sup_{u \in \tilde{S}_r} \int_\Omega \inf_{y \in X} \left(f(y) + \frac{1}{2}\|u(t) - y\|^2 \right) d\mu.$$

Proof. For each $x \in X$, put

$$\varphi(x) = \inf_{y \in X} \left(f(y) + \frac{1}{2}\|x - y\|^2 \right).$$

By classical results ([2]), the functional φ is C^1, one has

$$\varphi'(x) = x - (\text{id} + \partial f)^{-1}(x) \tag{7}$$

for all $x \in X$, and φ' is nonexpansive. From (7), we infer that the set of all fixed points of φ' coincides with $\partial f(0)$. As a consequence, by the choice of r, there is no fixed point of φ' lying in $\text{int}(B_r)$. Therefore, the conclusion follows directly from Theorem 1 applied to $T = \varphi$. ∎

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