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Boundedness of commutators of singular and potential operators in generalized grand Morrey spaces and some applications

by

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Abstract. In the setting of spaces of homogeneous type, it is shown that the commutator of Calderón–Zygmund type operators as well as the commutator of a potential operator with a BMO function are bounded in a generalized grand Morrey space. Interior estimates for solutions of elliptic equations are also given in the framework of generalized grand Morrey spaces.

1. Introduction. In 1992 T. Iwaniec and C. Sbordone [17], in their studies related to the integrability properties of the Jacobian in a bounded open set Ω, introduced a new type of function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. Their generalized version, $L^{p),\theta}(\Omega)$, appeared in L. Greco, T. Iwaniec and C. Sbordone [16]. Harmonic analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), has been intensively studied during the last years due to various applications; we mention e.g. [1, 8–12, 18, 21].

Recently in [35] a version of weighted grand Lebesgue spaces was introduced, adjusted for sets $\Omega \subseteq \mathbb{R}^n$ of infinite measure, where the integrability of $|f(x)|^{p-\varepsilon}$ at infinity is controlled by means of a weight; moreover, grand grand Lebesgue spaces were also considered, together with the study of classical operators of harmonic analysis in such spaces. Another idea of introducing "bilateral" grand Lebesgue spaces on sets of infinite measure was suggested in [24], where the structure of such spaces was investigated, but not operators; the spaces in [24] are two-parameter spaces with respect to the exponent p, with norm involving $\sup_{p_1<p<p_2}$.

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Morrey spaces $L^{p,\lambda}$ were introduced in 1938 by C. Morrey [28] in connection with regularity of solutions to partial differential equations, and provided a useful tool in the regularity theory of PDEs (for Morrey spaces we refer to the books [14, 23]; see also [31] where an overview of various generalizations may be found).

Recently, in the spirit of grand Lebesgue spaces, A. Meskhi [26, 27] introduced *grand Morrey spaces* (in [26] they were defined on quasi-metric measure spaces with doubling measure) and obtained results on the boundedness of the maximal operator, Calderón–Zygmund singular operators and Riesz potentials. The boundedness of commutators of singular and potential operators in grand Morrey spaces was treated by X. Ye [40]. Note that the "grandification procedure" was applied only to the parameter p.

This paper is a continuation of work begun in [30] and [22]; in the former, generalized grand Morrey spaces (called "grand grand Morrey spaces") were introduced and maximal and Calderón–Zygmund operators were studied in the framework of Euclidean spaces, whereas in the latter paper the boundedness of potential operators was studied in the framework of generalized grand Morrey spaces in the homogeneous and even in the nonhomogeneous case.

Notation:

- d_X denotes the diameter of the set X;
- c and C denote various absolute positive constants, which may have different values even in the same line;
- $A \sim B$ for positive A and B means that there exists $c > 0$ such that $c^{-1}A \le B \le cA$;
- $B(x,r) = \{y \in X : d(x,y) < r\}$;
- $A \lesssim B$ stands for $A \le CB$;
- $\hookrightarrow$ means continuous imbedding;
- $\fint_B f\,d\mu$ denotes the integral average of f, i.e. $\fint_B f\,d\mu := \frac{1}{\mu B}\int_B f\,d\mu$;
- $\mathcal{D}(X)$ denotes the set of L^∞ functions on X with compact support;
- p' stands for the conjugate exponent $1/p + 1/p' = 1$.

2. Preliminaries

2.1. Spaces of homogeneous type. Let $X := (X, d, \mu)$ be a topological space with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X,\mu)$ and d is a *quasimetric*, i.e. a non-negative real-valued function d on $X \times X$ which satisfies:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) there exists a constant $C_t > 0$ such that $d(x,y) \le C_t[d(x,z)+d(z,y)]$ for all $x, y, z \in X$, and

(iii) there exists a constant $C_s > 0$ such that $d(x, y) \le C_s d(y, x)$ for all $x, y \in X$.

Let μ be a positive measure on a σ-algebra of subsets of X which contains the d-balls $B(x, r)$. Everywhere we assume that all balls have finite measure, that is, $\mu B(x, r) < \infty$ for all $x \in X$ and $r > 0$, and that for every neighborhood V of $x \in X$, there exists $r > 0$ such that $B(x, r) \subset V$.

We say that the measure μ is *lower α-Ahlfors regular* if

$$\mu B(x, r) \ge c r^{\alpha},$$

and is *upper β-Ahlfors regular* (or satisfies the *growth condition of degree β*) if

$$\mu B(x, r) \le c r^{\beta},$$

where $\alpha, \beta, c > 0$ do not depend on x or r. When $\alpha = \beta$, the measure μ is simply called *α-Ahlfors regular.*

The condition

$$\mu B(x, 2r) \le C_d \mu B(x, r), \qquad C_d > 1,$$

on the measure μ with C_d independent of $x \in X$ and $0 < r < d_X$, is known as the *doubling condition.*

Iterating it, we obtain

$$\frac{\mu B(x, R)}{\mu B(y, r)} \le C_d \left(\frac{R}{r} \right)^{\log_2 C_d}, \qquad 0 < r \le R, \tag{2.1}$$

for all d-balls $B(x, R)$ and $B(y, r)$ with $B(y, r) \subset B(x, R)$.

The triplet (X, d, μ), with μ satisfying the doubling condition, is called a *space of homogeneous type*, abbreviated from now on as SHT. For some important examples of SHTs we refer e.g. to [5].

From (2.1) it follows that every homogeneous type space (X, d, μ) with finite measure is lower $(\log_2 C_d)$-Ahlfors regular.

Throughout the paper we will also assume that

$$\mu(B(x, R) \setminus B(x, r)) > 0 \tag{2.2}$$

for all $x \in X$ and r, R with $0 < r < R < d_X$. The reverse doubling condition, following from the doubling condition under certain restrictions, is well known (cf., for example, [39, p. 269]). For instance, when (2.2) is valid and (X, d, μ) is an SHT, the measure μ also satisfies the reverse doubling condition

$$\frac{\mu B(x, r)}{\mu B(x, R)} \le C \left(\frac{r}{R} \right)^{\gamma} \tag{2.3}$$

for appropriate positive constants C and γ. For other conditions ensuring the validity of the reverse doubling condition whenever the measure is doubling, see e.g. [32].

2.2. Generalized Lebesgue spaces. For $1 < p < \infty$, $\theta > 0$ and $0 < \varepsilon < p-1$ the *grand Lebesgue space* is the set of measurable functions for which

$$\|f\|_{L^{p),\theta}(X,\mu)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\theta/(p-\varepsilon)} \|f\|_{L^{p-\varepsilon}(X,\mu)} < \infty,$$

where

$$\|f\|^p_{L^p(X,\mu)} := \int_X |f(y)|^p \, d\mu(y).$$

For $\theta = 1$, we denote the space $L^{p),\theta}(X,\mu)$ simply by $L^{p)}(X,\mu)$.

When $\mu X < \infty$, for all ε, θ_1 and θ_2 satisfying the conditions $0 < \varepsilon < p-1$ and $0 < \theta_1 \le \theta_2$, we have

$$L^p_\omega(X,\mu) \hookrightarrow L^{p),\theta_1}_\omega(X,\mu) \hookrightarrow L^{p),\theta_2}_\omega(X,\mu) \hookrightarrow L^{p-\varepsilon}_\omega(X,\mu),$$

where ω is any Muckenhoupt weight (see Subsection 2.3).

For more properties of grand Lebesgue spaces, see [18].

2.3. Muckenhoupt weights. A weight ω (i.e. a non-negative locally integrable function) is in the *Muckenhoupt class* $A_\infty(X)$ if there are positive constants C and ε such that

$$\frac{\omega(E)}{\omega(B)} \le C \left(\frac{\mu(E)}{\mu(B)} \right)^\varepsilon$$

for all balls B and every measurable set $E \subset B$, where $\omega(E) = \int_E \omega \, d\mu$. The infimum of such C will be denoted by $[w]_{A_\infty}$. A weight ω is in the *Muckenhoupt class* $A_p(X)$ if there is a positive constant C such that

$$\left(\fint_B \omega \, d\mu \right) \left(\fint_B \omega^{-1/(p-1)} \, d\mu \right)^{p-1} \le C$$

for all balls B, and the infimum of such C will be denoted by $[w]_{A_p}$. A weight ω is in the *Muckenhoupt class* $A_1(X)$ if there is a constant C such that $M\omega(x) \le Cw(x)$, where M is the maximal operator (2.4). Note that the $A_p(X)$ classes increase with p, namely $A_1(X) \subsetneq A_p(X) \subsetneq A_q(X) \subsetneq A_\infty(X)$, $1 < p < q < \infty$. For general properties of Muckenhoupt weights we refer e.g. to [13] in the Euclidean case and [37] for spaces of homogeneous type.

2.4. Morrey spaces. For $1 \le p < \infty$ and $0 \le \lambda < 1$, the usual *Morrey space* $L^{p,\lambda}(X,\mu)$ is the set of all measurable functions such that

$$\|f\|_{L^{p,\lambda}(X,\mu)} := \sup_{\substack{x\in X \\ 0<r<d_X}} \left(\frac{1}{\mu B(x,r)^\lambda} \int_{B(x,r)} |f(y)|^p \, d\mu(y) \right)^{1/p} < \infty.$$

2.5. BMO space. The space of functions of *bounded mean oscillation*, denoted by $\mathrm{BMO}(X,\mu)$, is the set of all real-valued locally integrable functions such that

$$\|f\|_{\mathrm{BMO}(X,\mu)} = \sup_{\substack{x\in X\\ 0<r<d_X}} \frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\, d\mu(y) < \infty,$$

where $f_{B(x,r)}$ is the integral average over the ball $B(x,r)$. $\mathrm{BMO}(X,\mu)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathrm{BMO}(X,\mu)}$ when we regard its elements as equivalence classes of functions modulo additive constants.

Remark. The space $\mathrm{BMO}(X,\mu)$ can be given several equivalent norms:

(i) we have

$$\|f\|_{\mathrm{BMO}(X,\mu)} \sim \sup_{\substack{x\in X\\ 0<r<d_X}} \inf_{c\in\mathbb{R}} \frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y) - c|\, d\mu(y),$$

(ii) the John–Nirenberg inequality (see e.g. [38, Cor. 1.5, p. 203]) gives

$$\|f\|_{\mathrm{BMO}(X,\mu)} \sim \sup_{\substack{x\in X\\ 0<r<d_X}} \left(\frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p\, d\mu(y)\right)^{1/p}$$

for $1 < p < \infty$.

2.6. Maximal operators. In the following we always assume that f is a locally integrable function defined in X. We denote by Mf the *Hardy–Littlewood maximal operator*, given by

$$Mf(x) = \sup_{0<r<d_X} \fint_{B(x,r)} |f(y)|\, d\mu(y) \quad \text{for } x \in X. \tag{2.4}$$

From [27] we have the following boundedness result for Morrey spaces.

Lemma 2.1. *Let $1 < p < \infty$ and $0 \le \lambda < 1$. Then*

$$\|Mf\|_{L^{p,\lambda}(X,\mu)} \le (C \cdot C_d^{\lambda/p}(p')^{1/p} + 1)\|f\|_{L^{p,\lambda}(X,\mu)}$$

where the constant $C_d \ge 1$ comes from the doubling condition for μ, and C is a constant independent of p.

We denote by $M_s f$ the *maximal operator*

$$M_s f(x) := (M|f|^s)^{1/s}$$

for $1 \le s < \infty$. Using Lemma 2.1, it is easy to obtain the following boundedness result.

Lemma 2.2. *Let $1 < s < p < \infty$ and $0 \le \lambda < 1$. Then*

$$\|M_s f\|_{L^{p,\lambda}(X,\mu)} \le (C \cdot C_d^{\lambda s/p}((p/s)')^{s/p} + 1)\|f\|_{L^{p,\lambda}(X,\mu)}$$

where the constant $C_d \geq 1$ *comes from the doubling condition for* μ*, and* C *is a constant independent of* p*.*

When $\lambda = 0$, the statement above reduces to the well-known analogous result in the setting of L^p spaces.

2.7. Calderón–Zygmund singular operators. In this section we follow [27], in particular, making use of the definition of Calderón–Zygmund singular operators as integral operators

$$Tf(x) = \text{p.v.} \int\limits_X K(x,y) f(y)\, d\mu(y), \qquad f \in \mathcal{D}(X),$$

with the kernel $K : X \times X \setminus \{(x,x) : x \in X\} \to \mathbb{R}$ being a measurable function satisfying the conditions:

(i) $|K(x,y)| \leq C/\mu B(x, d(x,y))$, $x, y \in X$, $x \neq y$;
(ii) $|K(x_1,y) - K(x_2,y)| + |K(y,x_1) - K(y,x_2)|$

$$\leq Cw\left(\frac{d(x_2,x_1)}{d(x_2,y)}\right) \frac{1}{\mu B(x_2, d(x_2,y))}$$

for all x_1, x_2 and y with $d(x_2,y) \geq C d(x_1,x_2)$, where w is a positive nondecreasing function on $(0,\infty)$ which satisfies the Δ_2 condition $w(2t) \leq cw(t)$ $(t > 0)$ and the Dini condition $\int_0^1 w(t)/t\, dt < \infty$. We also assume that Tf exists almost everywhere on X in the principal value sense for all $f \in L^2(X)$ and that T is bounded in $L^2(X)$.

Such Calderón–Zygmund operators are bounded in Morrey spaces, as can be seen from the following proposition, proved in [27].

Proposition 2.3. *Let* $1 < p < \infty$ *and* $0 \leq \lambda < 1$*. Then*

$$\|Tf\|_{L^{p,\lambda}(X,\mu)} \leq C_{p,\lambda} \|f\|_{L^{p,\lambda}(X,\mu)}, \qquad f \in \mathcal{D}(X),$$

where

$$C_{p,\lambda} \leq c \begin{cases} \dfrac{p}{p-1} + \dfrac{p}{2-p} + \dfrac{p-\lambda+1}{1-\lambda} & \text{if } 1 < p < 2, \\ p + \dfrac{p}{p-2} + \dfrac{p-\lambda+1}{1-\lambda} & \text{if } p > 2, \end{cases}$$

with c *independent of* p *and* λ*.*

Remark. The Riesz–Thorin interpolation theorem and Lemma 4.1 of [27] imply that there is a positive constant C independent of p such that

$$\|T\|_{L^p \to L^p} \leq C, \qquad p \in [3/2, 5/2].$$

Following now the proof of Proposition 4.2 in [27] it can be deduced that there is a positive constant C independent of p and λ such that

$$\|T\|_{L^{p,\lambda} \to L^{p,\lambda}} \leq C\left(1 + \frac{p-\lambda-1}{1-\lambda}\right), \qquad p \in [3/2, 5/2].$$

2.8. Commutators. Let U be an operator and b a locally integrable function. We denote, indiscriminately, the *commutator* by $[b,U]f$ or $U_b f$ and define it as

$$[b,U]f := bU(f) - U(bf) =: U_b f.$$

Commutators are very useful when studying problems related to regularity of solutions of elliptic partial differential equations of second order (e.g., [2, 3]).

3. Generalized grand Morrey spaces and the reduction lemma. In this section we will assume that the measure μ is upper γ-Ahlfors regular. All the results stated in this section were proved in [22].

We introduce the following functional:

$$\Phi^{p,\lambda}_{\varphi,A}(f,s) := \sup_{0<\varepsilon<s} \varphi(\varepsilon)^{1/(p-\varepsilon)} \|f\|_{L^{p-\varepsilon,\lambda-A(\varepsilon)}(X,\mu)},$$

where s is a positive number and A is a non-negative function defined on $(0, p-1)$.

Definition 3.1 (Generalized grand Morrey spaces). Let $1 < p < \infty$, $0 \le \lambda < 1$, φ be a positive bounded function with $\lim_{t\to 0+} \varphi(t) = 0$ and A be a non-decreasing real-valued non-negative function with $\lim_{x\to 0+} A(x) = 0$. We denote by $L^{p),\lambda)}_{\varphi,A}(X,\mu)$ the space of measurable functions with finite norm

$$\|f\|_{L^{p),\lambda)}_{\varphi,A}(X)} := \Phi^{p,\lambda}_{\varphi,A}(f, s_{\max}), \qquad s_{\max} = \min\{p-1, a\}, \tag{3.1}$$

where $a = \sup\{x > 0 : A(x) \le \lambda\}$.

Remark. For appropriate φ, in the case $A \equiv 0, \lambda > 0$ we recover the grand Morrey spaces introduced by A. Meskhi [27], and when $\lambda = 0$, $A \equiv 0$ we have the grand Lebesgue spaces introduced in [16] (and in [17] in the case $\theta = 1$).

For fixed $p, \lambda, \varphi, A, f$ the function

$$s \mapsto \Phi^{p,\lambda}_{\varphi,A}(f,s)$$

is non-decreasing, but it is possible to estimate $\Phi^{p,\lambda}_{\varphi,A}(f,s)$ via $\Phi^{p,\lambda}_{\varphi,A}(f,\sigma)$ with $\sigma < s$ as follows.

Lemma 3.2. *For $0 < \sigma < s < s_{\max}$ we have*

$$\Phi^{p,\lambda}_{\varphi,A}(f,s) \le C\varphi(\sigma)^{-1/(p-\sigma)} \Phi^{p,\lambda}_{\varphi,A}(f,\sigma),$$

where C depends on γ, the parameters p, λ, φ, A and the diameter d_X, but does not depend on f, s or σ.

From Lemma 3.2 we immediately have

Lemma 3.3. *For $0 < \sigma < s_{\max}$, the norm defined in* (3.1) *satisfies*

$$\|f\|_{L^{p),\lambda)}_{\varphi,A}(X)} \le C \frac{\Phi^{p,\lambda}_{\varphi,A}(f,\sigma)}{\varphi(\sigma)^{1/(p-\sigma)}},$$

where C depends on γ, the parameters p, λ, φ, A and the diameter d_X, but does not depend on f or σ.

Lemma 3.4 (Extended reduction lemma). *Let U and Λ be operators* (*not necessarily sublinear*) *satisfying the following relation in Morrey spaces:*

$$\|Uf\|_{L^{q-\varepsilon,\lambda-A_2(\varepsilon)}(X)} \le C_{p-\varepsilon,\lambda-A_1(\varepsilon),q-\varepsilon,\lambda-A_2(\varepsilon)} \|\Lambda f\|_{L^{p-\varepsilon,\lambda-A_1(\varepsilon)}(X)}$$

for all sufficiently small $\varepsilon \in (0,\sigma]$, where $0 < \sigma < s_{\max}$. If

$$\sup_{0<\varepsilon<\sigma} C_{p-\varepsilon,\lambda-A_1(\varepsilon),q-\varepsilon,\lambda-A_2(\varepsilon)} < \infty$$

and

$$\sup_{0<\varepsilon<\sigma} \frac{\psi(\varepsilon)^{1/(q-\varepsilon)}}{\varphi(\varepsilon)^{1/(p-\varepsilon)}} < \infty,$$

then the relation is also valid in generalized grand Morrey spaces:

$$\|Uf\|_{L^{q),\lambda)}_{\psi,A_2}(X)} \le C\|\Lambda f\|_{L^{p),\lambda)}_{\varphi,A_1}(X)}$$

with

$$C = \frac{C_0}{\varphi(\sigma)^{1/(p-\sigma)}} \sup_{0<\varepsilon<\sigma} C_{p-\varepsilon,\lambda-A_1(\varepsilon),q-\varepsilon,\lambda-A_2(\varepsilon)},$$

where C_0 may depend on γ, p, λ, φ, A and d_X, but does not depend on σ or f.

Proof. The proof follows the same lines as for the case where Λ is the identity operator (see [22]). ∎

Using the reduction lemma we obtain the boundedness of maximal and Calderón–Zygmund operators in generalized grand Morrey spaces:

Theorem 3.5. *Let $1 < p < \infty$ and $0 \le \lambda < 1$. Then the Hardy–Littlewood maximal operator is bounded from $L^{p),\lambda)}_{\varphi,A}(X,\mu)$ to $L^{p),\lambda)}_{\psi,A}(X,\mu)$ if there exists a small σ such that*

$$\sup_{0<\varepsilon<\sigma} \psi(\varepsilon)^{1/(q-\varepsilon)}/\varphi(\varepsilon)^{1/(p-\varepsilon)} < \infty.$$

Theorem 3.6. *Let $1 < p < \infty$, $\theta > 0$ and $0 < \lambda < 1$. Then the Calderón–Zygmund operator T is bounded in the generalized grand Morrey space $L^{p),\lambda)}_{\theta,A}(X,\mu)$.*

4. Boundedness of commutators in generalized grand Morrey spaces

4.1. Commutators of Calderón–Zygmund operators. Before proving the main result of this subsection, we need some auxiliary results. The following theorem was proved in [29].

THEOREM 4.1. *Let $1 < p < \infty$, $\omega \in A_\infty(X)$ and $b \in \mathrm{BMO}(X,\mu)$. Then there is a constant C_p depending on the space (X,d,μ) and the $A_\infty(X)$ constant of ω such that*

$$\int_X |T_b f(x)|^p \omega(x)\, d\mu(x) \le C_p \|b\|^p_{\mathrm{BMO}(X,\mu)} \int_X (M^2 f(x))^p \omega(x)\, d\mu(x) \tag{4.1}$$

for all $f \in \mathcal{D}(X)$, where M^2 is the twice iterated Hardy–Littlewood maximal operator.

REMARK. Analyzing the proof of Theorem 4.1 in [29], when $\omega \in A_1(X)$, we see that the constant C_p in (4.1) has the property that for every p there exists η such that

$$\sup_{0<\varepsilon<\eta} C_{p-\varepsilon} < \infty.$$

COROLLARY 4.2. *Under the assumptions of Theorem 4.1,*

$$\int_X |T_b f(x)|^p \omega(x)\, d\mu(x) \le C \|b\|^p_{\mathrm{BMO}(X,\mu)} \int_X |f(x)|^p \omega(x)\, d\mu(x)$$

for all $\omega \in A_p(X)$ and $f \in \mathcal{D}(X)$.

THEOREM 4.3. *Let $1 < p < \infty$ and $0 < \lambda < 1$. Suppose that $b \in \mathrm{BMO}(X,\mu)$. Then*

$$\|T_b f\|_{L^{p,\lambda}(X,\mu)} \le C_{p,\lambda} \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}, \qquad f \in \mathcal{D}(X), \tag{4.2}$$

where the positive constant C depends only on (X,d,μ), p and λ.

Proof. For simplicity assume that d is a metric and let $B \equiv B(x,r)$. It is known that

$$(M\chi_B)^\delta \in A_1(X) \quad \text{for } 0 < \delta < 1$$

(the proof is the same as in the classical case, see [4]). Let $0 < \lambda < \delta < 1$. Using Corollary 4.2 and the definition of the classical Morrey space, we have

(assuming that $0 < r < d_X/2$)

$$\begin{aligned}(4.3)\qquad \int\limits_{B(x,r)} |T_b f(x)|^p\,d\mu(x) &\le C\int\limits_X |T_b f(x)|^p (M(\chi_B)(x))^\delta\,d\mu(x)\\ &\le C\|b\|^p_{\mathrm{BMO}(X,\mu)}\int\limits_X |f(x)|^p (M(\chi_B)(x))^\delta\,d\mu(x)\\ &= C\|b\|^p_{\mathrm{BMO}(X,\mu)}\Big[\Big(\int\limits_{B(x,r)} + \int\limits_{(B(x,r))^{\complement}}\Big)|f(x)|^p (M(\chi_B)(x))^\delta\,d\mu(x)\Big]\\ &\le C\|b\|^p_{\mathrm{BMO}(X,\mu)}\Big[\Big(\frac{1}{(\mu B)^\lambda}\int\limits_{B(x,r)} |f(x)|^p\,d\mu(x)\Big)(\mu B)^\lambda + \sum_{n=0}^m I_n\Big]\end{aligned}$$

where the last inequality comes from the fact that $M(\chi_B)(x) \le 1$ and $I_n := \int_{2^{n+1}B\setminus 2^n B} |f(x)|^p (M(\chi_B)(x))^\delta\,d\mu(x)$. Since $x \in 2^{n+1}B \setminus 2^n B$ implies $M(\chi_B)(x) \le C\mu B/\mu(2^{n+1}B)$ we get

$$(4.4)\qquad I_n \le \int\limits_{2^{n+1}B} |f(x)|^p \Big(\frac{\mu B}{\mu(2^{n+1}B)}\Big)^\delta\,d\mu(x).$$

By (4.3) and (4.4),

$$\begin{aligned}\int\limits_{B(x,r)} |T_b f(x)|^p\,d\mu(x) &\le C\|b\|^p_{\mathrm{BMO}(X,\mu)}\|f\|^p_{L^{p,\lambda}(X,\mu)}(\mu B)^\lambda\Big(1+\sum_{k=0}^\infty 2^{n_0 k(\lambda-\delta)}\Big)\\ &\le C\|b\|^p_{\mathrm{BMO}(X,\mu)}\|f\|^p_{L^{p,\lambda}(X,\mu)},\end{aligned}$$

where we have used the reverse doubling condition $\mu(2B) \ge 2^{n_0}\mu B$, for some constant n_0.

Finally, we have

$$\|T_b f\|_{L^{p,\lambda}(X,\mu)} \le C_{p,\lambda}\|b\|_{\mathrm{BMO}(X,\mu)}\|f\|_{L^{p,\lambda}(X,\mu)}, \qquad f \in \mathcal{D}(X),$$

where the positive constant $C_{p,\lambda}$ depends on p and λ. ■

Remark. Analyzing the proofs of Theorem 4.3 and the Remark after Theorem 4.1, we see that the constant $C_{p,\lambda}$ in (4.2) satisfies the following condition: for every p and λ there are η and σ such that

$$(4.5)\qquad \sup_{\substack{0<\varepsilon<\eta\\ 0<\alpha<\sigma}} C_{p-\varepsilon,\lambda-\alpha} < \infty.$$

Theorem 4.4. *Let* $1 < p < \infty$, $\theta > 0$ *and* $0 < \lambda < 1$. *Suppose* T *is a Calderón–Zygmund operator and* $b \in \mathrm{BMO}(X,\mu)$. *Then the commutator* $[b,T]$ *is bounded in* $L^{p),\lambda)}_{\theta,A}(X,\mu)$.

Proof. This follows from the extended reduction Lemma 3.4, Theorem 4.3 and relation (4.5). ■

4.2. Commutators of potential operators. Let $0 < \alpha < 1$ and let

$$I^\alpha f(x) = \int_X \frac{f(y)}{\mu B(x, d(x,y))^{1-\alpha}}\, d\mu(y)$$

be a potential operator.

The following lemma was shown in [40] by well-known arguments; we give a slightly modified proof for completeness and for the convenience of the reader.

LEMMA 4.5. *Let I^α be a potential operator, $1 < s < p < \infty$, $0 < \alpha < (1-\lambda)/p$, $0 \le \lambda < 1$ and $1/p - 1/q = \alpha/(1-\lambda)$. If $b \in \mathrm{BMO}(X,\mu)$, then there exists a constant $C_{p,\alpha,\lambda} > 0$ such that for all functions f with compact support,*

$$\|M([b, I^\alpha]f)\|_{L^{q,\lambda}(X,\mu)} \le C_{p,\alpha,\lambda} \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)},$$

where

$$C_{p,\alpha,\lambda} = C\big(C_d^{\lambda s/p}((p/s)')^{s/p} + 1\big)^{1+p/q}\bigg(1 + \frac{p}{1-\lambda-\alpha p}\bigg)[(p')^{1/q} + 1], \tag{4.6}$$

and C_d is the doubling constant.

Proof. For any ball $B = B(x,r) \subset X$ and any real number c where c does not depend on s, p, α, λ or C_d, we write

$$\begin{aligned}[b, I^\alpha]f(y) &= [b - c, I^\alpha]f(y) \\ &= (b-c)I^\alpha f(y) - I^\alpha((b-c)f\chi_{c_0B})(y) - I^\alpha((b-c)f\chi_{(c_0B)^\complement})(y) \\ &= \mathscr{I}_1(y) - \mathscr{I}_2(y) - \mathscr{I}_3(y),\end{aligned}$$

where c_0 is a constant depending on C_t and C_s, to be determined later. Then, by the sublinearity of the maximal operator, we have

$$M([b, I^\alpha]f)(x) \le M\mathscr{I}_1(x) + M\mathscr{I}_2(x) + M\mathscr{I}_3(x).$$

For $M\mathscr{I}_1(x)$, we have the pointwise estimate

$$M\mathscr{I}_1(x) \le C\|b\|_{\mathrm{BMO}(X,\mu)} M_s(I^\alpha f(x)),$$

which follows from Hölder's inequality. Taking Lemma 2.2 and the bound of $\|I^\alpha\|_{L^{p,\lambda}(X,\mu)\to L^{q,\lambda}(X,\mu)}$ (see [27, Lemma 4.5]) into account we have

$$\begin{aligned}\|M\mathscr{I}_1\|_{L^{q,\lambda}(X,\mu)} &\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|M_s(I^\alpha f)\|_{L^{q,\lambda}(X,\mu)} \\ &\lesssim (C_d^{\lambda s/p}((p/s)')^{s/p} + 1)\|b\|_{\mathrm{BMO}(X,\mu)} \|I^\alpha f\|_{L^{q,\lambda}(X,\mu)} \\ &\lesssim C_{p,\alpha,\lambda}\|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)},\end{aligned} \tag{4.7}$$

where $C_{p,\alpha,\lambda}$ is (4.6).

For $1 < s < p < \infty$, $0 < \alpha < (1-\lambda)/p$, there exist $s_0, s_1, t_0, t > 1$ such that $1/s_0 = 1/t_0 - \alpha$, $1/t_0 = 1/s_1 + 1/s$ and $s/t = \alpha p/(1-\lambda)$. By Hölder's inequality together with Jensen's inequality and the fact that I^α is of strong type (t_0, s_0) we have (remember that $B := B(x,r)$)

$$
\begin{aligned}
M\mathscr{I}_2(x) &\le \sup_{0<r<d_X} \Big(\fint_B |I^\alpha((b-c)f\chi_{c_0B})(y)|^{s_0}\, d\mu(y)\Big)^{1/s_0} \\
&\lesssim \sup_{0<r<d_X} \bigg(\frac{1}{\mu(B)^{1-t_0\alpha}} \int_{c_0B} |b(y)-c|^{t_0} |f(y)|^{t_0}\, d\mu(y)\bigg)^{1/t_0} \\
&\lesssim \sup_{0<r<d_X} \bigg(\frac{1}{\mu(B)^{1-t_0\alpha}} \int_{c_0B} |b(y)-c|^{t_0} |f(y)|^{t_0}\, d\mu(y)\bigg)^{1/t_0} \\
&\lesssim \sup_{0<r<d_X} \bigg(\frac{1}{\mu(B)^{1-t_0\alpha}} \int_{c_0B} |b(y)-c|^{t_0} |f(y)|^{t_0}\, d\mu(y)\bigg)^{1/t_0}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sup_{0<r<d_X} \Big(\fint_{c_0B} |b(y)-c|^{s_1}\, d\mu(y)\Big)^{1/s_1} \bigg(\frac{1}{\mu(c_0B)^{1-s\alpha}} \int_{c_0B} |f(y)|^s\, d\mu(y)\bigg)^{1/s} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \sup_{0<r<d_X} \mu(c_0B)^{\alpha-1/s} \Big(\int_{c_0B} |f(y)|^s\, d\mu(y)\Big)^{1/s-1/t} \\
&\qquad\qquad \times \Big[\mu(c_0B)^{1-s/p} \Big(\int_{c_0B} |f(y)|^p d\mu(y)\Big)^{s/p}\Big]^{1/t} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \sup_{0<r<d_X} \mu(c_0B)^{\alpha-1/s+1/t-s(1-\lambda)/(pt)} \Big(\int_{c_0B} |f(y)|^s\, d\mu(y)\Big)^{1/s-1/t} \\
&\qquad\qquad \times \bigg(\frac{1}{\mu(c_0B)^{\lambda}} \int_{c_0B} |f(y)|^p\, d\mu(y)\bigg)^{s/(pt)} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{1-\lambda}} \sup_{0<r<d_X} \Big(\fint_{c_0B} |f(y)|^s\, d\mu(y)\Big)^{\frac{1}{s}(1-s/t)} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{1-\lambda}} (M_s f(x))^{1-\frac{\alpha p}{1-\lambda}}.
\end{aligned}
$$

Consequently, by Lemma 2.2,

$$
\begin{aligned}
\|M\mathscr{I}_2\|_{L^{q,\lambda}(X,\mu)} &\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{1-\lambda}} \|(M_s f)^{1-\frac{\alpha p}{1-\lambda}}\|_{L^{q,\lambda}(X,\mu)} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{1-\lambda}} \|M_s f\|_{L^{p,\lambda}}^{p/q} \\
&\lesssim (C_{p,\alpha,\lambda})^{p/q} \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)},
\end{aligned}
\tag{4.8}
$$

where $C_{p,\alpha,\lambda}$ is as in (4.6).

By the reverse doubling condition (see (2.3)), there exist constants $0 < \alpha, \beta < 1$ such that for all $x \in X$ and small positive r, $\mu B(x, \alpha r) \le \beta \mu B(x,r)$. Let us take an integer m so that $\alpha^m d_X$ is sufficiently small.

Observe now that (see also [20, p. 929]) if $z \in B(x,r)$, then

$$B(x,r) \subset B(z, C_t(C_s+1)r) \subset B\big(x, C_t(C_t(C_s+1)+1)r\big);$$

we rewrite it simply as $B(x,r) \subset B(z,c_1r) \subset B(x,c_2r)$. Hence,

$$\begin{aligned}
\|M\mathscr{I}_3\|_{L^{q,\lambda}(X,\mu)} &\le \sup_{\substack{x\in X\\ 0<r<d_X}} \bigg(\frac{1}{\mu B(x,r)^\lambda} \int\limits_{B(x,r)} |M(\mathscr{I}_3)(y)|^q\, d\mu(y)\bigg)^{1/q} \\
&\lesssim \sup_{\substack{x\in X\\ 0<r<d_X}} \mu B(x,r)^{(1-\lambda)/q} \sup_{B\subset B(z,c_1r)} \frac{1}{\mu B(z,c_1r)} \int\limits_{B(z,c_1r)} |\mathscr{I}_3(y)|\, d\mu(y) \\
&\lesssim \sup_{B\subset B(z,c_1r)} \mu B(z,c_1r)^{(1-\lambda)/q-1} \int\limits_{B(z,c_1r)} |\mathscr{I}_3(y)|\, d\mu(y).
\end{aligned}$$

Further, notice that when c_0 is an appropriate constant, $B \subset B(z,c_1r)$, $y \in B(z,c_1r)$, $\alpha^m d(y,z) \le d(y,t) \le \alpha^{m+1} d(y,z)$ and $z \in (c_0B)^c$, then $d(x,t) > \bar{c}_0 r$, where $\bar{c}_0$ depends on C_t, C_s and c_0; it is also easy to check that there are positive constants b_1, b_2 and b_3 such that $B(y, b_1 d(y,t)) \subset B(x, b_2 d(x,t)) \subset B(y, b_3 d(y,t))$. Consequently, by using Fubini's theorem and Lemma 1.2 of [19] we have, for $y \in B(z,c_1r)$,

$$\begin{aligned}
\mathscr{I}_3(y) &\le \int\limits_{X\setminus c_0B} |(b(z)-c)f(z)| \mu B(y, d(y,z))^{\alpha-1}\, d\mu(z) \\
&\le C \int\limits_{X\setminus c_0B} |(b(z)-c)f(z)| \\
&\qquad \times \bigg(\int\limits_{B(y,\alpha^m d(y,z))\setminus B(y,\alpha^{m-1}d(y,z))} \mu B(y,d(y,t))^{\alpha-2}\, d\mu(t)\bigg)\, d\mu(z) \\
&\le C \int\limits_{X\setminus B(x,\bar{c}_0 r)} \mu B(y,d(y,t))^{\alpha-2} \\
&\qquad \times \bigg(\int\limits_{B(y,a^{-m}d(y,t))} |(b(z)-c)f(z)|\, d\mu(z)\bigg)\, d\mu(t) \\
&\le C\|b\|_{\mathrm{BMO}(X,\mu)} \int\limits_{X\setminus B(x,\bar{c}_0 r)} \mu B(y,d(y,t))^{\alpha-1} \\
&\qquad \times \bigg(\frac{1}{\mu B(y, a^{1-m}d(y,t))} \int\limits_{B(y,a^{1-m}d(y,t))} |f(z)|^p\, d\mu(z)\bigg)^{1/p} d\mu(t) \\
&\le C\|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)} \int\limits_{X\setminus B(x,\bar{c}_0 r)} \mu B(y,d(y,t))^{\alpha-(1-\lambda)/p-1}\, d\mu(t) \\
&\le C\mu B(x,r)^{\alpha-(1-\lambda)/p} \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}.
\end{aligned}$$

Thus applying the relation between $B(x,r)$ and $B(z,r)$ we find that

$$\begin{aligned} (4.9)\qquad & \|M\mathscr{I}_3\|_{L^{q,\lambda}(X,\mu)} \\ & \lesssim \|b\|_{\mathrm{BMO}(X,\mu)}\|f\|_{L^{p,\lambda}(X,\mu)} \sup_{B\subset B(z,c_1r)} \mu B(z,c_1r)^{(1-\lambda)/q+\alpha-(1-\lambda)/p} \\ & \lesssim \|b\|_{\mathrm{BMO}(X,\mu)}\|f\|_{L^{p,\lambda}(X,\mu)}. \end{aligned}$$

Gathering (4.7)–(4.9) it is easy to show that

$$\|M([b,I^\alpha]f)\|_{L^{q,\lambda}(X,\mu)} \le C_{p,\alpha,\lambda}\|b\|_{\mathrm{BMO}(X,\mu)}\|f\|_{L^{p,\lambda}(X,\mu)}. \ \blacksquare$$

Before proving the next result, we define the following auxiliary functions which where introduced in [22].

Definition 4.6 (auxiliary functions). On an interval $(0,\delta]$, δ small, we define the following functions:

$$\bar\phi(x) := p + \frac{(x-q)(1-\lambda+A_2(x))}{1-\lambda+A_2(x)-\alpha(x-q)}, \qquad \tilde\phi(x) := q - \frac{(p-x)(1-\lambda+A_1(x))}{1-\lambda+A_1(x)-\alpha(p-x)},$$

$$\bar A(x) := 1 - \frac{\alpha(x-q)}{1-\lambda+A_2(x)}, \qquad \tilde A(x) := \frac{1-\lambda+A_1(\eta)}{1-\lambda+A_1(\eta)-(p-\eta)\alpha},$$

$$\phi(x) := \bar\phi(x)^{\bar A(x)}, \qquad \Phi(x) := \tilde\phi(x)^{\tilde A(x)},$$

$$\psi(\varepsilon) := \phi(\varepsilon^{\theta_1}), \qquad \Psi(\varepsilon) := \Phi(\varepsilon^{\theta_1}),$$

for $\theta_1 > 0$.

Theorem 4.7. *Let I^α be a potential operator and let M be the maximal operator. Assume that $1<p<\infty$, $0<\alpha<(1-\lambda)/p$, $0<\lambda<1$, $1/p-1/q=\alpha/(1-\lambda)$. Suppose that $\theta_1>0$ and $\theta_2 \ge \theta_1[1+\alpha q/(1-\lambda)]$. Let A_1 and A_2 be continuous non-negative functions on $(0,p-1]$ and $(0,q-1]$ respectively satisfying the conditions:*

(i) *$A_2 \in C^1((0,\delta])$ for some positive $\delta>0$;*
(ii) *$\lim_{x\to 0+} A_2(x) = 0$;*
(iii) *$0 \le B := \lim_{x\to 0+} \frac{d}{dx}A_2(x) < \frac{(1-\lambda)^2}{\alpha q^2}$;*
(iv) *$A_1(\eta) = A_2(\bar\phi^{-1}(\eta))$, where $\bar\phi^{-1}$ is the inverse of $\bar\phi$ on $(0,\delta]$ for some $\delta>0$.*

If $b\in \mathrm{BMO}(X,\mu)$, then the operator $M([b,I^\alpha])$ is bounded from the space $L^{p),\lambda}_{\theta_1,A_1}(X,\mu)$ to $L^{q),\lambda}_{\theta_2,A_2}(X,\mu)$.

Proof. We note that it is enough to prove the theorem for $\theta_2=\theta_1(1+\alpha q/(1-\lambda))$ because $\varepsilon^{\theta_2} \le \varepsilon^{\theta_1(1+\alpha q/(1-\lambda))}$ for $\theta_2 > \theta_1[1+\alpha q/(1-\lambda)]$ and small ε. We also note that, by L'Hospital's rule, $\bar\phi(x)\sim x$ as $x\to 0+$ since $B<(1-\lambda)^2/(\alpha q^2)$. Moreover, $\bar\phi$ is invertible near 0, since $\frac{d\bar\phi}{dx}(x)>0$. Under the conditions of Theorem 4.7 the function A_1 is continuous on $(0,\delta]$ and $\lim_{x\to 0+}A_1(x)=0$. With all of the previous remarks taken into account, it is enough to prove the boundedness of $M([b,I^\alpha])$ from $L^{p),\lambda)}_{\theta_1,A_1}(X,\mu)$ to $L^{q),\lambda)}_{\psi,A_2}(X,\mu)$ since $\phi(x)\sim x^{1+\alpha q/(1-\lambda)}$, and consequently

$$\psi(x)=\phi(x^{\theta_1})\sim x^{\theta_1(1+\alpha q/(1-\lambda))}\qquad\text{as } x\to 0.$$

The case $\sigma<\varepsilon\le s_{\max}$, where $s_{\max}$ is from (3.1). Letting

$$I:=\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\left(\frac{1}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}}\int\limits_{B(x,r)}|M([b,I^\alpha]f)(y)|^{q-\varepsilon}\,d\mu(y)\right)^{\frac{1}{q-\varepsilon}}$$

we have

$$\begin{aligned}
I&\lesssim\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\mu B(x,r)^{\frac{A_2(\varepsilon)+1-\lambda}{q-\varepsilon}}\Big(\fint\limits_{B(x,r)}|M([b,I^\alpha]f)(y)|^{q-\sigma}\,d\mu(y)\Big)^{\frac{1}{q-\sigma}}\\
&\lesssim\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\mu B(x,r)^{\frac{A_2(\sigma)+1-\lambda}{q-\sigma}}\Big(\fint\limits_{B(x,r)}|M([b,I^\alpha]f)(y)|^{q-\sigma}\,d\mu(y)\Big)^{\frac{1}{q-\sigma}}\\
&\lesssim\Big(\sup_{\sigma\le\varepsilon\le s_{\max}}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\Big)\psi(\sigma)^{\frac{1}{\sigma-q}}\\
&\qquad\times\sup_{0<\varepsilon\le\sigma}\sup_{\substack{x\in X\\ r>0}}\left(\frac{\psi(\varepsilon)}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}}\int\limits_{B(x,r)}|M([b,I^\alpha]f)(y)|^{q-\varepsilon}\,d\mu(y)\right)^{\frac{1}{q-\varepsilon}},
\end{aligned}$$

where the first inequality comes from Hölder's inequality and the second one is due to the fact that A_2 is bounded on $[\sigma,q-1)$ and $x\mapsto(1-\lambda)/(q-x)$ is an increasing function. Hence, it is enough to consider the case $0<\varepsilon\le\sigma$.

The case $0<\varepsilon\le\sigma$. Let η and ε be chosen so that

$$\frac{1}{p-\eta}-\frac{1}{q-\varepsilon}=\frac{\alpha}{1-\lambda+A_2(\varepsilon)}. \tag{4.10}$$

Obviously, $\varepsilon\to 0$ if and only if $\eta\to 0$, and solving for η in (4.10) we obtain

$$\eta=p-\frac{(q-\varepsilon)(1-\lambda+A_2(\varepsilon))}{1-\lambda+A_2(\varepsilon)-\alpha(\varepsilon-q)}=\bar{\phi}(\varepsilon).$$

Letting

$$J:=\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\left(\frac{1}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}}\int\limits_{B(x,r)}|M([b,I^\alpha]f)(y)|^{q-\varepsilon}\,d\mu(y)\right)^{\frac{1}{q-\varepsilon}}$$

we have

$$J \lesssim C\psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \sup_{\substack{x\in X\\ r>0}} \bigg(\frac{1}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}} \int\limits_{B(x,r)} |f(y)|^{p-\eta}\, d\mu(y)\bigg)^{\frac{1}{p-\eta}}$$

$$\lesssim C\eta^{\frac{\theta_1}{\eta-p}}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \sup_{\substack{x\in X\\ r>0}} \bigg(\frac{\eta^{\theta_1}}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}} \int\limits_{B(x,r)} |f(y)|^{p-\eta}\, d\mu(y)\bigg)^{\frac{1}{p-\eta}}$$

$$\lesssim \|f\|_{L^{p),\lambda)}_{\theta_1,A_1}(X,\mu)},$$

where $C := C_{p-\eta,q-\varepsilon,\alpha,\lambda-A_2(\varepsilon)}$ is the constant from (4.6) and the first inequality is due to Lemma 4.5. The last inequality follows from $\eta = \bar{\phi}(\varepsilon)$. Since the constant in the last inequality is uniformly bounded with respect to ε, we obtain the desired boundedness of the operator. ∎

COROLLARY 4.8. *Let the assumptions of Theorem* 4.7 *be satisfied. Then the commutator* $[b, I^\alpha]$ *is bounded from* $L^{p),\lambda)}_{\theta_1,A_1}(X,\mu)$ *to* $L^{q),\lambda)}_{\theta_2,A_2}(X,\mu)$.

Proof. The result follows from the previous theorem and the inequality

$$\|[b, I^\alpha]f\|_{L^{q),\lambda)}_{\theta_2,A_2}(X,\mu)} \le \|M([b, I^\alpha]f)\|_{L^{q),\lambda)}_{\theta_2,A_2}(X,\mu)}. \ ∎$$

5. Interior estimates for elliptic equations. In this section we apply the main result of this paper to establish some interior estimates of solutions to non-divergence elliptic equations with VMO coefficients (see also [25] for related topics). Suppose $n \ge 3$ and Ω is an open set in $\mathbb{R}^n$. Let

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j},$$

where $a_{ij} = a_{ji}$ for $i, j = 1, \dots, n$, a.e. in Ω; assume that there exists $C > 0$ such that, for $y = (y_1, \dots, y_n) \in \mathbb{R}^n$,

$$C^{-1}|y|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)y_iy_j \le C|y|^2 \quad \text{for a.e. } x \in \Omega;$$

denote by $(A_{ij})_{n\times n}$ the inverse of the matrix $(a_{ij})_{n\times n}$. For $x \in \Omega$ and $y \in \mathbb{R}^n$, let

$$K(x,y) = \frac{1}{(n-2)C_n\sqrt{\det(a_{ij}(x))}}\Big(\sum_{i,j=1}^{n} A_{ij}(x)y_iy_j\Big)^{1-n/2},$$

$$K_i(x,y) = \frac{\partial}{\partial y_i}K(x,y), \qquad K_{ij}(x,y) = \frac{\partial^2}{\partial x_i \partial x_j}K(x,y).$$

We denote by VMO(Ω) the class of all locally integrable functions with vanishing mean oscillation introduced in [36] (used e.g. in [33] and [34]).

From [2, 7], we obtain the *interior representation formula*: if $a_{ij} \in \mathrm{VMO} \cap L^\infty(\Omega)$ and $u \in W_0^{2,r}(\Omega)$, $1 < r < \infty$ (see [2, 3, 15]), then

$$u_{x_i x_j}(x) = \text{p.v.} \int_B K_{ij}(x, x-y)\Big[\sum_{k,l=1}^n (a_{kl}(x) - a_{kl}(y))u_{x_k x_l}(y) + Lu(y)\Big]\,dy + Lu(x)\int_{|y|=1} K_i(x,y)y_j\,d\delta_y,$$

for a.e. $x \in B \subset \Omega$, where B is a ball in Ω. We also set

$$M := \max_{i,j=1,\dots,n}\ \max_{|\alpha|\le 2n} \|\partial^\alpha K_{ij}(x,y)/\partial y^\alpha\|_{L^\infty}.$$

To prove the next statement we need a local version of Theorem 4.3 (see also Theorem 2.4 in [2] or Theorem 2.13 in [3]).

Corollary 5.1. *Let $1 < p < \infty$ and let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that $a \in \mathrm{VMO} \cap L^\infty$. Assume that T is a Calderón–Zygmund operator defined on Ω and η is the VMO modulus of a. Then for any $\varepsilon > 0$, there exists a positive number $\rho = \rho(\varepsilon, \eta)$ such that for any balls B_r with $\Omega_r := B_r \cap \Omega \ne \emptyset$, $r \in (0, \rho)$ and all $f \in L_{\theta,A}^{p),\lambda)}(\Omega_r)$, we have*

$$\|[a,T]f\|_{L_{\theta,A}^{p),\lambda)}(\Omega_r)} \le C\varepsilon \|f\|_{L_{\theta,A}^{p),\lambda)}(\Omega_r)}.$$

Theorem 5.2. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that $1 < p, r < \infty$. Let $a_{ij} \in \mathrm{VMO}(\Omega) \cap L^\infty$, $i, j = 1, \dots, n$. Suppose that $\eta_{i,j}$ is the VMO modulus of a_{ij}; set $\eta = (\sum_{i,j=1}^n \eta_{i,j})^{1/2}$. Suppose also that $M < \infty$. Then there is a positive constant $\rho = \rho(n, r, p, \lambda, M, \theta, A, \eta)$ such that for all balls $B \subset \Omega$ with radius smaller than ρ, and all $u \in W_0^{2,r}(\Omega)$ with $\|Lu\|_{L_{\theta,A}^{p),\lambda)}(B)} < \infty$, we have $u_{x_i x_j} \in L_{\theta,A}^{p),\lambda)}(B)$, and there exists a positive constant $C = C(n, p, \lambda, \theta, M, A, \eta)$ such that*

$$\|u_{x_i x_j}\|_{L_{\theta,A}^{p),\lambda)}(B)} \le C\|Lu\|_{L_{\theta,A}^{p),\lambda)}(B)}.$$

Proof. It is easy to verify that K_{ij} satisfies the assumptions of Corollary 5.1 by the representation of $u_{x_i x_j}$ and the conditions of K_{ij}. Thus, from Corollary 5.1, we deduce, for any $\varepsilon > 0$,

$$\|u_{x_i x_j}\|_{L_{\theta,A}^{p),\lambda)}(B)} \le C\varepsilon \|u_{x_i x_j}\|_{L_{\theta,A}^{p),\lambda)}(B)} + C\|Lu\|_{L_{\theta,A}^{p),\lambda)}(B)}.$$

Choosing ε to be small enough (e.g. $\varepsilon < 1$), we then obtain

$$\|u_{x_i x_j}\|_{L_{\theta,A}^{p),\lambda)}(B)} \le (C/(1 - C\varepsilon))\|Lu\|_{L_{\theta,A}^{p),\lambda)}(B)}.$$

This finishes the proof. ■

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