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The Maurey extension property for Banach spaces with the Gordon–Lewis property and related structures

by

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Abstract. The main result of this paper states that if a Banach space X has the property that every bounded operator from an arbitrary subspace of X into an arbitrary Banach space of cotype 2 extends to a bounded operator on X, then every operator from X to an L_1-space factors through a Hilbert space, or equivalently $B(\ell_\infty, X^*) = \Pi_2(\ell_\infty, X^*)$. If in addition X has the Gaussian average property, then it is of type 2. This implies that the same conclusion holds if X has the Gordon–Lewis property (in particular X could be a Banach lattice) or if X is isomorphic to a subspace of a Banach lattice of finite cotype, thus solving the Maurey extension problem for these classes of spaces. The paper also contains a detailed study of the property of extending operators with values in ℓ_p-spaces, $1 \le p < \infty$.

Introduction. In 1974 Maurey [12] proved that if X is a Banach space of type 2, then every bounded operator from an arbitrary subspace of X to an arbitrary Banach space Y of cotype 2 admits a bounded extension from X to Y. Since then it has been an open problem whether this property known as the Maurey extension property characterizes Banach spaces of type 2. Since it follows from [14] that a Banach space with this property is of weak type 2, the answer to the problem is clearly affirmative for the class of spaces where weak type 2 is equivalent to type 2, e.g. rearrangement invariant function spaces.

The main result of this paper states that if a Banach space X has the Maurey extension property, then every bounded operator from X to an L_1-space factors through a Hilbert space. If in addition X has the Gaussian average property GAP (as defined in [2]), then it is of type 2. This implies that the answer to the problem is also affirmative for Banach spaces which have the Gordon–Lewis property, in particular Banach lattices, as well as for Banach spaces which are isomorphic to subspaces of Banach lattices of finite cotype.

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It is not known in general whether the condition $B(\ell_\infty, X^*) = \Pi_2(\ell_\infty, X^*)$ implies that X^* is of cotype 2 or equivalently in the case above that X is of type 2. It seems at the moment that GAP is the weakest known condition to ensure this for K-convex spaces. It should be noted that every space of type 2 has GAP.

We shall say that a Banach space X has M_p, $1 \le p < \infty$, if every bounded operator from a subspace of X to ℓ_p admits a bounded extension to X. Another major result of the paper states that M_p, $2 < p < \infty$, characterizes Hilbert spaces among Köthe function spaces on $[0,1]$. Finally we investigate M_p, $1 \le p \le 2$, in detail and prove that M_1 is equivalent to M_p, $1 < p < 2$, and that M_1 implies M_2.

It is an open problem whether M_2 implies M_1 and whether M_1 or M_2 imply the Maurey extension property.

We now wish to discuss the arrangement of this paper in greater detail.

In Section 1 of the paper we prove some general results on extensions of operators which are needed to prove the main results. Some of them are probably of interest in their own right. Section 2 is devoted to the main results stated above while Section 3 contains the investigation of the properties M_p, $1 \le p \le 2$, and the proof of the implications $M_1 \Leftrightarrow M_p$, $1 < p < 2$, and $M_1 \Rightarrow M_2$.

Acknowledgements. The authors are indebted to Nigel Kalton who drew our attention to the spaces $\ell_p(\delta, 2)$, $2 < p < \infty$, by using them to show that ℓ_p does not have M_r for $2 < p < r < \infty$. This subsequently led to the idea of the proof of our main result.

Spaces like $\ell_p(\delta, 2)$ were first considered by Rosenthal in his construction of new $\mathcal{L}_p$-spaces [20].

0. Notation and peliminaries. In this paper we shall use the notation and terminology commonly used in Banach space theory as it appears in [10], [11] and [21]. B_X will always denote the closed unit ball of the Banach space X.

If X and Y are Banach spaces, then $B(X,Y)$ ($B(X) = B(X,X)$) denotes the space of all bounded linear operators from X to Y and throughout the paper we shall identify $X \otimes Y$ with the space of all ω^*-continuous finite rank operators from X^* to Y in the canonical manner. Further, if $1 \le p < \infty$, we recall that an operator $T \in B(X,Y)$ is called *p-summing* if there exists a constant $K \ge 0$ so that for all finite sets $\{x_1, \ldots, x_n\} \subseteq X$ we have

$$\Big(\sum_{j=1}^{n} \|Tx_j\|^p\Big)^{1/p} \le K \sup\Big\{\Big(\sum_{j=1}^{n} |x^*(x_j)|^p\Big)^{1/p} \;\Big|\; x^* \in X^*,\ \|x^*\| \le 1\Big\}.$$

The space of all p-summing operators from X to Y is denoted by $\Pi_p(X,Y)$. If $T \in \Pi_p(X,Y)$, the *p-summing norm* $\pi_p(T)$ is defined to be the smallest constant K which can be used in the inequality above.

An operator $T \in B(X,Y)$ is called *p-integral* if there exist a probability measure μ, operators $A \in B(X, L_\infty(\mu))$ and $B(L_1(\mu), Y^{**})$ so that $T = BIA$ where I denote the formal identity operator from $L_\infty(\mu)$ to $L_1(\mu)$. We let $I_p(X,Y)$ denote the space of all p-integral operators from X to Y equipped with the *p-integral norm* i_p defined by $i_p(T) = \inf\{\|A\|\,\|B\|\}$ where the infimum is taken over all A and B satisfying the above.

An operator $T \in B(X,Y)$ is called *p-nuclear* if it admits a representation of the form $T = \sum_{j=1}^\infty x_j^* \otimes y_j$ where $(x_j^*) \subseteq X^*$ and $(y_j) \subseteq Y$ satisfy $\sum_{j=1}^\infty \|x_j^*\|^p < \infty$ and $\sum_{j=1}^\infty |y^*(y_j)|^{p'} < \infty$ for all $y^* \in Y^*$; here $1/p + 1/p' = 1$. We let $N_p(X,Y)$ denote the space of all p-nuclear operators from X to Y equipped with the *p-nuclear norm* ν_p defined by

$$\nu_p(T) = \inf\Big\{\Big(\sum_{j=1}^\infty \|x_j^*\|^p\Big)^{1/p} \sup\Big\{\Big(\sum_{j=1}^\infty |y^*(y_j)|^{p'}\Big)^{1/p'} \;\Big|\; \|y^*\| \le 1\Big\} \;\Big|\; T \text{ represented as above}\Big\}.$$

We recall that if $1 \le p \le \infty$, then an operator $T \in B(X,Y)$ is said to *factor through L_p* if it admits a factorization $T = BA$ where $A \in B(X, L_p(\mu))$ and $B \in B(L_p(\mu), Y)$ for some measure μ, and we denote the space of all operators which factor through L_p by $\Gamma_p(X,Y)$. If $T \in \Gamma_p(X,Y)$, then we define

$$\gamma_p(T) = \inf\{\|A\|\,\|B\| \mid T = BA,\ A \text{ and } B \text{ as above}\};$$

γ_p is a norm on $\Gamma_p(X,Y)$ turning it into a Banach space. All these spaces of operators are operator ideals and we refer to the above mentioned books, [4] and [8] for further details.

In the formulas of this paper we shall, as is customary, interpret π_∞ as the operator norm and i_∞ as the γ_∞-norm.

We let (r_n) denote the sequence of Rademacher functions on $[0,1]$ and recall that a Banach space X is said to be of *type* p, $1 \le p \le 2$ (respectively *cotype* p, $2 \le p < \infty$), if there is a constant $K \ge 1$ so that for all finite sets $\{x_1, \ldots, x_n\} \subseteq X$ we have

$$\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(t)x_j\Big\|^p dt\Big)^{1/p} \le K\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p}, \tag{0.1}$$

respectively

$$\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p} \le K\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(t)x_j\Big\|^p dt\Big)^{1/p}. \tag{0.2}$$

The smallest constant K which can be used in (0.1) (respectively (0.2)) is denoted by $K^p(X)$ (respectively $K_p(X)$).

A Banach space X is said to be of *weak type* 2 if there is a constant C and a δ, $0 < \delta < 1$, so that whenever $E \subseteq X$ is a subspace, $n \in \mathbb{N}$ and $T \in B(E, \ell_2^n)$, then there is an orthogonal projection P on ℓ_2^n of rank larger than δn and an operator $S \in B(X, \ell_2^n)$ with $Sx = PTx$ for all $x \in E$ and $\|S\| \le C\|T\|$.

Similarly X is called a *weak cotype* 2 *space* if there is a constant C and a δ, $0 < \delta < 1$, so that whenever $E \subseteq X$ is a finite-dimensional subspace, then there is a subspace $F \subseteq E$ so that $\dim F \ge \delta \dim E$ and $d(F, \ell_2^{\dim F}) \le C$.

Our definitions of weak type 2 and weak cotype 2 space are not the original ones, but are chosen out of the many equivalent characterizations given by Pisier [19].

Following [5] we shall say that a Banach space X has $\mathrm{GL}(p,q)$, $1 \le p, q \le \infty$, if there is a constant K so that for all Banach spaces Y and all $T \in X^* \otimes Y$ we have $i_q(T) \le K\pi_p(T^*)$. The smallest constant K which can be used in this inequality is denoted by $\mathrm{GL}_{p,q}(X)$. We note that $\mathrm{GL}(1,\infty)$ corresponds to the classical Gordon–Lewis property GL (see [6]). X is said to have the *Gordon–Lewis property* GL_2 if every 1-summing operator from X to a Hilbert space factors through an L_1-space.

If $n \in \mathbb{N}$ and $T \in B(\ell_2^n, X)$, then following [21, §12] we define the ℓ-*norm* of T by

$$\ell(T) = \Big(\int_{\ell_2^n} \|Tx\|^2 \, d\gamma(x) \Big)^{1/2}$$

where γ is the canonical Gaussian probability measure on ℓ_2^n.

A Banach space X is said to have the *Gaussian Average Property* (abbreviated GAP) (see [2]) if there is a constant K so that $\ell(T) \le K\pi_1(T^*)$ for every $T \in B(\ell_2^n, X)$ and every $n \in \mathbb{N}$.

We shall also need some notation on subspaces of Banach lattices and on operators with ranges in a Banach lattice. Recall that if X is a Banach space and L is a Banach lattice, then an operator $T \in B(X, L)$ is called *order bounded* [15] if there exists a $z \in L$, $z \ge 0$, so that

$$|Tx| \le \|x\| z \quad \text{for all } x \in X, \tag{0.3}$$

and the *order bounded norm* $\|T\|_m$ is defined by

$$\|T\|_m = \inf\{\|z\| \mid z \text{ can be used in (0.3)}\}.$$

We let $\mathcal{B}(X, L)$ denote the space of all order bounded operators from X to L equipped with the norm $\|\cdot\|_m$. It is readily seen to be a Banach space and a left ideal. We let $X^* \otimes_m L$ denote the closure of $X^* \otimes L$ in $\mathcal{B}(X, L)$ under the norm $\|\cdot\|_m$.

If X is a subspace of a Banach lattice L and $1 \le p < \infty$, then we shall say that X is *p-convex in L* (respectively *p-concave in L*) if there is a constant $K \ge 1$ so that for all finite sets $\{x_1, \ldots, x_n\} \subseteq X$ we have

$$\Big\|\Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p}\Big\| \le K\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p},$$

respectively

$$\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p} \le K\Big\|\Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p}\Big\|.$$

Note that these inequalities depend on the embedding of X into L. The lattice L is called *p-convex* (respectively *q-concave*) if the above inequalties hold for every finite set of vectors in L.

If E is a Banach space and $T \in B(E, X)$, then T is called *p-convex* if there exists a constant $K \ge 0$ so that for all finite sets $\{x_1, \ldots, x_n\} \subseteq E$ we have

$$\Big\|\Big(\sum_{j=1}^{n} |Tx_j|^p\Big)^{1/p}\Big\| \le K\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p}.$$

Concavity of an operator from a Banach lattice to a Banach space is defined in a similar manner.

1. Some basic results on extensions of operators. In this section we shall prove some general results on extensions of operators which will be useful for us in what follows. We start with the following localization theorem:

THEOREM 1.1. *Let X and Y be Banach spaces. Consider the statements*:

(i) *Every bounded operator from an arbitrary subspace of X into Y extends to a bounded operator from X to Y.*

(ii) *There is a constant $K \ge 1$ so that whenever $E \subseteq X$ is a finite-dimensional subspace*, *every $T \in B(E, Y)$ admits an extension $\widetilde{T} \in B(X, Y)$ with $\|\widetilde{T}\| \le K\|T\|$.*

Then (i) *implies* (ii) *and if Y is a dual space*, (ii) *implies* (i).

Proof. Assume first that (ii) does not hold. By induction we shall construct a sequence (E_n) of finite-dimensional subspaces of X, a sequence (F_n) of subspaces of X of finite codimension and a sequence $(T_n) \subseteq B(E_n, Y)$ with $\|T_n\| = 1$ for all $n \in \mathbb{N}$ so that the following conditions are satisfied:

(a) $F_n \cap \operatorname{span}\{E_j \mid 1 \le j \le n\} = \{0\}$ and the natural projection of $\operatorname{span}\{E_j \mid 1 \le j \le n\} \oplus F_n$ onto $\operatorname{span}\{E_j \mid 1 \le j \le n\}$ has norm less than or equal to 2 for all $n \in \mathbb{N}$.

(b) $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$.

(c) If $\widetilde{T}_n \in B(X,Y)$ is an extension of T_n, then $\|\widetilde{T}_1\| \geq 4$ and $\|\widetilde{T}_n\| \geq 2^{2n+1} \operatorname{codim} F_{n-1} + \operatorname{codim} F_{n-1}$ for all $n \geq 2$.

Since (ii) does not hold, we can for $n = 1$ choose a finite-dimensional subspace E_1 of X and a $T_1 \in B(E_1, Y)$ with $\|T_1\| = 1$ so that any bounded extension of T_1 to X has norm greater than or equal to 4. Let F_1 be a subspace of finite codimension so that $F_1^\perp$ is 2-norming over E_1 (F_1 can be chosen to be of codimension $5^{\dim E_1}$). Clearly $E_1 \cap F_1 = \{0\}$ and the natural projection of $E_1 \oplus F_1$ onto E_1 has norm less than or equal to 2.

Assume now that $E_1, \dots, E_n$, $F_1, \dots, F_n$ and $T_1, \dots, T_n$ have been constructed so that (a)–(c) hold. By assumption there is a finite-dimensional subspace $E_{n+1} \subseteq X$ and an operator $T_{n+1} \in B(E_{n+1}, Y)$ with $\|T_{n+1}\| = 1$ so that if $\widetilde{T}_{n+1} \in B(X,Y)$ is an extension of T_{n+1}, then

$$\|\widetilde{T}_{n+1}\| \geq 2^{2n+2} \operatorname{codim} F_n + \operatorname{codim} F_n,$$

which shows that (c) holds. If we choose a subspace $\widehat{F}_{n+1} \subseteq X$ so that $\widehat{F}_{n+1}^\perp$ is 2-norming over $\operatorname{span}\{E_j \mid 1 \leq j \leq n+1\}$ and put $F_{n+1} = \widehat{F}_{n+1} \cap F_n$, then clearly also (a) and (b) are satisfied.

Hence we have constructed the required sequences. Put now $G_1 = E_1$ and $G_{n+1} = E_{n+1} \cap F_n$ for all $n \geq 1$. By choosing an Auerbach basis for E_n/G_n we easily achieve that there is a subspace $H_n \subseteq E_n$ and a projection P_n of X onto H_n so that

$$E_n = G_n \oplus H_n \quad \text{for all } n \in \mathbb{N}, \tag{1.1}$$

$$P_n x = 0 \quad \text{for all } x \in G_n \text{ and all } n \in \mathbb{N}, \tag{1.2}$$

$$\|P_{n+1}\| \leq \operatorname{codim} F_n \quad \text{for all } n \in \mathbb{N}. \tag{1.3}$$

Let $n \geq 2$ and assume that $\widetilde{S}_n \in B(X,Y)$ is an extension of $T_{n|G_n}$. Put

$$\widetilde{T}_n = \widetilde{S}_n(I - P_n) + T_n P_n.$$

If $x \in E_n$, then

$$\widetilde{T}_n x = \widetilde{S}_n(x - P_n x) + T_n P_n x = T_n(x - P_n x) + T_n P_n = T_n x.$$

Hence $\widetilde{T}_n$ is an extension of T_n and therefore, by (c),

$$\|\widetilde{T}_n\| \geq 2^{2n+1} \operatorname{codim} F_{n-1} + \operatorname{codim} F_{n-1},$$

which in view of (1.3) clearly implies that

$$\|\widetilde{S}_n\| \geq 2^{2n}. \tag{1.4}$$

By construction (G_n) forms an infinite direct sum and we can therefore put

$$G = \bigoplus_{n=1}^\infty G_n.$$

We define $S \in B(G, Y)$ by

$$Sx = \sum_{n=1}^{\infty} 2^{-n} T_n x_n$$

for all $x \in G$ with $x = \sum_{n=1}^{\infty} x_n,\ x_n \in G_n$ for all $n \in \mathbb{N}$. (Actually $\|S\| \le 3$.) Then S does not have a bounded extension to X. Indeed, if $\widetilde{S} \in B(X, Y)$ is an extension, then $2^n \widetilde{S}$ is an extension of $T_{n|G_n}$ and therefore, by (1.4),

$$\|\widetilde{S}\| \ge 2^n \quad \text{for all } n \ge 2,$$

which is a contradiction. This shows that (i) implies (ii).

Assume next that (ii) holds and that Y is a dual space; let Z be a Banach space so that $Z^* = Y$. Further, let $F \subseteq X$ be a subspace and $T \in B(F, Z^*)$ with $\|T\| = 1$. For every finite-dimensional subspace $E \subseteq F$ we can by assumption find $\widetilde{T}_E = B(X, Z^*)$ so that

$$\widetilde{T}_E x = Tx \quad \text{for all } x \in E, \quad \|\widetilde{T}_E\| \le K.$$

By ω^*-compactness it follows that we can find a subnet $(\widetilde{T}_{E'})$ of $(\widetilde{T}_E)$ and an operator $\widetilde{T} \in B(X, Z^*)$ so that

$$\widetilde{T}_{E'} x \xrightarrow{\omega^*} \widetilde{T} x \quad \text{for all } x \in X.$$

Clearly $\widetilde{T}$ is an extension of T. ■

The following corollary is an immediate consequence of Theorem 1.1:

Corollary 1.2. *Let X, Y and Z be Banach spaces and assume that Z is finitely representable in X. If every bounded operator from an arbitrary subspace of X to Y^* extends to a bounded operator from the whole space to Y^*, then every bounded operator from an arbitrary subspace of Z to Y^* extends.*

Our next result shows that under certain conditions it is enough to consider extensions of finite rank operators.

Theorem 1.3. *Let X and Y be Banach spaces and $E \subseteq X$ a subspace. Assume that there is a constant K so that every $T \in E^* \otimes Y$ admits an extension $\widetilde{T} \in B(X, Y)$ with $\|\widetilde{T}\| \le K\|T\|$. If either E or Y has the λ-bounded approximation property, then every $T \in B(E, Y)$ admits an extension $\widetilde{T} \in B(X, Y^{**})$ with $\|\widetilde{T}\| \le K\lambda\|T\|$.*

Proof. Let $T \in B(E, Y)$. By assumption we can find a net $(T_\alpha)_{\alpha \in J} \subseteq E^* \otimes Y$ with $\|T_\alpha\| \le \lambda\|T\|$ for all α so that $T_\alpha x \to Tx$ for all $x \in E$. Let $\widetilde{T}_\alpha \in B(X, Y)$ denote an extension of T_α for each $\alpha \in J$ with

$$\|\widetilde{T}_\alpha\| \le K\|T_\alpha\| \le K\lambda\|T\|.$$

This immediately gives that there is a $\widetilde{T} \in B(X, Y^{**})$ with $\|\widetilde{T}\| \le K\lambda\|T\|$ and a subnet $(\widetilde{T}_{\alpha'})$ of $(\widetilde{T}_\alpha)$ so that

$$\widetilde{T}_{\alpha'}x \overset{\omega^*}{\to} \widetilde{T}x \quad \text{for all } x \in X.$$

Since clearly also $\widetilde{T}_{\alpha'}x \overset{\omega^*}{\to} Tx$ for all $x \in E$, it follows that $\widetilde{T}$ is the required extension. ■

We shall need:

LEMMA 1.4. *If E is an n-dimensional subspace of a Banach space X, then $(E \oplus \ell_2^n)_\infty$ is 12-isomorphic to a subspace of X.*

Proof. Let F be a subspace of X of finite codimension so that $F^\perp$ is 2-norming on E (F can be chosen so that $\operatorname{codim} F = 5^n$). By Dvoretzky's theorem F contains an n-dimensional subspace G with $d(G, \ell_2^n) \le 2$ and clearly $E \cap G = \{0\}$. It is readily verified that $(E \oplus \ell_2^n)_\infty$ is 12-isomorphic to $E \oplus G$. ■

The next result will be very useful for us in what follows:

THEOREM 1.5. *Let X and Y be Banach spaces and μ a measure. If every bounded operator from an arbitrary subspace of X to Y^* extends to a bounded operator from X to Y^*, then the same holds for every bounded operator from an arbitrary subspace of $X \oplus L_2(\mu)$ to Y^*.*

Proof. Let $E \subseteq (X \oplus L_2(\mu))_\infty$ be an arbitrary finite-dimensional subspace. Clearly there exists an $n \in \mathbb{N}$ so that we can find n-dimensional subspaces $G \subseteq X$ and $F \subseteq L_2(\mu)$ with $E \subseteq G \oplus F$. By Lemma 1.4, $G \oplus F$ and therefore also E is 12-isomorphic to a subspace of X. Hence $X \oplus L_2(\mu)$ is finitely representable in X and the conclusion follows from Corollary 1.2. ■

Finally we shall need the following proposition, the proof of which is obvious:

PROPOSITION 1.6. *Let X and Y be Banach spaces so that for every subspace $E \subseteq X$ every $T \in B(E, Y)$ admits an extension $\widetilde{T} \in B(X, Y)$. If Z is a quotient of X, then Z has the same property.*

2. The main results. We start with the following definition:

DEFINITION 2.1. (i) A Banach space X is said to have the *Maurey extension property* (MEP) if for any subspace $E \subseteq X$, any Banach space Y of cotype 2 and every $T \in B(E, Y)$ there exists an extension $\widetilde{T} \in B(X, Y)$ of T.

(ii) X is said to have M_p, $1 \le p \le \infty$, if the condition in (i) holds with $Y = \ell_p$.

Maurey [12] proved that if X is a Banach space of type 2, then it has MEP.

We need the following lemma:

LEMMA 2.2. *Let X be a Banach space with MEP. For every $\lambda \geq 1$ there exists a constant $C(\lambda) \geq 1$ so that every bounded operator T from an arbitrary finite-dimensional subspace E of X to an arbitrary Banach space Y of cotype λ admits an extension $\widetilde{T}$ from X to Y with $\|\widetilde{T}\| \leq C(\lambda)\|T\|$. If in addition Y is a dual space*, *then the above holds for any subspace E of X.*

Proof. Assume that the statement is not true. Then there exist a $\lambda \geq 1$, a sequence (Y_n) of Banach spaces of cotype 2 with $K_2(Y_n) \leq \lambda$, a sequence (E_n) of finite-dimensional subspaces of X and a sequence (T_n) of operators, $T_n \in B(E_n, Y_n)$, $\|T_n\| = 1$ for all $n \in \mathbb{N}$, so that every extension $\widetilde{T}_n \in B(X, Y_n)$ has $\|\widetilde{T}_n\| \geq n$. Put $Y = (\sum_{n=1}^{\infty} Y_n)_2$ and let, for every $n \in \mathbb{N}$, P_n denote the natural projection of Y onto Y_n. Clearly Y is of cotype 2 with $K_2(Y) \leq \lambda$.

Since X has MEP, it follows that (ii) of Theorem 1.1 holds. Hence for every n, T_n admits an extension $S_n \in B(X, Y)$ with $\|S_n\| \leq K$. If we put $\widetilde{T}_n = P_n S_n$, then $\widetilde{T}_n \in B(X, Y_n)$ is an extension of T_n with $\|\widetilde{T}_n\| \leq K$. This is a contradiction.

The second part of the lemma follows from the first part and Theorem 1.1 (ii)⇒(i) (or rather the proof of it!). ∎

A refined version of Theorem 1.1 will probably show that the lemma is true for all subspaces $E \subseteq X$ without assuming that Y is a dual space. We did not check this, however, since in our application the target space Y will be a reflexive space, even isomorphic to a Hilbert space.

It follows immediately from Theorem 1.1 that X has M_p if and only if there is a constant K so that for every finite-dimensional subspace $E \subseteq X$ every $T \in B(E, \ell_p)$ has an extension $\widetilde{T} \in B(X, \ell_p)$ with $\|\widetilde{T}\| \leq K\|T\|$. We let $M_p(X)$ denote the smallest constant which can be used here.

Using the above together with the local properties of L_p-spaces we find that in Definition 2.1 we can substitute ℓ_p with an arbitrary infinite-dimensional L_p-space.

The following result follows immediately from [14, Theorem 10]:

THEOREM 2.3. *If X is a Banach space with M_2*, *then it is of weak type* 2.

We shall postpone the investigation of the property M_p to the next section and turn to our main results. They state in short that MEP characterizes type 2 spaces among Banach spaces with the Gaussian average property and that M_p, $2 < p < \infty$, characterizes Hilbert spaces among Köthe function spaces on $[0, 1]$. Before we can prove it we need to define certain special spaces of cotype 2.

If μ is a probability measure and $0 < \delta < 1$, then we define the space $L_1(\mu;\delta L_2)$ by

$$L_1(\mu;\delta L_2) = \{(f,\delta f) \mid f \in L_2(\mu)\} \subseteq (L_1(\mu) \oplus L_2(\mu))_\infty.$$

Since $L_1(\mu) \oplus L_2(\mu)$ is isomorphic to a subspace of an L_1-space, it follows that $L_1(\mu;\delta L_2)$ is of cotype 2 with a constant C independent of δ. Note also that it is a sublattice of $L_1(\mu) \oplus L_2(\mu)$. It is a reflexive space since it is $1/\delta$-isomorphic to a Hilbert space.

We are now ready to prove:

Theorem 2.4. *If X is a Banach space with the Maurey extension property, then every operator from X to an arbitrary L_1-space factors through a Hilbert space* (*equivalently $B(\ell_\infty, X^*) = \Pi_2(\ell_\infty, X^*)$*).

Proof. Let X be a Banach space with MEP, let $(\Omega,\mathcal{S},\nu)$ be an arbitrary probability space and let $T \in B(X, L_1(\nu))$ be arbitrary with $\|T\| = 1$. From [11, Corollary 1.d.12] it follows that if we prove that T is a 2-convex operator, then $T \in \Gamma_2(X, L_1(\nu))$. Hence let $n \in \mathbb{N}$ and $\{x_1,\ldots,x_n\} \subseteq X$ with $h = (\sum_{j=1}^n |Tx_j|^2)^{1/2} \neq 0$. We may assume that $\|h\|_1 = 1$. Put $E = \operatorname{span}\{x_1,\ldots,x_n\}$, let $\Delta = \{t \in \Omega \mid h(t) > 0\}$ and define the probability measure μ on Δ by $d\mu = h d\nu$. Further we let M_h: $L_1(\Delta,\nu) \to L_1(\mu)$ denote the isometry given by

$$M_h(f) = fh^{-1} \quad \text{for all } f \in L_1(\Delta,\nu)$$

and define Φ: $E \to L_1(\mu)$ by $\Phi = M_h T$.

Since X has MEP and $L_1(\mu;\delta L_2)$, $0 < \delta < 1$, has cotype 2 with constant C it follows from Theorem 1.5 and Lemma 2.2 that there is a constant M independent of δ and μ so that every bounded operator S from a subspace of $(X \oplus L_2(\mu))_\infty$ to $L_1(\mu;\delta L_2)$ has an extension $\widetilde{S}$ to $(X \oplus L_2(\mu))_\infty$ with $\|\widetilde{S}\| \le M\|S\|$. Choose now δ so that $2CM\delta < 1$ and let $Z \subseteq (X \oplus L_2(\mu))_\infty$ be defined by

$$Z = \{(x, \delta\Phi(x)) \mid x \in E\}$$

(note that $\Phi(E) \subseteq L_\infty(\mu)$), define $I : Z \to L_1(\mu;\delta L_2)$ by

$$I(x,\delta\Phi(x)) = (\Phi(x), \delta\Phi(x)) \quad \text{for all } x \in E$$

and let $\widetilde{I}$: $(X \oplus L_2(\mu))_\infty \to L_1(\mu;\delta L_2)$ be an extension of I with $\|\widetilde{I}\| \le M\|I\| \le M$. For every $x \in E$ we now get

$$(\Phi(x), \delta\Phi(x)) = \widetilde{I}(x,0) + \delta\widetilde{I}(0,\Phi(x)).$$

Using this on the x_j's we obtain

$$(1,\delta) = \Big(\Big(\sum_{j=1}^n |\Phi(x_j)|^2\Big)^{1/2}, \delta\Big(\sum_{j=1}^n |\Phi(x_j)|^2\Big)^{1/2}\Big) \tag{2.1}$$

$$
\begin{aligned}
&= \Big(\sum_{j=1}^{n} |(\Phi(x_j), \delta\Phi(x_j))|^2\Big)^{1/2} \\
&= \Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0) + \delta\widetilde{I}(0, \Phi(x_j))|^2\Big)^{1/2} \\
&\le \Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0))|^2\Big)^{1/2} + \delta\Big(\sum_{j=1}^{n} |\widetilde{I}(0, \Phi(x_j))|^2\Big)^{1/2}.
\end{aligned}
$$

Taking norms on both sides of (2.1) we get

$$
\begin{aligned}
1 &\le \Big\|\Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2\Big)^{1/2}\Big\| + \delta\Big\|\Big(\sum_{j=1}^{n} |\widetilde{I}(0, \Phi(x_j))|^2\Big)^{1/2}\Big\| \\
&\le \Big\|\Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2\Big)^{1/2}\Big\| + \delta C\Big(\int_0^1 \Big\|\sum_{j=1}^{n} r_j(t)\widetilde{I}(0, \Phi(x_j))\Big\|^2 dt\Big)^{1/2} \\
&\le \Big\|\Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2\Big)^{1/2}\Big\| + \delta CM\Big(\int_0^1 \Big\|\sum_{j=1}^{n} r_j(t)(0, \Phi(x_j))\Big\|^2 dt\Big)^{1/2} \\
&= \Big\|\Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2\Big)^{1/2}\Big\| + \delta CM\Big\|\Big(0, \sum_{j=1}^{n} |\Phi(x_j)|^2\Big)^{1/2}\Big\| \\
&= \Big\|\Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2\Big)^{1/2}\Big\| + \delta CM.
\end{aligned}
$$

Hence

$$
\frac{1}{2} \le \Big\|\Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2\Big)^{1/2}\Big\|. \tag{2.2}
$$

Let now Q: $L_1(\mu) \oplus L_2(\mu) \to L_2(\mu)$ be the canonical projection onto the second coordinate. By the definition of the order in $L_1(\mu) \oplus L_2(\mu)$ we have

$$
\Big(\sum_{j=1}^{n} |Q\widetilde{I}(x_j, 0)|^2\Big)^{1/2} = Q\Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2\Big)^{1/2}.
$$

Assume now that

$$
\Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2\Big)^{1/2} = (g, \delta g) \quad \text{with } g \in L_2(\mu).
$$

If $\|(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2)^{1/2}\| = \|g\|_1$, then by (2.2),

$$
\frac{\delta}{2} \le \delta\Big\|\Big(\sum_{j=1}^{n} |\widetilde{I}(x_j, 0)|^2\Big)^{1/2}\Big\| = \delta\|g\|_1 \le \delta\|g\|_2 = \Big\|\Big(\sum_{j=1}^{n} |Q\widetilde{I}(x_j, 0)|^2\Big)^{1/2}\Big\|
$$

and if $\|(\sum_{j=1}^n |\widetilde{I}(x_j,0)|^2)^{1/2}\| = \delta\|g\|_2$, then

$$\frac{1}{2} \le \Big\|\Big(\sum_{j=1}^n |\widetilde{I}(x_j,0)|^2\Big)^{1/2}\Big\| = \Big\|\Big(\sum_{j=1}^n |Q\widetilde{I}(x_j,0)|^2\Big)^{1/2}\Big\|.$$

Using the fact that the range of $Q\widetilde{I}$ is a Hilbert space we obtain

$$\frac{\delta}{2} \le \Big\|\Big(\sum_{j=1}^n |Q\widetilde{I}(x_j,0)|^2\Big)^{1/2}\Big\| = \Big(\sum_{j=1}^n \|Q\widetilde{I}(x_j,0)\|^2\Big)^{1/2} \le M\Big(\sum_{j=1}^n \|x_j\|^2\Big)^{1/2}.$$

We have now verified that T is 2-convex with constant less than or equal to $2M\delta^{-1}$. ■

Theorem 2.4 immediately implies:

THEOREM 2.5. *Let X be a Banach space which satisfies one of the following conditions*:

(i) *X has the Gaussian average property.*

(ii) *X has the Gordon–Lewis property* GL_2 (*in particular X could be a Banach lattice*).

(iii) *X is isomorphic to a subspace of a Banach lattice of finite cotype.*

If X has the Maurey extension property, *then X is of type* 2.

Proof. Let X be a Banach space with MEP.

(i) If X has GAP, then it follows from Theorem 2.4 and [2, Theorem 1.10] that X is of type 2.

(ii) Since X has MEP, it is of finite cotype and if in addition it has GL_2, then it has GAP by [2, Theorem 1.3]. (ii) can also be derived directly from Theorem 2.4 and [18, Proposition 8.16].

(iii) If X is isomorphic to a subspace of a Banach lattice of finite cotype, then it has GAP by [2, Theorem 1.4]. ■

REMARK. It follows from [2] that every space of type 2 has GAP. Hence if there exists a Banach space with MEP and without GAP, then it cannot have type 2.

If a Banach space X has MEP, then every bounded operator from a subspace of X to a cotype 2 space Y with GL can be extended to X through a Hilbert space (as in Maurey's original result). Indeed, let E be a subspace of X and $T \in B(X,Y)$. Since E has MEP and Y has $\mathrm{GL}(1,2)$ by [3, Theorem 3.4], it follows from Theorem 2.4 and Theorem 3.6 in the next section that $T \in \Gamma_2(E,Y)$. Since X has MEP, the part of the factorization of T which goes into a Hilbert space can be extended to X.

Before we can prove our main result on M_p, $2 < p < \infty$, we need a sequence space equivalent of the spaces considered in Theorem 2.4.

If X, respectively Y, have unconditional normalized bases (x_n), respectively (y_n), then we say that (x_n) *dominates* (y_n) and write $(y_n) < (x_n)$ if the linear operator $T\colon \operatorname{span}(x_n) \to \operatorname{span}(y_n)$ defined by $Tx_n = y_n$ for all $n \in \mathbb{N}$ is bounded. If $1 \le q \le \infty$ and the unit vector basis of ℓ_q dominates (x_n), respectively is dominated by (x_n), then we shall say that (x_n) satisfies an *upper q-estimate*, respectively *lower q-estimate*.

If $1 \le q < \infty$ and (e_n) denotes the unit vector basis of ℓ_q, then for every $0 < \delta < 1$ we define the space $X(\delta, q)$ to be the closed linear span in $(X \oplus \ell_q)_\infty$ of the sequence $(x_j + \delta e_j)$.

The next theorem, which will be very useful for us in several contexts, states:

THEOREM 2.6. *Let X, respectively Y, be Banach spaces with normalized unconditional bases (x_n), respectively (y_n), and let $1 \le q < \infty$ be such that $(y_n) < (x_n)$ with constant K_1 and (y_n) satisfies an upper q-estimate with constant K_2. If for some $0 < \delta < 1$ the formal identity operator I_δ from $X(\delta, q)$ to $Y(\delta, q)$ extends to a bounded operator $\widetilde{I}_\delta$ from $(X \oplus \ell_q)_\infty$ to $Y(\delta, q)$ with $\|\widetilde{I}_\delta\| < \delta^{-1}$, then for all $(t_n) \subseteq \mathbb{R}$,*

$$\delta^2(1 - \|I_\delta\|\delta)\Big(\sum_{n=1}^{\infty} |t_n|^2\Big)^{1/2} \le \sqrt{2}\, K_2 \operatorname{ubc}(x_n)\Big\|\sum_{n=1}^{\infty} t_n x_n\Big\| \quad \text{if } 1 \le q \le 2, \tag{2.3}$$

$$\delta^2(1 - \|I_\delta\|\delta)\Big(\sum_{n=1}^{\infty} |t_n|^q\Big)^{1/q} \le K_2 \operatorname{ubc}(x_n)\Big\|\sum_{n=1}^{\infty} t_n x_n\Big\| \quad \text{if } 2 \le q \le \infty. \tag{2.4}$$

For example, (x_n) has a lower 2-estimate if $1 \le q \le 2$ and a lower q-estimate if $2 \le q < \infty$.

Proof. Since $\widetilde{I}_\delta$ extends I_δ, for all $n \in \mathbb{N}$ we have

$$y_n + \delta e_n = \widetilde{I}_\delta x_n + \delta \widetilde{I}_\delta e_n$$

and hence by the triangle inequality

$$1 - \|\widetilde{I}_\delta\|\delta \le \|\widetilde{I}_\delta x_n\| \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

Let $Q\colon (Y \oplus \ell_q)_\infty \to \ell_q$ be the canonical projection and let $T = Q\widetilde{I}_\delta$. Fix $n \in \mathbb{N}$ and let $(a_k) \subseteq \mathbb{R}$ be chosen so that

$$\widetilde{I}_\delta x_n = \sum_{k=1}^{\infty} a_k y_k + \delta \sum_{k=1}^{\infty} a_k e_k.$$

If $\|\widetilde{I}_\delta x_n\| = \delta(\sum_{k=1}^\infty |a_k|^q)^{1/q}$, then by (2.5),

$$1 - \|\widetilde{I}_\delta\|\delta \le \delta\Big(\sum_{k=1}^\infty |a_k|^q\Big)^{1/q} = \|Tx_n\| \tag{2.6}$$

and if $\|\widetilde{I}_\delta x_n\| = \Big\|\sum_{k=1}^\infty a_k y_k\Big\|$, we obtain

$$\delta(1 - \|\widetilde{I}_\delta\|\delta) \le \delta\Big\|\sum_{k=1}^\infty a_k y_k\Big\| \le K_2\delta\Big(\sum_{k=1}^\infty |a_k|^q\Big)^{1/q} = \|Tx_n\|. \tag{2.7}$$

Comparing (2.6) and (2.7) we deduce that for all $n \in \mathbb{N}$,

$$K_2^{-1}\delta(1 - \|\widetilde{I}_\delta\|\delta) \le \|Tx_n\|.$$

Let $r = \max(q, 2)$. Since ℓ_q is of cotype r, for all $n \in \mathbb{N}$ and all $(t_j)_{j=1}^n \subseteq \mathbb{R}$ we get

$$\begin{aligned} K_2^{-1}\delta(1 - \|\widetilde{I}\|\delta)\Big(\sum_{j=1}^n |t_j|^r\Big)^{1/r} &\le \Big(\sum_{j=1}^n |t_j|^r \|Tx_j\|^r\Big)^{1/r} \\ &\le C_q\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(t)t_j Tx_j\Big\|^r dt\Big)^{1/r} \\ &\le C_q\|T\|\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(t)t_j x_j\Big\|^r dt\Big)^{1/r} \\ &\le C_q\delta^{-1}\operatorname{ubc}(x_j)\Big\|\sum_{j=1}^n t_j x_j\Big\| \end{aligned}$$

where $C_q \le \sqrt{2}$ for $1 \le q < 2$ and $C_q = 2$ for $2 \le q < \infty$. This immediately gives (2.3) and (2.4). Note that our assumptions imply that $\delta < K_1^{-1}$. ■

Remark. Theorem 2.6 remains true if we assume that both X and Y are finite-dimensional.

Theorem 2.6 was inspired by Nigel Kalton, who used the spaces $\ell_p(\delta, 2)$ to prove that ℓ_p does not have M_r for $2 < p < r < \infty$. This subsequently led to the idea of the proof of Theorem 2.4. Spaces like $\ell_p(\delta, 2)$ were first considered by Rosenthal in his construction of new $\mathcal{L}_p$-spaces [20].

Before we go on we need a few facts about the spaces $\ell_p(\delta, 2)$, $p > 2$, which all go back to [20]. Hence let $2 < p < \infty$ and $0 < \delta < 1$. The space $L_p(0, \infty) \cap L_2(0, \infty)$ equipped with the maximum of the p-norm and the 2-norm is a rearrangement invariant function space on $[0, \infty[$ which is isomorphic to $L_p(0, 1)$ [11, Theorem 2.f.1]. In addition $\ell_p(\delta, 2)$ is isometric to a norm 1 complemented subspace of $L_p(0, \infty) \cap L_2(0, \infty)$. Indeed, it is readily seen that if we take a sequence $(I_k)_{k=1}^\infty$ of mutually disjoint intervals in $[0, \infty[$ each of length $\delta^{2p/(p-2)}$, then the closed linear span of $\{1_{I_k}\}$ is isometric to $\ell_p(\delta,2)$. This span is also norm 1 complemented since conditional expectations are norm 1 projections in $L_p(0,\infty)\cap L_2(0,\infty)$. Hence we have verified:

LEMMA 2.7. *Let $2<p<\infty$. There exists a constant C so that for all $\delta\in\,]0,1[$, $\ell_p(\delta,2)$ is C-isomorphic to a C-complemented subspace of $L_p(0,1)$.*

We need yet another lemma:

LEMMA 2.8. *If X is a Banach space with M_p for some $2<p<\infty$, then $\inf\{q\mid X$ has cotype $q\}<p$. In particular X has cotype p.*

Proof. Put $q_0=\inf\{q\mid X \text{ has cotype } q\}$. By [13], $L_{q_0}(0,1)$ is finitely representable in X and hence it has M_p by Corollary 1.2. If $p\le q_0$, then $L_p(0,1)$ is a quotient of $L_{q_0}(0,1)$ and hence it also has M_p by Proposition 1.6; this is a contradiction since $L_p(0,1)$ contains uncomplemented subspaces isomorphic to ℓ_p (see [20]). ∎

We are now ready to prove:

THEOREM 2.9. *If $2<p<\infty$ and X is a Banach space with M_p, then the following statements hold*:

(i) *For every $\lambda\ge 1$ there exists a constant $c(\lambda)$ so that whenever $(x_j)\subseteq X$ is a finite or infinite λ-unconditional normalized sequence then*

$$c(\lambda)\Big(\sum_j |a_j|^2\Big)^{1/2}\le\Big\|\sum_j a_jx_j\Big\|\qquad \textit{for all } (a_j)\subseteq\mathbb{R}. \tag{2.8}$$

(ii) *X is of weak type 2 and has property (H). If in addition X is a Banach lattice then it is a weak Hilbert space which satisfies a lower 2-estimate.*

Proof. (i) Let $n\in\mathbb{N}$, $\lambda\ge 1$ and let $(x_j)_{j=1}^n\subseteq X$ be a normalized λ-unconditional sequence. Since $([x_j]\oplus\ell_2^n)_\infty$ is 12-isomorphic to a subspace of X, it follows that $([x_j]\oplus\ell_2^n)_\infty$ has M_p with constant less than or equal to $12M_p(X)$. Combining this with Lemma 2.7 we find that every bounded operator T from a subspace of $([x_j]\oplus\ell_2^n)_\infty$ to any $\ell_p(\delta,2)$, $0<\delta<1$, has an extension $\widetilde{T}$ to $([x_j]\oplus\ell_2^n)_\infty$ with $\|\widetilde{T}\|\le 12C^2M_p(X)$. By Lemma 2.8, X has cotype p and hence the cotype constant of $([x_j]\oplus\ell_2^n)_\infty$ is less than or equal to $2K_p(X)$ and therefore the formal identity operator I_δ of $[x_j](\delta,2)$ into $\ell_p(\delta,2)$ has a norm less than or equal to $2K_p(X)$. If we now choose δ so that $24C^2k_p(X)M_p(X)\delta<1$, then it follows that I_δ has an extension to $([x_j]\oplus\ell_2^n)_\infty$ with norm less than δ^{-1}. Hence by Theorem 2.6 we get, for all $(t_j)_{j=1}^n\subseteq\mathbb{R}$,

$$\frac{\delta^2}{2}\Big(\sum_{j=1}^n |t_j|^2\Big)^{1/2}\le\lambda\Big\|\sum_{j=1}^n t_jx_j\Big\|,$$

which proves (2.8).

(ii) Since X has M_p, it also has M_2 (because L_p has a complemented subspace isomorphic to a Hilbert space) and hence X is of weak type 2. Combining this with (2.8) we deduce that there exists a constant $C(\lambda)$ so that if $(x_j)_{j=1}^n \subseteq X$ is λ-unconditional and normalized, then

$$c(\lambda)\sqrt{n} \le \Big\| \sum_{j=1}^n x_j \Big\| \le C(\lambda)\sqrt{n},$$

which proves that X has property (H).

If in addition X is a Banach lattice, then it follows from [17, Corollary 4.4] that X is a weak Hilbert space which by (2.8) satisfies a lower 2-estimate. ∎

Let us conclude this section with two corollaries.

COROLLARY 2.10. *Let X be a Köthe function space on* $[0,1]$. *If X has M_p for some p, $2 < p < \infty$, then X is lattice isomorphic to $L_2(0,1)$.*

Proof. It follows from Theorem 2.9 that X is a weak Hilbert space and hence by [16, Theorem 3], X is lattice isomorphic to $L_2(0,1)$. ∎

COROLLARY 2.11. *If X is a Banach lattice with an upper 2-estimate which has M_p for some p, $2 < p < \infty$, then X is isomorphic to a Hilbert space.*

3. The extension properties M_p, $1 \le p < \infty$. In this section we shall investigate the properties M_p in greater detail. Our first theorem gives a necessary and sufficient condition for an operator from a subspace of X to ℓ_p to extend to X.

THEOREM 3.1. *Let X be a Banach space, E a subspace of X and $T \in B(E, \ell_p)$, $1 \le p \le \infty$. Let Q be the natural quotient map of X^* onto E^*. The following statements are equivalent:*

(i) *T has an extension $\widetilde{T} \in B(X, \ell_p)$.*

(ii) *There is a constant $K \ge 1$ so that for all Banach spaces Z and all $S \in B(Z, E)$ with $S^*Q \in \Pi_p(X^*, Z^*)$, TS is p-integral with*

$$i_p(TS) \le K\pi_p(S^*Q). \tag{3.1}$$

Proof. Assume that (i) holds and let $\widetilde{T} \in B(X, \ell_p)$ be an extension. Since $\|\widetilde{T}\| = \gamma_p(\widetilde{T})$, it follows from [4, Theorem 9.11] that if Z is an arbitrary Banach space and $S \in B(Z, E)$ with $S^*Q \in \Pi_p(X^*, Z^*)$, then $\widetilde{T}S = TS$ is p-integral with

$$i_p(TS) = i_p(\widetilde{T}S) \le \|\widetilde{T}\|\pi_p(S^*Q),$$

which is (3.1) with $K = \|\widetilde{T}\|$.

Assume next that (ii) holds and define

$$\mathcal{N} = \{U \in N_1(\ell_p, X) \mid U(\ell_p) \subseteq E\}.$$

If we can prove that T acts as a bounded linear functional on $\mathcal{N}$ via trace duality, then since $N_1(\ell_p, X)^* = B(X, \ell_p^{**})$ it follows that T admits an extension $\widetilde{T} \in B(X, \ell_p)$.

Hence let $U \in \mathcal{N}$ be arbitrary and let $\varepsilon > 0$. From Kwapień's characterization of Γ_p^* (see [8]) it follows that there exist a Banach space Z, $A \in \Pi_{p'}(\ell_p, Z)$ and $S \in B(Z, E)$ with $S^*Q \in \Pi_p(X^*, Z^*)$ so that $U = SA$ and

$$\pi_{p'}(A)\pi_p(S^*Q) \le \nu_1(U) + \varepsilon.$$

Applying now (1.2) we obtain

$$|\operatorname{tr}(TU)| \le i_p(TS)\pi_{p'}(A) \le K\pi_p(S^*Q)\pi_{p'}(A) \le K(\nu_1(U) + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, this shows that T admits an extension $\widetilde{T}$ with $\|\widetilde{T}\| \le K$. ∎

In our next result we shall use Theorem 3.1 to give a necessary and sufficient condition for every operator from a given subspace of X to extend to X.

THEOREM 3.2. *Let E be a subspace of a Banach space X and $1 \le p \le \infty$. Further let Q denote the canonical quotient map of X^* onto E^*. The following statements are equivalent*:

(i) *Every $T \in B(E, \ell_p)$ extends to a $\widetilde{T} \in B(X, \ell_p)$.*

(ii) *There is a constant $K \ge 1$ so that every $T \in E^* \otimes \ell_p$ extends to a $\widetilde{T} \in B(X, \ell_p)$ with $\|\widetilde{T}\| \le K\|T\|$.*

(iii) *There exists a constant $K \ge 1$ so that for all Banach spaces Z, whenever $S \in B(E^*, Z)$ with $SQ \in \Pi_p(X^*, Z)$ then $S \in \Pi_p(E^*, Z)$ with*

$$\pi_p(S) \le K\pi_p(SQ). \tag{3.2}$$

Proof. In view of the open mapping theorem and Theorem 1.3 it is immediate that (i) and (ii) are equivalent. Hence assume that (ii) holds and let K be a constant from there. Let Z be an arbitrary Banach space and let $S \in B(E^*, Z)$ with $SQ \in \Pi_p(X^*, Z)$. Our assumption and [9] (see also [15]) imply that

$$\begin{aligned}&\sup\{\|TS^*\|_m \mid T \in B(E^{**}, \ell_p),\ \|T\| \le 1\}\\ &\qquad \le K \sup\{\|TS^*\|_m \mid T \in B(X^{**}, \ell_p), \|T\| \le 1\} = K\pi_p(SQ).\end{aligned}$$

Since the left hand side is finite, we can conclude that it is equal to $\pi_p(S)$. Hence $S \in \Pi_p(E^*, Z)$ with $\pi_p(S) \le K\pi_p(SQ)$.

Assume next that (iii) holds and let $T \in B(E, \ell_p)$ be arbitrary. We shall verify that (ii) of Theorem 3.1 holds. Hence let Z be an arbitrary Banach space and $S \in B(Z,E)$ with $S^*Q \in \Pi_p(X^{**}, Z^*)$. From (3.2) we conclude that $S^* \in \Pi_p(E^*, Z^*)$, and therefore by [9], TS is order bounded and hence also p-integral with

$$i_p(TS) \le \|TS\|_m \le \|T\|\pi_p(S^*) \le K\|T\|\pi_p(S^*Q).$$

Hence T admits an extension $\widetilde{T}$ to X with $\|\widetilde{T}\| \le K\|T\|$. ∎

Using the previous results we now obtain:

THEOREM 3.3. *Let X be a Banach space and $1 \le p \le \infty$. The following statements are equivalent.*

(i) *X has M_p.*

(ii) *There exists a constant $K \ge 1$ so that if E is an arbitrary subspace of X, Q_E is the canonical quotient map of X^* onto E^* and Z is an arbitrary Banach space, then for every $S \in B(E^*, Z)$ with $SQ \in \Pi_p(X^*, Z)$ we have $S \in \Pi_p(E^*, Z)$ with*

$$\pi_p(S) \le K\pi_p(SQ).$$

Proof. This follows immediately from Theorems 1.1 and 3.2. ∎

We now need the following lemma:

LEMMA 3.4. *If X is a Banach space with M_1, then has type p for some p, $1 < p \le 2$.*

Proof. Let X have M_1. If X is not of type greater than one, then by [13], ℓ_1 is finitely representable in X and hence it follows from Corollary 1.2 that ℓ_1 has M_1. By [1], ℓ_1 contains an uncomplemented subspace E isomorphic to ℓ_1; hence no isomorphism of E onto ℓ_1 can be extended to ℓ_1, which is a contradiction. ∎

We are now able to prove

THEOREM 3.5. *If X is a Banach space, then the following statements hold:*

(i) *If X has M_1, then it has M_2.*

(ii) *If $1 < p < 2$, then X has M_1 if and only if it has M_p.*

(iii) *If X has M_p for some p, $2 < p < \infty$, then it has M_2.*

Proof. (i) Let X have M_1. By Lemma 3.4 there is a $q > 1$ so that X has type q. Let $1 < p < q$. If $E \subseteq X$ is a subspace, then it follows from [13] that $\Pi_1(E^*, Z) = \Pi_p(E^*, Z)$ for every Banach space Z and hence we see from our assumption and Theorem 3.3 that X has M_p. Since $L_p(0,1)$ has a complemented subspace isomorphic to a Hilbert space, we conclude that X has M_2.

(ii) Let $1 < p < 2$ and assume first that X has M_1. By (i) and Theorem 2.3, X has type q for all $q < 2$ and hence we can argue as in (i) to conclude that X has M_p. Assume next that X has M_p. Again the argument of (i) shows that X has M_2 and is therefore of type q for all $q < 2$. If $E \subseteq X$ is a subspace and $T \in B(E, \ell_1)$, then $T \in \Gamma_p(E, \ell_1)$ and hence it can be extended to a bounded $\widetilde{T} \in B(X, \ell_1)$.

(iii) If $2 < p < \infty$, then $L_p(0,1)$ has a complemented subspace isomorphic to a Hilbert space and hence if X has M_p, it also has M_2. ∎

We shall now need the following factorization theorem which is a generalization of [18, Theorem 8.17].

THEOREM 3.6. *Let $1 \le p \le 2$ and let X and Y be Banach spaces. If $B(\ell_\infty, X^*) = \Pi_{p'}(\ell_\infty, X^*)$ and Y has* GL$(1,p)$, *then $B(X,Y) \subseteq \Gamma_p(X, Y^{**})$ and there exists a constant C_p so that*

$$\gamma_p(T) \le C_p \, \mathrm{GL}_{1,p}(Y) \|T\| \qquad \text{for all } T \in B(X,Y). \tag{3.3}$$

Proof. Let $T \in B(X,Y)$ be arbitrary. We shall use [4, Theorem 9.11] to show that $T \in \Gamma_p(X, Y^{**})$. To this end let Z be an arbitrary Banach space and $S \in B(Z, X)$ with $S^* \in \Pi_p(X^*, Z^*)$. The assumptions on X give that S^* is absolutely summing and since Y has GL$(1,p)$, we deduce that TS is p-integral with

$$i_p(TS) \le \mathrm{GL}_{1,p}(Y)\pi_1(S^*T^*) \le C_p \, \mathrm{GL}_{1,p}(Y)\pi_p(S^*)\|T\|$$

for a suitable constant C_p. This together with the above-mentioned theorem gives (3.3). ∎

COROLLARY 3.7. *Let p and X be as in Theorem* 3.6. *If Y is a complemented subspace of a p-concave Banach lattice Z, then $B(X,Y) = \Gamma_p(X,Y)$.*

Proof. It follows from [5] that Y has GL$(1,p)$ and since Z does not contain c_0, it follows from [11] that Z and hence also Y is complemented in its second dual. ∎

The next theorem is a direct consequence of Theorems 3.6 and 3.5.

THEOREM 3.8. *Let X be a Banach space with M_1 and Y a Banach space with* GL$(1,p)$ *where $1 \le p < 2$. If $E \subseteq X$ is a subspace, then every $T \in B(E,Y)$ extends to a $\widetilde{T} \in B(X, Y^{**})$ with*

$$\|\widetilde{T}\| \le M_p(X) \, \mathrm{GL}_{1,p}(Y) T_r(X) \|T\| \quad \text{for all } r \text{ with } p < r < 2. \tag{3.4}$$

Proof. Choose $p < r < 2$ and let $T \in B(E,Y)$. Since X (and hence E) has type r by Theorem 3.5, we deduce from Theorem 3.6 that $T \in \Gamma_p(E, Y^{**})$ with

$$\gamma_p(T) \le T_r(X) \, \mathrm{GL}_{1,p}(Y) \|T\|. \tag{3.5}$$

Since X also has M_p it follows from (3.5) that T can be extended to a $\widetilde{T} \in B(X, Y^{**})$ so that (3.4) holds. ∎

It is immediate from the definition of M_2 that the following holds:

Proposition 3.9. *Let X be a Banach space with M_2. For every finite-dimensional subspace $E \subseteq X$ there exists a projection P of X onto E with*

$$\|P\| \leq M_2(X) d(E, \ell_2^{\dim E}). \tag{3.6}$$

If X is a Banach space and there exists a constant K so that (3.6) holds with K in place of $M_2(X)$, then X is said to have the *Maurey projection property*. It follows from [18, Theorem 11.6] that a Banach space with this property is of weak type 2. We end this section with the following result:

Theorem 3.10. *Let X be a Köthe function space on* $[0,1]$ *with an unconditional basis. If X has the Maurey projection property*, *then it is of type* 2.

Proof. Since X has an unconditional basis, it follows from [7] that X is isomorphic to $X(\ell_2)$ $(= \ell_2 \otimes_m X)$. It therefore follows from from [19, Remark 11.8] that X being of weak type 2 is actually of type 2. ∎

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