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The growth speed of digits in infinite iterated function systems

by

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Abstract. Let $\{f_n\}_{n\geq 1}$ be an infinite iterated function system on $[0,1]$ satisfying the open set condition with the open set $(0,1)$ and let Λ be its attractor. Then to any $x \in \Lambda$ (except at most countably many points) corresponds a unique sequence $\{a_n(x)\}_{n\geq 1}$ of integers, called the digit sequence of x, such that

$$x = \lim_{n\to\infty} f_{a_1(x)} \circ \cdots \circ f_{a_n(x)}(1).$$

We investigate the growth speed of the digits in a general infinite iterated function system. More precisely, we determine the dimension of the set

$$\Big\{x \in \Lambda : a_n(x) \in B \ (\forall n \geq 1), \lim_{n\to\infty} a_n(x) = \infty\Big\}$$

for any infinite subset $B \subset \mathbb{N}$, a question posed by Hirst for continued fractions. Also we generalize Łuczak's work on the dimension of the set

$$\{x \in \Lambda : a_n(x) \geq a^{b^n} \text{ for infinitely many } n \in \mathbb{N}\}$$

with $a, b > 1$. We will see that the dimension of the sets above is tightly connected with the convergence exponent of the contraction ratios of the sequence $\{f_n\}_{n\geq 1}$.

1. Introduction. We first recall the definition of an infinite iterated function system. For a thorough study and foundations of the theory of infinite iterated function systems, one is referred to the works of Hanus, Mauldin, and Urbański [HaMU], Mauldin and Urbański [MU1, MU2] or their monograph [MU3]. Here we follow the notation used in [JorR] by Jordan and Rams.

Let $\{f_n\}_{n\geq 1}$ be a sequence of functions with $f_n : [0,1] \to [0,1]$ satisfying

(i) *Smoothness:* $f_n \in C^1$ for each $n \geq 1$;

(ii) *Contraction property:* there exists an integer m and a real number $0 < \rho < 1$ such that for any $(a_1, \ldots, a_m) \in \mathbb{N}^m$ and $x \in [0,1]$,

$$0 < |(f_{a_1} \circ \cdots \circ f_{a_m})'(x)| \leq \rho < 1;$$

(iii) *Open set condition:* for any $i \neq j \in \mathbb{N}$, $f_i((0,1)) \cap f_j((0,1)) = \emptyset$.

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Then we call $([0,1],\{f_n\}_{n\ge 1})$ or simply $\{f_n\}_{n\ge 1}$ an *infinite iterated function system* (iIFS). Clearly, there is a natural projection $\Pi : \mathbb{N}^{\mathbb{N}} \to [0,1]$ defined as

$$\Pi(\underline{a}) = \lim_{n\to\infty} f_{a_1} \circ \cdots \circ f_{a_n}(1)$$

for any $\underline{a} = \{a_n\}_{n\ge 1} \in \mathbb{N}^{\mathbb{N}}$. Let Λ be the attractor of the iIFS $\{f_n\}_{n\ge 1}$, i.e.,

$$\Lambda = \Pi(\mathbb{N}^{\mathbb{N}}).$$

To each $x \in \Lambda$ corresponds a sequence $\{a_n\}_{n\ge 1}$ of integers such that

$$x = \lim_{n\to\infty} f_{a_1} \circ \cdots \circ f_{a_n}(1).$$

We call $\{a_n\}_{n\ge 1}$ the *digit sequence* of x. It should be pointed out that the digit sequence of a point may not be unique. However, the open set condition (iii) guarantees that the points having more than one digit sequence are at most countably many. In the following, since we are concerned with the Hausdorff dimension, a countable set is negligible. Thus we can assume that to each $x \in \Lambda$ is attached a unique digit sequence, denoted by $\{a_n(x)\}_{n\ge 1}$.

Now we limit ourselves to the iIFS with some regularity properties.

(iv) *Regularity:* there exists a sequence $\{\xi_n\}_{n\ge 1}$ such that for any $\epsilon > 0$ there are $0 < c_1(\epsilon) \le 1 \le c_2(\epsilon)$ such that for all $n \in \mathbb{N}$ and $x \in [0,1]$,

$$\frac{c_1(\epsilon)}{\xi_n^{1+\epsilon}} \le |f_n'(x)| \le \frac{c_2(\epsilon)}{\xi_n^{1-\epsilon}}.$$

The system $\{f_n\}_{n\ge 1}$ satisfying conditions (i) to (iv) is named a *ξ-regular iIFS*. Moreover, it is called *Gauss-like* if

(v) $\overline{\bigcup_{n=1}^{\infty} f_n([0,1])} = [0,1]$, and $f_i(x) > f_j(x)$ whenever $i < j$.

We list two classical infinite iterated function systems closely connected with number theory:

- Continued fractions:
$$f_n(x) = \frac{1}{x+n}, \qquad x \in [0,1],\ n \in \mathbb{N}.$$
Then the digit sequence $\{a_n(x)\}_{n\ge 1}$ is just the partial quotients of x in its continued fraction expansion.
- Lüroth expansions:
$$f_n(x) = \frac{x}{n(n+1)} + \frac{1}{n+1}, \qquad x \in (0,1],\ n \in \mathbb{N}.$$
Then $\{a_n(x)\}_{n\ge 1}$ is just the digit sequence in the Lüroth series expansion of x.

When $\xi_n = n^{-d}$, a ξ-regular iIFS $\{f_n\}_{n\ge 1}$ is referred to as a *d-decaying system* by Jordan and Rams [JorR]. So both continued fractions and Lüroth expansion are 2-decaying systems.

Since these systems are *infinite* iterated function systems, unlike the finite case, the digit sequence $\{a_n(x)\}_{n\geq 1}$ can assume large values. Thus one wonders which growth speed the digit can have.

In the case of continued fractions, in 1941, Good [Go] studied the size of the set

$$\mathbb{E}_\infty := \{x \in [0,1) : a_n(x) \to \infty \text{ as } n \to \infty\}$$

and showed that its Hausdorff dimension (denoted by $\dim_{\mathrm{H}}$) is $1/2$. Hirst [Hir2] asked what happens when the partial quotients of points in $\mathbb{E}_\infty$ are further restricted to an infinite subset $B \subset \mathbb{N}$ of natural numbers, i.e., what is the dimension of the set

$$\{x \in [0,1) : a_n(x) \in B (n \geq 1) \text{ and } a_n(x) \to \infty \text{ as } n \to \infty\}.$$

(This is solved by the second and third named authors in [WanW, Wu].)

Besides the points with mild growth speed of their partial quotients, the points with much larger partial quotients are also paid much attention to. For any $a, b > 1$, define

$$\mathbb{E}(a,b) = \{x \in [0,1] : a_n(x) \geq a^{b^n} \text{ for infinitely many } n \in \mathbb{N}\}.$$

As one can see in [FengW], bounding $\dim_{\mathrm{H}} \mathbb{E}(a,b)$ from above constitutes the difficult part in getting the explicit dimension of $\mathbb{E}(a,b)$. After partial progress by Cusick [Cu], Hirst [Hir1], Moorthy [Mo] etc., this was solved by Łuczak [Lu] in 1997 by showing that

$$\dim_{\mathrm{H}} \mathbb{E}(a,b) \leq \frac{1}{1+b},$$

which is the exact dimension of $\mathbb{E}(a,b)$ by combining this bound with the main result in [FengW].

In the case of Lüroth expansions, Munday [Mun] showed that

$$\dim_{\mathrm{H}} \Big\{x \in (0,1] : \lim_{n\to\infty} a_n(x) = \infty\Big\} = 1/2.$$

(Munday studied the dimension of the above set in a more general setting than Lüroth expansion, called α-Lüroth expansion.)

For general d-decaying systems, Jordan and Rams [JorR] considered the dimension of the sets of points with increasing digits, i.e.

$$X_\Phi := \Pi\{\underline{a} \in \mathbb{N}^{\mathbb{N}} : a_{n+1} > \Phi(a_n) \text{ for all } n \in \mathbb{N}\}, \tag{1.1}$$

where $\Phi(n) \geq n$ for any $n \in \mathbb{N}$. They proved

THEOREM A (Jordan & Rams [JorR]).

- *When* $n \leq \Phi(n) \leq \beta n$ *for some* $\beta \geq 1$, $\dim_{\mathrm{H}} X_\Phi = 1/d$.
- *If* $\{f_n\}_{n\geq 1}$ *is a d-decaying Gauss-like system and* $\Phi(n) = n^\alpha$ *for some* $\alpha > 1$, *then* $\dim_{\mathrm{H}} X_\Phi = \frac{1}{1+\alpha(d-1)}$.

- *For any $d > 1$ and any strictly increasing function $\Phi : \mathbb{N} \to \mathbb{N}$ there exists a d-decaying system $\{f_n\}_{n\geq 1}$ such that* $\dim_{\mathrm{H}} X_\Phi = 1/d$.

In this note, we consider the dimension of the set defined by Hirst in the setting of a ξ-regular infinite iterated function system. At the same time, we generalize Łuczak's result to Gauss-like systems. We will see that the convergence exponent

$$s_0(B) = \inf\left\{ s \geq 0 : \sum_{n\in B} \frac{1}{\xi_n^s} < \infty \right\} \quad \text{with } B \subset \mathbb{N}$$

plays an important role in the dimension of the sets above. In fact, $1/2$ and $1/d$ are just the convergence exponents for the continued fractions and the d-decaying systems. We also mention that besides the sets above, the convergence exponent also plays part role in the multifractal analysis of Birkhoff averages in infinite iterated function systems (see [FJLR, FLM, KMS, LMW]). The attractor of an iIFS and its subsets considered here are like the Moran constructions but with countably many branches. For more about the Moran set construction, one is referred to a survey about Moran sets by Wen [Wen] and a recent work of Rempe-Gillen and Urbański [ReU].

Now we state the main results of this note.

Theorem 1.1. *Suppose that $\{f_n\}_{n\geq 1}$ is a ξ-regular iIFS. Let B be an infinite subset of natural numbers. Define*

$$\mathbb{E}(B) = \{x \in \Lambda : a_n(x) \in B\,(n \geq 1) \text{ and } a_n(x) \to \infty \text{ as } n \to \infty\}.$$

Then $\dim_{\mathrm{H}} \mathbb{E}(B) = s_0(B)$.

Theorem 1.2. *Suppose that $\{f_n\}_{n\geq 1}$ is a Gauss-like system and $\{\xi_n\}_{n\geq 1}$ is increasing. For any $a, b > 1$, let*

$$\mathbb{E}(a,b) = \{x \in \Lambda : a_n(x) \geq a^{b^n} \text{ for infinitely many } n \in \mathbb{N}\},$$
$$\tilde{\mathbb{E}}(a,b) = \{x \in \Lambda : a_n(x) \geq a^{b^n} \text{ for all } n \in \mathbb{N}\}.$$

Then

$$\dim_{\mathrm{H}} \tilde{\mathbb{E}}(a,b) \leq \dim_{\mathrm{H}} \mathbb{E}(a,b) \leq \frac{s_0(\mathbb{N})}{s_0(\mathbb{N}) + b(1 - s_0(\mathbb{N}))}. \tag{1.2}$$

For the lower bound of the dimension of $\tilde{\mathbb{E}}(a,b)$, two concrete systems are provided in Section 3 indicating that even if they share the same convergence exponent, the dimension of $\tilde{\mathbb{E}}(a,b)$ in different systems may be different. But on the other hand, there do exist cases such that the inequality in (1.2) is an equality, for example, for continued fractions [FengW] and d-decaying Gauss-like systems, by combining Theorem 1.2 above and the second result in Theorem A.

In [JorR] (see the third result in Theorem A), it was pointed out that if $\{f_n\}_{n\geq 1}$ is not Gauss-like, the dimension of $\tilde{\mathbb{E}}(a,b)$ may exceed the upper bound above. The results in Theorems 1.1 and 1.2 are also sharp in the following sense:

THEOREM 1.3. *Suppose that $\{f_n\}_{n\geq 1}$ is a Gauss-like system and $\{\xi_n\}_{n\geq 1}$ is increasing. For any function $\psi : \mathbb{N} \to \mathbb{R}^+$ with $\psi(n) \to \infty$ as $n \to \infty$, there exists $B \subset \mathbb{N}$ such that*

$$s_0(B) = s_0(\mathbb{N}) \quad \textit{but} \quad \dim_{\mathrm{H}} \mathbb{E}(B, \psi) = 0,$$

where $\mathbb{E}(B, \psi) = \{x \in \Lambda : a_n(x) \in B$ and $a_n(x) \geq \psi(n)$ for all $n \in \mathbb{N}\}$.

We end this note by a result indicating that Theorem A cannot be extended to general Gauss-like systems without d-decaying assumptions.

THEOREM 1.4. *For any function $\psi : \mathbb{N} \to \mathbb{R}^+$ with $\psi(n) \to \infty$ as $n \to \infty$ and $s_0 \in [0, 1]$, there exists a Gauss-like system $\{f_n\}_{n\geq 1}$ such that*

$$s_0(\mathbb{N}) = s_0 \quad \textit{but} \quad \dim_{\mathrm{H}} \mathbb{E}(\psi) = 0,$$

where $\mathbb{E}(\psi) = \{x \in \Lambda : a_n(x) \geq \psi(n)$ for all $n \in \mathbb{N}\}$.

If Theorem A is true for Gauss-like systems without d-decaying assumptions, one would have the dimension of X_Φ defined in (1.1) equal to s_0 if one takes $\Phi(n) = n$. However, if we let $\psi(n) = n/2$, it is clear that $X_\Phi \subset E(\psi)$, which gives $\dim_{\mathrm{H}} E(\psi) \geq s_0$, contradicting $\dim_{\mathrm{H}} E(\psi) = 0$.

2. Proof of Theorem 1.1. We begin with some notation. For each $a_1, \ldots, a_n \in \mathbb{N}$, define

$$I_n(a_1, \ldots, a_n) = \{x \in \Lambda : a_k(x) = a_k,\ 1 \leq k \leq n\},$$

i.e., the collection of points whose digit sequence begins with $a_1, \ldots, a_n$. We call $I_n(a_1, \ldots, a_n)$ an nth *order basic interval.*

Since $f_k \in C^1$ for each $k \geq 1$, it follows that the length of an nth order basic interval satisfies

$$\prod_{k=1}^{n} \frac{c_1(\epsilon)}{\xi_{a_k}^{1+\epsilon}} \leq |I_n(a_1, \ldots, a_n)| \leq \prod_{k=1}^{n} \frac{c_2(\epsilon)}{\xi_{a_k}^{1-\epsilon}}. \tag{2.1}$$

This estimate is essential to all the arguments below.

The proof of Theorem 1.1 is divided into two parts, for the upper bound and the lower bound.

2.1. Upper bound. Recall the set

$$\mathbb{E}(B) = \{x \in \Lambda : a_n(x) \in B\ (n \geq 1) \text{ and } a_n(x) \to \infty \text{ as } n \to \infty\}.$$

Write $B = \{b_1, b_2, \ldots\}$. The upper bound of $\dim_{\mathrm{H}} \mathbb{E}(B)$ can be determined by a standard covering argument.

Fix $M \in \mathbb{N}$. For any $x \in \mathbb{E}(B)$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, $a_n(x) \geq M$. Thus

$$\mathbb{E}(B) \subset \bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \{x \in \Lambda : a_n(x) \in B,\, a_n(x) \geq M,\, \forall n > N\}.$$

If we write, for each $a_1, \ldots, a_N \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}_M(B; a_1, \ldots, a_N) = \{x \in \Lambda : a_k(x) = a_k,\, 1 \leq k \leq N, \\ a_n(x) \in B,\, a_n(x) \geq M,\, \forall n > N\},\end{aligned}$$

then

$$\mathbb{E}(B) \subset \bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcup_{a_1,\ldots,a_N \in \mathbb{N}} \mathbb{E}_M(B; a_1, \ldots, a_N).$$

The desired result on the dimension of $\mathbb{E}(B)$ will follow if we can show that, for any $\epsilon > 0$,

$$\dim_{\mathrm{H}} \mathbb{E}_M(B; a_1, \ldots, a_N) \leq \frac{s_0(B)(1+\epsilon)}{1-\epsilon} =: s \tag{2.2}$$

when M is large enough.

Now fix $\epsilon > 0$. Recall that $c_1(\epsilon), c_2(\epsilon)$ appear in the definition of a ξ-regular iIFS. By the definition of $s_0 = s_0(B)$, the convergence exponent of $\{\xi_b\}_{b \in B}$, one has

$$\sum_{b \in B} 1/\xi_b^{s_0(1+\epsilon)} < \infty.$$

So, one can choose an integer $M_0 = M_0(\epsilon)$ sufficiently large ensuring that

$$\sum_{b \in B,\, b \geq M_0} \frac{c_2(\epsilon)}{\xi_b^{s_0(1+\epsilon)}} \leq 1. \tag{2.3}$$

To show (2.2), we search for a cover of $\mathbb{E}_M(B; a_1, \ldots, a_N)$. In fact, its natural cover is sufficient to get (2.2). More precisely, for each $n \geq N$, the family of basic intervals

$$\{I_n(a_1, \ldots, a_N, a_{N+1}, \ldots, a_n) : a_k \in B,\, a_k \geq M,\, N < k \leq n\}$$

covers $\mathbb{E}_M(B; a_1, \ldots, a_N)$. Hence the s-dimensional Hausdorff measure of $\mathbb{E}_M(B; a_1, \ldots, a_N)$ can be estimated as

$$\begin{aligned}\mathcal{H}^s(\mathbb{E}_M(B; a_1, \ldots, a_N)) &\leq \liminf_{n\to\infty} \sum_{a_k \in B,\, a_k \geq M,\, N<k\leq n} |I_n(a_1, \ldots, a_n)|^s \\ &\leq \liminf_{n\to\infty} \sum_{a_k \in B,\, a_k \geq M,\, N<k\leq n} \prod_{k=1}^{n} \left(\frac{c_2(\epsilon)}{\xi_{a_k}^{1-\epsilon}}\right)^s \\ &\leq \prod_{k=1}^{N} \left(\frac{c_2(\epsilon)}{\xi_{a_k}^{1-\epsilon}}\right)^s \liminf_{n\to\infty} \left(\sum_{b \in B,\, b \geq M} \frac{c_2(\epsilon)}{\xi_b^{s_0(1+\epsilon)}}\right)^{n-N}.\end{aligned}$$

By (2.3), when $M \geq M_0$, we have

$$\mathcal{H}^s(\mathbb{E}_M(B; a_1, \ldots, a_N)) \leq \prod_{k=1}^{N} \left(\frac{c_2(\epsilon)}{\xi_{a_k}^{1-\epsilon}} \right)^s < \infty.$$

This implies directly that, when $M \geq M_0$,

$$\dim_{\mathrm{H}} \mathbb{E}_M(B; a_1, \ldots, a_N) \leq \frac{s_0(1+\epsilon)}{1-\epsilon}.$$

2.2. Lower bound. From now on until the end of this note, for simplicity of notation, we assume that $m = 1$ in (ii) of the definition of an iIFS, i.e., for each $n \geq 1$ and $x \in [0, 1]$,

$$|f_n'(x)| \leq \rho < 1.$$

Thus there is a natural bound on the length of a basic interval:

$$|I_n(a_1, \ldots, a_n)| \leq \rho^n. \tag{2.4}$$

For the lower bound of $\dim_{\mathrm{H}} \mathbb{E}(B)$, we will construct a Cantor subset consisting of points whose digits grow to infinity as slow as a "snail".

First, we give some preliminaries. Without loss of generality, we assume that $s_0(B) > 0$. Fix $\epsilon > 0$. Then we fix a real $s > 0$ such that

$$s(1+\epsilon)^2 < s_0(B). \tag{2.5}$$

Now we define a sequence $\{r_k\}_{k \geq 1}$ of integers. For each $k \geq 1$, choose r_k to be the smallest integer such that

$$\sum_{n=k}^{r_k} \left(\frac{c_1(\epsilon)}{\xi_{b_n}^{1+\epsilon}} \right)^{s(1+\epsilon)} \geq 3.$$

After the sequence $\{r_k\}_{k \geq 1}$ has been defined, we choose another integer sequence $\{n_k\}_{k \geq 1}$ as follows. For each $k \geq 1$, choose an integer n_k so large that

$$\max\{(\xi_{b_i}\xi_{b_j})^{1+\epsilon} : k \leq i \leq r_{k+1},\ k \leq j \leq r_{k+2}\} \leq \rho^{-\epsilon n_k} c_1(\epsilon)^2. \tag{2.6}$$

Then we define two sequences $\{u_n, v_n\}_{n \geq 1}$ of integers with $u_n \to \infty$ as $n \to \infty$ in a very slow manner:

$$\begin{cases} u_n = 1,\ v_n = r_1 & \text{when } 1 \leq n < n_1; \\ u_n = k,\ v_n = r_k & \text{when } n_k \leq n < n_{k+1}. \end{cases} \tag{2.7}$$

Such design ensures the following

PROPOSITION 2.1. *For every $(a_1, \ldots, a_{n+1}, a_{n+2}) \in B^{n+2}$ with $b_{u_{n+1}} \leq a_{n+1} \leq b_{v_{n+1}}$, $b_{u_{n+2}} \leq a_{n+2} \leq b_{v_{n+2}}$ and $n_k \leq n < n_{k+1}$, one has*

$$|I_{n+2}(a_1, \ldots, a_{n+1}, a_{n+2})| \geq |I_n(a_1, \ldots, a_n)|^{1+\epsilon}. \tag{2.8}$$

Proof. It is clear that when $n_k \le n < n_{k+1}$ and $a_{n+1}, a_{n+2} \in B$,

$$\begin{aligned} a_{n+1} &= b_i \quad \text{for some } k \le i \le r_{k+1}, \\ a_{n+2} &= b_j \quad \text{for some } k \le j \le r_{k+2}. \end{aligned}$$

Thus we have

$$\begin{aligned} |I_{n+2}(a_1, \ldots, a_{n+1}, a_{n+2})| &\ge |I_n(a_1, \ldots, a_n)| \cdot \frac{c_1(\epsilon)}{\xi_{b_i}^{1+\epsilon}} \cdot \frac{c_1(\epsilon)}{\xi_{b_j}^{1+\epsilon}} \\ &\ge |I_n(a_1, \ldots, a_n)| \cdot \rho^{\epsilon n_k} \qquad \text{(by (2.6))}. \end{aligned}$$

Then the desired result follows from (2.4). ∎

Now we construct a Cantor subset of $\mathbb{E}(B)$ as

$$\overline{\mathbb{F}} = \{x \in \Lambda : a_n(x) \in B \text{ and } b_{u_n} \le a_n(x) \le b_{v_n} \text{ for all } n \ge 1\}.$$

By (2.7), we know that $\overline{\mathbb{F}} \subset \mathbb{E}(B)$ and the digits of points in $\overline{\mathbb{F}}$ grow to infinity but as slow as one wishes.

To allow some gap between those basic intervals with nonempty intersection with $\overline{\mathbb{F}}$, we refine it to another Cantor subset $\mathbb{F}$ of $\mathbb{E}(B)$ as follows.

- *The first level of* $\mathbb{F}$: The first level $\mathbb{F}_1$ of the Cantor set $\mathbb{F}$ is defined as

$$\mathbb{F}_1 = \{I_1(a_1) : b_{u_1} \le a_1 \le b_{v_1},\ a_1 \in B\}.$$

- *The second level of* $\mathbb{F}$: Level 2 is composed of sublevels for each $I_1(a_1)$ in $\mathbb{F}_1$. Fix $I_1(a_1)$ in $\mathbb{F}_1$. The sublevel $\mathbb{F}_2(I_1(a_1))$ is constructed as follows. Define

$$\overline{\mathbb{F}}_2(I_1(a_1)) = \{I_2(a_1, a_2) : b_{u_2} \le a_2 \le b_{v_2},\ a_2 \in B\}.$$

  Now $\mathbb{F}_2(I_1(a_1))$ is defined by eliminating from $\overline{\mathbb{F}}_2(I_1(a_1))$ the leftmost and the rightmost elements of $I_1(a_1)$. Then the second level $\mathbb{F}_2$ of the Cantor set $\mathbb{F}$ is defined as

$$\mathbb{F}_2 = \bigcup_{I_1(a_1) \in \mathbb{F}_1} \mathbb{F}_2(I_1(a_1)).$$

- *From the nth level to the* $(n+1)$*th level*: Suppose that the nth level $\mathbb{F}_n$ of the Cantor set $\mathbb{F}$ has been defined, and it is a collection of basic intervals of order n. Now we follow the construction procedure of the second level to obtain the $(n+1)$th level. For each $I_n(a_1, \ldots, a_n) \in \mathbb{F}_n$, define

$$\begin{aligned} \overline{\mathbb{F}}_{n+1}(I_n(a_1, \ldots, a_n)) = \{I_{n+1}(a_1, \ldots, a_n, a_{n+1}) :& \\ b_{u_{n+1}} \le a_{n+1} \le b_{v_{n+1}},\ a_{n+1} \in B\}&. \end{aligned}$$

  Then the sublevel $\mathbb{F}_{n+1}(I_n(a_1, \ldots, a_n))$ is defined by eliminating from $\overline{\mathbb{F}}_{n+1}(I_n(a_1, \ldots, a_n))$ the leftmost and the rightmost elements of

$I_n(a_1, \ldots, a_n)$. Then the $(n+1)$th level $\mathbb{F}_{n+1}$ of the Cantor set $\mathbb{F}$ is defined as

$$\mathbb{F}_{n+1} = \bigcup_{I_n(a_1,\ldots,a_n) \in \mathbb{F}_n} \mathbb{F}_{n+1}(I_n(a_1, \ldots, a_n)).$$

Then we have

$$\mathbb{F} = \bigcap_{n \geq 1} \bigcup_{I_n \in \mathbb{F}_n} I_n.$$

For each $n \in \mathbb{N}$, an element in $\mathbb{F}_n$ is called a *cylinder* of order n.

Now we show that $\dim_{\mathrm{H}} \mathbb{F} \geq s$, where s is given in (2.5). This is done by showing that for any given covering system $\mathcal{U}$ of $\mathbb{F}$, we always have

$$\sum_{U \in \mathcal{U}} |U|^s \geq 1. \tag{2.9}$$

STEP 1. By the property of the Hausdorff measure in $\mathbb{R}^1$, we need only consider covers by open intervals. Since $\mathbb{F}$ is compact, the covering $\mathcal{U}$ can be assumed to contain only finitely many open intervals. Then add the endpoints to each element in $\mathcal{U}$ to make them closed intervals. Moreover, these intervals can be shortened to make their endpoints belong to $\mathbb{F}$.

In this way, we get a new cover of $\mathbb{F}$. But in the process, the sum in (2.9) is not increased. So we still denote by $\mathcal{U}$ the cover after the above modifications.

STEP 2. Now, we argue that the covering system $\mathcal{U}$ can be transformed to a cover by cylinders.

For each $U \in \mathcal{U}$ with nonempty intersection with $\mathbb{F}$, let n be the largest integer such that U intersects only one cylinder $I_n \in \mathbb{F}_n$. Thus, there will exist $a_1, \ldots, a_n \in B$ and $b_{u_{n+1}} \leq \ell \neq r \leq b_{v_{n+1}}$, $\ell, r \in B$ such that

$$I_{n+1}(a_1, \ldots, a_n, \ell) \in \mathbb{F}_{n+1}, \quad I_{n+1}(a_1, \ldots, a_n, r) \in \mathbb{F}_{n+1}$$

and

$$U \cap I_{n+1}(a_1, \ldots, a_n, \ell) \neq \emptyset, \quad U \cap I_{n+1}(a_1, \ldots, a_n, r) \neq \emptyset.$$

Since the endpoints of U are in $\mathbb{F}$, they are also in $\mathbb{F}_{n+2}$. Thus, the length of U is larger than the gap between

$$I_{n+1}(a_1, \ldots, a_n, \ell) \cap \mathbb{F}_{n+2} \quad \text{and} \quad I_{n+1}(a_1, \ldots, a_n, r) \cap \mathbb{F}_{n+2}.$$

Notice that when defining $\mathbb{F}_{n+2}(I_{n+1})$ from $\overline{\mathbb{F}}_{n+2}(I_{n+1})$, we eliminate the two cylinders lying in the edge of $\overline{\mathbb{F}}_{n+2}(I_{n+1})$. Thus, the gap must be larger than one of the four deleted cylinders in $\overline{\mathbb{F}}_{n+2}(I_{n+1}(a_1, \ldots, a_n, \ell))$ and $\overline{\mathbb{F}}_{n+2}(I_{n+1}(a_1, \ldots, a_n, r))$. It follows that

$$|U| \geq |I_{n+2}(a_1, \ldots, a_n, a_{n+1}, a_{n+2})|$$

for some $b_{u_{n+1}} \le a_{n+1} \le b_{v_{n+1}}$, $b_{u_{n+2}} \le a_{n+2} \le b_{v_{n+2}}$. By (2.8), we get

$$|U| \ge |I_n(a_1, \dots, a_n)|^{1+\epsilon}.$$

Denote by $\mathcal{V}$ the collection of I_n's corresponding to all $U \in \mathcal{U}$. Assume $\mathcal{V}$ is maximal in the sense that if $I \in \mathcal{V}$, there is no $I' \in \mathcal{V}$ such that $I \subset I'$ and $I \neq I'$. Then we get a new finite cover of $\mathbb{F}$ by cylinders, and

$$\sum_{u \in \mathcal{U}} |U|^s \ge \sum_{I_n \in \mathcal{V}} |I_n|^{s(1+\epsilon)}. \tag{2.10}$$

Step 3. We show that the cylinders composing the new cover can be replaced by cylinders of smaller order without increasing the sum (2.9).

Suppose that the largest order of cylinders in $\mathcal{V}$ is ℓ. Then there exists $I_\ell(a_1, \dots, a_\ell) \in \mathcal{V}$. Since each cylinder $I_\ell(a_1, \dots, \bar{a}_\ell) \in \mathbb{F}_\ell(I_{\ell-1}(a_1, \dots, a_{\ell-1}))$ contains infinitely many points in $\mathbb{F}$, all the cylinders in $\mathbb{F}_\ell(I_{\ell-1}(a_1, \dots, a_{\ell-1}))$ must be members of $\mathcal{V}$, i.e.

$$\mathbb{F}_\ell(I_{\ell-1}(a_1, \dots, a_{\ell-1})) \subset \mathcal{V}.$$

Now we replace the subset $\mathbb{F}_\ell(I_{\ell-1}(a_1, \dots, a_{\ell-1}))$ of $\mathcal{V}$ by $I_{\ell-1}(a_1, \dots, a_{\ell-1})$. If there are still other cylinders of order ℓ in $\mathcal{V}$, with the same method, we replace them by their mother cylinder of order $\ell - 1$. Since $\mathcal{V}$ contains only finitely many elements, the procedure above will terminate after finitely many steps. Finally, we arrive at a new cover $\mathcal{V}_1$ of $\mathbb{F}$ consisting of cylinders of order at most $\ell - 1$.

It should also be noticed that

$$\begin{aligned}
&\sum_{I_\ell(a_1,\dots,a_{\ell-1},a_\ell) \in \mathbb{F}_\ell(I_{\ell-1}(a_1,\dots,a_{\ell-1}))} |I_\ell(a_1, \dots, a_{\ell-1}, a_\ell)|^{s(1+\epsilon)} \\
&\quad \ge |I_{\ell-1}(a_1, \dots, a_{\ell-1})|^{s(1+\epsilon)} \cdot \Bigg(\sum_{\substack{b_{u_\ell} \le a_\ell \le b_{v_\ell} \\ a_\ell \in B}} \left(\frac{c_1(\epsilon)}{\xi_{a_\ell}^{1+\epsilon}} \right)^{s(1+\epsilon)} - 2 \Bigg) \\
&\quad \ge |I_{\ell-1}(a_1, \dots, a_{\ell-1})|^{s(1+\epsilon)}.
\end{aligned}$$

So, we have

$$\sum_{I \in \mathcal{V}} |I|^{s(1+\epsilon)} \ge \sum_{I \in \mathcal{V}_1} |I|^{s(1+\epsilon)}.$$

Thus, we can replace the cover $\mathcal{V}$ by the new cover $\mathcal{V}_1$ consisting of cylinders of the largest order $\ell - 1$ without increasing the quantity in (2.10).

Continuing this process, we finally get a cover consisting of cylinders of order 1. Note also that

$$\sum_{\substack{b_{u_1} \le a_1 \le b_{v_1} \\ a_1 \in B}} |I_1(a_1)|^{s(1+\epsilon)} \ge 1.$$

Thus we come to the assertion (2.9).

By now we have proven $\dim_{\mathrm{H}} \mathbb{F} \geq s$. The arbitrariness of $\epsilon > 0$ enables us to conclude that

$$\dim_{\mathrm{H}} \mathbb{E}(B) \geq s_0(B).$$

3. Proof of Theorem 1.2. From now on, the systems we deal with are all *Gauss-like*. We assume that $s_0(\mathbb{N}) < 1$ otherwise the right side of (1.2) equals 1. Recall that when $\{\xi_n\}_{n\geq 1}$ is monotone, its convergence exponent can be expressed as

$$s_0 := s_0(\mathbb{N}) = \limsup_{n\to\infty} \frac{\log n}{\log \xi_n}. \tag{3.1}$$

Then for any $0 < \epsilon < (1 - s_0)/2$, there exists a constant $c_3 = c_3(\epsilon) > 1$ such that

$$|f_n'(x)| \leq \left(\frac{c_3}{n}\right)^{\frac{1}{s_0+\epsilon}}, \qquad \forall x \in [0,1],\ \forall n \geq 1.$$

According to (2.1), the length of a basic interval satisfies

$$|I_n(a_1, \ldots, a_n)| \leq \prod_{k=1}^{n} \left(\frac{c_3}{a_k}\right)^{\frac{1}{s_0+\epsilon}}. \tag{3.2}$$

For any $x \in \Lambda$, write

$$Q_n(x) = a_1(x) \cdots a_n(x).$$

We begin with a property of $Q_n(x)$ when $x \in \mathbb{E}(a, b)$.

LEMMA 3.1. *For any $x \in \mathbb{E}(a, b)$ and $1 < d < b$, the inequality*

$$a_{n+1}(x) \geq \max\{a^{2d^n}, Q_n(x)^{d-1}\}$$

holds for infinitely many $n \in \mathbb{N}$.

Proof. The idea is due to Łuczak [Lu]. For any $M \in \mathbb{N}$ and any $x \in \mathbb{E}(a, b)$, there exists an integer $k > M$ with $(b/d)^k > 2/(d - 1)$ such that

$$Q_M(x) < a^{b^k d^{M-k}} \quad \text{and} \quad a_k(x) \geq a^{b^k}.$$

Put $g(n) = a^{b^k d^{n-k}}$. Then

$$Q_M(x) < g(M) \quad \text{and} \quad Q_k(x) \geq a_k(x) \geq a^{b^k} = g(k).$$

So, there exists an integer n with $M \leq n < k$ such that

$$Q_n(x) < g(n) \quad \text{and} \quad Q_{n+1}(x) \geq g(n + 1).$$

On one hand,

$$g(n + 1) = g(n)^d.$$

On the other hand, since $(b/d)^k > 2/(d - 1)$, we have

$$g(n + 1) = g(n)^{d-1} g(n) = a^{b^k d^{n-k}(d-1)} g(n) > a^{2d^n} g(n).$$

Thus $Q_{n+1}(x) > \max\{Q_n(x)^d, Q_n(x)a^{2d^n}\}$. As a result, by the definition of $Q_n(x)$, we obtain

$$a_{n+1}(x) \geq \max\{Q_n(x)^{d-1}, a^{2d^n}\}. \ \blacksquare$$

Lemma 3.1 enables us to conclude that for any $1 < d < b$,

$$\mathbb{E}(a,b) \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in \Lambda : a_{n+1}(x) \geq \max\{Q_n(x)^{d-1}, a^{2d^n}\}\}.$$

For each block of digits $(a_1, \ldots, a_n) \in \mathbb{N}^n$, we write

$$J_n(a_1, \ldots, a_n) = \bigcup_{a_{n+1} \geq \max\{Q_n^{d-1}, a^{2d^n}\}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}),$$

where $Q_n = a_1 \cdots a_n$. Then for each $N \geq 1$,

$$\mathbb{E}(a,b) \subset \bigcup_{n=N}^{\infty} \bigcup_{a_1,\ldots,a_n} J_n(a_1, \ldots, a_n).$$

This gives us a collection of covers of $\mathbb{E}(a,b)$.

Let $N_1 = N_1(\epsilon)$ be an integer such that for all $n \geq N_1$,

$$\frac{2c_3^{\frac{n+1}{s_0+\epsilon}}}{1/(s_0+\epsilon) - 1} \leq (a^{d^n})^{\frac{\epsilon}{(s_0+\epsilon)(s_0+2\epsilon)}}, \qquad \frac{\epsilon \log a}{n^2} d^n \geq 1 + \log \sum_{i \geq 1} \frac{1}{i^{1+\epsilon}}. \tag{3.3}$$

Then we are led to estimate the length of $J_n(a_1, \ldots, a_n)$ for each $(a_1, \ldots, a_n) \in \mathbb{N}^n$ with $n \geq N_1$. Since the system is Gauss-like, we have

$$\begin{aligned} |J_n(a_1, \ldots, a_n)| &= \sum_{a_{n+1} \geq \max\{Q_n^{d-1}, a^{2d^n}\}} |I_{n+1}(a_1, \ldots, a_n, a_{n+1})| \\ &\leq \sum_{a_{n+1} \geq \max\{Q_n^{d-1}, a^{2d^n}\}} c_3^{\frac{n+1}{s_0+\epsilon}} \left(\frac{1}{Q_n}\right)^{\frac{1}{s_0+\epsilon}} \left(\frac{1}{a_{n+1}}\right)^{\frac{1}{s_0+\epsilon}} \\ &\leq \frac{2c_3^{\frac{n+1}{s_0+\epsilon}}}{1/(s_0+\epsilon) - 1} \cdot Q_n^{-\frac{1}{s_0+\epsilon}} \cdot (\max\{Q_n^{d-1}, a^{2d^n}\})^{1-\frac{1}{s_0+\epsilon}} \\ &\leq Q_n^{-\frac{1}{s_0+\epsilon}} \cdot (\max\{Q_n^{d-1}, a^{2d^n}\})^{1-\frac{1}{s_0+2\epsilon}}. \end{aligned} \tag{3.4}$$

It is easy to observe that

$$\max\{Q_n^{d-1}, a^{2d^n}\} \geq \begin{cases} Q_n^{d-1} & \text{when } Q_n \geq a^{\frac{1}{n}d^n}, \\ a^{2d^n} \geq Q_n^n \cdot a^{d^n} & \text{when } Q_n < a^{\frac{1}{n}d^n}. \end{cases}$$

Thus when $Q_n \geq a^{\frac{1}{n}d^n}$, we have

$$|J_n(a_1, \ldots, a_n)| \leq Q_n^{-\frac{1}{s_0+\epsilon}} \cdot Q_n^{(d-1)(1-\frac{1}{s_0+2\epsilon})} =: Q_n^{-t_1}. \tag{3.5}$$

When $Q_n < a^{\frac{1}{n}d^n}$, we have

$$|J_n(a_1,\ldots,a_n)| \le (Q_n^n \cdot a^{d^n})^{1-\frac{1}{s_0+2\epsilon}} =: (Q_n^n \cdot a^{d^n})^{-t_2}. \tag{3.6}$$

Now we estimate (3.5) and (3.6) respectively but with the same method. For any $t > 1$, define a set function μ_t on basic intervals as

$$\mu_t(I_n(a_1,\ldots,a_n)) = e^{-t\sum_{j=1}^n \log a_j - nP(t)},$$

where $e^{P(t)} = \sum_{i\ge 1} i^{-t}$ is the classical Riemann zeta function. Then μ_t can be extended to a Borel probability measure supported on Λ.

When $Q_n \ge a^{\frac{1}{n}d^n}$,

$$\begin{aligned}
(Q_n^{-t_1})^{\frac{1+2\epsilon}{t_1}} &\le e^{-(1+\epsilon)\sum_{j=1}^n \log a_j - \frac{\epsilon}{n}d^n \log a} \\
&\le e^{-(1+\epsilon)\sum_{j=1}^n \log a_j - nP(1+\epsilon)-n} \qquad \text{(by (3.3))} \\
&= e^{-n}\mu_{1+\epsilon}(I_n(a_1,\ldots,a_n)).
\end{aligned}$$

When $Q_n < a^{\frac{1}{n}d^{n+1}}$,

$$\begin{aligned}
(Q_n^{-nt_2}a^{-d^n t_2})^{\frac{1+\epsilon}{nt_2}} &= e^{-(1+\epsilon)\sum_{j=1}^n \log a_j - \frac{1+\epsilon}{n}d^n \log a} \\
&\le e^{-(1+\epsilon)\sum_{j=1}^n \log a_j - nP(1+\epsilon)-n} \qquad \text{(by (3.3))} \\
&= e^{-n}\mu_{1+\epsilon}(I_n(a_1,\ldots,a_n)).
\end{aligned}$$

Clearly $\frac{1+2\epsilon}{t_1} \ge \frac{1+\epsilon}{nt_2}$ when n is large enough. As a result, we have

$$|J_n(a_1,\ldots,a_n)|^{\frac{1+2\epsilon}{t_1}} \le e^{-n}\mu_{1+\epsilon}(I_n(a_1,\ldots,a_n)). \tag{3.7}$$

So the $\frac{1+2\epsilon}{t_1}$-dimensional Hausdorff measure of $\mathbb{E}(a,b)$ can be estimated as

$$\begin{aligned}
\mathcal{H}^{\frac{1+2\epsilon}{t_1}}(\mathbb{E}(a,b)) &\le \sum_{n=N}^{\infty}\sum_{a_1,\ldots,a_n} |J_n(a_1,\ldots,a_n)|^{\frac{1+2\epsilon}{t_1}} \\
&\le \sum_{n=N}^{\infty} e^{-n} \sum_{a_1,\ldots,a_n} \mu_{1+\epsilon}(I_n(a_1,\ldots,a_n)) \le \frac{1}{e-1}.
\end{aligned}$$

Thus,

$$\dim_{\mathrm{H}} \mathbb{E}(a,b) \le \frac{1+2\epsilon}{t_1} = \frac{1+2\epsilon}{\frac{1}{s_0+\epsilon} + (d-1)\big(\frac{1}{s_0+2\epsilon}-1\big)}.$$

By the arbitrariness of ϵ and d, we obtain

$$\dim_{\mathrm{H}} \mathbb{E}(a,b) \le \frac{s_0}{s_0+b(1-s_0)}.$$

Remark on the lower bound of $\dim_{\mathrm{H}} \tilde{\mathbb{E}}(a,b)$**.** For a d-decaying Gauss-like system, by taking $\Phi(n) = n^b$ in X_Φ, we know that

$$X_\Phi \subset \tilde{\mathbb{E}}(a,b).$$

So we will have

$$\dim_{\mathrm{H}} \tilde{\mathbb{E}}(a,b) = \frac{1}{1+b(d-1)} = \frac{s_0}{s_0+b(1-s_0)}.$$

But for a general ξ-regular iIFS, two systems may share the same convergence exponent, but the dimension of $\dim_{\mathrm{H}} \tilde{\mathbb{E}}(a,b)$ may be different. To see this, we construct a system with convergence exponent $1/2$ the same as that of the continued fraction but with a different dimension of $\dim_{\mathrm{H}} \tilde{\mathbb{E}}(a,b)$.

First we define a sequence $\{\xi_n\}_{n\geq 1}$ such that $\xi_n \asymp n^2$ for most $n \in \mathbb{N}$ but which jumps very high at some positions. More precisely, take

$$\xi_n = \begin{cases} 4c_0 & \text{when } n=1, \\ 2^{2^{k!+1}} c_0 & \text{when } 2^{2^{(k-1)!}} \leq n < 2^{2^{k!}},\ \forall k \in \mathbb{N}, \end{cases} \tag{3.8}$$

where

$$c_0 = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{2^{2^{k!}} - 2^{2^{(k-1)!}}}{2^{2^{k!+1}}}.$$

Secondly, we construct linear functions $\{f_n\}_{n\geq 1}$ with respective slopes $\{1/\xi_n\}_{n\geq 1}$ which are arranged from right to left, side by side, to ensure this is a Gauss-like system. More precisely, take $f_1(x) = 1 - \frac{x}{4c_0}$ and

$$f_n(x) = \left(1 - \frac{1}{4c_0} - \sum_{i=1}^{k-1} \frac{2^{2^{i!}} - 2^{2^{(i-1)!}}}{2^{2^{i!+1}} c_0} - \frac{n - 2^{2^{(k-1)!}}}{2^{2^{k!+1}} c_0}\right) - \frac{x}{2^{2^{k!+1}} c_0}$$

when $2^{2^{(k-1)!}} \leq n < 2^{2^{k!}}$ for any $x \in [0,1]$.

It is easy to see that $\{f_n\}_{n\geq 1}$ is a Gauss-like system and the convergence exponent of $\{\xi_n\}_{n\geq 1}$ is

$$s_0(\mathbb{N}) = \limsup_{n\to\infty} \frac{\log n}{\log \xi_n} = \limsup_{k\to\infty} \frac{\log 2^{2^{k!}}}{\log 2^{2^{k!+1}}} = \frac{1}{2}.$$

By taking $a=b=2$ in $\widetilde{\mathbb{E}}(a,b)$, we will show that $\dim_{\mathrm{H}} \widetilde{\mathbb{E}}(2,2) = 0$ and not $1/3$ as in the system of continued fractions.

First we will show by the same method as in Lemma 3.1 that

Lemma 3.2. *For any $M \geq 1$ and $x \in \widetilde{\mathbb{E}}(2,2)$,*

$$\xi_{a_{n+1}(x)} \geq (\xi_{a_1(x)} \cdots \xi_{a_n(x)})^M \quad \textit{for infinitely many } n \in \mathbb{N}. \tag{3.9}$$

Proof. Let $d > 1$ be such that $2^d > M+1$. For any $x \in \widetilde{\mathbb{E}}(2,2)$ and any $m > 1$, there exists $k > d$ with $k! > m$ such that

$$\xi_{a_1(x)} \cdots \xi_{a_m(x)} < 2^{2^{dk!}(M+1)^{m-k!}}.$$

By the choice of ξ in (3.8), we have $\xi_{a_{k!}(x)} \geq 2^{2^{(k+1)!+1}} c_0$ since $a_{k!}(x) \geq 2^{2^{k!}}$. Thus

$$\xi_{a_1(x)} \cdots \xi_{a_{k!}(x)} \geq 2^{2^{(k+1)!+1}} c_0 \geq 2^{2^{(k+1)!}} > 2^{2^{dk!}}.$$

Then put

$$f(n) = 2^{2^{dk!(M+1)^{n-k!}}}.$$

So there exists n with $m \le n \le k!$ such that

$$\xi_{a_1(x)} \cdots \xi_{a_n(x)} < f(n) \quad \text{and} \quad \xi_{a_1(x)} \cdots \xi_{a_{n+1}(x)} \ge f(n+1). \ \blacksquare$$

Now we give an upper estimate of the dimension of the set defined by (3.9), i.e., the set

$$\mathbb{E}_M := \{x \in \varLambda : \xi_{a_{n+1}(x)} \ge (\xi_{a_1(x)} \cdots \xi_{a_n(x)})^M \text{ for infinitely many } n \in \mathbb{N}\}.$$

For each $a_1, \ldots, a_n \in \mathbb{N}$, define

$$\widetilde{J}_n(a_1, \ldots, a_n) = \bigcup_{a_{n+1}\colon \xi_{a_{n+1}} \ge (\xi_{a_1} \cdots \xi_{a_n})^M} I_{n+1}(a_1, \ldots, a_{n+1}).$$

Clearly, for each $N \in \mathbb{N}$, $\mathbb{E}_M$ is covered by the collection of sets

$$\bigcup_{n=N}^{\infty} \bigcup_{a_1, \ldots, a_n} \widetilde{J}_n(a_1, \ldots, a_n).$$

The length of $\widetilde{J}_n$ satisfies

$$\begin{aligned} |\widetilde{J}_n(a_1, \ldots, a_n)| &= \sum_{a_{n+1}\colon \xi_{a_{n+1}} \ge (\xi_{a_1} \cdots \xi_{a_n})^M} |I_{n+1}(a_1, \ldots, a_n, a_{n+1})| \\ &= |I_n(a_1, \ldots, a_n)| \sum_{j\colon \xi_j \ge (\xi_{a_1} \cdots \xi_{a_n})^M} \frac{1}{\xi_j}. \end{aligned}$$

To estimate the last sum, define

$$j_0 = \min\{j : \xi_j \ge (\xi_{a_1} \cdots \xi_{a_n})^M\}.$$

Let k_0 be the integer such that

$$2^{2^{(k_0-1)!}} \le j_0 < 2^{2^{k_0!}}.$$

Recalling the definition of ξ_n in (3.8), we have

$$\begin{aligned} \sum_{j\colon \xi_j \ge (\xi_{a_1} \cdots \xi_{a_n})^M} \frac{1}{\xi_j} &= \sum_{j=j_0}^{\infty} \frac{1}{\xi_j} = \frac{2^{2^{k_0!}} - j_0}{2^{2^{k_0!+1}}} + \sum_{k=k_0+1}^{\infty} \frac{2^{2^{k!}} - 2^{2^{(k-1)!}}}{2^{2^{k!+1}}} \\ &\le \sum_{k=k_0}^{\infty} \frac{2^{2^{k!}} - 2^{2^{(k-1)!}}}{2^{2^{k!+1}}} \le \sum_{k=k_0}^{\infty} \frac{1}{2^{2^{k!}}} \le \frac{4}{3} \cdot \frac{1}{2^{2^{k_0!}}}. \end{aligned} \tag{3.10}$$

Note that $2^{2^{k_0!+1}} = \xi_{j_0} \ge (\xi_{a_1} \cdots \xi_{a_n})^M$. So

$$\sum_{j\colon \xi_j \ge (\xi_{a_1} \cdots \xi_{a_n})^M} \frac{1}{\xi_j} \le \frac{4}{3} (\xi_{a_1} \cdots \xi_{a_n})^{-M/2}.$$

As a result,

$$\begin{aligned}|\widetilde{J}_n(a_1,\ldots,a_n)| &\le \tfrac{4}{3}|I_n(a_1,\ldots,a_n)|\cdot(\xi_{a_1}\cdots\xi_{a_n})^{-M/2}\\ &= \tfrac{4}{3}|I_n(a_1,\ldots,a_n)|^{1+M/2}.\end{aligned}$$

Therefore, for any $\epsilon > 0$,

$$\begin{aligned}\mathcal{H}^{\frac{1+\epsilon}{1+M/2}}(\mathbb{E}_M) &\le \liminf_{N\to\infty}\sum_{n=N}^{\infty}\sum_{a_1,\ldots,a_n}|\widetilde{J}_n(a_1,\ldots,a_n)|^{\frac{1+\epsilon}{1+M/2}}\\ &\le \left(\frac{4}{3}\right)^{\frac{1+\epsilon}{1+M/2}}\liminf_{N\to\infty}\sum_{n=N}^{\infty}\sum_{a_1,\ldots,a_n}|I_n(a_1,\ldots,a_n)|^{1+\epsilon} < \infty.\end{aligned}$$

So, $\dim_{\mathrm{H}}\mathbb{E}_M \le \frac{1}{1+M/2}$. Thus we conclude that

$$\dim_{\mathrm{H}}\widetilde{\mathbb{E}}(2,2) \le \dim_{\mathrm{H}}\mathbb{E}_M \le \frac{1}{1+M/2} \to 0 \quad \text{as } M\to\infty.$$

4. Proof of Theorem 1.3. Recall (3.1). Denote by $\mathcal{N}$ the subset of $\mathbb{N}$ such that

$$s_0 = \lim_{n\to\infty,\, n\in\mathcal{N}}\frac{\log n}{\log \xi_n}.$$

Now we construct a subset $B\subset\mathbb{N}$ such that $\{\xi_n : n\in B\}$ also has convergence exponent s_0. This is done by defining B part by part. Fix $a>1$.

Define $m_1 = 0$ and pick $\ell_1 \gg 1$ such that $m_1+\ell_1\in\mathcal{N}$. Then let the first part of B be

$$B_1 = \{m_1+1,\ldots,m_1+\ell_1\}.$$

Assume the $(k-1)$th part of B has been defined, say

$$B_{k-1} = \{m_{k-1}+1,\ldots,m_{k-1}+\ell_{k-1}\}.$$

Now we define the kth part of B as follows. Since $\psi(n)\to\infty$ as $n\to\infty$, choose an integer n_k such that

$$\psi(n_k) > m_{k-1}+\ell_{k-1}. \tag{4.1}$$

Then choose another integer $m_k > m_{k-1}+\ell_{k-1}$ such that

$$m_k \ge a^{k^{n_k}}. \tag{4.2}$$

Finally, choose a third integer ℓ_k such that

$$m_k+\ell_k\in\mathcal{N} \quad\text{and}\quad \frac{\ell_k}{m_k+\ell_k} \ge 1-\frac{1}{k}. \tag{4.3}$$

Then define

$$B_k = \{m_k+1,\ldots,m_k+\ell_k\}$$

and

$$B = \{n \in B_k : k \ge 1\} = \{\underbrace{m_1+1,\dots,m_1+\ell_1}_{B_1},\underbrace{m_2+1,\dots,m_2+\ell_2}_{B_2},\dots\}.$$

By noticing that the $(\ell_1+\cdots+\ell_k)$th term in $\{\xi_n : n \in B\}$ is $\xi_{m_k+\ell_k}$, the convergence exponent of $\{\xi_n : n \in B\}$ can be estimated as

$$\begin{aligned} s_0(B) &\ge \limsup_{k\to\infty} \frac{\log(\ell_1+\cdots+\ell_k)}{\log \xi_{m_k+\ell_k}} \\ &\ge \limsup_{k\to\infty} \frac{\log \ell_k}{\log \xi_{m_k+\ell_k}} = \limsup_{k\to\infty} \frac{\log(m_k+\ell_k)}{\log \xi_{m_k+\ell_k}} \qquad \text{(by (4.3))} \\ &= s_0. \end{aligned}$$

Now we show that for the set B constructed above, the set

$$\mathbb{E}(B,\psi) = \{x \in \Lambda : a_n(x) \in B, a_n(x) \ge \psi(n),\ \forall n \ge 1\}$$

has Hausdorff dimension 0.

More precisely, for any $x \in \mathbb{E}(B,\psi)$, when $n = n_k$ is chosen as in (4.1), we have

$$a_{n_k}(x) \ge \psi(n_k) > m_{k-1} + \ell_{k-1}.$$

Since $a_n(x)$ is also in B and the first term in B larger than $m_{k-1}+\ell_{k-1}$ is m_k+1, we have

$$a_{n_k}(x) \ge m_k + 1 \ge a^{k^{n_k}}.$$

Thus for any $M > 1$,

$$\mathbb{E}(B,\psi) \subset \{x \in [0,1] : a_n(x) \ge a^{M^n} \text{ for infinitely many } n\}.$$

By Theorem 1.2, we have

$$\dim_{\mathrm{H}} \mathbb{E}(B,\psi) \le \frac{s_0}{s_0 + M(1-s_0)} \to 0 \qquad \text{as } M \to \infty.$$

5. Proof of Theorem 1.4. In this section, since several systems will be considered at the same time, we use Λ_f and $a_n(x,f)$ for the attractor and the nth digit of x in the system $\{f_n\}_{n\ge1}$. Let $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n) \ge 2$ for all $n \in \mathbb{N}$ and $\psi(n) \to \infty$ as $n \to \infty$. Define

$$\mathbb{E}_f(\psi) = \{x \in \Lambda_f : a_n(x,f) \ge \psi(n) \text{ for all } n \in \mathbb{N}\}.$$

Without loss of generality, suppose that $s_0 > 0$. We take, for example,

$$\xi_n = \begin{cases} n^{1/s_0} \displaystyle\sum_{k=1}^{\infty} k^{-1/s_0} & \text{when } 0 < s_0 < 1, \\ n(\log 2n)^2 \displaystyle\sum_{k=1}^{\infty} \frac{1}{k(\log 2k)^2} & \text{when } s_0 = 1. \end{cases}$$

This ensures that the convergence exponent of $\{\xi_n\}_{n\ge1}$ is s_0 and $\sum_{n=1}^{\infty}\xi_n^{-1} = 1$.

Take

$$f_n(x) = \frac{x}{\xi_n} + \sum_{k=n+1}^{\infty}\frac{1}{\xi_k} \quad \text{for any } x\in[0,1].$$

Then $\{f_n\}_{n\ge1}$ is a Gauss-like system with convergence exponent s_0. By Theorem 1.3, there exists $B\subset\mathbb{N}$ such that

$$s_0(B) = s_0 \quad \text{and} \quad \dim_{\mathrm{H}}\mathbb{E}_f(B,\psi) = 0, \tag{5.1}$$

where

$$\mathbb{E}_f(B,\psi) = \{x\in\Lambda_f : a_n(x,f)\in B \text{ and } a_n(x,f)\ge\psi(n),\ \forall n\in\mathbb{N}\}.$$

Write $B = \{b_1, b_2, \ldots\}$. For each $n\ge1$, let $g_n(x) := f_{b_n}(x)$. It is clear that $\Lambda_g\subset\Lambda_f$ and the digit sequences $a_n(x,g)$ and $a_n(x,f)$ for $x\in\Lambda_g$ satisfy

$$b_{a_n(x,g)} = a_n(x,f) \quad \text{for all } n\ge1. \tag{5.2}$$

Thus, for any x in

$$\mathbb{E}_g(\psi) := \{x\in\Lambda_g : a_n(x,g)\ge\psi(n) \text{ for all } n\in\mathbb{N}\},$$

one has

$$a_n(x,f)\in B, \qquad a_n(x,f) = b_{a_n(x,g)} \ge a_n(x,g)\ge\psi(n).$$

This gives $\mathbb{E}_g(\psi)\subset\mathbb{E}_f(B,\psi)$. So,

$$\dim_{\mathrm{H}}\mathbb{E}_g(\psi) = 0.$$

Now there is only one step left since what we required is a Gauss-like system. This can be done by translating and then expanding the functions $\{g_n\}_{n\ge1}$ in the following way.

(1) *Translation.* For each $n\ge1$, translate g_n to

$$\widetilde{h}_n(x) = g_n(x) + \sum_{k=n}^{\infty}|g_k([0,1])| - \max_{y\in[0,1]} g_n(y) = \frac{x}{\xi_{b_n}} + \sum_{k=n+1}^{\infty}\frac{1}{\xi_{b_k}}.$$

This means that the gaps between

$$\min\{g_n(x) : x\in[0,1]\} \quad \text{and} \quad \max\{g_{n+1}(x) : x\in[0,1]\}$$

are reduced to zero for each $n\ge1$ and $\lim_{n\to\infty}\widetilde{h}_n(x) = 0$.

In the process, the distances between points in Λ_g get contracted. So,

$$\dim_{\mathrm{H}}\mathbb{E}_{\widetilde{h}}(\psi) \le \dim_{\mathrm{H}}\mathbb{E}_g(\psi) = 0. \tag{5.3}$$

(2) *Expanding.*

$$h_n(x) = \begin{cases} \Big(1 - \sum_{k=2}^{\infty}\frac{1}{\xi_{b_k}}\Big)x + \sum_{k=2}^{\infty}\frac{1}{\xi_{b_k}}, & n = 1,\\ \widetilde{h}_n(x), & n\ge2, \end{cases}$$

i.e., all functions remain the same but the first one which gets expanded to ensure that

$$\overline{\bigcup_{n\geq 1} h_n([0,1])} = [0,1].$$

Clearly the system $\{h_n\}_{n\geq 1}$ has convergence exponent $s_0(B)$, which equals s_0 by (5.1).

Recall that at the very beginning of this section, we assume that $\psi(n) \geq 2$ for all $n \in \mathbb{N}$. Thus

$$\mathbb{E}_h(\psi) = \mathbb{E}_{\widetilde{h}}(\psi) \tag{5.4}$$

since only the functions $\{h_n\}_{n\geq 2}$, $\{\widetilde{h}_n\}_{n\geq 2}$ are involved.

By (5.3) and (5.4), we complete the proof.

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