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Uniqueness of minimal projections onto two-dimensional subspaces

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Abstract. We prove that minimal projections from L_p $(1 < p < \infty)$ onto any two-dimensional subspace are unique. This result complements the theorems of W. Odyniec ([OL, Theorem I.1.3], [O3]). We also investigate the minimal number of norming points for such projections.

0. Introduction. W. Odyniec ([OL, Theorem I.1.3], [O3]) proved that minimal projections of norm greater than one from a three-dimensional Banach space onto any of its two-dimensional subspace are unique. This result can be generalized neither to the subspaces of codimension one nor to the subspaces of dimension two, unless additional assumptions on the space are considered.

However, as proved by Odyniec ([OL, Theorem I.2.22], [O1, O2]) every subspace of codimension one in L_p $(1 < p < \infty)$ has a unique minimal projection.

In this paper we complete the picture by showing that every two-dimensional subspace of L_p $(1 < p < \infty)$ has a unique minimal projection. Specifically we prove the following theorem:

THEOREM 0.1. *Let V be a two-dimensional subspace of an $L_p(\mu)$ $(1 < p < \infty)$. Then there is a unique minimal projection from X onto V.*

We prove this theorem in Section 1. The proof depends on the number of norming points (and functionals) for minimal projections. In Section 2 we investigate one particular minimal projection and its norming pairs.

The rest of this section is devoted to general remarks and necessary definitions.

It is well known (see [IS] and [CMO]) that for every finite-dimensional subspace V of a Banach space X there exists a minimal projection.

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The problem of finding a minimal projection and related problems received the attention of many mathematicians (see [BP, CF, CL, CM1, CM2, CHM, CMO, CP, F, KTJ, R]) and it turned out to be easier in L_1 spaces than in L_p spaces (mostly due to Theorem 1 in [CM2] which can be effectively applied in L_1).

The problem of uniqueness of minimal projection, however, is not well understood yet. It is clear that subspaces of L_1 usually lack uniqueness (see [CM1]) though the classical Fourier projection onto trigonometric polynomials is unique in L_1 as well as in the space of continuous functions (compare [CHM, FMW]). For necessary and sufficient conditions for the uniqueness of minimal projections onto two-dimensional subspaces of ℓ_∞^n see [L3].

As far as we know, for $1 < p < \infty$ there is no example of subspaces of L_p (finite-dimensional or finite-codimensional) for which a minimal projection is not unique. Even the uniqueness of minimal projections onto trigonometric polynomials is not known.

To the best of our knowledge the only results in this direction are the previously mentioned theorem of Odyniec and the theorem of H. B. Cohen and F. E. Sullivan which states that if a minimal projection in L_p ($1 < p < \infty$) has norm one then it is unique (see [CS]). In particular all one-dimensional subspaces of L_p ($1 < p < \infty$) have unique minimal projection. We hope that Theorem 0.1 is a modest contribution to this list.

It is worth mentioning that the result of W. Odyniec has been improved by G. Lewicki ([L3, Theorem 2.6.11]) by showing that a minimal projection of norm greater than one from a three-dimensional real Banach space onto any two-dimensional subspace is in fact strongly unique.

Let us introduce some basic notions, definitions and facts used in this paper. Let $S(X)$ and $B(X)$ denote the unit sphere and unit ball of a Banach space X.

A projection P from X onto a subspace V is called *minimal* if it has the smallest possible norm, i.e.,

$$\|P\| = \lambda(V,X) = \inf\{\|Q\| : Q \text{ is a projection from } X \text{ onto } V\}. \tag{0.1}$$

The constant $\lambda(V,X)$ is called the *relative projection constant.*

Definition 0.2. A functional $f \in S(X^*)$ is a *norming functional* for a projection $P : X \to V$ iff $\|f \circ P\| = \|P\|$.

It is well known that if V is finite-dimensional then P has norming functionals (see [OL, Lemma III.2.1]).

Definition 0.3. A point $x \in S(X)$ is a *norming point* for a projection $P : X \to V$ iff $\|P(x)\| = \|P\|$.

If X is a reflexive space and V is finite-dimensional then P has a norming functional f and since the functional $f \circ P$ attains its norm, P has a norming point (this is not so in general Banach spaces: the Fourier projection does not have a norming point in the space of continuous functions, see [OL, Lemma I.2.7]).

DEFINITION 0.4. A pair (f, x) is called a *norming pair* for a projection P iff $f(Px) = \|P\|$. A set of all norming pairs for a projection P is denoted by $\mathcal{E}(P)$.

As usual, for $g \in X^*$ and $y \in X$, the symbol $g \otimes y$ denotes the one-dimensional operator from X to X given by $g \otimes y(x) = g(x)y$.

For the sake of completeness we state the Rudin Theorem which will be used for proving minimality of a projection given in Section 2.

DEFINITION 0.5. Suppose that a Banach space X and a topological group G are related in the following manner: to every $s \in G$ corresponds a continuous linear operator $T_s : X \to X$ such that

$$T_e = I, \quad T_{st} = T_s T_t \quad (s \in G,\, t \in G).$$

Under these conditions, G is said to *act as a group of linear operators* on X.

DEFINITION 0.6. A map $L : X \to X$ *commutes with* G if $T_g L T_g^{-1} = L$ for every $g \in G$.

THEOREM 0.7 (Rudin, [W, III.B.13]). *Let X be a Banach space and V a complemented subspace, i.e., $\mathcal{P}(X, V) \neq \emptyset$. Let G be a compact group which acts as a group of linear operators on X such that*

(1) *$T_g(x)$ is a continuous function of g for every $x \in X$,*
(2) *$T_g(V) \subset V$ for all $g \in G$.*
(3) *T_g is an isometry for all $g \in G$.*

Furthermore, assume that there exists only one projection $P : X \to V$ which commutes with G. Then this projection is minimal.

Once we know that there is only one projection P commuting with G it can be easily found: fix any projection Q from X onto V; then

$$P(x) = \int_G T_g Q T_{g^{-1}}(x)\, dg \quad \text{for } x \in X.$$

This theorem, however, does not imply that this projection is the unique minimal projection as there could be projections which do not commute with G but still have a minimal norm (see [S], [L1]).

1. Proof of Theorem 0.1

LEMMA 1.1. *Let V be a two-dimensional subspace of a Banach space X. Let $x \in S(X) \setminus V$. Then for any $\alpha > 0$ there exists a projection Q from X onto V such that $\|Q(x)\| = \alpha$.*

Proof. Let $v_1, v_2 \in S(V)$ be a basis for V. Since x, v_1, v_2 are linearly independent, using the Hahn–Banach theorem we can choose

$$f_1 \in X^* \text{ such that } f_1(v_1) = 1 \text{ and } f_1|_{\operatorname{span}\{x,v_2\}} = 0$$

and

$$f_2 \in X^* \text{ such that } f_2(v_2) = 1 \text{ and } f_2|_{\operatorname{span}\{\frac{1}{\alpha}x - v_2, v_1\}} = 0.$$

We have chosen f_1 and f_2 such that

$$\begin{aligned} f_1(x) &= 0, & f_1(v_1) &= 1, & f_1(v_2) &= 0, \\ f_2(x) &= \alpha, & f_2(v_1) &= 0, & f_2(v_2) &= 1. \end{aligned} \tag{1.1}$$

Now take

$$Q = f_1 \otimes v_1 + f_2 \otimes v_2 : X \to V.$$

From (1.1), $Q(v_1) = v_1$ and $Q(v_2) = v_2$ so Q is a projection, and from (1.1),

$$Q(x) = f_1(x)v_1 + f_2(x)v_2 = \alpha v_2,$$

hence $\|Q(x)\| = \alpha$. ∎

Theorem 1.2. *Let V be a two-dimensional subspace of a uniformly convex Banach space X. Let P be a minimal projection from X onto V. Then there exist at least two linearly independent norming points for P.*

Proof. Since uniformly convex spaces are reflexive and P is a compact operator and every compact operator attains its norm in a reflexive Banach space, P has at least one norming point. If $\|P\| = 1$ then the statement is obvious. Now, suppose that $\pm x_0 \in S(X)$ are the only norming points for P. From Lemma 1.1 there is a projection Q from X onto V such that

$$\|Q(\pm x_0)\| \le 1/2$$

and by continuity we can find $\varepsilon > 0$ such that

$$\|Q(x)\| < 1 \quad \text{for any } x \in B(x_0, \varepsilon) \cup B(-x_0, \varepsilon). \tag{1.3}$$

We now claim that there exists $\eta > 0$ such that

$$\|P(x)\| < \|P\| - \eta \quad \text{for any } x \notin B(x_0, \varepsilon) \cup B(-x_0, \varepsilon) \text{ and } x \in S(X). \tag{1.4}$$

Indeed, otherwise for any $1/n$ we can find $x_n \in S(X)$ such that $x_n \notin B(x_0, \varepsilon) \cup B(-x_0, \varepsilon)$ and $\|P(x_n)\| \to \|P\|$. The sequence $\{P(x_n)\}$ is contained in the two-dimensional space V, so choosing a subsequence if necessary, we can assume that $P(x_n) \to y_0$; since $\|P(x_n)\| \to \|P\|$, we have $\|y_0\| = \|P\|$. Since uniformly convex spaces have the Banach–Saks property (see [D, Theorem III.7.1]) and the sequence $\{x_n\}$ is bounded in norm, we can choose a subsequence $\{x_{n_k}\}$ whose arithmetic means converge in norm, i.e.,

$$y_k := \frac{x_{n_1} + \cdots + x_{n_k}}{k} \to y.$$

We will show that y is a norming point for P (of course $y \neq x_0$ and $y \neq -x_0$, hence a contradiction). First observe that since $\|x_n\| = 1$, we have $\|y_k\| \leq 1$, which implies $\|y\| \leq 1$. Now

$$P(y_k) = \frac{P(x_{n_1}) + \cdots + P(x_{n_k})}{k} \to P(y);$$

but $P(x_{n_k}) \to y_0$, hence also $P(y_k) \to y_0$. Therefore $\|y_0\| = \|P\|$ implies $\|P(y)\| = \|P\|$, so y is a norming point for P different from $\pm x_0$, contrary to the assumption that $\pm x_0$ are the only norming points for P.

Now for every $t \in (0,1)$ consider the projection

$$R_t = tQ + (1-t)P : X \to V.$$

If $x \in B(x_0, \varepsilon) \cup B(-x_0, \varepsilon)$ and $x \in S(X)$ then by (1.3),

$$\begin{aligned} \|R_t(x)\| &= \|tQ(x) + (1-t)P(x)\| \\ &\leq t\|Q(x)\| + (1-t)\|P(x)\| \\ &< t\|P\| + (1-t)\|P\| = \|P\|. \end{aligned} \tag{1.5}$$

If $x \notin B(x_0, \varepsilon) \cup B(-x_0, \varepsilon)$ and $x \in S(X)$ then by (1.4),

$$\begin{aligned} \|R_t(x)\| = \|tQ(x) + (1-t)P(x)\| &\leq t\|Q(x)\| + (1-t)\|P(x)\| \\ &< t\|Q\| + (1-t)(\|P\| - \eta) \\ &= t(\|Q\| - \|P\| + \eta) + (\|P\| - \eta), \end{aligned} \tag{1.6}$$

and the right hand side tends to $\|P\| - \eta$ as t tends to zero. Therefore for t_0 sufficiently small, by (1.5) and (1.6),

$$\|R_{t_0}\| < \|P\|,$$

which contradicts minimality of P. ∎

THEOREM 1.3. *Let V be a two-dimensional subspace of a smooth and uniformly convex space X. Then there is a unique minimal projection from X onto V.*

Proof. Assume that there are two different minimal projections, say P_1 and P_2. Then $Q = (P_1 + P_2)/2$ is also a minimal projection (since $\|Q\| \leq \|(P_1 + P_2)/2\| \leq (\|P_1\| + \|P_2\|)/2 \leq \lambda(V, X)$). Now take any $(f, x) \in \mathcal{E}(Q)$ (see Definition 0.4) and compute

$$\lambda(V,X) = f(Qx) = \tfrac{1}{2}f(P_1x) + \tfrac{1}{2}f(P_2x) \leq \tfrac{1}{2}\lambda(V,X) + \tfrac{1}{2}\lambda(V,X) = \lambda(V,X);$$

therefore, since $f(P_ix) \leq \|P_i\| = \lambda(V, X)$,

$$f(P_1x) = \lambda(V,X) = \|P_1\| \quad \text{and } f(P_2x) = \lambda(V,X) = \|P_2\|.$$

As a consequence we have

$$\mathcal{E}(Q) \subset \mathcal{E}(P_1), \quad \mathcal{E}(Q) \subset \mathcal{E}(P_2), \tag{1.7}$$

i.e., any norming pair for Q is also a norming pair for P_1 and P_2.

Since Q is a minimal projection, by Theorem 1.2, there are two linearly independent norming points x_1 and x_2 for Q. Let (f_1, x_1) and (f_2, x_2) be corresponding norming pairs for Q. Observe that

$$f_1|_{V^*}, f_2|_{V^*} \quad \text{are linearly independent.} \tag{1.8}$$

Indeed, if not then $f_1 = \pm f_2$ and $f_1(Qx_1) = f_1(Q(\pm x_2)) = \|Q\|$. Hence

$$f_1\left(Q\left(\frac{x_1 + (\pm x_2)}{2}\right)\right) = \|Q\|,$$

so $\|(x_1 + (\pm x_2))/2\| = 1$, which is not possible if X is strictly convex.

From (1.7),

$$f_i(P_1 x_i) = \|P_1\|, \quad f_i(P_2 x_i) = \|P_2\|.$$

Therefore

$$(P_1^* f_i)(x_i) = \|P_1\| = \lambda(V, X), \quad (P_2^* f_i)(x_i) = \|P_2\| = \lambda(V, X).$$

It now follows that $(P_1^* f_i)/\|P_1\|$ and $(P_2^* f_i)/\|P_2\|$ are two norming functionals for x_i. Since X is smooth they have to be equal. Hence

$$P_1^* f_i = P_2^* f_i,$$

and since the $f_i|_{V^*}$ span V^* ((1.8)) we have $P_1^* = P_2^*$. Hence $P_1 = P_2$. ∎

COROLLARY 1.4. *Let V be a two-dimensional subspace of $L_p(\mu)$ with $1 < p < \infty$. Then there is a unique minimal projection from $L_p(\mu)$ onto V (this covers both classical cases $L_p[0,1]$ and ℓ_p).*

2. Norming pairs.

It was seen in the previous section that there are at least two linearly independent norming points for a minimal projection onto a two-dimensional subspace. In this section we show that there are at least six norming points, altogether, for such a projection. We show, by means of an example, that the number six cannot be increased.

THEOREM 2.1. *A minimal projection from $L_p(\mu)$ (with $1 < p < \infty$) onto a two-dimensional subspace has at least six different norming functionals $\pm f_1, \pm f_2, \pm f_3$. Moreover the set of restrictions to V^* of these functionals, $\pm f_1|_{V^*}, \pm f_2|_{V^*}, \pm f_3|_{V^*}$, contains six different elements.*

Proof. By [OL, Theorem III.2.8 and Remark III.2.9], every set C such that

$$C \cup -C = \{\text{the set of norming functionals of } P \text{ restricted to } V^*\}$$

and

$$C \cap -C = \emptyset$$

is linearly dependent over V^*. But by Theorem 1.2 and reasoning as in the proof of Theorem 1.3 (see (1.8)) we have at least two norming functionals for P which are linearly independent over V^*. Hence C has to contain at least three elements. ∎

THEOREM 2.2. *A minimal projection from $L_p(\mu)$ (with $1 < p < \infty$) onto a two-dimensional subspace has at least six different norming points $\pm x_1, \pm x_2, \pm x_3$.*

Proof. By the previous theorem there are three norming functionals for P such that

$$f_1|_{V^*}, f_2|_{V^*}, f_3|_{V^*} \quad \text{are three different functionals.} \tag{2.1}$$

To these functionals correspond three norming points x_1, x_2, x_3. Let

$$g_i = \frac{f_i \circ P}{\|P\|}.$$

By (2.1), g_1, g_2, g_3 are three different functionals on X^* of norm one. Also

$$g_i(x_i) = 1.$$

Now if $x_i = x_j$ (for some $i \neq j \in \{1, 2, 3\}$) then g_i and g_j are norming functionals for the same point $x = x_i = x_j$. Since the $L_p(\mu)$ is smooth, that would imply $g_i = g_j$, a contradiction. Hence x_1, x_2, x_3 are all diferent. ∎

THEOREM 2.3. *Let P be a minimal projection from ℓ_p^3 onto a two-dimensional subspace V. Let $W = \{x \in \ell_p^n : (x_1, x_2, x_3) \in V$ and $x_4 = \cdots = x_n = 0\}$. Take a projection Q from ℓ_p^n onto W defined by*

$$Q(x_1, x_2, x_3, x_4, \ldots, x_n) = (P(x_1, x_2, x_3), 0, \ldots, 0).$$

Then Q is also a minimal projection having the same number of norming points and norming functionals as P.

Proof. By the very construction of Q, if $x = (x_1, \ldots, x_n)$ is a norming point for Q then $x_4 = \cdots = x_n = 0$. If $f = (f_1, \ldots, f_n)$ is a norming functional for Q then by the form of norming functionals (i.e., $f_i = \operatorname{sgn}(a_i) \cdot |a_i|^{p/q}$) and the form of Q we get $f_4 = \cdots = f_n = 0$. Hence $\|Q\| = \|P\|$, and moreover

$$x = (x_1, \ldots, x_n) \text{ is a norming point for } Q$$
$$\Updownarrow$$
$$x = (x_1, x_2, x_3) \text{ is a norming point for } P$$

and

$$f = (f_1, \ldots, f_n) \text{ is a norming functional for } Q$$
$$\Updownarrow$$
$$f = (f_1, f_2, f_3) \text{ is a norming functional for } P.$$

Since $L : \ell_p^n \to \ell_p^3$ given by $L(x_1, \ldots, x_n) = (x_1, x_2, x_3, 0, \ldots, 0)$ is a norm one projection, by [OL, Proposition I.3.1] the projection Q is also minimal. ∎

Now we will compute the norm, all norming points and all norming functionals for a particular minimal projection.

THEOREM 2.4. *Let $f = (1,1,1) \in \ell_q^3$ be a representation of a functional. Then $P : \ell_p^3 \to \ker f$ given by*

$$P = \mathrm{Id} - \tfrac{1}{3}(1,1,1) \otimes (1,1,1) \tag{2.2}$$

is a minimal projection for any $1 \le p \le \infty$.

Proof. First we will prove that P given by (2.2) is indeed a minimal projection. We will use the Rudin Theorem. Observe that the operators

$$L_\sigma(x_1, x_2, x_3) := (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \tag{2.3}$$

(where σ is any permutation of $\{1,2,3\}$) are isometries in ℓ_p^3. Furthermore

$$L_\sigma(\ker(1,1,1)) \subset \ker(1,1,1). \tag{2.4}$$

Now according to Theorem 0.7 it is enough to prove that P is the only projection which commutes with L_σ.

Any projection $Q : \ell_p^3 \to \ker(1,1,1)$ is given by

$$Qx = x - (1,1,1) \otimes (v_1, v_2, v_3), \quad \text{where} \quad v_1 + v_2 + v_3 = 1. \tag{2.5}$$

If Q commutes with L_σ, then

$$((1,1,1) \otimes (v_1,v_2,v_3)) \circ L_\sigma = L_\sigma((1,1,1) \otimes (v_1,v_2,v_3)).$$

Taking the value at $x = (x_1, x_2, x_3)$ on both sides of the above equality results in

$$\Big(\sum_{i=1}^3 x_i, \sum_{i=1}^3 x_i, \sum_{i=1}^3 x_i\Big) \cdot (v_1, v_2, v_3) = \Big(\sum_{i=1}^3 x_i, \sum_{i=1}^3 x_i, \sum_{i=1}^3 x_i\Big) \cdot (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}),$$

for any $\sigma \in S_3$ and any $x = (x_1, x_2, x_3)$. Therefore $v_1 = v_2 = v_3$ and since $v_1 + v_2 + v_3 = 1$ we have

$$v_1 = v_2 = v_3 = 1/3.$$

Hence $Q = P$. On the other hand, it is easy to see that P indeed commutes with L_σ. Therefore P is minimal. ∎

Now we will restrict ourselves to $p = 4$.

THEOREM 2.5. *Let $p = 4$. Then the minimal projection from Theorem* 2.4 (*see* (2.2)) *has exactly six norming points*

$$
(2.6)\qquad
\begin{aligned}
x_0 &= \frac{1}{(2+2^{4/3})^{1/4}}\,(2^{1/3},-1,-1), &\quad x_3 &= -x_0,\\
x_1 &= \frac{1}{(2+2^{4/3})^{1/4}}\,(-1,2^{1/3},-1), &\quad x_4 &= -x_1,\\
x_2 &= \frac{1}{(2+2^{4/3})^{1/4}}\,(-1,-1,2^{1/3}), &\quad x_5 &= -x_2,
\end{aligned}
$$

and exactly six norming functionals

$$
(2.7)\qquad
\begin{aligned}
f_0 &= \frac{1}{(2+2^{4})^{3/4}}\,(2^{3},-1,-1), &\quad f_3 &= -f_0,\\
f_1 &= \frac{1}{(2+2^{4})^{3/4}}\,(-1,2^{3},-1), &\quad f_4 &= -f_1,\\
f_2 &= \frac{1}{(2+2^{4})^{3/4}}\,(-1,-1,2^{3}), &\quad f_5 &= -f_3.
\end{aligned}
$$

Moreover,

$$
(2.8)\qquad \|P\| = \lambda(\ker(1,1,1),\ell_p^3) = \tfrac{1}{3}(1+2^3)^{1/4}(1+2^{1/3})^{3/4}.
$$

Proof. The projection P from (2.2) is given by

$$
P(x_1,x_2,x_3) = \tfrac{1}{3}(2x_1-x_2-x_3,-x_1+2x_2-x_3,-x_1-x_2+2x_3),
$$

therefore the problem of finding its norm and all norming points is equivalent to finding the maximum of the function

$$
(2.9)\qquad h(x_1,x_2,x_3) = \left(\frac{2x_1-x_2-x_3}{3}\right)^4 + \left(\frac{-x_1+2x_2-x_3}{3}\right)^4 + \left(\frac{-x_1-x_2+2x_3}{3}\right)^4
$$

in the set $x_1^4+x_2^4+x_3^4=1$, and finding all points at which this maximum is attained.

Let

$$
(2.10)\qquad
\begin{aligned}
z_1 &= \frac{2x_1-x_2-x_3}{3}, &\qquad z_2 &= \frac{-x_1+2x_2-x_3}{3},\\
z_3 &= \frac{-x_1-x_2+2x_3}{3}, &\qquad d &= \frac{x_1+x_2+x_3}{3}.
\end{aligned}
$$

Then the above problem is equivalent to finding the maximum and all points at which this maximum is attained for the following function:

$$
(2.11)\qquad f(z_1,z_2,z_3,d) = z_1^4+z_2^4+z_3^4
$$

in the set

$$
(z_1+d)^4+(z_2+d)^4+(z_3+d)^4 = 1 \quad\text{and}\quad z_1+z_2+z_3=0.
$$

Using the standard Lagrange multiplier method we construct the function

$$\begin{aligned}\varphi(z_1,z_2,z_3,d) = z_1^4+z_2^4+z_3^4 \\ -\lambda_1((z_1+d)^4+(z_2+d)^4+(z_3+d)^4) \\ -\lambda_2(z_1+z_2+z_3)\end{aligned}$$

and in particular we find that z_1,z_2,z_3 have to satisfy the equations

$$g(z_1)=g(z_2)=g(z_3)=0, \quad \text{where} \quad g(x)=4x^3-4\lambda_1(x+d)^3-\lambda_2. \tag{2.12}$$

Now assume that z_1,z_2,z_3 are distinct. Then by (2.12), z_1,z_2,z_3 will be three distinct zeros of g. That implies $\lambda_1\neq 0$ (in that case g has only one zero), $\lambda_1\neq 1$ (in that case g is a polynomial of degree 2, hence has at most two zeros) and

$$g(x)=(4-4\lambda_1)(x-z_1)(x-z_2)(x-z_3). \tag{2.13}$$

Now comparing the coefficients of g in (2.12) and (2.13) gives

$$z_1+z_2+z_3=\frac{3\lambda_1 d}{1-\lambda_1}.$$

On the other hand, $z_1+z_2+z_3=0$, hence $d=0$. But clearly a four-tuple $(z_1,z_2,z_3,0)$ is not a maximum point of the function (2.11) since $f(z_1,z_2,z_3,0)=1$. Thus we proved that

$$z_1=z_2 \quad \text{or} \quad z_2=z_3 \quad \text{or} \quad z_3=z_1,$$

which is equivalent to

$$x_1=x_2 \quad \text{or} \quad x_2=x_3 \quad \text{or} \quad x_3=x_1. \tag{2.14}$$

By symmetry it is enough to let $x_2=x_3$. Letting $x_2=x_3$ in (2.9) we have to find the maximum (and all points at which this maximum is attained) of the function

$$h(x_1,x_2)=\frac{2+2^4}{3^4}(x_1-x_2)^4 \quad \text{in the set} \quad x_1^4+2x_2^4=1.$$

This can be easily solved by using Lagrange multipliers and together with (2.14) it leads to (2.6). Note that (2.7) follows immediately from (2.6). ∎

Using Theorems 2.3 and 2.5 we obtain

COROLLARY 2.6. *For any ℓ_4^n we can construct a two-dimensional subspace V of ℓ_4^n such that the minimal projection P from ℓ_4^n onto V has only six norming functionals and six norming points.*

REMARK 2.7. Theorem 2.2 is not true if the word "minimal" is dropped: we can easily find a projection (not minimal of course) which has only two different norming points. For instance,

$$Q=\mathrm{Id}-(1,1,1)\otimes(0,0,1)$$

has only two norming points $\pm(1/2^{1/p},1/2^{1/p},0)$.

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