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Conjugacy for Fourier–Bessel expansions

by

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Abstract. We define and investigate the conjugate operator for Fourier–Bessel expansions. Weighted norm and weak type $(1,1)$ inequalities are proved for this operator by using a local version of the Calderón–Zygmund theory, with weights in most cases more general than A_p weights. Also results on Poisson and conjugate Poisson integrals are furnished for the expansions considered. Finally, an alternative conjugate operator is discussed.

1. Introduction and statement of results. Given $\nu > -1$ consider the differential operator

$$L_\nu = -\left(\frac{d^2}{dx^2} + \frac{1/4-\nu^2}{x^2}\right), \tag{1.1}$$

initially defined on the space $C_c^\infty(0,1)$. It is a positive and symmetric operator in $L^2((0,1),dx)$. The functions $\{\psi_n^\nu\}_{n\ge1}$,

$$\psi_n^\nu(x) = d_{n,\nu}(\lambda_{n,\nu}x)^{1/2}J_\nu(\lambda_{n,\nu}x), \qquad d_{n,\nu} = \sqrt{2}\,|\lambda_{n,\nu}^{1/2}J_{\nu+1}(\lambda_{n,\nu})|^{-1},$$

where $\{\lambda_{n,\nu}\}_{n\ge1}$ denotes the sequence of the successive positive zeros of the Bessel function $J_\nu(z)$, are eigenfunctions of L_ν corresponding to the eigenvalues $\lambda_{n,\nu}^2$,

$$L_\nu\psi_n^\nu = \lambda_{n,\nu}^2\psi_n^\nu,$$

and form a complete orthonormal system in $L^2((0,1),dx)$; see [13, Chapter XVII] for a comprehensive study of Fourier–Bessel expansions.

In particular,

$$\psi_n^{-1/2}(x) = \sqrt{2}\cos(\pi(n-1/2)x), \qquad \psi_n^{1/2}(x) = \sqrt{2}\sin(\pi n x),$$

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for $n = 1, 2, \ldots$. It may be easily checked that the operator $\mathcal{L}_\nu$ given by

$$\mathcal{L}_\nu\Big(\sum_{n=1}^{\infty}\langle f, \psi_n^\nu\rangle\psi_n^\nu\Big) = \sum_{n=1}^{\infty}\lambda_{n,\nu}^2\langle f, \psi_n^\nu\rangle\psi_n^\nu$$

on the domain

$$\mathrm{Dom}(\mathcal{L}_\nu) = \Big\{f \in L^2((0,1), dx) : \sum_{n=1}^{\infty}|\lambda_{n,\nu}^2\langle f, \psi_n^\nu\rangle|^2 < \infty\Big\},$$

with $\langle f, \psi_n^\nu\rangle = \int_0^1 f(x)\psi_n^\nu(x)\, dx$, is a self-adjoint extension of L_ν, has the discrete spectrum $\{\lambda_{n,\nu}^2 : n = 1, 2, \ldots\}$ and admits the spectral decomposition

$$\mathcal{L}_\nu f = \sum_{n=1}^{\infty}\lambda_{n,\nu}^2\mathcal{P}_n f, \qquad f \in \mathrm{Dom}(\mathcal{L}_\nu),$$

where $\mathcal{P}_n f = \langle f, \psi_n^\nu\rangle\psi_n^\nu$ are the spectral projections (the inclusion $C_{\mathrm{c}}^\infty(0,1) \subset \mathrm{Dom}(\mathcal{L}_\nu)$ is a consequence of [6, Lemma 2.2]). Notice that for ν such that $0 < |\nu| < 1$, the operators L_ν and $L_{-\nu}$ are identical but $\mathcal{L}_\nu \neq \mathcal{L}_{-\nu}$.

Let

$$\delta_\nu = -\frac{d}{dx} + \frac{\nu + 1/2}{x}$$

denote the derivative associated with L_ν. Formally, we define the conjugate operator by

$$\mathcal{R}_\nu = \delta_\nu(\mathcal{L}_\nu)^{-1/2}.$$

This definition is motivated by the fact that the (formal) adjoint of δ_ν in $L^2((0,1),\, dx)$ is

$$\delta_\nu^* = \frac{d}{dx} + \frac{\nu + 1/2}{x}$$

and a direct computation then shows that

$$L_\nu = \delta_\nu^*\delta_\nu.$$

The precise definition of $\mathcal{R}_\nu$ is the following. Since the spectrum of $\mathcal{L}_\nu$ is separated from zero, $\mathcal{L}_\nu^{-1/2}$ is a bounded operator on $L^2((0,1), dx)$ given by

$$\mathcal{L}_\nu^{-1/2} f = \sum_{n=1}^{\infty}\frac{1}{\lambda_{n,\nu}}\langle f, \psi_n^\nu\rangle\psi_n^\nu, \qquad f \in L^2((0,1), dx).$$

A calculation that uses (2.1) (see Section 2) also shows that

$$\delta_\nu\psi_n^\nu = \lambda_{n,\nu}\widetilde{\psi}_n^\nu, \qquad \delta_\nu^*\widetilde{\psi}_n^\nu = \lambda_{n,\nu}\psi_n^\nu, \tag{1.2}$$

where

$$\widetilde{\psi}_n^\nu(x) = d_{n,\nu}(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x). \tag{1.3}$$

Since the system $\{\widetilde{\psi}_n^\nu\}_{n\ge1}$, $\nu>-1$, is an orthonormal basis in $L^2((0,1),dx)$ (cf. Lemma 2.4), we define

$$\mathcal{R}_\nu f=\sum_{n=1}^{\infty}\langle f,\psi_n^\nu\rangle\widetilde{\psi}_n^\nu,\qquad f\in L^2((0,1),dx). \tag{1.4}$$

(The series on the right converges in $L^2((0,1),dx)$ and defines a bounded operator there.) In other words, the conjugate operator is furnished by the mapping $\psi_n^\nu\mapsto\widetilde{\psi}_n^\nu$. If $\nu=-1/2$, then $\psi_n^{-1/2}(x)=\sqrt{2}\cos(\pi(n-1/2)x)$; moreover, $\lambda_{n,-1/2}=\pi(n-1/2)$, hence a calculation gives

$$\widetilde{\psi}_n^{-1/2}(x)=\sqrt{2}\sin(\pi(n-1/2)x).$$

Therefore, as the corresponding conjugate operator we recover the operator determined by the mapping

$$\cos(\pi(n-1/2)x)\mapsto\sin(\pi(n-1/2)x),$$

which differs slightly from the classical conjugate operator C_{e} for trigonometric expansions of even functions on $(-1,1)$, i.e. the operator given by $\cos(\pi nx)\mapsto\sin(\pi nx)$ (cf. [1, p. 100]).

Given a weight function $w(x)$ on $(0,1)$, consider the following set of conditions (p' denotes the conjugate to p, $1/p+1/p'=1$):

$$\sup_{0<r<1}\Big(\int_r^1 w(x)^p x^{-p(\nu+3/2)}\,dx\Big)^{1/p}\Big(\int_0^r w(x)^{-p'}x^{p'(\nu+1/2)}\,dx\Big)^{1/p'}<\infty, \tag{1.5}$$

$$\sup_{0<r<1}\Big(\int_0^r w(x)^p x^{p(\nu+3/2)}\,dx\Big)^{1/p}\Big(\int_r^1 w(x)^{-p'}x^{-p'(\nu+5/2)}\,dx\Big)^{1/p'}<\infty, \tag{1.6}$$

$$\sup_{0<u<v<\min\{1,2u\}}\frac{1}{v-u}\Big(\int_u^v w(x)^p\,dx\Big)^{1/p}\Big(\int_u^v w(x)^{-p'}\,dx\Big)^{1/p'}<\infty. \tag{1.7}$$

For a weight w satisfying (1.7) we write $w^p\in A_{p,\mathrm{loc}}$ and say that w^p is a *local* A_p *weight*. The left side of (1.7) is then called the $A_{p,\mathrm{loc}}$ *norm* of w^p. We allow $1\le p<\infty$ when considering conditions (1.5)–(1.7). Here and later on, for $p'=\infty$ the above integrals have the usual interpretation. For example, the second factor in (1.5) is taken as $\operatorname{ess\,sup}_{x\in(0,r)}[w(x)^{-1}x^{\nu+1/2}]$. It is easily seen that for a power weight function $w(x)=x^a$, $a\in\mathbb{R}$, (1.5) is satisfied if and only if $a<-1/p+(\nu+3/2)$, (1.6) is satisfied if and only if $a>-(\nu+3/2)-1/p$, and (1.7) is satisfied for each $a\in\mathbb{R}$. The condition (1.5) is necessary and sufficient for the weighted Hardy inequality

$$\int_0^1\Big|w(x)x^{-(\nu+3/2)}\int_0^x f(t)\,dt\Big|^p\,dx\le C\int_0^1|w(x)x^{-(\nu+1/2)}f(x)|^p\,dx \tag{1.8}$$

to hold, while the condition (1.6) is necessary and sufficient for

$$(1.9)\qquad \int_0^1 \Big| w(x)x^{\nu+3/2}\int_x^1 f(t)\,dt\Big|^p\,dx \le C\int_0^1 |w(x)x^{\nu+5/2}f(x)|^p\,dx$$

to be satisfied; this follows from [9, Theorems 1 and 2]. The local A_p condition (1.7) for w^p is, for $1<p<\infty$, sufficient for the estimate

$$(1.10)\qquad \int_0^1 |Tf(x)w(x)|^p\,dx \le C\int_0^1 |f(x)w(x)|^p\,dx$$

to hold, where T represents a local Calderón–Zygmund operator (see [7, Definition 3.2], cf. also [11, Definition 4.2]). In the case $p=1$ the condition (1.7) is sufficient for the weighted weak type $(1,1)$ inequality

$$(1.11)\qquad \int_{\{0<x<1\,:\,|Tf(x)|>\lambda\}} w(x)\,dx \le \frac{C}{\lambda}\int_0^1 |f(x)|w(x)\,dx, \qquad \lambda>0,$$

to hold. These estimates for local Calderón–Zygmund operators are contained in [7, Theorem 3.3] (see also [11, Section 4]).

Finally, note that if a weight w on $(0,1)$ satisfies any of the conditions (1.5)–(1.7) then either $w\equiv 0$ or $w(x)>0$ x-a.e. (here the convention $0\cdot\infty = 0$ is used), and the same applies to the conditions (1.14) and (1.15).

Throughout the paper we use a fairly standard notation. Thus, for a *weight* w on $(0,1)$ (a nonnegative measurable function such that $w(x)<\infty$ x-a.e.) we write $L^p(w)$ and $L^{1,\infty}(w)$ to denote the weighted L^p and weighted weak L^1 spaces (with respect to Lebesgue measure dx) that consist of all functions f on $(0,1)$ for which

$$\|f\|_{L^p(w)} = \Big(\int_0^1 |f(x)w(x)|^p\,dx\Big)^{1/p} < \infty,$$

or

$$\|f\|_{L^{1,\infty}(w)} = \sup_{t>0}\Big(t\int_{\{0<x<1\,:\,|f(x)|>t\}} w(x)\,dx\Big) < \infty,$$

respectively. If $w\equiv 1$, we simply write L^p or $L^{1,\infty}$. By P_r and Q_r, $0<r<1$, we denote the usual Poisson and conjugate Poisson kernels,

$$P_r(x) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n\cos(nx) = \frac{1-r^2}{2(1-2r\cos x + r^2)},$$

$$Q_r(x) = \sum_{n=1}^{\infty} r^n\sin(nx) = \frac{r\sin x}{1-2r\cos x + r^2}.$$

Notice that for $x\ne 2k\pi$, $k\in\mathbb{Z}$, $\lim_{r\to 1^-} P_r(x) = 0$ and $\lim_{r\to 1^-} Q_r(x) = \frac{1}{2}\cot(x/2)$.

We write $g \sim \sum_{n=1}^{\infty} a_n \psi_n^\nu$ to indicate that the Fourier–Bessel expansion of g exists and a_n represents its nth coefficient, $a_n = \langle g, \psi_n^\nu \rangle$; this, in particular, means that $\int_0^1 |g(x)\psi_n^\nu(x)|\,dx < \infty$. The analogous convention holds for other orthonormal bases that appear later on, for instance $\{\widetilde{\psi}_n^\nu\}_{n\ge 1}$.

With this notation, the main results of the paper are the following.

THEOREM 1.1. *Let $\nu > -1$ and $1 < p < \infty$. Let $w(x)$ be a weight that satisfies the conditions* (1.5)–(1.7). *Then*

$$\Big(\int_0^1 |\mathcal{R}_\nu f(x) w(x)|^p\,dx\Big)^{1/p} \le C\Big(\int_0^1 |f(x)w(x)|^p\,dx\Big)^{1/p} \tag{1.12}$$

for all $f \in L^2 \cap L^p(w)$. Consequently, $\mathcal{R}_\nu$ extends uniquely to a bounded linear operator on $L^p(w)$. Using the same symbol $\mathcal{R}_\nu$ to denote this extension, if in addition w satisfies the conditions that result from (1.5) *and* (1.6) *by replacing ν by $\nu+1$, then*

$$\mathcal{R}_\nu f \sim \sum_{n=1}^{\infty} \langle f, \psi_n^\nu \rangle \widetilde{\psi}_n^\nu, \qquad f \in L^p(w). \tag{1.13}$$

In order to treat weighted weak type $(1,1)$ inequalities for $\mathcal{R}_\nu$, for a given weight function $w(x)$ on $(0,1)$, consider the following set of conditions:

$$\sup_{0<r<1} \bigg(\int_r^1 \Big(\frac{r}{x}\Big)^{\delta} \frac{w(x)}{x^{\nu+3/2}}\,dx\bigg)\bigg(\operatorname*{ess\,sup}_{x\in(0,r)} \frac{x^{\nu+1/2}}{w(x)}\bigg) < \infty, \tag{1.14}$$

$$\sup_{0<r<1} \bigg(\int_0^r \Big(\frac{x}{r}\Big)^{\delta} x^{\nu+3/2} w(x)\,dx\bigg)\bigg(\operatorname*{ess\,sup}_{x\in(r,1)} \frac{1}{x^{\nu+5/2}w(x)}\bigg) < \infty. \tag{1.15}$$

In (1.14) and (1.15) we assume that there exists a positive δ such that the corresponding quantities are finite. It is easily seen that for a power weight function $w(x) = x^a$, $a \in \mathbb{R}$, (1.14) is satisfied if and only if $a \le \nu + 1/2$, and (1.15) is satisfied if and only if $a \ge -(\nu + 5/2)$. Let P_η, Q_η, η real, denote the Hardy operators acting on functions defined on $(0,1)$:

$$P_\eta f(x) = x^{-\eta}\int_0^x f(t)\,dt, \qquad Q_\eta f(x) = x^{-\eta}\int_x^1 f(t)\,dt, \qquad 0 < x < 1.$$

The condition (1.14) is necessary and sufficient for the inequality

$$\int_{\{0<x<1\,:\,|P_{\nu+3/2}f(x)|>\lambda\}} w(x)\,dx \le \frac{C}{\lambda}\int_0^1 |f(x)|x^{-(\nu+1/2)}w(x)\,dx, \qquad \lambda > 0, \tag{1.16}$$

to hold; this follows from [2, Theorem 2] taken with $p = q = 1$, $\eta = \nu + 3/2 > 0$, $U(x) = w(x)$ and $V(x) = x^{-(\nu+1/2)}w(x)$ for $x \in (0,1)$, and $U(x) =$

$V(x)=0$ for $x\geq 1$. The condition (1.15) is necessary and sufficient for

$$\int_{\{0<x<1\,:\,|Q_{-(\nu+3/2)}f(x)|>\lambda\}} w(x)\,dx \leq \frac{C}{\lambda}\int_0^1 |f(x)|x^{\nu+5/2}w(x)\,dx, \qquad \lambda>0, \tag{1.17}$$

to hold; this follows from [2, Theorems 4 and 5] taken with $p=q=1$, $\eta=-(\nu+3/2)$, $U(x)=w(x)$ and $V(x)=x^{\nu+5/2}w(x)$ for $x\in(0,1)$, and $U(x)=V(x)=0$ for $x\geq 1$.

THEOREM 1.2. *Let $\nu>-1$ and $w(x)$ be a weight that satisfies the conditions* (1.14), (1.15), *and* (1.7) *with $p=1$. Then*

$$\int_{\{0<x<1\,:\,|\mathcal{R}_\nu f(x)|>\lambda\}} w(x)\,dx \leq \frac{C}{\lambda}\int_0^1 |f(x)|w(x)\,dx, \qquad \lambda>0,$$

for all $f\in L^2\cap L^1(w)$. Consequently, $\mathcal{R}_\nu$ extends uniquely to a bounded linear operator from $L^1(w)$ to $L^{1,\infty}(w)$.

The proofs of our main results, Theorems 1.1 and 1.2, rely on subtle estimates of the kernel $R_\nu(x,y)$ associated to the operator $\mathcal{R}_\nu$ (see Proposition 3.3), and on an application of the aforementioned local Calderón–Zygmund theory. This theory, described in [11], has been adapted to the present setting in [7]. We stress that in the case $\nu>1/2$, when $R_\nu(x,y)$ is a standard Calderón–Zygmund kernel (see Proposition 3.3), restricting the kernel to the local region

$$\mathcal{D}_3=\{(x,y)\in(0,1)\times(0,1) : x/2<y<3x/2\},$$

i.e. treating $\mathcal{R}_\nu$ by means of the local Calderón–Zygmund theory, brings an advantage at least when $1<p<\infty$. Then more weights are allowed since outside $\mathcal{D}_3$, i.e. on the regions $\mathcal{D}_1=\{(x,y) : 0<y\leq x/2\}$ and $\mathcal{D}_2=\{(x,y) : \min\{1,3x/2\}\leq y<1\}$, weighted Hardy inequalities are applied. Here are the details. Recall that the (*global*) A_p *condition* for w^p, $1\leq p<\infty$, is

$$\sup_{0\leq u<v\leq 1}\frac{1}{v-u}\Big(\int_u^v w(x)^p\,dx\Big)^{1/p}\Big(\int_u^v w(x)^{-p'}\,dx\Big)^{1/p'}<\infty. \tag{1.18}$$

Here, as in (1.7), the second integral is understood as $\operatorname{ess\,sup}_{x\in(u,v)}[w^{-1}(x)]$ for $p=1$. Clearly, the (global) A_p condition implies (1.7). We showed in [7, Proposition 2.4] that for $\nu\geq-1/2$ if w satisfies (1.18) then it satisfies (1.5) and (1.6) if $p>1$, or (1.14) and (1.15) if $p=1$. Therefore, taking into account the remarks concerning power weights, it follows that in the case $\nu>-1/2$ in Theorems 1.1 and 1.2 we are considering a range of weights substantially wider than the classical range of A_p weights. On the other hand, we showed in [7, Proposition 2.5] that for $\nu = -1/2$ and $1 < p < \infty$, if w satisfies (1.5) and (1.6) then it satisfies (1.18). Thus, in the case $\nu = -1/2$, in Theorem 1.1 we consider precisely the range of A_p weights.

A theory of Riesz transforms for the differential operator L_ν considered as a positive symmetric operator on $C_c^\infty(0,\infty) \subset L^2((0,\infty), dx)$ has recently been developed in [3] by Betancor, Buraczewski, Fariña, Martínez and Torrea (for $\nu \geq -1/2$); in a slightly different setting the same problem was investigated in [5]. A self-adjoint extension of this operator is realized in terms of the Hankel transform $\mathcal{H}_\nu$. Since for a given $\nu > -1$ the Fourier–Bessel expansions with respect to $\{\psi_n^\nu\}_{n\geq 1}$ may be viewed as discrete analogues of the (continuous) Hankel transform $\mathcal{H}_\nu$, it follows that, in some sense, the results of the present paper can be considered as a discrete counterpart of the results of [3]. A difference between [3] and our paper is that in [3] the relevant operators are defined as singular integral operators while here they are initially defined as bounded operators on L^2.

In [10, Section 18] Muckenhoupt and E. M. Stein outlined a theory of conjugacy for Fourier–Bessel expansions in a setting different from ours. For the system $\{\phi_n^\nu\}_{n\geq 1}$, $\phi_n^\nu(x) = \psi_n^\nu(x)x^{-(\nu+1/2)}$, complete and orthonormal in $L^2((0,1), x^{2\nu+1}\,dx)$, $\nu \geq -1/2$, they suggested the mapping $f \mapsto \widetilde{f}$,

$$\widetilde{f}(x) = -x\sum_{n=1}^{\infty}\langle f, \phi_n^\nu\rangle_{L^2((0,1),\,x^{2\nu+1}\,dx)}\phi_n^{\nu+1}(x),$$

as the appropriate conjugate operator for Fourier–Bessel expansions; in other words, the conjugate operator is furnished by the mapping $\phi_n^\nu \mapsto -x\phi_n^{\nu+1}$ (note that $\{-x\phi_n^{\nu+1}\}_{n\geq 1}$ is an orthonormal basis in $L^2((0,1), x^{2\nu+1}\,dx)$). In that setting the underlying differential operator is

$$L_{(\nu)} = -\left(\frac{d^2}{dx^2} + \frac{2\nu+1}{x}\frac{d}{dx}\right).$$

The structure of the paper is as follows. In Section 2 we gather necessary facts and tools that are used later on and prove a number of lemmas. Section 3 is devoted to proving estimates of the auxiliary kernel $R_\nu(r,x,y)$ and its gradient, and then defining the conjugate kernel $R_\nu(x,y)$ as the limit $\lim_{r\to 1^-} R_\nu(r,x,y)$, $x \neq y$, and proving similar estimates for it. The main results of this section are contained in Propositions 3.1 and 3.2; proving them we heavily exploit the techniques developed in our previous papers [6] and [7]. In Section 4 some results about Poisson and conjugate Poisson integrals are stated and proved. The proofs of the main results are given in Section 5. Finally, in Section 6 we provide a definition of an alternative conjugate operator and state, without proofs, some results concerning them.

2. Preliminaries. The Bessel function J_ν satisfies

$$J_\nu'(t) = -\frac{\nu}{t} J_\nu(t) + J_{\nu-1}(t), \qquad J_\nu'(t) = \frac{\nu}{t} J_\nu(t) - J_{\nu+1}(t). \tag{2.1}$$

The following asymptotics will be used (see [8, p. 122]):

$$\sqrt{z}\, J_\nu(z) = \sum_{j=0}^{M} \left(\frac{A_{\nu,j}}{z^j} \sin z + \frac{B_{\nu,j}}{z^j} \cos z \right) + H_M(z), \tag{2.2}$$

where $M = 0, 1, \ldots$ and $|H_M(z)| \le C z^{-(M+1)}$, $z \to \infty$. At $z = 0^+$ one has

$$J_\nu(z) = O(z^\nu), \qquad z \to 0^+. \tag{2.3}$$

Given $\nu > -1$ the following pointwise estimates also hold:

$$|\psi_n^\nu(x)| \le C \begin{cases} (nx)^{\nu+1/2}, & 0 < x \le n^{-1}, \\ 1, & n^{-1} < x < 1, \end{cases} \tag{2.4}$$

$$|\widetilde{\psi}_n^\nu(x)| \le C \begin{cases} (nx)^{\nu+3/2}, & 0 < x \le n^{-1}, \\ 1, & n^{-1} < x < 1. \end{cases} \tag{2.5}$$

We will also use the fact that

$$\lambda_{n,\nu} = O(n), \qquad d_{n,\lambda} = O(1). \tag{2.6}$$

Moreover, Poisson's integral formula will be helpful:

$$J_\nu(z) = C_\nu z^\nu \int_0^1 (1-t^2)^{\nu-1/2} \cos(zt)\, dt, \qquad \nu > -1/2. \tag{2.7}$$

LEMMA 2.1. *Let $\nu > -1$ and $f \in L^p(w)$, where $1 \le p < \infty$ and w satisfies* (1.5) *and, in addition, the conditions* (1.6) *and* (1.7) *if $p > 1$, or* (1.15) *and* (1.7) *if $p = 1$. Then the coefficients $\langle f, \psi_n^\nu \rangle$ exist and satisfy*

$$\langle f, \psi_n^\nu \rangle = O(n^\tau) \tag{2.8}$$

with some $\tau = \tau(\nu, p, w)$. The analogous statement holds for the system $\{\widetilde{\psi}_n^\nu\}_{n\ge1}$ provided w satisfies (1.7) *and the conditions that result either from* (1.5) *and* (1.6) *if $p > 1$ or from* (1.14) *and* (1.15) *if $p = 1$, upon replacing ν by $\nu + 1$.*

Proof. Using (2.4) gives

$$\int_0^1 |f(x)\psi_n^\nu(x)|\, dx \le C n^{\nu+1/2} \int_0^{1/n} |f(x)| x^{\nu+1/2}\, dx + C \int_{1/n}^1 |f(x)|\, dx.$$

We shall show that the two integrals are finite and, in addition, $O(n^\tau)$.

By Hölder's inequality we obtain (assuming for simplicity $n \ge 2$)

$$\int_0^{1/n} |f(x)| x^{\nu+1/2}\, dx \le \|f\|_{L^p(w)} \Big(\int_0^{1/2} w(x)^{-p'} x^{p'(\nu+1/2)} \Big)^{1/p'}$$

(if $p=1$ the last integral becomes $\operatorname{ess\,sup}_{x\in(0,1/2)}[w(x)^{-1}x^{\nu+1/2}]$). By taking $r=1/2$ either in (1.5) or in (1.14) the very last integral turns out to be a finite constant, hence $\int_0^{1/n}|f(x)|x^{\nu+1/2}\,dx=O(1)$.

For the second relevant integral, by using Hölder's inequality we get

$$\int_{1/n}^{1}|f(x)|\,dx\le\|f\|_{L^p(w)}\Big(\int_{1/n}^{1}w(x)^{-p'}\,dx\Big)^{1/p'}$$

(if $p=1$ the last quantity becomes $\operatorname{ess\,sup}_{x\in(1/n,1)}[w(x)^{-1}]$). Since

$$\int_{1/2}^{1}w(x)^{-p'}\,dx\le\int_{1/2}^{1}w(x)^{-p'}x^{-p'(\nu+5/2)}\,dx$$

(or $\le\operatorname{ess\,sup}_{x\in(1/2,1)}[w(x)^{-1}x^{-(\nu+5/2)}]$ if $p=1$), and the last quantity is finite by taking $r=1/2$ in (1.6) (or in (1.15) if $p=1$), it is sufficient to consider $\int_{1/n}^{1/2}w(x)^{-p'}\,dx$ (or $\operatorname{ess\,sup}_{x\in(1/n,1/2)}[w(x)^{-1}]$ if $p=1$). We have

$$\int_{1/n}^{1/2}w(x)^{-p'}\,dx\le(\max\{2,n\})^{p'(\nu+1/2)}\int_0^{1/2}w(x)^{-p'}x^{p'(\nu+1/2)}\,dx$$

if $1<p<\infty$, or

$$\operatorname*{ess\,sup}_{x\in(1/n,1/2)}[w(x)^{-1}]\le(\max\{2,n\})^{\nu+1/2}\operatorname*{ess\,sup}_{x\in(0,1/2)}\frac{x^{\nu+1/2}}{w(x)}$$

if $p=1$. By taking $r=1/2$ either in (1.5) or in (1.14) if $p=1$, both outermost quantities are finite constants, hence $\int_{1/n}^{1}|f(x)|\,dx=O(n^{\tau})$.

The proof of the statement concerning the system $\{\widetilde{\psi}_n^\nu\}_{n\ge1}$ is completely analogous, so we omit it. ∎

LEMMA 2.2. *Let $\nu>-1$, $1<p<\infty$ and suppose that w satisfies the conditions* (1.5)–(1.7). *Then $\psi_n^\nu\in L^{p'}(w^{-1})$, $n=1,2,\ldots,$ and*

$$\|\psi_n^\nu\|_{L^{p'}(w^{-1})}=O(n^{\tau}) \tag{2.9}$$

with some $\tau=\tau(\nu,p,w)$. The analogous statement holds for the system $\{\widetilde{\psi}_n^\nu\}_{n\ge1}$ provided w satisfies (1.7) *and the conditions resulting from* (1.5) *and* (1.6) *upon replacing ν by $\nu+1$.*

Proof. Using (2.4) gives

$$\int_0^1|\psi_n^\nu(x)w(x)^{-1}|^{p'}\,dx\le Cn^{p'(\nu+1/2)}\int_0^{1/n}x^{p'(\nu+1/2)}w(x)^{-p'}\,dx+C\int_{1/n}^{1}w(x)^{-p'}\,dx.$$

Now repeat the arguments from the proof of Lemma 2.1. ∎

Lemma 2.3. *Let $\nu > -1$. The functions $\{\widetilde{\psi}_n^\nu\}_{n\geq 1}$ given by* (1.3) *are eigenfunctions of the differential operator $L_{\nu+1}$ corresponding to the eigenvalues $\{\lambda_{n,\nu}^2\}_{n\geq 1}$,*

$$L_{\nu+1}\widetilde{\psi}_n^\nu = \lambda_{n,\nu}^2\widetilde{\psi}_n^\nu.$$

Proof. A straightforward calculation gives

$$L_{\nu+1} = \delta_\nu\delta_\nu^*. \tag{2.10}$$

The claim follows by using the above and (1.2). ∎

Lemma 2.4. *Let $\nu > -1$. The functions $\{\widetilde{\psi}_n^\nu\}_{n\geq 1}$ given by* (1.3) *form an orthonormal basis in L^2.*

Proof. We recall Lommel's formula (see [13, Ch. 5, p. 134], where the name "Lommel's formula" is not used, but it commonly appears in the literature, cf. [12])

$$\int_0^1 xJ_{\nu+1}(ax)J_{\nu+1}(bx)\,dx = \begin{cases} \dfrac{aJ_{\nu+1}'(a)J_{\nu+1}(b) - bJ_{\nu+1}'(b)J_{\nu+1}(a)}{b^2-a^2}, & a\neq b,\\[2mm] \dfrac{1}{2}J_{\nu+1}'(a)^2+\dfrac{1}{2}\left(1-\dfrac{(\nu+1)^2}{a^2}\right)J_{\nu+1}(a)^2, & a=b. \end{cases}$$

We also recall the following facts from the theory of Dini series (see [13, p. 134]). Given $\alpha > -1$ and $\varrho\in\mathbb{R}$, the functions

$$\theta_n^{\alpha,\varrho}(x) = b_n\sqrt{x}\,J_\alpha(\mu_n x), \qquad b_n^{-2} = \int_0^1 (\theta_n^{\alpha,\varrho}(x))^2\,dx,$$

$n = 1, 2, \ldots$, where $\{\mu_n\}_{n\geq 1}$ denotes the sequence of successive positive zeros of the equation

$$xJ_\alpha'(x) + \varrho J_\alpha(x) = 0, \tag{2.11}$$

form an orthonormal system in L^2 for $\alpha > -1$; moreover, the system $\{\theta_n^{\alpha,\varrho}\}_{n\geq 1}$ is complete if $\alpha+\varrho > 0$, and if $\alpha+\varrho = 0$, it becomes complete after adjoining the function $\theta_0^{\alpha,\varrho}(x) = \sqrt{2(\alpha+1)}\,x^{\alpha+1/2}$.

Now consider (2.11) with $\alpha = \nu+1$ (then $\alpha > -1$). By the identity

$$xJ_{\nu+1}'(x) + (\nu+1)J_{\nu+1}(x) = xJ_\nu(x)$$

the equation (2.11) can be rewritten as

$$xJ_\nu(x) + (\varrho-\nu-1)J_{\nu+1}(x) = 0.$$

Taking $\varrho = \nu+1$ (note that $\varrho+\alpha = 2(\nu+1) > 0$) one obtains $\mu_n = \lambda_{n,\nu}$, hence the functions

$$\theta_n^{\nu+1,\nu+1}(x) = k_n\widetilde{\psi}_n^\nu(x)$$

form an orthonormal and complete system in L^2; using Lommel's formula, one shows that $k_n = 1$, which finishes the proof. ∎

We will also extensively use the following simplified version of a combination of [6, Lemmas 4.1 and 4.2]. The proof of these lemmas is based on (2.2) and good asymptotics of the sequences $\{\lambda_{n,\nu}\}_{n\geq 1}$ and $\{d_{n,\nu}\}_{n\geq 1}$ (more subtle than those in (2.6)).

LEMMA 2.5. *Let $\nu > -1$, ℓ be a nonnegative integer and γ be a real number. Then each of the four functions*

$$d_{n,\nu}^2 \lambda_{n,\nu}^{\gamma} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\lambda_{n,\nu}(x \pm y)), \qquad n = 1, 2, \ldots, \tag{2.12}$$

is a sum of sixteen terms of the form

$$n^{\gamma} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi n(x \pm y)) E_{\gamma,\ell}(n, x, y),$$

where

$$E_{\gamma,\ell}(n, x, y) = \sum_{k=0}^{\ell} \frac{A_k(x, y)}{n^k} + q_n^{(\ell)}(x, y),$$

and $A_k(x, y)$, $k = 0, 1, \ldots, \ell$, $q_n^{(\ell)}(x, y)$, $n = 1, 2, \ldots$, are functions such that $|A_k(x, y)| \leq C$, $|q_n^{(\ell)}(x, y)| \leq C n^{-\ell-1}$, $0 < x, y < 1$, with a constant $C = C_{\nu,\ell,\gamma}$.

The lemma follows by taking $\mu = \nu$, $m = j = 0$ in [6, Lemmas 4.1 and 4.2] (the functions $A_k(x, y)$ now incorporate some bounded functions that appear in those lemmas).

Another useful fact is taken from [6, Proposition 4.3] (see (4.12) at the end of the proof).

LEMMA 2.6. *We have*

$$\Bigg| \sum_{n=[1/|t|]}^{\infty} \frac{r^n}{n} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (nt) \Bigg| \leq C, \qquad 0 < |t| < 3\pi/2, \tag{2.13}$$

with C not depending on $0 < r < 1$ and t.

We will frequently use, without further mention, the fact that

$$\sum_{n=1}^{N} n^{\varrho} = \begin{cases} O(N^{\varrho+1}) & \text{for } \varrho > -1, \\ O(\log N) & \text{for } \varrho = -1, \end{cases}$$

and

$$\sum_{n=N}^{\infty} n^{\varrho} = O(N^{\varrho+1}), \qquad \varrho < -1.$$

By $\langle f, g \rangle$ we shall mean $\int_0^1 f(x)\overline{g(x)}\,dx$ whenever the integral makes sense.

3. Estimates of the kernel. We first define the kernel $R_\nu(r,x,y)$, $0<r<1$,

$$R_\nu(r,x,y)=\sum_{n=1}^{\infty} r^n\widetilde{\psi}_n^\nu(x)\psi_n^\nu(y),\qquad 0<x,y<1,$$

associated with the integral operator

$$\mathcal{R}_{\nu,r}f(x)=\int_0^1 R_\nu(r,x,y)f(y)\,dy=\sum_{n=1}^{\infty} r^n\langle f,\psi_n^\nu\rangle\widetilde{\psi}_n^\nu(x),$$

and prove the following:

PROPOSITION 3.1. *Let* $\nu>-1$. *Then*

$$|R_\nu(r,x,y)|\le C\begin{cases} x^{-\nu-3/2}y^{\nu+1/2}, & 0<y\le x/2,\\ |x-y|^{-1}, & x/2<y<\min\{1,3x/2\},\\ x^{\nu+3/2}y^{-\nu-5/2}, & \min\{1,3x/2\}\le y<1,\end{cases}\tag{3.1}$$

with C *independent of* $0<r<1$, x *and* y. *Consequently*, *if* $\nu\ge -1/2$ *then*

$$|R_\nu(r,x,y)|\le C|x-y|^{-1},\qquad 0<x,y<1.$$

Proof. The last statement is a straightforward consequence of (3.1). To prove (3.1) we shall consider three cases determined by the right side of this estimate.

CASE 1: $0<y\le x/2$. We split the series defining $R_\nu(r,x,y)$ into

$$\begin{aligned} A&=\sum_{n=1}^{N-1} r^n\widetilde{\psi}_n^\nu(x)\psi_n^\nu(y)\\ &=\sum_{n=1}^{N-1} r^n d_{n,\nu}^2(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x)\cdot(\lambda_{n,\nu}y)^{1/2}J_\nu(\lambda_{n,\nu}y),\\ B&=\sum_{n=N}^{\infty} r^n\widetilde{\psi}_n^\nu(x)\psi_n^\nu(y)\\ &=\sum_{n=N}^{\infty} r^n d_{n,\nu}^2(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x)\cdot(\lambda_{n,\nu}y)^{1/2}J_\nu(\lambda_{n,\nu}y),\end{aligned}$$

where $N=[1/x]$. Using (2.6) and (2.3) we obtain

$$\begin{aligned}|A|&\le\sum_{n=1}^{N-1} d_{n,\nu}^2|(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x)|\,|(\lambda_{n,\nu}y)^{1/2}J_\nu(\lambda_{n,\nu}y)|\\ &\le C(xy)^{1/2}\sum_{n=1}^{N-1} n|J_{\nu+1}(\lambda_{n,\nu}x)|\,|J_\nu(\lambda_{n,\nu}y)|\\ &\le Cx^{\nu+3/2}y^{\nu+1/2}\sum_{n=1}^{N-1} n^{2\nu+2}\le Cx^{-\nu-3/2}y^{\nu+1/2}.\end{aligned}$$

To get the same estimate for $|B|$ it is enough to show that for $0 < r < 1$, $0 < x < 1$, $0 < y \le x/2$ and $\nu > -1/2$,

$$\Big|\sum_{n=N}^{\infty} r^n \widetilde{\psi}_n^{\nu}(x) d_{n,\nu} \lambda_{n,\nu}^{\nu+1/2} \cos(\lambda_{n,\nu} y)\Big| \le C x^{-\nu-3/2}, \tag{3.2}$$

and the analogous estimate with the exponents $\nu+1/2$ and $-\nu-3/2$ replaced by $(\nu+2)+1/2$ and $-(\nu+2)-3/2$ respectively (the latter is needed in the case $-1 < \nu \le -1/2$ only). Indeed, using (3.2) and Poisson's formula (2.7) applied to $J_\nu(\lambda_{n,\nu} y)$ gives, for $\nu > -1/2$,

$$\begin{aligned} |B| &= C_\nu \Big|\sum_{n=N}^{\infty} r^n \widetilde{\psi}_n^{\nu}(x) d_{n,\nu} (\lambda_{n,\nu} y)^{\nu+1/2} \int_0^1 (1-t^2)^{\nu-1/2} \cos(\lambda_{n,\nu} y t)\, dt\Big| \\ &\le C y^{\nu+1/2} \int_0^1 (1-t^2)^{\nu-1/2} \Big|\sum_{n=N}^{\infty} r^n \widetilde{\psi}_n^{\nu}(x) d_{n,\nu} \lambda_{n,\nu}^{\nu+1/2} \cos(\lambda_{n,\nu} y t)\Big|\, dt \\ &\le C x^{-\nu-3/2} y^{\nu+1/2}. \end{aligned}$$

In the case $-1 < \nu \le -1/2$, applying the identity

$$J_\nu(z) = -J_{\nu+2}(z) + \frac{2(\nu+1)}{z} J_{\nu+1}(z)$$

gives

$$\begin{aligned} B = &- \sum_{n=N}^{\infty} r^n \widetilde{\psi}_n^{\nu}(x) d_{n,\nu} (\lambda_{n,\nu} y)^{1/2} J_{\nu+2}(\lambda_{n,\nu} y) \\ &+ 2(\nu+1) \sum_{n=N}^{\infty} r^n \widetilde{\psi}_n^{\nu}(x) d_{n,\nu} (\lambda_{n,\nu} y)^{-1/2} J_{\nu+1}(\lambda_{n,\nu} y). \end{aligned}$$

Now, using Poisson's formula (2.7) for $J_{\nu+1}(\lambda_{n,\nu} y)$ and $J_{\nu+2}(\lambda_{n,\nu} y)$ (together with the assumption $y \le x/2$ in the first summand) and applying (3.2) we obtain the result.

Proving (3.2) (the proof of its counterpart with the aforementioned replacements in exponents is completely analogous, hence we do not treat it separately) we use (2.2) to expand $(\lambda_{n,\nu} x)^{1/2} J_{\nu+1}(\lambda_{n,\nu} x)$ and choose M to be the unique nonnegative integer satisfying $M - 1 \le \nu + 1/2 < M$. It is then clear that

$$\Big|\sum_{n=N}^{\infty} r^n \widetilde{\psi}_n^{\nu}(x) d_{n,\nu} \lambda_{n,\nu}^{\nu+1/2} \cos(\lambda_{n,\nu} y)\Big| \le C \sum_{j=0}^{M} x^{-j} (|\mathcal{C}_j| + |\mathcal{S}_j|) + G_M, \tag{3.3}$$

where

$$\begin{aligned} \begin{Bmatrix} \mathcal{S}_j \\ \mathcal{C}_j \end{Bmatrix} &= \sum_{n=N}^{\infty} r^n d_{n,\nu}^2 \lambda_{n,\nu}^{-j+\nu+1/2} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\lambda_{n,\nu}(x \pm y)), \quad j = 0, 1, \ldots, M, \\ G_M &= \sum_{n=N}^{\infty} d_{n,\nu}^2 |H_M(\lambda_{n,\nu} x)| \lambda_{n,\nu}^{\nu+1/2}. \end{aligned}$$

Then G_M is well controlled. Indeed, using (2.6) and $M > \nu + 1/2$ gives

$$G_M \le Cx^{-(M+1)} \sum_{n=N}^{\infty} n^{-M-1/2+\nu} \le Cx^{-(M+1)} N^{-M+1/2+\nu} \le Cx^{-\nu-3/2}.$$

Taking into account (3.3), to finish the proof of (3.2) it remains to check that both $|\mathcal{S}_j|$ and $|\mathcal{C}_j|$ are bounded by $Cx^{j-\nu-3/2}$. It follows from Lemma 2.5 that for given $j = 0, 1, \ldots, M$, $\mathcal{S}_j$ and $\mathcal{C}_j$ are sums of sixteen series of the form

$$\sum_{n=N}^{\infty} r^n n^{-j+\nu+1/2} E_{-j+\nu+1/2, M-j}(n, x, y) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi n(x \pm y)). \tag{3.4}$$

It is therefore clear that our task is reduced to estimating the absolute value of each of the series in (3.4) by $Cx^{j-\nu-3/2}$. Given $j = 0, \ldots, M$, we use the expression for $E_{-j+\nu+1/2, M-j}(n, x, y)$ from Lemma 2.5 to show that the absolute value of

$$R_{j,k} = \sum_{n=N}^{\infty} r^n n^{-j-k+\nu+1/2} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi n(x \pm y)) \tag{3.5}$$

is, for $k = 0, \ldots, M-j$, bounded by $Cx^{j-\nu-3/2}$, and

$$\left| \sum_{n=N}^{\infty} r^n n^{-j+\nu+1/2} q_n^{(M-j)}(x, y) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi n(x \pm y)) \right| \le Cx^{j-\nu-3/2}.$$

For the term involving $q_n^{(M-j)}(x, y)$, using the fact that $-M - 1/2 + \nu < -1$ gives

$$\begin{aligned} \left| \sum_{n=N}^{\infty} r^n n^{-j+\nu+1/2} q_n^{(M-j)}(x, y) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi n(x \pm y)) \right| &\le C \sum_{n=N}^{\infty} n^{-M-1/2+\nu} \\ &\le Cx^{M-\nu-1/2}, \end{aligned}$$

which is enough for our purpose.

The hypothesis made on M shows that $-j - k + \nu + 1/2 > -1$ for $j = 0, \ldots, M$ and $k = 0, \ldots, M-j$ when $M - 1 < \nu + 1/2$, and the same is true for $j = 0, \ldots, M-1$ and $k = 0, \ldots, M-j-1$ when $M - 1 = \nu + 1/2$. Hence, in these cases,

$$\begin{aligned} \left| \sum_{n=1}^{N-1} r^n n^{-j-k+\nu+1/2} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi n(x \pm y)) \right| &\le C \sum_{n=1}^{N} n^{-j-k+\nu+1/2} \\ &\le Cx^{j+k-\nu-3/2} \le Cx^{j-\nu-3/2}. \end{aligned}$$

Consequently, in (3.5) we can extend the sum to start from $n = 1$ and then use [6, Lemma 3.3] to estimate the complete sum. Thus,

$$|\widetilde{R}_{j,k}| = \left|\sum_{n=1}^{\infty} r^n n^{-j-k+\nu+1/2} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} (\pi n(x \pm y))\right| \le C x^{j-\nu-3/2}.$$

This completes the estimate of $R_{j,k}$, $k = 0, 1, \dots, M-j$, except for the cases of $R_{j,M-j}$ when $M-1 = \nu + 1/2$ for $j = 0, \dots, M$. In these exceptional cases we have to show that $|R_{j,M-j}| \le C x^{j-M}$. Since $R_{j,M-j}$ takes the form of the series in (2.13) with $t = \pi(x \pm y)$, Lemma 2.6 and the fact that $N = [1/x] \sim [1/x+y] \sim [1/|x-y|]$ give the bound $|R_{j,M-j}| \le C \le C x^{j-M}$.

Case 2: $x/2 < y < \min\{1, 3x/2\}$. We use (2.2) with $M = 1$ to expand the functions $(\lambda_{n,\nu} x)^{1/2} J_{\nu+1}(\lambda_{n,\nu} x)$ and $(\lambda_{n,\nu} y)^{1/2} J_\nu(\lambda_{n,\nu} y)$. Then, taking $N = [1/x] \sim [1/y]$, we write $R_\nu(r, x, y)$ as the sum

$$F(r, x, y) + \sum_{j,l=0}^{1} x^{-j} y^{-l} O_{j,l}(r, x, y) + J_1(r, x, y) + J_2(r, x, y) + G_1(r, x, y),$$

where

$$F(r, x, y) = \sum_{n=1}^{N-1} r^n d_{n,\nu}^2 (\lambda_{n,\nu} x)^{1/2} J_{\nu+1}(\lambda_{n,\nu} x) \cdot (\lambda_{n,\nu} y)^{1/2} J_\nu(\lambda_{n,\nu} y),$$

and, for the remainder sum that starts from $n = N$, the $O_{j,l}$ terms capture the part that comes from the main parts of the aforementioned expansions and are sums of four terms of the form

$$D_{j,l} \sum_{n=N}^{\infty} r^n d_{n,\nu}^2 \lambda_{n,\nu}^{-j-l} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} (\lambda_{n,\nu} x) \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} (\lambda_{n,\nu} y),$$

($D_{j,l}$ is a product of $A_{\nu+1,j}$ or $B_{\nu+1,j}$ and $A_{\nu,l}$ or $B_{\nu,l}$ depending on the choice of sine or cosine); J_1 gathers the part that comes from the main parts of the second expansion and the remainder of the first one, hence its absolute value is bounded by

$$\begin{aligned}|J_1(r, x, y)| \le{} & C \left| \sum_1^2 \sum_{n=N}^{\infty} r^n d_{n,\nu}^2 H_1(\lambda_{n,\nu} x) \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} (\lambda_{n,\nu} y) \right| \\ & + C y^{-1} \left| \sum_1^2 \sum_{n=N}^{\infty} r^n d_{n,\nu}^2 \lambda_{n,\nu}^{-1} H_1(\lambda_{n,\nu} x) \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} (\lambda_{n,\nu} y) \right|\end{aligned}$$

(the symbol $\sum_1^2$ indicates that we add two series, one for the choice of the sine and the other for the cosine); J_2 acts as J_1 but with the position of both expansions switched, and its absolute value is controlled by

$$|J_2(r,x,y)| \le C\Big|\sum_1^2 \sum_{n=N}^{\infty} r^n d_{n,\nu}^2 \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}x)H_1(\lambda_{n,\nu}y)\Big| \\ + Cx^{-1}\Big|\sum_1^2 \sum_{n=N}^{\infty} r^n d_{n,\nu}^2 \lambda_{n,\nu}^{-1} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}x)H_1(\lambda_{n,\nu}y)\Big|;$$

and finally G_1 captures the part that comes from the remainders,

$$G_1(r,x,y) = \sum_{n=N}^{\infty} r^n d_{n,\nu}^2 H_1(\lambda_{n,\nu}x)H_1(\lambda_{n,\nu}y).$$

We will now analyze separately each of the summands in the above decomposition of $R_\nu(r,x,y)$ and bound them by $C|x-y|^{-1}$.

For $F(r,x,y)$, using (2.3) and (2.6) we have

$$|F(r,x,y)| \le Cx^{\nu+3/2}y^{\nu+1/2}\sum_{n=1}^{N-1} n^{2\nu+2} \\ \le Cx^{2\nu+2}N^{2\nu+3} \le Cx^{-1},$$

which is dominated by $C|x-y|^{-1}$ in the region considered.

For $J_1(r,x,y)$ (the same reasoning works for $J_2(r,x,y)$), using $H_1(z) = O(z^{-2})$, $z \ge 1$, and again (2.3) and (2.6), shows that

$$|J_1(r,x,y)| \le Cx^{-2}\Big(\sum_{n=N}^{\infty} n^{-2} + y^{-1}\sum_{n=N}^{\infty} n^{-3}\Big) \\ \le Cx^{-2}(N^{-1} + y^{-1}N^{-2}) \le Cx^{-1}.$$

In a similar way we show that

$$|G_1(r,x,y)| \le C(xy)^{-2}\sum_{n=N}^{\infty} n^{-4} \le Cx^{-4}N^{-3} \le Cx^{-1}.$$

The remainder of the proof consists in a more delicate analysis of the $x^{-j}y^{-l}O_{j,l}(r,x,y)$ terms. We start with the $x^{-1}y^{-1}O_{1,1}(x,y)$ term. It is clear that

$$|x^{-1}y^{-1}O_{1,1}(r,x,y)| \le Cx^{-2}\sum_{n=N}^{\infty} n^{-2} \le Cx^{-2}N^{-1} \le Cx^{-1}.$$

Using Lemma 2.5 with $\gamma = -1$ and $\ell = 0$ yields $|x^{-1}O_{1,0}(r,x,y)| \le C|x-y|^{-1}$ once we show that

$$\frac{1}{x}\Big|\sum_{n=N}^{\infty} \frac{r^n}{n} E_{-1,0}(n,x,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y))\Big| \le C\,\frac{1}{x}\log\Big(\frac{2x}{|x-y|}\Big).$$

The form of $E_{-1,0}$ reduces this task to showing the estimates

$$\Big|\sum_{n=N}^{\infty}\frac{r^n}{n}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y))\Big|\le C\log\bigg(\frac{2x}{|x-y|}\bigg) \tag{3.6}$$

and

$$\Big|\sum_{n=N}^{\infty}\frac{r^n}{n}\,q_n^{(0)}(x,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y))\Big|\le C,$$

where $|q_n^{(0)}(x,y)|\le Cn^{-1}$. The very last series is absolutely convergent and the bound follows. The estimate (3.6) is the same as [6, (5.3)] and was proved there. The estimate for $y^{-1}O_{0,1}(r,x,y)$ follows analogously.

It remains to consider the case of $O_{0,0}(r,x,y)$. Using Lemma 2.5 with $\gamma=0$ and $\ell=1$ shows that each of the four terms of $O_{0,0}(r,x,y)$ is a sum of sixteen terms of the form

$$\sum_{n=N}^{\infty} r^n\bigg(A_0+\frac{A_1(x,y)}{n}+q_n^{(1)}(x,y)\bigg)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y)), \tag{3.7}$$

where $|q_n^{(1)}(x,y)|\le Cn^{-2}$ for $0<x,y<1$. The expression in (3.7) equals the expression in (5.4) of [6] corresponding to the case $s=0$. We proved in [6] (cf. the proof of [6, Proposition 5.1]) that this expression equals

$$u(x,y)P_r(\pi(x-y))+v(x,y)Q_r(\pi(x-y)),$$

where u and v are bounded functions on $(0,1)\times(0,1)$, plus some terms whose absolute values are bounded by either $C\log(2x/|x-y|)$ or $C(2-x-y)^{-1}$. Each of the aforementioned bounds is stronger than $C|x-y|^{-1}$; in addition also $P_r(\pi(x-y))$ as well as $|Q_r(\pi(x-y))|$ are bounded by $C|x-y|^{-1}$ for $0<x,y<1$. Hence the estimate $|O_{0,0}(r,x,y)|\le C|x-y|^{-1}$ follows.

Case 3: $\min\{1,3x/2\}\le y<1$. We split the series defining $R_\nu(r,x,y)$ into A and B (as in the case $0<y\le x/2$) but this time we set $N=[1/y]$. Then we get

$$\begin{aligned}|A|&\le\sum_{n=1}^{N-1} d_{n,\nu}^2|(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x)|\,|(\lambda_{n,\nu}y)^{1/2}J_\nu(\lambda_{n,\nu}y)|\\ &\le C(xy)^{1/2}\sum_{n=1}^{N-1} n|J_{\nu+1}(\lambda_{n,\nu}x)|\,|J_\nu(\lambda_{n,\nu}y)|\\ &\le Cx^{\nu+3/2}y^{\nu+1/2}\sum_{n=1}^{N-1} n^{2\nu+2}\le Cx^{\nu+3/2}y^{-\nu-5/2}.\end{aligned}$$

To get the analogous estimate of $|B|$ it is enough to show that for $0<r<1$,

$0 < x \le 2y/3$, $0 < y < 1$ and $\nu > -1$,

$$\Big|\sum_{n=N}^{\infty} r^n d_{n,\nu}\lambda_{n,\nu}^{\nu+3/2}\cos(\lambda_{n,\nu}x)\psi_n^\nu(y)\Big| \le Cy^{-\nu-5/2}. \tag{3.8}$$

Indeed, using (3.8) and Poisson's formula (2.7) applied to $J_{\nu+1}(\lambda_{n,\nu}x)$ gives, for $\nu > -1$,

$$\begin{aligned}|B| &= C_{\nu+1}\Big|\sum_{n=N}^{\infty} r^n d_{n,\nu}(\lambda_{n,\nu}x)^{\nu+3/2}\int_0^1 (1-t^2)^{\nu+1/2}\cos(\lambda_{n,\nu}xt)\,dt\,\psi_n^\nu(y)\Big|\\ &\le Cx^{\nu+3/2}\int_0^1 (1-t^2)^{\nu+1/2}\Big|\sum_{n=N}^{\infty} r^n d_{n,\nu}\lambda_{n,\nu}^{\nu+3/2}\cos(\lambda_{n,\nu}xt)\psi_n^\nu(y)\Big|\,dt\\ &\le Cx^{\nu+3/2}y^{-\nu-5/2}.\end{aligned}$$

Proving (3.8) we use (2.2) to expand $(\lambda_{n,\nu}y)^{1/2}J_\nu(\lambda_{n,\nu}y)$ and choose M to be the positive integer satisfying $M-1 \le \nu+3/2 < M$. It is then clear that

$$\Big|\sum_{n=N}^{\infty} r^n \cos(\lambda_{n,\nu}x)d_{n,\nu}\lambda_{n,\nu}^{\nu+3/2}\psi_n^\nu(y)\Big| \le C\sum_{j=0}^{M} y^{-j}(|\mathcal{C}_j|+|\mathcal{S}_j|)+G_M, \tag{3.9}$$

where

$$\begin{Bmatrix}\mathcal{S}_j\\ \mathcal{C}_j\end{Bmatrix} = \sum_{n=N}^{\infty} r^n d_{n,\nu}^2\lambda_{n,\nu}^{-j+\nu+3/2}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}(x\pm y)),$$

$j = 0,1,\dots,M$, and

$$G_M = \sum_{n=N}^{\infty} d_{n,\nu}^2|H_M(\lambda_{n,\nu}y)|\lambda_{n,\nu}^{\nu+3/2}.$$

Then G_M is well controlled. Indeed, using (2.6) gives

$$G_M \le Cy^{-(M+1)}\sum_{n=N}^{\infty} n^{-M+1/2+\nu} \le Cy^{-(M+1)}N^{-M+3/2+\nu} \le Cy^{-\nu-5/2}.$$

Taking into account (3.9), to finish the proof of (3.8) it remains to check that both $|\mathcal{S}_j|$ and $|\mathcal{C}_j|$ are bounded by $Cy^{j-\nu-5/2}$. It follows from Lemma 2.5 that for given $j = 0,1,\dots,M$, $\mathcal{S}_j$ and $\mathcal{C}_j$ are sums of sixteen series of the form

$$\sum_{n=N}^{\infty} r^n n^{-j+\nu+3/2}E_{-j+\nu+3/2,M-j}(n,x,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y)). \tag{3.10}$$

It is therefore clear that our task is reduced to estimating the absolute value of each of the series in (3.10) by $Cy^{j-\nu-5/2}$. Given $j = 0,\dots,M$, we use the expression for $E_{-j+\nu+3/2,M-j}(n,x,y)$ from Lemma 2.5 to show that the absolute value of

$$(3.11)\qquad R_{j,k}=\sum_{n=N}^{\infty} r^n n^{-j-k+\nu+3/2}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y)),$$

is, for $k=0,\dots,M-j$, bounded by $Cy^{j-\nu-5/2}$ and

$$\Big|\sum_{n=N}^{\infty} r^n n^{-j+\nu+3/2} q_n^{(M-j)}(x,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y))\Big|\le Cy^{j-\nu-5/2}.$$

For the term involving $q_n^{(M-j)}(x,y)$, using the fact that $-M+1/2+\nu<-1$ gives

$$\begin{aligned}\Big|\sum_{n=N}^{\infty} r^n n^{-j+\nu+3/2} q_n^{(M-j)}(x,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y))\Big| &\le C\sum_{n=N}^{\infty} n^{-M+1/2+\nu}\\ &\le C^{M-\nu-3/2},\end{aligned}$$

which is enough for our purpose. The hypothesis made on M shows that $-j-k+\nu+3/2>-1$ for $j=0,\dots,M$ and $k=0,\dots,M-j$ when $M-1<\nu+3/2$, and the same is true for $j=0,\dots,M-1$ and $k=0,\dots,M-j-1$ when $M-1=\nu+3/2$. Hence, in these cases,

$$\begin{aligned}\Big|\sum_{n=1}^{N-1} r^n n^{-j-k+\nu+3/2}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y))\Big| &\le C\sum_{n=1}^{N-1} n^{-j-k+\nu+3/2}\\ &\le Cy^{j+k-\nu-5/2}\le Cy^{j-\nu-5/2}.\end{aligned}$$

Consequently, in (3.5) we can extend the sum to start from $n=1$ and then use [6, Lemma 3.3] to estimate the complete sum. Thus,

$$|\widetilde{R}_{j,k}|=\Big|\sum_{n=1}^{\infty} r^n n^{-j-k+\nu+3/2}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y))\Big|\le Cy^{j-\nu-5/2}.$$

This completes the estimate of $R_{j,k}$, $k=0,1,\dots,M-j$, except the cases of $R_{j,M-j}$ when $M-1=\nu+3/2$ for $j=0,\dots,M$. In these exceptional cases an argument analogous to that from the end of the proof of Case 1 applies.

This finishes considering Case 3 and completes the proof of Proposition 3.1. ∎

PROPOSITION 3.2. *Let $\nu>-1$. Then*

$$(3.12)\qquad |\nabla_{x,y}R_\nu(r,x,y)|\le C|x-y|^{-2},\qquad x/2<y<\min\{1,3x/2\},$$

with C independent of $0<r<1$, x and y. Moreover, if $\nu\ge 1/2$ then

$$|\nabla_{x,y}R_\nu(r,x,y)|\le C|x-y|^{-2},\qquad 0<x,y<1.$$

Proof. We use (2.1) (see also (1.2)) to find that

$$\frac{d\widetilde{\psi}_n^\nu(x)}{dx} = -\frac{2\nu+1}{2x}\,\widetilde{\psi}_n^\nu(x) + \lambda_{n,\nu}\psi_n^\nu(x),$$
$$\frac{d\psi_n^\nu(y)}{dy} = \frac{2\nu+1}{2y}\,\psi_n^\nu(y) - \lambda_{n,\nu}\widetilde{\psi}_n^\nu(y).$$

In this way (in both cases, exchanging summation and differentiation is easily seen to be possible)

$$\frac{\partial R_\nu}{\partial x}(r,x,y) = -\frac{2\nu+1}{2x}\,R_\nu(r,x,y) + \sum_{n=1}^\infty r^n\lambda_{n,\nu}\psi_n^\nu(x)\psi_n^\nu(y), \tag{3.13}$$

$$\frac{\partial R_\nu}{\partial y}(r,x,y) = \frac{2\nu+1}{2y}\,R_\nu(r,x,y) - \sum_{n=1}^\infty r^n\lambda_{n,\nu}\widetilde{\psi}_n^\nu(x)\widetilde{\psi}_n^\nu(y). \tag{3.14}$$

For the first summands on the right of (3.13) and (3.14), using (3.1), it is clear that

$$\left|\frac{2\nu+1}{x}\,R_\nu(r,x,y)\right| \le \frac{C}{x|x-y|} \le \frac{C}{|x-y|^2},$$

and the same estimate holds for $\left|\frac{2\nu+1}{y}\,R_\nu(r,x,y)\right|$.

To treat the second summands we define

$$\begin{aligned} R_\nu^{(1)}(r,x,y) &= \sum_{n=1}^\infty r^n\lambda_{n,\nu}\psi_n^\nu(x)\psi_n^\nu(y) \\ &= \sum_{n=1}^\infty r^n d_{n,\nu}^2\lambda_{n,\nu}(\lambda_{n,\nu}x)^{1/2}J_\nu(\lambda_{n,\nu}x)\cdot(\lambda_{n,\nu}y)^{1/2}J_\nu(\lambda_{n,\nu}y) \end{aligned}$$

and

$$\begin{aligned} \widetilde{R}_\nu^{(1)}(r,x,y) &= \sum_{n=1}^\infty r^n\lambda_{n,\nu}\widetilde{\psi}_n^\nu(x)\widetilde{\psi}_n^\nu(y) \\ &= \sum_{n=1}^\infty r^n d_{n,\nu}^2\lambda_{n,\nu}(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x)\cdot(\lambda_{n,\nu}y)^{1/2}J_{\nu+1}(\lambda_{n,\nu}y), \end{aligned}$$

and proceed analogously to the proof of (3.1) in $x/2 < y < \min\{1, 3x/2\}$. Actually, we shall consider the case of $\widetilde{R}_\nu^{(1)}(r,x,y)$ only since treating $R_\nu^{(1)}(r,x,y)$ is completely analogous.

Now, we use the asymptotic expansion (2.2) with $M = 2$, to expand the functions $(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x)$ and $(\lambda_{n,\nu}y)^{1/2}J_{\nu+1}(\lambda_{n,\nu}y)$ and take $N = [1/x] \sim [1/y]$ to write $\widetilde{R}_\nu^{(1)}(r,x,y)$ as the sum

$$F(r,x,y) + \sum_{j,l=0}^{2} x^{-j}y^{-l}O_{j,l}(r,x,y) + J_1(r,x,y) + J_2(r,x,y) + G_2(r,x,y).$$

Here

$$F(r,x,y)=\sum_{n=1}^{N-1} r^n d_{n,\nu}^2 \lambda_{n,\nu}(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x)\cdot(\lambda_{n,\nu}y)^{1/2}J_{\nu+1}(\lambda_{n,\nu}y),$$

and, for the remainder sum that starts from $n=N$, the $O_{j,l}$ terms capture the part that comes from the main parts of the aforementioned expansions and are sums of four terms of the form

$$D_{j,l}\sum_{n=N}^{\infty} r^n d_{n,\nu}^2 \lambda_{n,\nu}^{-j-l+1}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}(x\pm y));$$

J_1 gathers the part that comes from the main parts of the second expansion and the remainder of the first one, hence

$$\begin{aligned}|J_1(r,x,y)|\le C&\left|\sum_1^2\sum_{n=N}^{\infty} r^n d_{n,\nu}^2\lambda_{n,\nu}H_2(\lambda_{n,\nu}x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}y)\right|\\ &+Cy^{-1}\left|\sum_1^2\sum_{n=N}^{\infty} r^n d_{n,\nu}^2 H_2(\lambda_{n,\nu}x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}y)\right|\\ &+Cy^{-2}\left|\sum_1^2\sum_{n=N}^{\infty} r^n d_{n,\nu}^2\lambda_{n,\nu}^{-1} H_2(\lambda_{n,\nu}x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}y)\right|;\end{aligned}$$

J_2 acts as J_1 but with the position of both expansions switched, and

$$\begin{aligned}|J_2(r,x,y)|\le C&\left|\sum_1^2\sum_{n=N}^{\infty} r^n d_{n,\nu}^2\lambda_{n,\nu}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}x)H_2(\lambda_{n,\nu}y)\right|\\ &+Cx^{-1}\left|\sum_1^2\sum_{n=N}^{\infty} r^n d_{n,\nu}^2\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}x)H_2(\lambda_{n,\nu}y)\right|\\ &+Cx^{-2}\left|\sum_1^2\sum_{n=N}^{\infty} r^n d_{n,\nu}^2\lambda_{n,\nu}^{-1}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}x)H_2(\lambda_{n,\nu}y)\right|;\end{aligned}$$

and finally G_2 captures the part that comes from the remainders,

$$G_2(r,x,y)=\sum_{n=N}^{\infty} r^n d_{n,\nu}^2\lambda_{n,\nu}H_2(\lambda_{n,\nu}x)H_2(\lambda_{n,\nu}y).$$

We will now analyze separately each of the summands in the above decomposition of $\widetilde{R}_\nu^{(1)}(r,x,y)$ and bound them by $C|x-y|^{-2}$.

For $F(r,x,y)$, using (2.3) and (2.6), we have

$$|F(r,x,y)|\le C(xy)^{\nu+3/2}\sum_{n=1}^{N-1} n^{2\nu+4}\le Cx^{2\nu+3}N^{2\nu+5}\le Cx^{-2},$$

which is dominated by $C|x-y|^{-2}$ in the region considered.

For $J_1(r,x,y)$ (the same reasoning works for $J_2(r,x,y)$), using $H_2(z) = O(z^{-3})$, $z \geq 1$, and again (2.3) and (2.6), shows that

$$|J_1(r,x,y)| \leq Cx^{-3}\Big(\sum_{n=N}^{\infty} n^{-2} + y^{-1}\sum_{n=N}^{\infty} n^{-3} + y^{-2}\sum_{n=N}^{\infty} n^{-4}\Big)$$
$$\leq Cx^{-3}(N^{-1} + y^{-1}N^{-2} + y^{-2}N^{-3}) \leq Cx^{-2}.$$

In a similar way we show that

$$|G_2(r,x,y)| \leq C(xy)^{-3}\sum_{n=N}^{\infty} n^{-5} \leq Cx^{-6}N^{-4} \leq Cx^{-2}.$$

The remainder of the proof is an analysis of the $x^{-j}y^{-l}O_{j,l}(r,x,y)$ terms. We start with the $x^{-2}y^{-2}O_{2,2}(x,y)$ term. It is clear that

$$|x^{-2}y^{-2}O_{2,2}(r,x,y)| \leq Cx^{-4}\sum_{n=N}^{\infty} n^{-3} \leq Cx^{-4}N^{-2} \leq Cx^{-2}.$$

The same bound is obtained for $|x^{-2}y^{-1}O_{2,1}(x,y)|$ and $|x^{-1}y^{-2}O_{1,2}(x,y)|$.

The estimate of $|x^{-2}O_{2,0}(r,x,y)|$ by $C|x-y|^{-2}$ uses Lemma 2.5 with $\gamma = -1$ and $\ell = 0$, and is essentially contained in the estimate of $|x^{-1}O_{1,0}(r,x,y)|$ already discussed when proving (3.1) in the region $x/2 < y < \min\{1, 3x/2\}$. The estimates of $|y^{-2}O_{0,2}(r,x,y)|$ and $|x^{-1}y^{-1}O_{1,1}(r,x,y)|$ follow analogously.

The estimate of $|x^{-1}O_{1,0}(r,x,y)|$ by $C|x-y|^{-2}$ uses Lemma 2.5 with $\gamma = 0$ and $\ell = 1$, and is essentially contained in the estimate of $|O_{0,0}(r,x,y)|$ already discussed when proving (3.1) in the relevant region. The estimate of $|y^{-1}O_{0,1}(r,x,y)|$ follows analogously.

It remains to consider the case of $O_{0,0}(r,x,y)$. We use Lemma 2.5 with $\gamma = 1$ and $\ell = 2$ to conclude that each of the four terms of $O_{0,0}(r,x,y)$ is a sum of sixteen terms of the form

$$\sum_{n=N}^{\infty} r^n n\left(A_0 + \frac{A_1(x,y)}{n} + \frac{A_2(x,y)}{n^2} + q_n^{(2)}(x,y)\right)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x \pm y)), \tag{3.15}$$

where $|q_n^{(2)}(x,y)| \leq Cn^{-3}$ for $0 < x, y < 1$. It is immediate to see that the series resulting from taking into account the remainder $q_n^{(2)}(x,y)$ is absolutely convergent, hence its absolute value is bounded by a constant. The series resulting from taking into account A_1 and A_2 were already (implicitly) discussed and are bounded by $C|x-y|^{-2}$ in the region considered. The series resulting from taking into account A_0 was discussed in [7] (cf. the proof of [7, Proposition 4.1]), and was also shown to be bounded by $C|x-y|^{-2}$.

This finishes estimating $\widetilde{R}_\nu^{(1)}$, hence proving the first part of the proposition. To prove the second part we first consider the first summands on the right of (3.13) and (3.14). Since $|R_\nu(r,x,y)| \le C|x-y|^{-1}$, $0<x,y<1$, for $0<y\le x/2$ we have

$$\left|\frac{2\nu+1}{x}\,R_\nu(r,x,y)\right| \le \frac{C}{x|x-y|} \le \frac{C}{|x-y|^2},$$

while for $\min\{1,3x/2\}\le y<1$ we obtain, by using the bottom line of (3.1),

$$\left|\frac{2\nu+1}{x}\,R_\nu(r,x,y)\right| \le C\left(\frac{x}{y}\right)^{\nu+1/2}\frac{1}{y^2} \le \frac{C}{|x-y|^2}.$$

Similarly, for $\min\{1,3x/2\}\le y<1$ we get

$$\left|\frac{2\nu+1}{y}\,R_\nu(r,x,y)\right| \le \frac{C}{y|x-y|} \le \frac{C}{|x-y|^2},$$

while for $0<y\le x/2$, by using the top line of (3.1) we obtain

$$\left|\frac{2\nu+1}{y}\,R_\nu(r,x,y)\right| \le C\left(\frac{y}{x}\right)^{\nu-1/2}\frac{1}{x^2} \le \frac{C}{|x-y|^2}.$$

To treat the second summands on the right of (3.13) and (3.14) we proceed analogously to the proof of (3.1) in the regions $0<y\le x/2$ and $\min\{1,3x/2\}\le y<1$, obtaining the bounds

$$|R_\nu^{(1)}(r,x,y)| \le C\begin{cases} x^{-\nu-5/2}y^{\nu+1/2}, & 0<y\le x/2,\\ x^{\nu+1/2}y^{-\nu-5/2}, & \min\{1,3x/2\}\le y<1,\end{cases}$$
$$|R_\nu^{(2)}(r,x,y)| \le C\begin{cases} x^{-\nu-7/2}y^{\nu+3/2}, & 0<y\le x/2,\\ x^{\nu+3/2}y^{-\nu-7/2}, & \min\{1,3x/2\}\le y<1.\end{cases}$$

It is easily seen that for $\nu\ge 1/2$ this is sufficient to bound $|R_\nu^{(i)}(r,x,y)|$, $i=1,2$, by $C|x-y|^{-2}$ in the regions considered. This finishes the proof of the proposition. ∎

Proposition 3.3. *Let $\nu>-1$. Then for every $x\ne y$, $0<x,y<1$, the limit*

$$R_\nu(x,y) = \lim_{r\to1^-} R_\nu(r,x,y) = \lim_{r\to1^-}\sum_{n=1}^{\infty} r^n\widetilde{\psi}_n^\nu(x)\psi_n^\nu(y)$$

exists and satisfies

$$|R_\nu(x,y)| \le C\begin{cases} x^{-\nu-3/2}y^{\nu+1/2}, & 0<y\le x/2,\\ |x-y|^{-1}, & x/2<y<\min\{1,3x/2\},\\ x^{\nu+3/2}y^{-\nu-5/2}, & \min\{1,3x/2\}\le y<1,\end{cases} \tag{3.16}$$

and

$$|\nabla R_\nu(x,y)| \le C|x-y|^{-2}, \qquad x/2<y<\min\{1,3x/2\}. \tag{3.17}$$

Moreover, if $\nu \ge -1/2$ *then the middle estimate of* (3.16) *holds for* $0 < x, y < 1$, *and the same is true for* (3.17) *if* $\nu \ge 1/2$; *in all cases* C *is independent of* x *and* y.

Proof. Once we prove the existence of the limit, the required estimates follow directly from Propositions 3.1 and 3.2. More precisely, justifying (3.17) also requires the identity

$$\frac{\partial}{\partial y}\big(\lim_{r\to 1^-} R_\nu(r,x,y)\big) = \lim_{r\to 1^-}\frac{\partial}{\partial y} R_\nu(r,x,y) \tag{3.18}$$

and a similar one for $\partial/\partial x$. Assuming for a moment that $\lim_{r\to 1^-} R_\nu(r,x,y)$ exists, what is still needed to prove (3.18) is the fact that for fixed $0<x<1$, the convergence on the right of (3.18) is locally uniform in y. Using (3.14) it is sufficient to check that for given $0<x<1$, the convergence of $R_\nu(r,x,y)$ and $\widetilde{R}_\nu^{(1)}(r,x,y)$ as $r\to 1^-$ is locally uniform in y. For $R_\nu(r,x,y)$ this will be explained below in the proof of the existence of $\lim_{r\to 1^-} R_\nu(r,x,y)$. For $\widetilde{R}_\nu^{(1)}(r,x,y)$ the argument is essentially the same, so we omit the details (a look into the proof of Proposition 3.2 is helpful). Analogous comments apply when $\partial/\partial y$ in (3.18) is replaced by $\partial/\partial x$.

We expand the functions $(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x)$ and $(\lambda_{n,\nu}y)^{1/2}J_\nu(\lambda_{n,\nu}y)$ by using (2.2) with $M=1$ to get

$$R_\nu(r,x,y) = \sum_{j,l=0}^{1} x^{-j}y^{-l}O_{j,l}(r,x,y) + J_1(r,x,y) + J_2(r,x,y) + G_1(r,x,y).$$

Here the $O_{j,l}$ terms capture the part that comes from the main parts of the aforementioned expansions and are linear combinations of terms of the form

$$\sum_{n=1}^{\infty} r^n d_{n,\nu}^2 \lambda_{n,\nu}^{-j-l} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}x) \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}y);$$

J_1 gathers the part that comes from the main parts of the second expansion and the remainder of the first one, hence it is a linear combination of terms of the form

$$y^{-\delta}\sum_{n=1}^{\infty} r^n d_{n,\nu}^2 \lambda_{n,\nu}^{-\delta} H_1(\lambda_{n,\nu}x) \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}y), \qquad \delta = 0,1;$$

J_2 acts as J_1 but with the position of both expansions switched, hence it is a linear combination of terms of the form

$$x^{-\delta}\sum_{n=1}^{\infty} r^n d_{n,\nu}^2 \lambda_{n,\nu}^{-\delta} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}x) H_1(\lambda_{n,\nu}y), \qquad \delta = 0,1;$$

and finally G_1 captures the part that comes from the remainders,

$$G_1(r,x,y)=\sum_{n=1}^{\infty} r^n d_{n,\nu}^2 H_1(\lambda_{n,\nu}x)H_1(\lambda_{n,\nu}y).$$

Due to the bound $H_1(z)=O(z^{-2})$, $z\geq 1$, it is evident that each of the series as in $G_1(r,x,y)$ or in the terms entering either J_1 or J_2, but with the factor r^n removed, is absolutely convergent since, for sufficiently large n, either $|H_1(\lambda_{n,\nu}x)|\leq C(xn)^{-2}$ or $|H_1(\lambda_{n,\nu}y)|\leq C(yn)^{-2}$ applies (or both). Thus the corresponding expressions converge as $r\to 1^-$. In addition, the convergence is locally uniform in y. It is therefore sufficient to analyze the $O_{j,l}$ terms. Given $j,l\in\{0,1\}$ we use Lemma 2.5 with $\ell=1$ and $\gamma=-j-l$. Then $O_{j,l}$ can be written as a linear combination of terms of the form

$$\sum_{n=1}^{\infty} r^n\bigg(A_0+\frac{A_1(x,y)}{n}+q_n^{(1)}(x,y)\bigg)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y)),$$

where $|q_n^{(1)}(x,y)|\leq Cn^{-2}$. Splitting the last series into three expressions corresponding to A_0, A_1/n and $q_n^{(1)}$ we see that the expression corresponding to $q_n^{(1)}$ converges as $r\to 1^-$ due to the fact that the series as in this expression, but with the factor r^n removed, is absolutely convergent; in addition the convergence is locally uniform in y. The first two expressions also converge as $r\to 1^-$, locally uniformly in y (see the proof of [7, Proposition 4.2]).

The proof of the proposition is complete. ∎

Remark 3.4. In the case $\nu=-1/2$, we have

$$R_{-1/2}(x,y)=\frac{\sin\frac{\pi}{2}x\,\cos\frac{\pi}{2}y\,\big(\cos^2\frac{\pi}{2}x+\sin^2\frac{\pi}{2}y\big)}{\sin\frac{\pi}{2}(x+y)\,\sin\frac{\pi}{2}(x-y)}+\sin\frac{\pi}{2}x\,\cos\frac{\pi}{2}y.$$

This is because, as a direct calculation shows,

$$\begin{aligned}
&R_{-1/2}(r,x,y)\\
&\quad=\cos\Big(\frac{\pi}{2}(x+y)\Big)Q_r(\pi(x+y))-\sin\Big(\frac{\pi}{2}(x+y)\Big)\Big(P_r(\pi(x+y))-\frac{1}{2}\Big)\\
&\qquad+\cos\Big(\frac{\pi}{2}(x-y)\Big)Q_r(\pi(x-y))-\sin\Big(\frac{\pi}{2}(x-y)\Big)\Big(P_r(\pi(x-y))-\frac{1}{2}\Big),
\end{aligned}$$

hence

$$\begin{aligned}
&R_{-1/2}(x,y)\\
&\quad=\cos\Big(\frac{\pi}{2}(x+y)\Big)\frac{1}{2\tan\big(\frac{\pi}{2}(x+y)\big)}+\cos\Big(\frac{\pi}{2}(x-y)\Big)\frac{1}{2\tan\big(\frac{\pi}{2}(x-y)\big)}\\
&\qquad+\frac{1}{2}\sin\Big(\frac{\pi}{2}(x+y)\Big)+\frac{1}{2}\sin\Big(\frac{\pi}{2}(x-y)\Big).
\end{aligned}$$

An application of trigonometric identities then gives the required equality. The fact that $R_{-1/2}(x,y)$ is a C^1 function on $(0,1)\times(0,1)\setminus\{x=y\}$ and satisfies estimates consistent with those of Proposition 3.3 now follows by inspection.

Finally, we show that the kernel $R_\nu(x,y)$ is associated with $\mathcal{R}_\nu$ in the sense of Calderón–Zygmund theory.

PROPOSITION 3.5. *Let $f,g\in C_c^\infty(0,1)$ have disjoint supports. Then*

$$\langle\mathcal{R}_\nu f,g\rangle=\int_0^1\int_0^1 R_\nu(x,y)f(y)\overline{g(x)}\,dy\,dx. \tag{3.19}$$

Proof. Let $g=\sum_{n=1}^\infty\langle g,\widetilde{\psi}_n^\nu\rangle\widetilde{\psi}_n^\nu$ (recall that the system $\widetilde{\psi}_n^\nu$ is an orthonormal basis in L^2). Since, by definition, $\mathcal{R}_\nu f=\sum_{n=1}^\infty\langle f,\psi_n^\nu\rangle\widetilde{\psi}_n^\nu$, Parseval's identity (for the system $\{\widetilde{\psi}_n^\nu\}_{n\ge1}$) gives

$$\langle\mathcal{R}_\nu f,g\rangle=\sum_{n=1}^\infty\langle f,\psi_n^\nu\rangle\overline{\langle g,\widetilde{\psi}_n^\nu\rangle}. \tag{3.20}$$

We will show that the right sides of (3.19) and (3.20) coincide. Denote by $F(x,y)$ the function from Proposition 3.3 that majorizes $|R_\nu(x,y)|$; then it is clear that

$$\int_0^1\int_0^1|F(x,y)f(y)\overline{g(x)}|\,dy\,dx<\infty.$$

Therefore the dominated convergence theorem justifies the second equality in the following chain of equalities:

$$\begin{aligned}\langle\mathcal{R}_\nu f,g\rangle&=\int_0^1\int_0^1\lim_{r\to1^-}R_\nu(r,x,y)f(y)\overline{g(x)}\,dy\,dx\\&=\lim_{r\to1^-}\int_0^1\int_0^1R_\nu(r,x,y)f(y)\overline{g(x)}\,dy\,dx\\&=-\lim_{r\to1^-}\int_0^1\mathcal{R}_{\nu,r}f(x)\overline{g(x)}\,dx=-\lim_{r\to1^-}\sum_{n=1}^\infty r^n\langle f,\psi_n^\nu\rangle\overline{\langle g,\widetilde{\psi}_n^\nu\rangle}.\end{aligned}$$

The third equality is explained in the proof of [6, Theorem 1.1], the fourth one is a consequence of [6, (1.10)] and Parseval's identity. Finally, since by [6, Lemma 2.2] (and its slight modification for the system $\{\widetilde{\psi}_n^\nu\}_{n\ge1}$) the series $\sum_{n=1}^\infty\langle f,\psi_n^\nu\rangle\overline{\langle g,\widetilde{\psi}_n^\nu\rangle}$ converges, the last limit equals the right side of (3.20). ∎

4. Poisson and conjugate Poisson integrals. The Poisson semigroup $\{P_t^\nu\}_{t\ge0}$ associated with $\mathcal{L}_\nu$ is, by the spectral theorem, given on L^2 by

$$P_t^\nu = e^{-t(\mathcal{L}_\nu)^{1/2}}.$$

For $f \in L^2$ with the expansion $f = \sum_{n=1}^\infty \langle f, \psi_n^\nu\rangle \psi_n^\nu$, we then have

$$P_t^\nu f = \sum_{n=1}^\infty e^{-t\lambda_{n,\nu}} \langle f, \psi_n^\nu\rangle \psi_n^\nu$$

(convergence in L^2).

We extend this definition by defining, for an appropriate f with the expansion $f \sim \sum_{n=1}^\infty \langle f, \psi_n^\nu\rangle \psi_n^\nu$, its *Poisson integral* $f^\nu(x,t)$ by

$$f^\nu(x,t) = \sum_{n=1}^\infty e^{-t\lambda_{n,\nu}} \langle f, \psi_n^\nu\rangle \psi_n^\nu(x), \quad 0<x<1,\ t>0. \tag{4.1}$$

We also define the *conjugate Poisson integral* $\widetilde{f}^\nu(x,t)$ of f by

$$\widetilde{f}^\nu(x,t) = \sum_{n=1}^\infty e^{-t\lambda_{n,\nu}} \langle f, \psi_n^\nu\rangle \widetilde{\psi}_n^\nu(x), \quad 0<x<1,\ t>0. \tag{4.2}$$

LEMMA 4.1. *Let $\nu > -1$ and $f \in L^p(w)$, where $1 \le p < \infty$ and w satisfies* (1.7) *and, in addition,* (1.5) *and* (1.6) *if $p>1$, or* (1.14) *and* (1.15) *if $p=1$. Then the Poisson and conjugate Poisson integrals of f given by* (4.1) *and* (4.2) *are well defined C^∞ functions on $(0,1)\times(0,\infty)$, harmonic in the sense that they satisfy the differential equations*

$$(\partial_t^2 - L_{\nu,x}) f^\nu(x,t) = 0, \quad (\partial_t^2 - L_{\nu+1,x})\widetilde{f}^\nu(x,t) = 0. \tag{4.3}$$

Moreover, $\widetilde{f}^\nu(x,t)$ and $f^\nu(x,t)$ are related by the "Cauchy–Riemann type" equations

$$\frac{\partial}{\partial t}\widetilde{f}^\nu(x,t) = \delta_{\nu,x} f^\nu(x,t), \quad \frac{\partial}{\partial t} f^\nu(x,t) = \delta_{\nu,x}^* \widetilde{f}^\nu(x,t). \tag{4.4}$$

Proof. Lemma 2.1 ensures the existence of the coefficients $\langle f, \psi_n^\nu\rangle$ and, together with (2.4) and (2.5), shows that $f^\nu(x,t)$ and $\widetilde{f}^\nu(x,t)$ are well defined, i.e., the relevant series converge. The fact that $f^\nu(x,t)$ and $\widetilde{f}^\nu(x,t)$ are twice differentiable and satisfy (4.3) follows from term by term differentiation of the defining series ((2.10) and the second identity in (1.2) are helpful). C^∞ is then a consequence of the fact that the operators $\partial_t^2 - L_{\nu,x}$ and $\partial_t^2 - L_{\nu+1,x}$ are hypoelliptic on $(0,1)\times(0,\infty)$. The identities (4.4) follow by differentiating term by term the defining series and using (1.2). ∎

It may be easily checked that for $f \in L^p(w)$, $1 \le p < \infty$, where w satisfies the assumptions of Lemma 4.1, f^ν and $\widetilde{f}^\nu$ given by (4.1) and (4.2) have the following integral form:

$$(4.5)\qquad f^\nu(x,t)=\int_0^1 P^\nu(t,x,y)f(y)\,dy,\qquad \widetilde{f}^\nu(x,t)=\int_0^1 \widetilde{P}^\nu(t,x,y)f(y)\,dy,$$

where

$$P^\nu(t,x,y)=\sum_{n=1}^\infty e^{-t\lambda_{n,\nu}}\psi_n^\nu(x)\psi_n^\nu(y),\qquad \widetilde{P}^\nu(t,x,y)=\sum_{n=1}^\infty e^{-t\lambda_{n,\nu}}\widetilde{\psi}_n^\nu(x)\psi_n^\nu(y).$$

For $\nu=\pm1/2$, a calculation shows that with $r=e^{-\pi t}$ one has

$$P^{1/2}(t,x,y)=P_r(\pi(x-y))-P_r(\pi(x+y)),$$
$$\widetilde{P}^{-1/2}(t,x,y)=\frac{1}{\sqrt{r}}\,R_{-1/2}(r,x,y)$$

(see the lines following Remark 3.4 for the explicit form of $R_{-1/2}(r,x,y)$).

5. Proofs of the main results. We define the integral operators $\mathcal{R}_\nu^1$ and $\mathcal{R}_\nu^2$ by

$$\mathcal{R}_\nu^1 f(x)=\int_0^{x/2} R_\nu(x,y)f(y)\,dy,\qquad \mathcal{R}_\nu^2 f(x)=\int_{\min\{1,3x/2\}}^1 R_\nu(x,y)f(y)\,dy.$$

By taking $p=2$ and $w(x)\equiv1$ in (1.8) and (1.9) it follows that $\mathcal{R}_\nu^1$ and $\mathcal{R}_\nu^2$ are bounded on L^2 (see the computations in the proof of Theorem 1.1 below). Thus

$$\mathcal{R}_\nu^3=\mathcal{R}_\nu-\mathcal{R}_\nu^1-\mathcal{R}_\nu^2$$

is also bounded on L^2. Moreover, by Proposition 3.5, $\mathcal{R}_\nu^3$ is associated with the kernel $R_\nu(x,y)\chi_{\mathcal{D}_3}(x,y)$, which, by Propositions 3.1 and 3.2, is a local Calderón–Zygmund kernel. Thus $\mathcal{R}_\nu^3$ is a local Calderón–Zygmund operator.

Proof of Theorem 1.1. By using the weighted Hardy inequality (1.8) we obtain

$$\begin{aligned}\int_0^1 |w(x)\mathcal{R}_\nu^1 f(x)|^p\,dx&=\int_0^1 \Big|w(x)\int_0^{x/2} R_\nu(x,y)f(y)\,dy\Big|^p\,dx\\&\le C\int_0^1\Big(w(x)x^{-\nu-3/2}\int_0^{x/2} y^{\nu+1/2}|f(y)|\,dy\Big)^p\,dx\\&\le C\int_0^1|w(x)f(x)|^p\,dx.\end{aligned}$$

Similarly, using the weighted Hardy inequality (1.9) we get

$$\int_0^1|w(x)\mathcal{R}_\nu^2 f(x)|^p\,dx\le C\int_0^1|w(x)f(x)|^p\,dx.$$

Finally, the corresponding $L^p(w)$ inequality for $\mathcal{R}_\nu^3$ is a consequence of (1.10) (see [7, Theorem 3]). Thus (1.12) follows.

To prove (1.13) we fix $f \in L^p(w)$ and choose a sequence $f_k \in L^2 \cap L^p(w)$ such that $f_k \to f$ in $L^p(w)$ as $k \to \infty$. Then, by the very definition, $\mathcal{R}_\nu f = \lim_{k\to\infty} \mathcal{R}_\nu f_k$ (convergence in $L^p(w)$). Since $\mathcal{R}_\nu f \in L^p(w)$ and the aforementioned modifications of (1.5) and (1.6) hold, $\mathcal{R}_\nu f$ has an expansion with respect to $\{\widetilde{\psi}_n^\nu\}_{n\ge 1}$ (see Lemma 2.1). In addition, for any $n = 1, 2, \ldots$, the mapping $g \mapsto \langle g, \widetilde{\psi}_n^\nu\rangle$ is a bounded functional on $L^p(w)$ (see Lemma 2.2). Therefore $\langle \mathcal{R}_\nu f_k, \widetilde{\psi}_n^\nu\rangle \to \langle \mathcal{R}_\nu f, \widetilde{\psi}_n^\nu\rangle$ as $k \to \infty$. On the other hand, since $g \mapsto \langle g, \psi_n^\nu\rangle$ is also a bounded functional on $L^p(w)$ (see Lemma 2.2), we have $\langle f_k, \psi_n^\nu\rangle \to \langle f, \psi_n^\nu\rangle$ as $k \to \infty$. But by (1.4), $\langle \mathcal{R}_\nu f_k, \widetilde{\psi}_n^\nu\rangle = -\langle f_k, \psi_n^\nu\rangle$, hence (1.13) follows. ∎

Proof of Theorem 1.2. Argue as in the first part of the proof of Theorem 1.1 but using (1.16), (1.17) and (1.11) instead of (1.8), (1.9) and (1.10). ∎

6. An alternative conjugacy mapping. Let

$$\widehat{\delta}_\nu = \frac{d}{dx} + \frac{\nu - 1/2}{x}$$

denote an alternative derivative associated with L_ν. One can easily check that the (formal) adjoint of $\widehat{\delta}_\nu$ in L^2 is

$$\widehat{\delta}_\nu^* = -\frac{d}{dx} + \frac{\nu - 1/2}{x},$$

and a direct computation then shows that $\widehat{\delta}_\nu^*\widehat{\delta}_\nu = L_\nu$. Hence, another possible formal definition of the conjugate operator is

$$\widehat{\mathcal{R}}_\nu = \widehat{\delta}_\nu(\mathcal{L}_\nu)^{-1/2}.$$

A calculation also shows that

$$\widehat{\delta}_\nu\psi_n^\nu = \lambda_{n,\nu}\widehat{\psi}_n^\nu, \qquad \widehat{\delta}_\nu^*\widehat{\psi}_n^\nu = \lambda_{n,\nu}\psi_n^\nu,$$

where

$$\widehat{\psi}_n^\nu(x) = d_{n,\nu}(\lambda_{n,\nu}x)^{1/2}J_{\nu-1}(\lambda_{n,\nu}x). \tag{6.1}$$

An analogue of Lemma 2.4 now reads:

LEMMA 6.1. *Let $\nu > 0$. The functions $\{\widehat{\psi}_n^\nu\}_{n\ge 0}$, where $\widehat{\psi}_0^\nu(x) = \sqrt{2\nu}\,x^{\nu-1/2}$ and, for $n \ge 1$, $\widehat{\psi}_n^\nu$ are given by* (6.1), *form an orthonormal basis in L^2.*

Proof. We use the facts and notation of the proof of Lemma 2.4, and consider (2.11) with $\alpha = \nu - 1$ (then $\alpha > -1$). By the identity

$$aJ'_{\nu-1}(a) - (\nu - 1)J_{\nu-1}(a) = -aJ_\nu(a)$$

the equation (2.11) can be rewritten as

$$-xJ_\nu(x) + (\varrho + \nu - 1)J_{\nu-1}(x) = 0.$$

Taking $\varrho = -\nu + 1$ (note that $\varrho + \alpha = 0$) one obtains $\mu_n = \lambda_{n,\nu}$ and the functions

$$\theta_n^{\nu-1,-\nu+1}(x) = k_n \widehat{\psi}_n^{\nu}(x)$$

form an orthonormal system in L^2. The system becomes complete upon adding the function $\sqrt{2\nu}\, x^{\nu-1/2}$. Now, using Lommel's formula, we can show that $k_n = 1$, and the proof is complete. ∎

Thus we define

$$\widehat{\mathcal{R}}_\nu f = \sum_{n=1}^{\infty} \langle f, \psi_n^\nu \rangle \widehat{\psi}_n^\nu, \qquad f \in L^2. \tag{6.2}$$

(The series on the right converges in L^2.) That means that $\widehat{\mathcal{R}}_\nu$ is furnished by the mapping $\psi_n^\nu \mapsto \widehat{\psi}_n^\nu$. In the particular case $\nu = 1/2$, as the corresponding conjugate operator we recover C_{o}, the classic conjugacy mapping for trigonometric expansions of odd functions on $(-1,1)$ (cf. [1, p. 100]),

$$C_{\mathrm{o}} : \sin(\pi n x) \mapsto \cos(\pi n x).$$

Given $\nu > 0$, the following pointwise estimates hold:

$$|\widehat{\psi}_n^\nu(x)| \le C \begin{cases} (nx)^{\nu-1/2}, & 0 < x \le n^{-1}, \\ 1, & n^{-1} < x < 1. \end{cases} \tag{6.3}$$

Lemma 6.2. *Let $\nu > 0$. The statement analogous to* (2.8) *from Lemma* 2.1 *holds for the system $\{\widehat{\psi}_n^\nu\}_{n\ge 0}$ provided w satisfies* (1.7) *and the conditions that result either from* (1.5) *and* (1.6) *if $p > 1$, or from* (1.14) *and* (1.15) *if $p = 1$, upon replacing ν by $\nu - 1$.*

Lemma 6.3. *Let $\nu > 0$. The statement analogous to* (2.9) *from Lemma* 2.2 *holds for the system $\{\widehat{\psi}_n^\nu\}_{n\ge 0}$ provided w satisfies* (1.7) *and the conditions resulting from* (1.5) *and* (1.6) *upon replacing ν by $\nu - 1$.*

It may be checked that for the kernel defined by

$$\widehat{R}_\nu(r,x,y) = \sum_{n=1}^{\infty} r^n \widehat{\psi}_n^\nu(x) \psi_n^\nu(y),$$

the analogues of Propositions 3.1 and 3.2 hold. More precisely, the estimate in (3.1), corresponding to the case $\min\{1, 3x/2\} \le y < 1$ has to be replaced by $Cx^{\nu-1/2}y^{-\nu-1/2}$. Consequently, the result corresponding to Proposition 3.3 now reads:

Proposition 6.4. *Let $\nu > 0$. Then for every $x \ne y$, $0 < x, y < 1$, the limit*

$$\widehat{R}_\nu(x,y) = \lim_{r\to 1^-} \widehat{R}_\nu(r,x,y) = \lim_{r\to 1^-} \sum_{n=1}^{\infty} r^n \widehat{\psi}_n^\nu(x)\psi_n^\nu(y)$$

exists and satisfies

$$|\widehat{R}_\nu(x,y)| \le C \begin{cases} x^{-\nu-3/2}y^{\nu+1/2}, & 0<y\le x/2,\\ |x-y|^{-1}, & x/2<y<\min\{1,3x/2\},\\ x^{\nu-1/2}y^{-\nu-1/2}, & \min\{1,3x/2\}\le y<1. \end{cases}$$

Consequently, if $\nu \ge 1/2$ *then*

$$|\widehat{R}_\nu(x,y)| \le C|x-y|^{-1}, \qquad 0<x,y<1.$$

Moreover,

$$|\nabla \widehat{R}_\nu(x,y)| \le C|x-y|^{-2}, \qquad x/2<y<\min\{1,3x/2\}$$

(*all estimates hold with* C *independent of* x *and* y).

REMARK 6.5. In the case $\nu = 1/2$, we have

$$\widehat{R}_{1/2}(x,y) = \frac{\sin(\pi y)}{\cos(\pi y)-\cos(\pi x)}. \tag{6.4}$$

This is because, as a direct calculation shows,

$$\widehat{R}_{1/2}(r,x,y) = Q_r\big(\pi(x+y)\big) - Q_r(\pi(x-y))$$

hence

$$\widehat{R}_{1/2}(x,y) = \frac{1}{2}\left(\frac{1}{\tan\big(\frac{\pi}{2}(x+y)\big)} - \frac{1}{\tan\big(\frac{\pi}{2}(x-y)\big)}\right)$$

and thus (6.4) follows. The fact that $\widehat{R}_{1/2}(x,y)$ is a C^1 function on $(0,1)\times(0,1)\setminus\{x=y\}$ and satisfies estimates consistent with those of Proposition 6.4 now follows by inspection (note, however, that the restriction on the range of x and y in the gradient estimate is essential). We also mention that $\widehat{R}_{1/2}(x,y)$ is the kernel of the operator C_{o} (cf. [1, p. 100]).

Similarly, the result corresponding to Proposition 3.5 is the following.

PROPOSITION 6.6. *Let* $f,g \in C_{\mathrm{c}}^\infty(0,1)$ *have disjoint supports. Then*

$$\langle \widehat{\mathcal{R}}_\nu f, g\rangle = \int_0^1\!\!\int_0^1 \widehat{R}_\nu(x,y) f(y)\overline{g(x)}\,dy\,dx.$$

We now state results concerning $\widehat{\mathcal{R}}_\nu$, analogous to those in Theorems 1.1 and 1.2.

THEOREM 6.7. *Let* $\nu>0$ *and* $1<p<\infty$. *Let* $w(x)$ *be a weight that satisfies* (1.7), *and also* (1.5) *and* (1.6) *with* ν *replaced by* $\nu-1$. *Then*

$$\Big(\int_0^1 |\widehat{\mathcal{R}}_\nu f(x) w(x)|^p\,dx\Big)^{1/p} \le C\Big(\int_0^1 |f(x)w(x)|^p\,dx\Big)^{1/p} \tag{6.5}$$

for all $f \in L^2\cap L^p(w)$. *Consequently,* $\widehat{\mathcal{R}}_\nu$ *extends uniquely to a bounded linear operator on* $L^p(w)$. *Using the same symbol* $\widehat{\mathcal{R}}_\nu$ *to denote this extension, if in* *addition w satisfies* (1.5) *and* (1.6), *then*

$$\mathcal{R}_\nu f \sim \sum_{n=1}^{\infty} \langle f, \psi_n^\nu \rangle \widehat{\psi}_n^\nu, \qquad f \in L^p(w).$$

THEOREM 6.8. *Let $\nu > 0$ and $w(x)$ be a weight that satisfies* (1.7) *with $p = 1$, and* (1.14) *and* (1.15) *with ν replaced by $\nu - 1$. Then*

$$\int\limits_{\{0<x<1\,:\,|\widehat{\mathcal{R}}_\nu f(x)|>\lambda\}} w(x)\,dx \le \frac{C}{\lambda}\int_0^1 |f(x)|w(x)\,dx, \qquad \lambda > 0, \tag{6.6}$$

for all $f \in L^2 \cap L^1(w)$. Consequently, $\widehat{\mathcal{R}}_\nu$ extends uniquely to a bounded linear operator from $L^1(w)$ to $L^{1,\infty}(w)$.

The conditions imposed on w in Theorem 6.7 for $\nu = 1/2$ are (1.7) and

$$\sup_{0<r<1} \Big(\int_r^1 w(x)^p x^{-p}\,dx\Big)^{1/p}\Big(\int_0^r w(x)^{-p'}\,dx\Big)^{1/p'} < \infty, \tag{6.7}$$

$$\sup_{0<r<1} \Big(\int_0^r w(x)^p x^{p}\,dx\Big)^{1/p}\Big(\int_r^1 w(x)^{-p'}x^{-2p'}\,dx\Big)^{1/p'} < \infty. \tag{6.8}$$

It was proved in [1, Theorem 3] that in the case $\nu = 1/2$, (6.5) holds if and only if

$$\begin{aligned}\Big(\int_u^v w(x)^p\,dx\Big)^{1/p}&\Big(\int_u^v w(x)^{-p'}(x(1-x))^{p'}\,dx\Big)^{1/p'} \\ &\le C(v^2-u^2)(2-(u+v)), \qquad 0 \le u < v \le 1.\end{aligned} \tag{6.9}$$

Therefore, it follows from our Theorem 6.7 and [1, Theorem 3] that a weight w satisfying (1.7), (6.7) and (6.8) must satisfy (6.9). We cannot, however, expect an equivalence of the set of conditions (1.7), (6.7) and (6.8), with the condition (6.9): Theorem 6.7 was stated for general ν, hence it does not take into account the fact that for $\nu = 1/2$ the corresponding kernel vanishes at $y = 1$.

Similarly, the conditions imposed on w in Theorem 6.8 for $\nu = 1/2$ are (1.7) with $p = 1$ and

$$\sup_{0<r<1} \Big(\int_r^1 \Big(\frac{r}{x}\Big)^{\delta}\frac{w(x)}{x}\,dx\Big)\Big(\operatorname*{ess\,sup}_{x\in(0,r)} \frac{1}{w(x)}\Big) < \infty, \tag{6.10}$$

$$\sup_{0<r<1} \Big(\int_0^r \Big(\frac{x}{r}\Big)^{\delta} x w(x)\,dx\Big)\Big(\operatorname*{ess\,sup}_{x\in(r,1)} \frac{1}{x^2 w(x)}\Big) < \infty \tag{6.11}$$

(in (6.10) and (6.11) we assume that there exists a positive δ such that the corresponding quantities are finite).

It was proved in [1, Theorem 3] that in the case $\nu = 1/2$, (6.6) holds if and only if the weight w satisfies (6.9) with $p = 1$. Therefore, it follows from our Theorem 6.8 and [1, Theorem 3] that a weight w satisfying (1.7) with $p = 1$, (6.10) and (6.11) must satisfy (6.9) with $p = 1$. The remarks concerning the lack of equivalence between the set of conditions (1.7) with $p = 1$, (6.10) and (6.11), with the condition (6.9) considered for $p = 1$, also apply in this case.

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