STUDIA MATHEMATICA 217 (2) (2013)

# Doubly commuting submodules of the Hardy module over polydiscs

by

Jaydeb Sarkar (Bangalore), Amol Sasane (London) and Brett D. Wick (Atlanta, GA)

**Abstract.** In this note we establish a vector-valued version of Beurling's theorem (the Lax–Halmos theorem) for the polydisc. As an application of the main result, we provide necessary and sufficient conditions for the "weak" completion problem in $H^\infty(\mathbb{D}^n)$.

**1. Introduction and statement of main results.** In [B], Beurling described all the invariant subspaces for the operator $M_z$ of "multiplication by $z$" on the Hilbert space $H^2(\mathbb{D})$ of the disc. In [L], Peter Lax extended Beurling's result to the (finite-dimensional) vector-valued case (while also considering the Hardy space of the half-plane). Lax's vectorial case proof was further extended to infinite-dimensional vector spaces by Halmos (see [NF]). The characterization of $M_z$-invariant subspaces obtained is the following famous result.

Theorem 1.1 (Beurling–Lax–Halmos). *Let $\mathcal{S}$ be a closed nonzero subspace of $H^2_{E_*}(\mathbb{D})$. Then $\mathcal{S}$ is invariant under multiplication by $z$ if and only if there exists a Hilbert space $E$ and an inner function $\Theta \in H^\infty_{E\to E_*}(\mathbb{D})$ such that $\mathcal{S} = \Theta H^2_E(\mathbb{D})$.*

For $n \in \mathbb{N}$ and $E_*$ a Hilbert space, $H^2_{E_*}(\mathbb{D}^n)$ is the set of all $E_*$-valued holomorphic functions in the polydisc $\mathbb{D}^n$, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ (with boundary $\mathbb{T}$) such that

$$\|f\|_{H^2_{E_*}(\mathbb{D}^n)} := \sup_{0<r<1} \Big( \int_{\mathbb{T}^n} \|f(r\mathbf{z})\|^2_{E_*} \, d\mathbf{z}\Big)^{1/2} < \infty.$$

On the other hand, if $\mathcal{L}(E, E_*)$ denotes the set of all continuous linear transformations from $E$ to $E_*$, then $H^\infty_{E\to E_*}(\mathbb{D}^n)$ denotes the set of all $\mathcal{L}(E, E_*)$-

2010 *Mathematics Subject Classification*: Primary 46J15; Secondary 47A15, 30H05, 47A56.
*Key words and phrases*: invariant subspace, shift operator, doubly commuting, Hardy algebra on the polydisc, completion problem.

DOI: 10.4064/sm217-2-5  

valued holomorphic functions with

$$\|f\|_{H^\infty_{E\to E_*}(\mathbb{D}^n)} := \sup_{\mathbf{z}\in\mathbb{D}^n} \|f(\mathbf{z})\|_{\mathcal{L}(E,E_*)} < \infty.$$

An operator-valued function $\Theta \in H^\infty_{E\to E_*}(\mathbb{D}^n)$ is *inner* if its pointwise boundary values are isometries a.e.:

$$(\Theta(\zeta))^*\Theta(\zeta) = I_E \quad \text{for almost all } \zeta \in \mathbb{T}^n.$$

A natural question is then to ask about an analogue of Theorem 1.1 in the case of several variables, for example for the Hardy space $H^2_{E_*}(\mathbb{D}^n)$. It is known that in general, a Beurling–Lax–Halmos type characterization of subspaces of this Hardy space is not possible [R]. It is, however, easy to see that $H^2_{E_*}(\mathbb{D}^n)$, when $n > 1$, has the *doubly commuting* property, that is, for all $1 \le i < j \le n$,

$$M^*_{z_i} M_{z_j} = M_{z_j} M^*_{z_i}.$$

We impose this additional assumption on submodules of $H^2_{E_*}(\mathbb{D}^n)$ and call that class of submodules doubly commuting submodules. More precisely:

Definition 1.2. A commuting family of bounded linear operators $\{T_1, \ldots, T_n\}$ on some Hilbert space $\mathcal{H}$ is said to be *doubly commuting* if

$$T_i T^*_j = T^*_j T_i \quad \text{for all } 1 \le i, j \le n \text{ and } i \ne j.$$

A closed subspace $\mathcal{S}$ of $H^2_E(\mathbb{D}^n)$ is said to be a *doubly commuting submodule* if $\mathcal{S}$ is a submodule, that is, $M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i$, and the family $\{R_{z_1}, \ldots, R_{z_n}\}$ of module multiplication operators, where

$$R_{z_i} := M_{z_i}|_{\mathcal{S}} \quad \text{for all } 1 \le i \le n,$$

is doubly commuting, that is,

$$R_{z_i} R^*_{z_j} = R^*_{z_j} R_{z_i} \quad \text{for all } i \ne j \text{ in } \{1, \ldots, n\}.$$

In this note we completely characterize the doubly commuting submodules of $H^2_{E_*}(\mathbb{D}^n)$. This result is an analogue of the classical Beurling–Lax–Halmos theorem.

Theorem 1.3. *Let $\mathcal{S}$ be a closed nonzero subspace of $H^2_{E_*}(\mathbb{D}^n)$. Then $\mathcal{S}$ is a doubly commuting submodule if and only if there exists a Hilbert space $E$ with $E \subseteq E_*$, where the inclusion is up to unitary equivalence, and an inner function $\Theta \in H^\infty_{E\to E_*}(\mathbb{D}^n)$ such that*

$$\mathcal{S} = M_\Theta H^2_E(\mathbb{D}^n).$$

In the special scalar case $E_* = \mathbb{C}$ and when $n = 2$ (the bidisc), this characterization was obtained by Mandrekar [M], and the proof given there relies on the Wold decomposition for two variables [S]. Our proof is based on the more natural language of Hilbert modules and a generalization of Wold decomposition for doubly commuting isometries [Sa].

As an application of this theorem, we can establish a version of the "Weak" Completion Property for the algebra $H^\infty(\mathbb{D}^n)$. Suppose that $E \subset E_c$. Recall that the *Completion Problem* for $H^\infty(\mathbb{D}^n)$ is the problem of characterizing the functions $f \in H^\infty_{E\to E_c}(\mathbb{D}^n)$ such that there exists an invertible function $F \in H^\infty_{E_c\to E_c}(\mathbb{D}^n)$ with $F|_E = f$.

In the case of $H^\infty(\mathbb{D})$, the Completion Problem was settled by Tolokonnikov in [To]. In that paper, it was pointed out that there is a close connection between the Completion Problem and the characterization of invariant subspaces of $H^2(\mathbb{D})$. Using Theorem 1.3 we then have the following analogue of the results in [To].

THEOREM 1.4 (Tolokonnikov's lemma for the polydisc). *Let* $f \in H^\infty_{E\to E_c}(\mathbb{D}^n)$ *with* $E \subset E_c$ *and* $\dim E, \dim E_c < \infty$. *Then the following statements are equivalent:*

(i) *There exists* $g \in H^\infty_{E_c\to E}(\mathbb{D}^n)$ *such that* $gf \equiv I$ *in* $\mathbb{D}^n$ *and the operators* $M_{z_1}, \dots, M_{z_n}$ *doubly commute on* $\ker M_g$.
(ii) *There exists* $F \in H^\infty_{E_c\to E_c}(\mathbb{D}^n)$ *such that* $F|_E = f$, $F|_{E_c\ominus E}$ *is inner, and* $F^{-1} \in H^\infty_{E_c\to E_c}(\mathbb{D}^n)$.

REMARK 1.5. Theorem 1.4 for the polydisc is different from Tolokonnikov's lemma in the disc in which one does not demand that the completion $F$ has the property that $F|_{E_c\ominus E}$ is inner. But, from the proof of Tolokonnikov's lemma in the case of the disc (see [N]), one can see that the following statements are equivalent for $f \in H^\infty_{E\to E_c}(\mathbb{D})$ with $E \subset E_c$ and $\dim E < \infty$:

(i) There exists a function $g \in H^\infty_{E_c\to E}(\mathbb{D})$ such that $gf \equiv I$ in $\mathbb{D}$.
(ii) There exists a function $F \in H^\infty_{E_c\to E_c}(\mathbb{D})$ such that $F|_E = f$ and $F^{-1} \in H^\infty_{E_c\to E_c}(\mathbb{D})$.
(ii′) There exists a function $F \in H^\infty_{E_c\to E_c}(\mathbb{D})$ such that $F|_E = f$, $F|_{E_c\ominus E}$ is inner, and $F^{-1} \in H^\infty_{E_c\to E_c}(\mathbb{D})$.

In the polydisc case it is unclear how the conditions (II) and (II′) below are related:

(II) There exists a function $F \in H^\infty_{E_c\to E_c}(\mathbb{D}^n)$ such that $F|_E = f$ and $F^{-1} \in H^\infty_{E_c\to E_c}(\mathbb{D}^n)$.
(II′) There exists a function $F \in H^\infty_{E_c\to E_c}(\mathbb{D}^n)$ such that $F|_E = f$, $F|_{E_c\ominus E}$ is inner, and $F^{-1} \in H^\infty_{E_c\to E_c}(\mathbb{D}^n)$.

We refer to the Completion Problem in (II) as the *Strong Completion Problem*, while the one in (II′) is the *Weak Completion Problem.* Whether the two are equivalent is an open problem.

We also remark that in the disc case, Tolokonnikov's lemma was proved by Sergei Treil [T] without any assumptions about the finite dimensionality of $E, E_c$. However, our proof of Theorem 1.4 relies on Lemma 3.1, whose validity we do not know without assuming the finite dimensionality of $E$ and $E_c$.

EXAMPLE 1.6. As a simple illustration of Theorem 1.4, take $n = 3$, $\dim E = 1$, $\dim E_c = 3$ and

$$f := \begin{bmatrix} e^{z_1} \\ e^{z_2} \\ e^{z_3} \end{bmatrix} \in (H^\infty(\mathbb{D}^3))^{3\times 1}.$$

With $g := [e^{-z_1} \quad 0 \quad 0] \in (H^\infty(\mathbb{D}^2))^{1\times 3}$, we see that $gf = 1$. We have

$$\begin{aligned} \ker M_g &= \left\{ \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \in (H^2(\mathbb{D}^3))^{3\times 1} : e^{-z_1}\varphi_1 = 0 \right\} \\ &= \left\{ \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \in (H^2(\mathbb{D}^3))^{3\times 1} : \varphi_1 = 0 \right\} = \Theta(H^2(\mathbb{D}^2))^{2\times 1}, \end{aligned}$$

where $\Theta$ is the inner function

$$\Theta := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \in (H^\infty(\mathbb{D}^3))^{3\times 2}.$$

As $\Theta$ is inner, it follows from Theorem 1.3 that $M_{z_1}, M_{z_2}, M_{z_3}$ doubly commute on the submodule $\Theta(H^2(\mathbb{D}^3))^{2\times 1} = \ker M_g$. Hence $f$ can be completed to an invertible matrix. In fact, with

$$F := [f \quad \Theta] = \begin{bmatrix} e^{z_1} & 0 & 0 \\ e^{z_2} & 1 & 0 \\ e^{z_3} & 0 & 1 \end{bmatrix},$$

one can easily see that $F$ is invertible as an element of $(H^\infty(\mathbb{D}^3))^{3\times 3}$.

In Section 2 we give a proof of Theorem 1.3, and subsequently, in Section 3, we use this theorem to study the Weak Completion Problem for $H^\infty(\mathbb{D}^n)$, providing a proof of Theorem 1.4.

**2. Beurling–Lax–Halmos theorem for the polydisc.** In this section we present a complete characterization of "reducing submodules" and a proof of the Beurling–Lax–Halmos theorem for doubly commuting submodules of $H^2_E(\mathbb{D}^n)$.

Recall that a closed subspace $\mathcal{S} \subseteq H^2_E(\mathbb{D}^n)$ is said to be a *reducing submodule* of $H^2_E(\mathbb{D}^n)$ if $M_{z_i}\mathcal{S}, M^*_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \dots, n$.

We start by reviewing some definitions and well-known facts about the vector-valued Hardy space over polydisc. For more details about reproducing kernel Hilbert spaces over domains in $\mathbb{C}^n$, we refer the reader to [DMS]. Let

$$\mathbb{S}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{j=1}^{n} (1 - \overline{w}_j z_j)^{-1} \qquad ((\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{D}^n \times \mathbb{D}^n)$$

be the Cauchy kernel on $\mathbb{D}^n$. Then for some Hilbert space $E$, the kernel function $\mathbb{S}_E$ of $H^2_E(\mathbb{D}^n)$ is given by

$$\mathbb{S}_E(\boldsymbol{z}, \boldsymbol{w}) = \mathbb{S}(\boldsymbol{z}, \boldsymbol{w}) I_E \qquad ((\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{D}^n \times \mathbb{D}^n).$$

In particular, $\{\mathbb{S}(\cdot, \boldsymbol{w})\eta : \boldsymbol{w} \in \mathbb{D}^n, \eta \in E\}$ is a *total subset* for $H^2_E(\mathbb{D}^n)$, that is,

$$\overline{\operatorname{span}}\{\mathbb{S}(\cdot, \boldsymbol{w})\eta : \boldsymbol{w} \in \mathbb{D}^n, \eta \in E\} = H^2_E(\mathbb{D}^n),$$

where $\mathbb{S}(\cdot, \boldsymbol{w}) \in H^2(\mathbb{D}^n)$ and

$$\mathbb{S}(\cdot, \boldsymbol{w})(\boldsymbol{z}) = \mathbb{S}(\boldsymbol{z}, \boldsymbol{w}) \qquad \text{for all } \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

Moreover, for all $f \in H^2_E(\mathbb{D}^n)$, $\boldsymbol{w} \in \mathbb{D}^n$ and $\eta \in E$

$$\langle f, \mathbb{S}(\cdot, \boldsymbol{w})\eta \rangle_{H^2_E(\mathbb{D}^n)} = \langle f(\boldsymbol{w}), \eta \rangle_E.$$

Note also that for the multiplication operator $M_{z_i}$ on $H^2_E(\mathbb{D}^n)$,

$$M^*_{z_i}(\mathbb{S}(\cdot, \boldsymbol{w})\eta) = \bar{w}_i(\mathbb{S}(\cdot, \boldsymbol{w})\eta) \qquad \text{for all } \boldsymbol{w} \in \mathbb{D}^n,\ \eta \in E \text{ and } 1 \le i \le n.$$

We also have

$$\mathbb{S}^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{0 \le i_1 < \cdots < i_l \le n} (-1)^l z_{i_1} \cdots z_{i_l} \bar{w}_{i_1} \cdots \bar{w}_{i_l} \qquad \text{for all } \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

For $H^2_E(\mathbb{D}^n)$ we set

$$\mathbb{S}^{-1}_E(\boldsymbol{M_z}, \boldsymbol{M_z}) := \sum_{0 \le i_1 < \cdots < i_l \le n} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M^*_{z_{i_1}} \cdots M^*_{z_{i_l}}.$$

The following lemma is well-known in the study of reproducing kernel Hilbert spaces.

Lemma 2.1. *Let $E$ be a Hilbert space. Then*

$$\mathbb{S}^{-1}_E(\boldsymbol{M_z}, \boldsymbol{M_z}) = P_E,$$

*where $P_E$ is the orthogonal projection of $H^2_E(\mathbb{D}^n)$ onto the space of all constant functions.*

*Proof.* For all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$ and $\eta, \zeta \in E$ we have

$$
\begin{aligned}
&\langle \mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z})(\mathbb{S}(\cdot, \boldsymbol{z})\eta), \mathbb{S}(\cdot, \boldsymbol{w})\zeta\rangle_{H_E^2(\mathbb{D}^n)} \\
&= \Big\langle \sum_{0 \le i_1 < \cdots < i_l \le n} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M_{z_{i_1}}^* \cdots M_{z_{i_l}}^* (\mathbb{S}(\cdot, \boldsymbol{z})\eta), \mathbb{S}(\cdot, \boldsymbol{w})\zeta \Big\rangle_{H_E^2(\mathbb{D}^n)} \\
&= \sum_{0 \le i_1 < \cdots < i_l \le n} (-1)^l \langle M_{z_{i_1}}^* \cdots M_{z_{i_l}}^* (\mathbb{S}(\cdot, \boldsymbol{z})\eta), M_{z_{i_1}}^* \cdots M_{z_{i_l}}^* (\mathbb{S}(\cdot, \boldsymbol{w})\zeta)\rangle_{H_E^2(\mathbb{D}^n)} \\
&= \sum_{0 \le i_1 < \cdots < i_l \le n} (-1)^l \bar{z}_{i_1} \cdots \bar{z}_{i_l} w_{i_1} \cdots w_{i_l} \langle \mathbb{S}(\cdot, \boldsymbol{z}), \mathbb{S}(\cdot, \boldsymbol{w})\rangle_{H^2(\mathbb{D}^n)} \langle \eta, \zeta\rangle_E \\
&= \mathbb{S}^{-1}(\boldsymbol{w}, \boldsymbol{z})\mathbb{S}(\boldsymbol{w}, \boldsymbol{z})\langle \eta, \zeta\rangle_E = \langle \eta, \zeta\rangle_E = \langle P_E \mathbb{S}(\cdot, \boldsymbol{z})\eta, \mathbb{S}(\cdot, \boldsymbol{w})\zeta\rangle_{H_E^2(\mathbb{D}^n)}.
\end{aligned}
$$

Since $\{\mathbb{S}(\cdot, \boldsymbol{z})\eta : \boldsymbol{z} \in \mathbb{D}^n, \eta \in E\}$ is a total subset of $H_E^2(\mathbb{D}^n)$, we conclude that, $\mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z}) = P_E$. ∎

We now characterize the reducing submodules of $H_E^2(\mathbb{D}^n)$.

PROPOSITION 2.2. *Let $\mathcal{S}$ be a closed subspace of $H_E^2(\mathbb{D}^n)$. Then $\mathcal{S}$ is a reducing submodule of $H_E^2(\mathbb{D}^n)$ if and only if*

$$\mathcal{S} = H_{E_*}^2(\mathbb{D}^n) \quad \textit{for some closed subspace } E_* \textit{ of } E.$$

*Proof.* Let $\mathcal{S}$ be a reducing submodule of $H_E^2(\mathbb{D}^n)$, that is,

$$M_{z_i} P_{\mathcal{S}} = P_{\mathcal{S}} M_{z_i} \quad \text{for all } 1 \le i \le n.$$

By Lemma 2.1,

$$P_E P_{\mathcal{S}} = \mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z}) P_{\mathcal{S}} = P_{\mathcal{S}} \mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z}) = P_{\mathcal{S}} P_E.$$

In particular, $P_{\mathcal{S}} P_E$ is an orthogonal projection and

$$P_{\mathcal{S}} P_E = P_E P_{\mathcal{S}} = P_{E_*},$$

where $E_* := E \cap \mathcal{S}$. Hence, for any $f = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}} \in \mathcal{S}$, where $a_{\boldsymbol{k}} \in E$ for all $\boldsymbol{k} \in \mathbb{N}^n$, we have

$$f = P_{\mathcal{S}} f = P_{\mathcal{S}} \Big( \sum_{\boldsymbol{k} \in \mathbb{N}^n} M_z^{\boldsymbol{k}} a_{\boldsymbol{k}} \Big) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} M_z^{\boldsymbol{k}} P_{\mathcal{S}} a_{\boldsymbol{k}}.$$

But $P_{\mathcal{S}} a_{\boldsymbol{k}} = P_{\mathcal{S}} P_E a_{\boldsymbol{k}} \in E_*$. Consequently, $M_z^{\boldsymbol{k}} P_{\mathcal{S}} a_{\boldsymbol{k}} \in H_{E_*}^2(\mathbb{D}^n)$ for all $\boldsymbol{k} \in \mathbb{N}^n$ and hence $f \in H_{E_*}^2(\mathbb{D}^n)$. That is, $\mathcal{S} \subseteq H_{E_*}^2(\mathbb{D}^n)$. For the reverse inclusion, it is enough to observe that $E_* \subseteq \mathcal{S}$ and that $\mathcal{S}$ is a reducing submodule.

The converse part is immediate. ∎

Let $\mathcal{S}$ be a doubly commuting submodule of $H_E^2(\mathbb{D}^n)$. Then

$$R_{z_i} R_{z_i}^* = M_{z_i} P_{\mathcal{S}} M_{z_i}^* P_{\mathcal{S}} = M_{z_i} P_{\mathcal{S}} M_{z_i}^*$$

implies that $R_{z_i}R_{z_i}^*$ is the orthogonal projection of $\mathcal{S}$ onto $z_i\mathcal{S}$ and hence $I_{\mathcal{S}} - R_{z_i}R_{z_i}^*$ is the orthogonal projection of $\mathcal{S}$ onto $\mathcal{S} \ominus z_i\mathcal{S}$, that is,

$$I_{\mathcal{S}} - R_{z_i}R_{z_i}^* = P_{\mathcal{S}\ominus z_i\mathcal{S}} \quad \text{for all } i = 1, \dots, n.$$

Define

$$\mathcal{W}_i = \operatorname{ran}(I_{\mathcal{S}} - R_{z_i}R_{z_i}^*) = \mathcal{S} \ominus z_i\mathcal{S} \quad \text{for all } i = 1, \dots, n,$$
$$\mathcal{W} = \bigcap_{i=1}^{n} \mathcal{W}_i.$$

By double commutativity of $\mathcal{S}$ (also see [Sa]),

$$(I_{\mathcal{S}} - R_{z_i}R_{z_i}^*)(I_{\mathcal{S}} - R_{z_j}R_{z_j}^*) = (I_{\mathcal{S}} - R_{z_j}R_{z_j}^*)(I_{\mathcal{S}} - R_{z_i}R_{z_i}^*)$$

for all $i \neq j$. Therefore $\{(I_{\mathcal{S}} - R_{z_i}R_{z_i}^*)\}_{i=1}^n$ is a family of commuting orthogonal projections and hence

$$\mathcal{W} = \bigcap_{i=1}^{n} \mathcal{W}_i = \bigcap_{i=1}^{n} (\mathcal{S} \ominus z_i\mathcal{S}) = \bigcap_{i=1}^{n} \operatorname{ran}(I_{\mathcal{S}} - R_{z_i}R_{z_i}^*) \tag{2.1}$$
$$= \operatorname{ran}\Big(\prod_{i=1}^{n} (I_{\mathcal{S}} - R_{z_i}R_{z_i}^*)\Big).$$

Now we present a wandering subspace theorem concerning doubly commuting submodules of $H_E^2(\mathbb{D}^n)$. It is a consequence of a several variables analogue of the classical Wold decomposition theorem, obtained by Gaşpar and Suciu [GS]. We provide a direct proof (see also Corollary 3.2 in [Sa]).

THEOREM 2.3. *Let $\mathcal{S}$ be a doubly commuting submodule of $H_E^2(\mathbb{D}^n)$. Then*

$$\mathcal{S} = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \oplus\, z^{\boldsymbol{k}}\mathcal{W}.$$

*Proof.* First, note that if $\mathcal{M}$ is a submodule of $H_E^2(\mathbb{D}^n)$ then

$$\bigcap_{k\in\mathbb{N}} R_{z_i}^{*k}\mathcal{M} \subseteq \bigcap_{k\in\mathbb{N}} M_{z_i}^{*k} H_E^2(\mathbb{D}^n) = \{0\} \quad \text{for each } i = 1, \dots, n.$$

Therefore $R_{z_i}$ is a shift, that is, the unitary part $\bigcap_{k\in\mathbb{N}} R_{z_i}^{*k}\mathcal{M}$ in the Wold decomposition (cf. [NF], [Sa]) of $R_{z_i}$ on $\mathcal{M}$ is trivial for all $i = 1, \dots, n$. Moreover, if $\mathcal{S}$ is doubly commuting then

$$R_{z_i}(I_{\mathcal{S}} - R_{z_j}R_{z_j}^*) = (I_{\mathcal{S}} - R_{z_j}R_{z_j}^*)R_{z_i} \quad \text{for all } i \neq j.$$

Therefore $\mathcal{W}_j$ is an $R_{z_i}$-reducing subspace for all $i \neq j$. Note also that for all $1 \leq m < n$,

$$\begin{aligned}\bigcap_{i=1}^{m+1} \mathcal{W}_i &= \operatorname{ran}\Big(\prod_{i=1}^{m+1} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)\Big) \\ &= \operatorname{ran}\Big(\prod_{i=1}^{m}(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) - R_{z_{m+1}} R_{z_{m+1}}^* \prod_{i=1}^{m}(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)\Big) \\ &= \operatorname{ran}\Big(\prod_{i=1}^{m}(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) - R_{z_{m+1}} \prod_{i=1}^{m}(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) R_{z_{m+1}}^*\Big) \\ &= (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m) \ominus z_{m+1}(\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m).\end{aligned}$$

We use induction to prove that for all $2 \leq m \leq n$,

$$\mathcal{S} = \sum_{\boldsymbol{k} \in \mathbb{N}^m} \oplus\, z^{\boldsymbol{k}} (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m).$$

First, by the Wold decomposition theorem for the shift $R_{z_1}$ on $\mathcal{S}$ we have

$$\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus\, R_{z_1}^{k_1} \mathcal{W}_1 = \sum_{k_1 \in \mathbb{N}} \oplus\, z_1^{k_1} \mathcal{W}_1.$$

Again by applying Wold decomposition for $R_{z_2}|_{\mathcal{W}_1} \in \mathcal{L}(\mathcal{W}_1)$ we obtain

$$\mathcal{W}_1 = \sum_{k_2 \in \mathbb{N}} \oplus\, R_{z_2}^{k_2} (\mathcal{W}_1 \ominus z_2 \mathcal{W}_1) = \sum_{k_2 \in \mathbb{N}} \oplus\, z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2),$$

and hence

$$\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus\, z_1^{k_1} \Big( \sum_{k_2 \in \mathbb{N}} \oplus\, m z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2) \Big) = \sum_{k_1, k_2 \in \mathbb{N}} \oplus\, z_1^{k_1} z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2).$$

Finally, suppose

$$\mathcal{S} = \sum_{\boldsymbol{k} \in \mathbb{N}^m} \oplus\, z^{\boldsymbol{k}} (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m)$$

for some $m < n$. Then we again apply the Wold decomposition of the isometry

$$R_{z_{m+1}}|_{\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m} \in \mathcal{L}(\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m)$$

to obtain

$$\begin{aligned}\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m &= \sum_{k_{m+1} \in \mathbb{N}} \oplus\, z_{m+1}^{k_{m+1}} \big( (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m) \ominus z_{m+1} \mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m \big) \\ &= \sum_{k_{m+1} \in \mathbb{N}} \oplus\, z_{m+1}^{k_{m+1}} (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m \cap \mathcal{W}_{m+1}),\end{aligned}$$

which yields

$$\mathcal{S} = \sum \oplus\, z^{\boldsymbol{k}} (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_{m+1}).$$

This completes the proof. ∎

*Proof of Theorem 1.3.* By Theorem 2.3 we have

$$\mathcal{S} = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \oplus\, z^{\boldsymbol{k}} \Big( \bigcap_{i=1}^{n} \mathcal{W}_i \Big). \tag{2.2}$$

Now define the Hilbert space $E$ by

$$E = \bigcap_{i=1}^{n} \mathcal{W}_i,$$

and the linear operator $V : H^2_E(\mathbb{D}^n) \to H^2_{E_*}(\mathbb{D}^n)$ by

$$V\Big( \sum_{\boldsymbol{k}\in\mathbb{N}^n} a_{\boldsymbol{k}} z^{\boldsymbol{k}} \Big) = \sum_{\boldsymbol{k}\in\mathbb{N}^n} M_z^{\boldsymbol{k}} a_{\boldsymbol{k}},$$

where $\sum_{\boldsymbol{k}\in\mathbb{N}^n} a_{\boldsymbol{k}} z^{\boldsymbol{k}} \in H^2_E(\mathbb{D}^n)$ and $a_{\boldsymbol{k}} \in E$ for all $\boldsymbol{k} \in \mathbb{N}^n$. Observe that

$$\Big\| \sum_{\boldsymbol{k}\in\mathbb{N}^n} M_z^{\boldsymbol{k}} a_{\boldsymbol{k}} \Big\|^2_{H^2_{E_*}(\mathbb{D}^n)} = \Big\| \sum_{\boldsymbol{k}\in\mathbb{N}^n} z^{\boldsymbol{k}} a_{\boldsymbol{k}} \Big\|^2_{H^2_{E_*}(\mathbb{D}^n)} = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \|z^{\boldsymbol{k}} a_{\boldsymbol{k}}\|^2_{H^2_{E_*}(\mathbb{D}^n)},$$

where the last equality follows from the orthogonal decomposition of $\mathcal{S}$ in (2.2). Therefore,

$$\begin{aligned} \Big\| \sum_{\boldsymbol{k}\in\mathbb{N}^n} M_z^{\boldsymbol{k}} a_{\boldsymbol{k}} \Big\|^2_{H^2_{E_*}(\mathbb{D}^n)} &= \sum_{\boldsymbol{k}\in\mathbb{N}^n} \|a_{\boldsymbol{k}}\|^2_{H^2_{E_*}(\mathbb{D}^n)} = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \|a_{\boldsymbol{k}}\|^2_E \\ &= \Big\| \sum_{\boldsymbol{k}\in\mathbb{N}^n} z^{\boldsymbol{k}} a_{\boldsymbol{k}} \Big\|^2_{H^2_E(\mathbb{D}^n)}, \end{aligned}$$

and hence $V$ is an isometry. Moreover, for all $\boldsymbol{k} \in \mathbb{N}^n$ and $\eta \in E$ we have

$$V M_{z_i}(z^{\boldsymbol{k}}\eta) = V(z^{\boldsymbol{k}+\boldsymbol{e}_i}\eta) = M_z^{\boldsymbol{k}+\boldsymbol{e}_i}\eta = M_{z_i}(M_z^{\boldsymbol{k}}\eta) = M_{z_i} V(z^{\boldsymbol{k}}\eta),$$

that is, $V M_{z_i} = M_{z_i} V$ for all $i = 1, \dots, n$. Hence $V$ is a module map. Therefore, $V = M_\Theta$ for some bounded holomorphic function $\Theta \in H^\infty_{E\to E_*}(\mathbb{D}^n)$ (cf. [BLTT, p. 655]). Moreover, since $V$ is an isometry, we have

$$M_\Theta^* M_\Theta = I_{H^2_E(\mathbb{D}^n)},$$

that is, $\Theta$ is an inner function. Also since $M_{z_i} E \subseteq \mathcal{S}$ for all $i = 1, \dots, n$ we have $\operatorname{ran} V \subseteq \mathcal{S}$ and by (2.2) also $\mathcal{S} \subseteq \operatorname{ran} V$. Hence

$$\operatorname{ran} V = \operatorname{ran} M_\Theta = \mathcal{S},$$

that is,

$$\mathcal{S} = \Theta H^2_E(\mathbb{D}^n).$$

Finally, for all $i = 1, \dots, n$, we have

$$\mathcal{S} \ominus z_i\mathcal{S} = \Theta H^2_E(\mathbb{D}^n) \ominus z_i \Theta H^2_E(\mathbb{D}^n) = \{\Theta f : f \in H^2_E(\mathbb{D}^n),\ M_{z_i}^* \Theta f = 0\},$$

and hence by (2.1),

$$E = \bigcap_{i=1}^{n} \mathcal{W}_i = \bigcap_{i=1}^{n} (\mathcal{S} \ominus z_i \mathcal{S}) = \{\Theta f : M_{z_i}^* \Theta f = 0, f \in H_E^2(\mathbb{D}^n), \forall i = 1, \ldots, n\}$$
$$\subseteq \{g \in H_{E_*}^2(\mathbb{D}^n) : M_{z_i}^* g = 0, \forall i = 1, \ldots, n\} = E_*,$$

that is, $E \subseteq E_*$.

To prove the converse, let $\mathcal{S} = M_\Theta H_E^2(\mathbb{D}^n)$ be a submodule of $H_{E_*}^2(\mathbb{D}^n)$ for some inner function $\Theta \in H_{E \to E_*}^\infty(\mathbb{D}^n)$. Then

$$P_\mathcal{S} = M_\Theta M_\Theta^*,$$

and hence for all $i \neq j$,

$$\begin{aligned} M_{z_i} P_\mathcal{S} M_{z_j}^* &= M_{z_i} M_\Theta M_\Theta^* M_{z_j}^* = M_\Theta M_{z_i} M_{z_j}^* M_\Theta^* = M_\Theta M_{z_j}^* M_{z_i} M_\Theta^* \\ &= M_\Theta M_{z_j}^* M_\Theta^* M_\Theta M_{z_i} M_\Theta^* = M_\Theta M_\Theta^* M_{z_j}^* M_{z_i} M_\Theta M_\Theta^* \\ &= P_\mathcal{S} M_{z_j}^* M_{z_i} P_\mathcal{S}. \end{aligned}$$

This implies

$$R_{z_j}^* R_{z_i} = P_\mathcal{S} M_{z_j}^* P_\mathcal{S} M_{z_i}|_\mathcal{S} = P_\mathcal{S} M_{z_j}^* M_{z_i}|_\mathcal{S} = M_{z_i} P_\mathcal{S} M_{z_j}^* = R_{z_i} R_{z_j}^*,$$

that is, $\mathcal{S}$ is a doubly commuting submodule. ■

**3. Tolokonnikov's lemma for the polydisc.** We will need the following lemma, which is a polydisc version of a similar result proved in the case of the disc in Nikolski's book [N, pp. 44–45]. Here we use the notation $M_g$ for the multiplication operator on $H_E^2$ induced by $g \in H_{E \to E_*}^\infty$.

LEMMA 3.1 (Lemma on local rank). *Let $E, E_c$ be Hilbert spaces with $\dim E, \dim E_c < \infty$. Let $g \in H_{E_c \to E}^\infty(\mathbb{D}^n)$ be such that*

$$\ker M_g = \{h \in H_{E_c}^2(\mathbb{D}^n) : g(z)h(z) \equiv 0\} = \Theta H_{E_a}^2(\mathbb{D}^n),$$

*where $E_a$ is a Hilbert space and $\Theta$ is an $\mathcal{L}(E_a, E_c)$-valued inner function. Then*

$$\dim E_c = \dim E_a + \operatorname{rank} g,$$

*where* $\operatorname{rank} g := \max_{\zeta \in \mathbb{D}^n} \operatorname{rank} g(\zeta)$.

*Proof.* We have $\ker M_g = \{h \in H_{E_c}^2(\mathbb{D}^n) : gh \equiv 0\}$. If $\zeta \in \mathbb{D}^n$, then let

$$[\ker M_g](\zeta) := \{h(\zeta) : h \in \ker M_g\}.$$

We claim that $[\ker M_g](\zeta) = \Theta(\zeta) E_a$. Indeed, let $v \in [\ker M_g](\zeta)$. Then $v = h(\zeta)$ for some element $h \in \ker M_g = \Theta H_{E_a}^2(\mathbb{D}^n)$. So $h = \Theta\varphi$ for some

$\varphi \in H^2_{E_a}(\mathbb{D}^n)$. In particular, $v = h(\zeta) = \Theta(\zeta)\varphi(\zeta)$, where $\varphi(\zeta) \in E_a$. So

$$[\ker M_g](\zeta) \subset \Theta(\zeta)E_a. \tag{3.1}$$

On the other hand, if $w \in \Theta(\zeta)E_a$, then $w = \Theta(\zeta)x$, where $x \in E_a$. Consider the constant function $\mathbf{x}$ mapping $\mathbb{D} \ni \mathbf{z} \mapsto x \in E_a$. Clearly $\mathbf{x} \in H^2_{E_a}(\mathbb{D}^n)$. So $h := \Theta\mathbf{x} \in \Theta H^2_{E_a}(\mathbb{D}^n) = \ker M_g$. Hence $w = \Theta(\zeta)x = (\Theta\mathbf{x})(\zeta) = h(\zeta)$, and so $w \in [\ker M_g](\zeta)$. So we also have

$$\Theta(\zeta)E_a \subset [\ker M_g](\zeta). \tag{3.2}$$

Our claim that $[\ker M_g](\zeta) = \Theta(\zeta)E_a$ follows from (3.1) and (3.2).

Suppose that $v \in [\ker M_g](\zeta)$ for some $\zeta \in \mathbb{D}^n$. Then $v = h(\zeta)$ for some $h \in \ker M_g$. Thus $gh \equiv 0$ in $\mathbb{D}^n$, and in particular $g(\zeta)v = g(\zeta)h(\zeta) = 0$. Thus $v \in \ker g(\zeta)$. So $[\ker M_g](\zeta) \subset \ker g(\zeta)$. Hence $\dim [\ker M_g](\zeta) \leq \dim\ker g(\zeta)$, and consequently

$$\dim \Theta(\zeta)E_a = \dim [\ker M_g](\zeta) \leq \dim\ker g(\zeta) = \dim E_c - \operatorname{rank} g(\zeta),$$

where the last equality follows from the Rank-Nullity Theorem. Since $\Theta$ is inner, its boundary values satisfy $\Theta(\zeta)^*\Theta(\zeta) = I_{E_c}$ for almost all $\zeta \in \mathbb{T}^n$. So there is an open set $U \subset \mathbb{D}^n$ such that for all $\zeta \in U$,

$$\dim E_a = \dim \Theta(\zeta)E_a.$$

But from the definition of the rank of $g$, we know that there is a $\zeta_* \in \mathbb{D}^n$ such that $k := \operatorname{rank} g = \operatorname{rank} g(\zeta_*)$. So there is a $k \times k$ submatrix of $g(\zeta_*)$ which is invertible. Now look at the determinant of this $k \times k$ submatrix of $g$. This is a holomorphic function, and so it cannot be identically zero in the open set $U$. So there must exist a point $\zeta_1 \in U \subset \mathbb{D}^n$ such that $\operatorname{rank} g = \operatorname{rank} g(\zeta_1)$ and $\dim E_a = \dim \Theta(\zeta_1)E_a$. Hence $\dim E_a \leq \dim E_c - \operatorname{rank} g$.

For the proof of the opposite inequality, consider a principal minor $g_1(\zeta_1)$ of the matrix of the operator $g(\zeta_1)$ (with respect to any two fixed bases in $E_c$ and $E$ respectively). Then $\det g_1 \in H^\infty$, $\det g_1 \not\equiv 0$. Let $E_c = E_{c,1} \oplus E_{c,2}$, $E = E_1 \oplus E_2$ ($\dim E_{c,1} = \dim E_1 = \operatorname{rank} g(\zeta_1)$) be the decompositions corresponding to this minor, and let

$$g(\zeta) = \begin{bmatrix} g_1(\zeta) & g_2(\zeta) \\ \gamma_1(\zeta) & \gamma_2(\zeta) \end{bmatrix}, \quad \zeta \in \mathbb{D}^n,$$

be the matrix representation of $g(\zeta)$ with respect to this decomposition. Owing to our assumption on the rank, it follows that there is a matrix function $\zeta \mapsto W(\zeta)$ such that

$$[\gamma_1(\zeta) \quad \gamma_2(\zeta)] = W(\zeta)[g_1(\zeta) \quad g_2(\zeta)].$$

So $\gamma_2(\zeta) = W(\zeta)g_2(\zeta) = \big(\gamma_1(\zeta)(g_1(\zeta))^{-1}\big)g_2(\zeta)$. Thus with $g_1^{\mathrm{co}} := (\det g_1)g_1^{-1}$, we have

$$\gamma_2 \det g_1 = \gamma_1 g_1^{\mathrm{co}} g_2,$$

and using this we get the inclusion $M_{\Omega} H^2_{E_{c,2}}(\mathbb{D}^n) \subset \ker M_g$, where $\Omega \in H^{\infty}_{E_{c,2}\to E_c}(\mathbb{D}^n)$ is given by

$$\Omega = \begin{bmatrix} g_1^{\mathrm{co}} g_2 \\ -\det g_1 \end{bmatrix}.$$

We have $\operatorname{rank} \Omega = \dim E_{c,2} = \dim E_c - \operatorname{rank} g = \dim \ker(g(\zeta_1))$. Consequently, $\dim [\ker M_g](\zeta_1) \geq \dim \ker(g(\zeta_1))$. ∎

We now turn to the extension of Tolokonnikov's Lemma to the polydisc.

*Proof of Theorem 1.4.* (ii)⇒(i): If $g := P_E F^{-1}$, then $gf = I$. It only remains to show that the operators $M_{z_1}, \ldots, M_{z_n}$ are doubly commuting on $\ker M_g$. Let $\Theta$, $\Gamma$ be such that

$$F = [f \quad \Theta] \quad \text{and} \quad F^{-1} = \begin{bmatrix} g \\ \Gamma \end{bmatrix}.$$

Since $FF^{-1} = I_{E_c}$, it follows that $fg + \Theta\Gamma = I_{E_c}$. Thus if $h \in H^2_{E_c}(\mathbb{D}^n)$ is such that $gh = 0$, then $\Theta(\Gamma h) = h$, and so $h \in \Theta H^2_{E_c \ominus E)}(\mathbb{D}^n)$. Hence $\ker M_g \subset \operatorname{ran} M_{\Theta}$. Also, since $F^{-1}F = I$, it follows that $g\Theta = 0$, and so $\operatorname{ran} M_{\Theta} \subset \ker M_g$. So $\ker M_g = \operatorname{ran} M_{\Theta} = \Theta H^2_{E_c \ominus E}(\mathbb{D}^2)$. By Theorem 1.3, the operators $M_{z_1}, \ldots, M_{z_n}$ doubly commute on $\ker M_g$.

(i)⇒(ii): Let

$$\mathcal{S} := \{h \in H^2_{E_c}(\mathbb{D}^n) : g(z)h(z) \equiv 0\} = \ker g.$$

$\mathcal{S}$ is a closed nonzero invariant subspace of $H^2_{E_c}(\mathbb{D}^n)$. Also, by assumption, $M_{z_1}, \ldots, M_{z_n}$ are doubly commuting operators on $\mathcal{S}$. Then by Theorem 1.3, there exists an auxiliary Hilbert space $E_a$ and an inner function $\widetilde{\Theta}$ with values in $\mathcal{L}(E_a, E_c)$ with $\dim E_a \leq \dim E_c$ such that

$$\mathcal{S} = \widetilde{\Theta} H^2_{E_a}(\mathbb{D}^n).$$

By the lemma on local rank, $\dim E_a = \dim E_c - \operatorname{rank} g = \dim E_c - \dim E = \dim(E_c \ominus E)$. Let $U$ be a (constant) unitary operator from $E_c \ominus E$ to $E_a$ and define $\Theta := \widetilde{\Theta} U$. Then $\Theta$ is inner, and we have $\ker g = \Theta H^2_{E_c \ominus E}(\mathbb{D}^n)$. To get $F \in H^{\infty}_{E_c \to E_c}(\mathbb{D}^n)$ define the function $F$ for $z \in \mathbb{D}^n$ by

$$F(z)e := \begin{cases} f(z)e & \text{if } e \in E, \\ \Theta(z)e & \text{if } e \in E_c \ominus E. \end{cases}$$

We note that $F \in H^{\infty}(\mathbb{D}^n)$ and $F|_E = f$. We now show that $F$ is invertible.

With this in mind, we first observe that

$$(I - fg)H^2_{E_c}(\mathbb{D}^n) \subset \Theta H^2_{E_c \ominus E}(\mathbb{D}^n) = \ker M_g.$$

This follows since $g(I - fg)h = gh - gh = 0$ for all $h \in H^2_{E_c}(\mathbb{D}^n)$. Thus we see that $\Theta^*(I - fg) \in H^\infty_{E_c \to E_c \ominus E}(\mathbb{D}^n)$. Now, define $\Omega = g \oplus \Theta^*(I - fg)$. Clearly $\Omega \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$. Next, note that

$$F\Omega = fg + \Theta\Theta^*(I - fg) = I.$$

Similarly,

$$\begin{aligned}\Omega F &= gf\mathbb{P}_E + \Theta^*(I - fg)(f\mathbb{P}_E + \Theta\mathbb{P}_{E_c \ominus E}) \\ &= \mathbb{P}_E + \Theta^*(f\mathbb{P}_E - fgf\mathbb{P}_E + \Theta\mathbb{P}_{E_c \ominus E}) \\ &= \mathbb{P}_E + \Theta^*\Theta\mathbb{P}_{E_c \ominus E} = I.\end{aligned}$$

Thus $F^{-1} \in H^\infty(\mathbb{D}^n; E_c \to E_c)$. ∎

**Acknowledgements.** The authors thank the anonymous referee for the careful review, for help in improving the presentation of the paper, and also for suggesting Example 1.6.

The research of B. D. Wick was supported in part by National Science Foundation DMS grants # 1001098 and # 955432.

## References

[BLTT] J. Ball, W. S. Li, D. Timotin and T. Trent, *A commutant lifting theorem on the polydisc*, Indiana Univ. Math. J. 48 (1999), 653–675.

[B] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1948), 17 pp.

[DMS] R. Douglas, G. Misra and J. Sarkar, *Contractive Hilbert modules and their dilations*, Israel J. Math. 187 (2012), 141–165.

[GS] D. Gaşpar and N. Suciu, *Wold decompositions for commutative families of isometries*, An. Univ. Timişoara Ser. Şti. Mat. 27 (1989), 31–38.

[L] P. D. Lax, *Translation invariant spaces*, Acta Math. 101 (1959), 163–178.

[M] V. Mandrekar, *The validity of Beurling theorems in polydiscs*, Proc. Amer. Math. Soc. 103 (1988), 145–148.

[N] N. K. Nikolski, *Treatise on the Shift Operator*, Grundlehren Math. Wiss. 273, Springer, Berlin, 1986.

[R] W. Rudin, *Function Theory in Polydiscs*, W. A. Benjamin, New York, 1969.

[Sa] J. Sarkar, *Wold decomposition for doubly commuting isometries*, arXiv:1304.7454.

[S] M. Słociński, *On the Wold-type decomposition of a pair of commuting isometries*, Ann. Polon. Math. 37 (1980), 255–262.

[NF] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.

[To] V. Tolokonnikov, *Extension problem to an invertible matrix*, Proc. Amer. Math. Soc. 117 (1993), 1023–1030.

[T] S. Treil, *An operator corona theorem*, Indiana Univ. Math. J. 53 (2004), 1763–1780.

Jaydeb Sarkar
Indian Statistical Institute
Statistics and Mathematics Unit
8th Mile, Mysore Road
Bangalore, 560059, India
E-mail: jay@isibang.ac.in, jaydeb@gmail.com
URL: http://www.isibang.ac.in/~jay/

Amol Sasane
Mathematics Department
London School of Economics
Houghton Street
London WC2A 2AE, U.K.
E-mail: sasane@lse.ac.uk

Brett D. Wick
School of Mathematics
Georgia Institute of Technology
686 Cherry Street
Atlanta, GA 30332-0160, U.S.A.
E-mail: wick@math.gatech.edu
URL: www.math.gatech.edu/~wick

*Received February 22, 2013*
*Revised version May 22, 2013* (7753)