STUDIA MATHEMATICA 168 (3) (2005)

Differentiability of the g-Drazin inverse

by

J. J. Koliha (Melbourne) and V. Rakočević (Niš)

Abstract. If $A(z)$ is a function of a real or complex variable with values in the space $B(X)$ of all bounded linear operators on a Banach space X with each $A(z)$ g-Drazin invertible, we study conditions under which the g-Drazin inverse $A^{\mathsf{D}}(z)$ is differentiable. From our results we recover a theorem due to Campbell on the differentiability of the Drazin inverse of a matrix-valued function and a result on differentiation of the Moore–Penrose inverse in Hilbert spaces.

1. Introduction and preliminaries. The Drazin inverse defined originally for semigroups in [4] in 1958 is an important theoretical and practical tool in algebra and analysis. When $\mathcal{A}$ is an algebra and $a \in \mathcal{A}$, then $b \in \mathcal{A}$ is the *Drazin inverse* of a if

$$(1.1) \qquad ab = ba, \quad bab = b, \quad aba = a + u \quad \text{where } u \text{ is nilpotent.}$$

It was observed by Harte [7, 8] and by the first author in [11] that in Banach algebras it is more natural to replace the nilpotent element u in (1.1) by a quasinilpotent element. If u in (1.1) is allowed to be quasinilpotent, we call b the *g-Drazin inverse* of a.

The g-Drazin inverse introduced in [11] is a useful construct that finds its applications in a number of areas. In the present paper we concentrate on the g-Drazin inverse in the Banach algebra $B(X)$ of bounded linear operators, and continue the investigation of the continuity of the g-Drazin inverse [14] by studying its differentiability. For matrices, this was studied by Campbell [1] and Hartwig and Shoaf [9]. Drazin [5] considered differentiation of the conventional Drazin inverse in associative rings, using a general derivation in the ring.

We can briefly describe the contents of this paper as follows: If $A(z)$ is a function of a real or complex variable with values in the space of all bounded linear operators on a Banach space with each $A(z)$ g-Drazin invertible, we study the conditions under which the g-Drazin inverse $A^{\mathsf{D}}(z)$ is

2000 *Mathematics Subject Classification*: 47A60, 47A05, 47A10.

Key words and phrases: bounded linear operator, g-Drazin inverse, differentiable g-Drazin inverse.

differentiable. From our results we recover a theorem due to Campbell on the differentiability of the Drazin inverse of a matrix-valued function and a result on differentiation of the Moore–Penrose inverse in Hilbert spaces.

By $B(X)$ we denote the Banach algebra of all bounded linear operators acting on the complex Banach space X with the usual operator norm. By $\varrho(T)$, $\sigma(T)$ and $r(T)$ we denote the resolvent set, the spectrum and the spectral radius of $T \in B(X)$, respectively. We also write $\sigma_0(T)$ for $\sigma(T)\setminus\{0\}$. The sets of all isolated and accumulation spectral points of T are denoted by $\operatorname{iso}\sigma(T)$ and $\operatorname{acc}\sigma(T)$. If $\lambda \in \varrho(T)$, then $R(\lambda; T) = (\lambda I - T)^{-1}$ is the resolvent of T. We recall [12] that $0 \in \operatorname{iso}\sigma(T)$ if and only if there exists a nonzero projection $P \in B(X)$ such that

$$AP = PA \text{ is quasinilpotent and } A + P \text{ is invertible;}$$

P is the *spectral projection of* T at 0, and is denoted by A^π [12, Theorem 1.2].

Definition 1.1 (Koliha [11, Definition 2.3]). An operator $A \in B(X)$ is *g-Drazin invertible* if there exists $B \in B(X)$ such that

$$(1.2)\qquad AB = BA, \qquad BAB = B, \qquad ABA = A + U, \quad \text{where } r(U) = 0.$$

The operator B is called the *g-Drazin inverse of* A, denoted by A^{D}. The *Drazin index* $i(A)$ *of* A is 0 if A is invertible, k if A is not invertible and U is nilpotent of index k, and ∞ otherwise. Definition 1.1 with $i(A)$ finite coincides with the definition of the conventional Drazin inverse (see [3, 4, 10]). An operator A has a conventional Drazin inverse if and only if 0 is at most a pole of the resolvent of A; A has the g-Drazin inverse if and only if $0 \notin \operatorname{acc}\sigma(A)$ ([11, Theorem 4.2], [12, Theorem 1.2]).

We need a representation of A^{D} in terms of the holomorphic calculus for A. A *cycle* is a formal linear combination Γ of loops with integral coefficients; Γ is a *Cauchy cycle* relative to the pair (Ω, K), where K is a compact subset of a nonempty open set $\Omega \subset \mathbb{C}$, if $\Gamma \subset \Omega \setminus K$, $\operatorname{ind}(\Gamma, \lambda) = 0$ for all $\lambda \notin \Omega$ and $\operatorname{ind}(\Gamma, \mu) = 1$ for all $\mu \in K$. The existence of a Cauchy cycle relative to any such pair (Ω, K) is proved in [16, Theorem 13.5]. By [11, Theorem 4.4],

$$(1.3)\qquad A^{\mathsf{D}} = \frac{1}{2\pi i}\int_{\Gamma} \lambda^{-1} R(\lambda; A)\, d\lambda,$$

where Γ is a Cauchy cycle relative to $(\mathbb{C}\setminus\{0\}, \sigma_0(A))$. (In the case that A is quasinilpotent, the formula is interpreted in the following way: As $\sigma_0(A) = \emptyset$, Γ can be any cycle in $\mathbb{C}\setminus\{0\}$ with $\operatorname{ind}(\Gamma, 0) = 0$. The integral in (1.3) is zero, which agrees with $A^{\mathsf{D}} = 0$.)

Below we will use the following perturbation result involving operator resolvents which follows from [6, Lemma VII.6.3].

LEMMA 1.2. *Let $A, A(z) \in B(X)$ for all z in some neighborhood U of z_0, and let $\|A(z) - A\| \to 0$ as $z \to z_0$. If K is a compact subset of the complex plane contained in the resolvent sets of A and $A(z)$ for all $z \in U$, then*

$$\lim_{z\to z_0} R(\lambda; A(z)) = R(\lambda; A) \quad \textit{uniformly for } \lambda \in K. \tag{1.4}$$

2. Differentiability properties of the g-Drazin inverse. In this section, U denotes an open interval in $\mathbb{R}$ or an open subset of $\mathbb{C}$, z_0 a point in U, and $A : U \to B(X)$ an operator-valued function. By $A'(z)$ we denote the derivative of $A(z)$ at z, and by $A^{\mathsf{D}}(z)$ the g-Drazin inverse $A(z)^{\mathsf{D}}$. Our main result on the differentiability of the g-Drazin inverse is given in the following theorem:

THEOREM 2.1. *Let A be a $B(X)$-valued function defined on U such that $A(z)$ is g-Drazin invertible for all $z \in U$, and differentiable at $z_0 \in U$. Then $A^{\mathsf{D}}(z)$ is differentiable at z_0 if and only if $A^{\mathsf{D}}(z)$ is continuous at z_0. In this case the derivative $(A^{\mathsf{D}})'(z_0)$ is given by*

$$(A^{\mathsf{D}})'(z_0) = \frac{1}{2\pi i}\int_{\Gamma} \lambda^{-1} R(\lambda; A(z_0))A'(z_0)R(\lambda; A(z_0))\, d\lambda, \tag{2.1}$$

where Γ is a Cauchy cycle relative to $(\mathbb{C} \setminus \{0\}, \sigma_0(A(z_0)))$.

Proof. Assume that $A^{\mathsf{D}}(z)$ is continuous at z_0. From [14, Theorem 4.1] (see equation (2.5) below) it follows that there exist $r>0$ and $\delta_1>0$ such that

$$0 < |\lambda| < r \ \Rightarrow\ \lambda \in \varrho(A(z)) \text{ whenever } |z - z_0| < \delta_1. \tag{2.2}$$

Let $\Omega = \{\lambda : |\lambda| > r\}$, and let Ω_1 be a bounded open set with $\sigma_0(A(z_0)) \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$. From the upper semicontinuity of the spectrum it follows that there exists $\delta \in (0, \delta_1)$ such that the sets $\sigma_0(A(z))$ are contained in Ω_1 whenever $|z - z_0| < \delta$. (The cases $\sigma_0(A(z)) = \emptyset$ or $\sigma_0(A(z_0)) = \emptyset$ are not excluded.) There exists a Cauchy cycle Γ relative to $(\Omega, \bar{\Omega}_1)$, and

$$A^{\mathsf{D}}(z) = \frac{1}{2\pi i}\int_{\Gamma} \lambda^{-1} R(\lambda; A(z))\, d\lambda, \qquad |z - z_0| < \delta, \tag{2.3}$$

by (1.3). Consider the existence of the limit

$$\lim_{z\to z_0} \frac{A^{\mathsf{D}}(z) - A^{\mathsf{D}}(z_0)}{z - z_0}.$$

Using the second resolvent equation, we get

$$\begin{aligned}\frac{A^{\mathsf{D}}(z) - A^{\mathsf{D}}(z_0)}{z - z_0} &= \frac{1}{z - z_0}\,\frac{1}{2\pi i}\int_{\Gamma} \lambda^{-1}[R(\lambda; A(z)) - R(\lambda; A(z_0))]\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma} \lambda^{-1} R(\lambda; A(z))\,\frac{A(z) - A(z_0)}{z - z_0}\, R(\lambda; A(z_0))\, d\lambda.\end{aligned}$$

In view of Lemma 1.2,

$$\lim_{z\to z_0} R(\lambda; A(z))\,\frac{A(z)-A(z_0)}{z-z_0} = R(\lambda; A(z_0))A'(z_0)$$

uniformly for $\lambda \in \Gamma$. Hence (2.1) follows.

The converse is clear. ∎

We note that Theorem 4.1 of [14] holds when sequences are replaced by functions of z; that theorem gives twelve conditions equivalent to the continuity of $A^{\mathsf{D}}(z)$ at z_0. For the sake of completeness we restate four of these conditions relevant to the present investigation. Under the hypotheses of Theorem 2.1, $A^{\mathsf{D}}(z) \to A^{\mathsf{D}}(z_0)$ as $z \to z_0$ if and only if any of the following conditions holds:

$$\sup\{\|A^{\mathsf{D}}(z)\| : |z-z_0| < \delta\} < \infty \quad \text{for some } \delta > 0, \tag{2.4}$$

$$\sup\{r(A^{\mathsf{D}}(z)) : |z-z_0| < \delta\} < \infty \quad \text{for some } \delta > 0, \tag{2.5}$$

$$A^{\mathsf{D}}(z)A(z) \to A^{\mathsf{D}}(z_0)A(z_0) \quad \text{as } z \to z_0, \tag{2.6}$$

$$A^{\pi}(z) \to A^{\pi}(z_0) \quad \text{as } z \to z_0. \tag{2.7}$$

We take this opportunity to correct a mistake in [14, Theorem 4.1]: Conditions (4.14) and (4.15) of that theorem should be

$$C_n \to C \quad \text{and} \quad \gamma(C_n) \to \gamma(C)$$

and

$$C_n \to C \quad \text{and} \quad \inf_n \gamma(C_n) > 0,$$

respectively, where $\gamma(A)$ denotes the reduced minimum modulus of an operator $A \in B(X)$.

Note 2.2. The preceding argument works with appropriate interpretation in the case that $r(A(z_0)) = 0$.

Note 2.3. Hartwig and Shoaf [9, (3.10)] used holomorphic calculus to give a formula for the derivative of the Drazin inverse of a complex matrix in terms of the spectral components of $A(z)$.

In the case that the operators $A(z)$ have the conventional Drazin inverse and the indices of $A(z)$ are uniformly bounded, we are able to obtain a stronger result.

Theorem 2.4. *Let A be a $B(X)$-valued function defined on U such that $A(z)$ is g-Drazin invertible for all $z \in U$ and differentiable at $z_0 \in U$. If the indices $i(A(z))$ are uniformly bounded and the spectral projections $A^{\pi}(z)$ are of finite rank, then $A^{\mathsf{D}}(z)$ is differentiable at z_0 if and only if there exists $\delta > 0$ such that*

$$\operatorname{rank} A^{\pi}(z) = \operatorname{rank} A^{\pi}(z_0) \quad \textit{whenever} \quad |z-z_0| < \delta.$$

Proof. This follows from Theorem 2.1 and [14, Theorem 5.1]. ∎

From the preceding theorem we recover the main result of [1] on the differentiability of the matrix Drazin inverse. The *core part* $C(z)$ of $A(z)$ is defined by $C(z) = A(z)(I - A^{\pi}(z))$; the *core rank* of $A(z)$ is the rank of $C(z)$.

Corollary 2.5 (Campbell [1, Theorem 4]). *Let A be a $p \times p$ matrix-valued function defined on U and differentiable at $z_0 \in U$. Then $A^{\mathsf{D}}(z)$ is differentiable at z_0 if and only if the core rank of $A(z)$ is constant in some neighborhood of z_0.*

Proof. This follows from Theorem 2.4 and the result for the core rank of $A(z)$ which states that $\operatorname{rank} C(z) = p - \operatorname{rank} A^{\pi}(z)$. ∎

Let us remark that our approach differs from the one adopted by Campbell in [1], who derived his theorem from the known differentiation result for the Moore–Penrose inverse and from the relation between the Drazin inverse A^{D} of a $p \times p$ matrix A and the Moore–Penrose inverse $A^{\dagger}$ of A:

$$A^{\mathsf{D}} = A^p (A^{2p+1})^{\dagger} A^p.$$

3. Series expansion for $(A^{\mathsf{D}})'$. Let U be an open interval in $\mathbb{R}$ or an open set in $\mathbb{C}$, and $A(z)$ an operator-valued function on U satisfying the hypotheses of Theorem 2.1 such that $A^{\mathsf{D}}(z)$ is continuous at z_0. To simplify notation, we write A, A^{D}, A', A^{π} for $A(z_0)$, $A^{\mathsf{D}}(z_0)$, $A'(z_0)$, $A^{\pi}(z_0)$. Then (2.2) holds, and we pick $R > \max\{r, r(A)\}$. In formula (2.1) we choose $\Gamma = \omega_R - \omega_r$, where $\omega_{\varrho}(s) = \varrho \exp(is)$ for any $\varrho > 0$, $s \in [0, 2\pi]$. It can be verified that Γ is a Cauchy cycle relative to the pair $(\mathbb{C} \setminus \{0\}, \sigma_0(A))$.

According to (2.1),

$$(3.1)\qquad (A^{\mathsf{D}})' = \frac{1}{2\pi i} \int_{\omega_R} \lambda^{-1} R(\lambda; A) A' R(\lambda; A)\, d\lambda - \frac{1}{2\pi i} \int_{\omega_r} \lambda^{-1} R(\lambda; A) A' R(\lambda; A)\, d\lambda.$$

Since $R(\lambda; A) = O(|\lambda|^{-1})$ as $|\lambda| \to \infty$, $\| \int_{\omega_R} \lambda^{-1} R(\lambda; A) A' R(\lambda; A)\, d\lambda \| = O(R^{-2})$ as $R \to \infty$. This shows that

$$\frac{1}{2\pi i} \int_{\omega_R} \lambda^{-1} R(\lambda; A) A' R(\lambda; A)\, d\lambda = 0.$$

By assumption, $0 \notin \operatorname{acc} \sigma(A)$; in view of [11, Theorem 5.1] there exists $r_0 > 0$ such that

$$R(\lambda; A) = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n A^{\pi} - \sum_{n=0}^{\infty} \lambda^n (A^{\mathsf{D}})^{n+1} =: U_{\lambda} - V_{\lambda}$$

for $0 < |\lambda| < r_0$. If $0 < \varrho < \min(r, r_0)$, then

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{\omega_r}\lambda^{-1}R(\lambda;A)A'R(\lambda;A)\,d\lambda &= \frac{1}{2\pi i}\int_{\omega_\varrho}\lambda^{-1}(U_\lambda-V_\lambda)A'(U_\lambda-V_\lambda)\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\omega_\varrho}\lambda^{-1}U_\lambda A'U_\lambda\,d\lambda+\frac{1}{2\pi i}\int_{\omega_\varrho}\lambda^{-1}V_\lambda A'V_\lambda\,d\lambda\\
&\quad -\frac{1}{2\pi i}\int_{\omega_\varrho}\lambda^{-1}U_\lambda A'V_\lambda\,d\lambda-\frac{1}{2\pi i}\int_{\omega_\varrho}\lambda^{-1}V_\lambda A'U_\lambda\,d\lambda\\
&= \sum_{m,n=0}^{\infty}A^\pi A^m A'A^n A^\pi\,\frac{1}{2\pi i}\int_{\omega_\varrho}\lambda^{-m-n-3}\,d\lambda\\
&\quad +\sum_{m,n=0}^{\infty}(A^{\mathsf D})^{m+1}A'(A^{\mathsf D})^{n+1}\,\frac{1}{2\pi i}\int_{\omega_r}\lambda^{m+n-1}\,d\lambda\\
&\quad -\sum_{m,n=0}^{\infty}A^\pi A^m A'(A^{\mathsf D})^{n+1}\,\frac{1}{2\pi i}\int_{\omega_r}\lambda^{-m+n-2}\,d\lambda\\
&\quad -\sum_{m,n=0}^{\infty}(A^{\mathsf D})^{n+1}A'A^m A^\pi\,\frac{1}{2\pi i}\int_{\omega_r}\lambda^{-m+n-2}\,d\lambda\\
&= A^{\mathsf D}A'A^{\mathsf D}-\sum_{n=0}^{\infty}A^\pi A^n A'(A^{\mathsf D})^{n+2}-\sum_{n=0}^{\infty}(A^{\mathsf D})^{n+2}A'A^nA^\pi
\end{aligned}
$$

as $\int_{\omega_\varrho}\lambda^k\,d\lambda$ is equal to $2\pi i$ if $k=-1$ and to 0 otherwise. Substituting this into (3.1) we get the following result.

THEOREM 3.1. *Let A be a $B(X)$-valued function defined on U such that $A(z)$ is g-Drazin invertible for all $z\in U$ and differentiable at $z_0\in U$. If $A^{\mathsf D}$ is continuous at z_0, then*

$$
(A^{\mathsf D})'=-A^{\mathsf D}A'A^{\mathsf D}+\sum_{n=0}^{\infty}A^\pi A^nA'(A^{\mathsf D})^{n+2}+\sum_{n=0}^{\infty}(A^{\mathsf D})^{n+2}A'A^nA^\pi, \tag{3.2}
$$

where A, $A^{\mathsf D}$, A', A^π stand for $A(z_0)$, $A^{\mathsf D}(z_0)$, $A'(z_0)$, $A^\pi(z_0)$, respectively.

In the case that the Drazin indices $i(A(z))$ are finite and uniformly bounded, the preceding theorem subsumes the differentiation formula of Campbell [1, Theorem 2]; the summation then becomes finite. Let us observe that Campbell's proof is based on the differentiation of the defining equations in the case that A has the Drazin index 1, that is, on the differentiation of the equations

$$AA^{\mathsf D}A=A,\qquad A^{\mathsf D}AA^{\mathsf D}=A^{\mathsf D},\qquad AA^{\mathsf D}=A^{\mathsf D}A.$$

Hartwig and Shoaf obtained Campbell's formula from a difference relation [9, (4.16)]. Under the assumption of finite and uniformly bounded indices, formula (3.2) formally agrees with Drazin's result [5, Theorem 2], which is derived for the conventional Drazin inverse in associative rings.

We note that if $i(A) \le 1$, formula (3.2) reduces to

$$(A^{\mathsf{D}})' = -A^{\mathsf{D}}A'A^{\mathsf{D}} + A^{\pi}A'(A^{\mathsf{D}})^2 + (A^{\mathsf{D}})^2A'A^{\pi}. \tag{3.3}$$

For matrices this yields [1, Theorem 1].

If A satisfies the hypotheses of Theorem 2.1 and A^{D} is continuous at z_0, equation (3.3) can be used to describe $(A^{\mathsf{D}})'$ in terms of the derivative C' of the core part of A, bearing in mind that C has Drazin index not exceeding one:

$$\begin{aligned}(A^{\mathsf{D}})' = (C^{\mathsf{D}})' &= -C^{\mathsf{D}}C'C^{\mathsf{D}} + C^{\pi}C'(C^{\mathsf{D}})^2 + (C^{\mathsf{D}})^2C'C^{\pi} \\ &= -A^{\mathsf{D}}C'A^{\mathsf{D}} + A^{\pi}C'(A^{\mathsf{D}})^2 + (A^{\mathsf{D}})^2C'A^{\pi};\end{aligned}$$

it is known that $A^{\mathsf{D}} = C^{\mathsf{D}}$ and $A^{\pi} = C^{\pi}$.

4. The Moore–Penrose inverse of Hilbert space operators. For H a complex Hilbert space and $A \in B(H)$ it is well known that

$$\begin{aligned}A \text{ has closed range } &\Leftrightarrow\ A^*A \text{ has closed range } \Leftrightarrow\ AA^* \text{ has closed range} \\ &\Leftrightarrow\ 0 \notin \operatorname{acc}\sigma(A^*A)\ \Leftrightarrow\ 0 \notin \operatorname{acc}\sigma(AA^*).\end{aligned}$$

For a closed range operator $A \in B(H)$ we can give a definition of the Moore–Penrose inverse $A^{\dagger}$ of A in terms of the Drazin inverse (see [13, Theorem 2.5]):

$$A^{\dagger} = (A^*A)^{\mathsf{D}}A^* = A^*(AA^*)^{\mathsf{D}}. \tag{4.1}$$

This equation enables us to obtain results on the continuity and differentiability of the Moore–Penrose inverse using our results on the g-Drazin inverse. (For the continuity of the Moore–Penrose inverse see, for instance, [15].)

THEOREM 4.1. *Let A be a $B(X)$-valued function defined on a real interval J differentiable at $t_0 \in J$ with $A(t)$ closed range operators for all $t \in J$. Write $B(t) = A^*(t)A(t)$ and $E(t) = A(t)A^*(t)$ for all $t \in J$. Then the following conditions are equivalent*:

(i) *$B^{\mathsf{D}}(t)$ is continuous at t_0.*
(ii) *$E^{\mathsf{D}}(t)$ is continuous at t_0.*
(iii) *$B^{\mathsf{D}}(t)$ is differentiable at t_0.*
(iv) *$E^{\mathsf{D}}(t)$ is differentiable at t_0.*
(v) *$A^{\dagger}(t)$ is differentiable at t_0.*
(vi) *$A^{\dagger}(t)$ is continuous at t_0.*
(vii) *$A^{\dagger}(t)A(t)$ is continuous at t_0.*
(viii) *$A(t)A^{\dagger}(t)$ is continuous at t_0.*
(ix) *$\|A^{\dagger}(t)\|$ is bounded in some neighborhood of t_0.*

Proof. (i)⇒(iii)⇒(v)⇒(vi)⇒(vii)⇒(i). The first four implications are justified by Theorem 2.1, by the product rule for differentiation applied to $A^\dagger(t) = B^{\mathsf{D}}(t)A^*(t)$, by the relation between differentiability and continuity, and by the continuity of the multiplication in $B(X)$, respectively. The last implication follows when we observe that if (vii) holds, then $A^\dagger(t)A(t) = (A^*(t)A(t))^{\mathsf{D}}(A^*(t)A(t)) = I - A^\pi(t)$ is continuous at t_0. Then (i) is true by Theorem 2.1 (equivalent condition (2.7)).

(ii)⇒(iv)⇒(v)⇒(vi)⇒(viii)⇒(ii) is proved by a symmetrical argument.

Condition (ix) is equivalent to (vi) when we use the inequality

$$\|A^\dagger(t) - A^\dagger(t_0)\| \le 3\max\{\|A^\dagger(t)\|^2, \|A^\dagger(t_0)\|^2\}\|A(t) - A(t_0)\|$$

(see [2, Theorem 10.4.5]). ■

NOTE 4.2. We note that in the proof of the implication (iii)⇒(v) the differentiability of $A^*(t)$ follows from the differentiability of $A(t)$ via the identity

$$\frac{dA^*(t)}{dt} = \left(\frac{dA(t)}{dt}\right)^*,$$

which holds only when t is real. The preceding theorem, unlike Theorem 2.1, does not hold for complex differentiation.

References

[1] S. L. Campbell, *Differentiation of the Drazin inverse*, SIAM J. Appl. Math. 30 (1976), 703–707.

[2] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*, Pitman, London, 1979.

[3] S. R. Caradus, *Generalized Inverses and Operator Theory*, Queen's Papers in Pure and Appl. Math. 50, Queen's Univ., Kingston, 1978.

[4] M. P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly 65 (1958), 506–514.

[5] —, *Differentiation of generalized inverses*, in: Recent Applications of Generalized Inverses, S. L. Campbell (ed.), Res. Notes in Math. 66, Pitman, Boston, 1982, 138–144.

[6] N. Dunford and J. T. Schwartz, *Linear Operators I*: *General Theory*, Interscience, New York, 1958.

[7] R. E. Harte, *Spectral projections*, Irish Math. Soc. Newsletter 11 (1984), 10–15.

[8] —, *On quasinilpotents in rings*, Panamer. Math. J. 1 (1991), 10–16.

[9] R. E. Hartwig and J. Shoaf, *On the derivative of the Drazin inverse of a complex matrix*, SIAM J. Math. Anal. 10 (1979), 207–216.

[10] C. F. King, *A note on Drazin inverses*, Pacific J. Math. 70 (1977), 383–390.

[11] J. J. Koliha, *A generalized Drazin inverse*, Glasgow Math. J. 38 (1996), 367–381.

[12] —, *Isolated spectral points*, Proc. Amer. Math. Soc. 124 (1996), 3417–3424.

[13] —, *The Drazin and Moore–Penrose inverse in C^*-algebras*, Math. Proc. Roy. Irish Acad. 99A (1999), 17–27.

[14] J. J. Koliha and V. Rakočević, *Continuity of the Drazin inverse II*, Studia Math. 131 (1998), 167–177.
[15] V. Rakočević, *On the continuity of the Moore–Penrose inverse in C^*-algebras*, Math. Montisnigri 2 (1993), 89–92.
[16] W. Rudin, *Real and Complex Analysis*, 2nd ed., McGraw-Hill, New York, 1974.

Department of Mathematics and Statistics
University of Melbourne
Melbourne, VIC 3010, Australia
E-mail: j.koliha@ms.unimelb.edu.au

Faculty of Science and Mathematics
Višegradska 33
18000 Niš, Serbia-Montenegro
E-mail: vrakoc@bankerinter.net

Received July 12, 1998
Revised version February 11, 2005 (4145)