STUDIA MATHEMATICA 191 (2) (2009)

Extenders for vector-valued functions

by

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Abstract. Given a subset A of a topological space X, a locally convex space Y, and a family $\mathcal{C}$ of subsets of Y we study the problem of the existence of a linear $\mathcal{C}$-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$, which is a linear operator extending bounded continuous functions $f : A \to C \subset Y$, $C \in \mathcal{C}$, to bounded continuous functions $\bar{f} = u(f) : X \to C \subset Y$. Two necessary conditions for the existence of such an extender are found in terms of a topological game, which is a modification of the classical strong Choquet game. The results obtained allow us to characterize reflexive Banach spaces as the only normed spaces Y such that for every closed subset A of a GO-space X there is a $\mathcal{C}$-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$ for the family $\mathcal{C}$ of closed convex subsets of Y. Also we obtain a characterization of Polish spaces and of weakly sequentially complete Banach lattices in terms of extenders.

Introduction. In this paper, given a subspace A of a topological space X and a locally convex [ordered] space Y we study the problem of existence (or rather non-existence) of a linear [monotone] operator that extends bounded continuous Y-valued functions from A to X. The results obtained have a dual nature: on the one hand, selecting a suitable pair (X,A) we can characterize certain important properties of locally convex spaces Y (like reflexivity, finite-dimensionality or weak sequential completeness) in terms of extenders (see Theorems 4.1, 5.1, 9.1), and on the other hand, selecting a suitable locally convex space Y, we can characterize topological properties of the pair (X,A) in terms of extenders (see Theorems 3.2, 7.1).

By definition, a (linear) operator $u : F(A,Y) \to F(X,Y)$ defined on a (linear) subspace $F(A,Y) \subset Y^A$ and taking values in a (linear) subspace

2000 *Mathematics Subject Classification*: 46A25, 46A40, 46A55, 46B09, 46B40, 46B42, 47B65, 54C20, 54F05, 60B05, 91A44, 91A80.
Key words and phrases: linear extender, conv-extender, $\overline{\text{conv}}$-extender, monotone extender, (countably) semireflexive locally convex space, reflexive Banach space, Polish space, weakly sequential complete Banach lattice, GO-space, strong Choquet game, Michael line.

DOI: 10.4064/sm191-2-2

$F(X,Y) \subset Y^X$ is called a (*linear*) *extender* if for any function $f \in F(A,Y)$ the function $\bar{f} = u(f) \in F(X,Y)$ extends f in the sense that $\bar{f}|A = f$.

If the space Y is partially ordered, then so are the function spaces Y^A and Y^X. In this case we define an extender $u : F(A,Y) \to F(X,Y)$ to be *monotone* if $u(f) \le u(g)$ for any functions $f \le g$ in $F(A,Y)$.

Given a collection $\mathcal{C}$ of subsets of Y we define an extender $u : F(A,Y) \to F(X,Y)$ to be a $\mathcal{C}$-*extender* if $u(F(A,Y) \cap C^A) \subset C^X$ for every $C \in \mathcal{C}$. This is equivalent to saying that for every function $f : A \to Y$ the image $\bar{f}(X)$ of the extended function $\bar{f} : X \to Y$ lies in the $\mathcal{C}$-hull

$$\mathrm{hull}_{\mathcal{C}}(f(A)) = \bigcap\{C \in \mathcal{C} : f(A) \subset C\}$$

of $f(A)$ in Y (here we assume that $\bigcap \emptyset = Y$). Three collections $\mathcal{C}$ of subsets of Y will be of special importance for us:

- $\mathrm{conv}(Y)$, the collection of convex subsets of Y,
- $\overline{\mathrm{conv}}(Y)$, the collection of closed convex subsets of Y,
- $\mathrm{wcc}(Y)$, the collection of weakly compact convex subsets of Y.

If $Y = Z^*$ is a dual space, then we shall also consider the collection

- $\overline{\mathrm{conv}}^*(Y)$ of all convex subsets of Y, closed in the weak-star topology of Y.

The corresponding $\mathcal{C}$-extenders will be called conv-, $\overline{\mathrm{conv}}$-, wcc-, and $\overline{\mathrm{conv}}^*$-extenders.

The inclusions $\mathrm{wcc}(Y) \subset \overline{\mathrm{conv}}(Y) \subset \mathrm{conv}(Y)$ yield the trivial implications:

$$\text{conv-extender} \;\Rightarrow\; \overline{\text{conv}}\text{-extender} \;\Rightarrow\; \text{wcc-extender}.$$

In the role of linear subspaces $F(X,Y)$ we shall consider the spaces:

- $l_\infty(X,Y)$ of all bounded functions from X to Y,
- $C(X,Y)$ of all continuous functions from X to Y,
- $C_A(X,Y)$ of all functions $f : X \to Y$ that are continuous on a subset $A \subset X$,
- $C_\infty(X,Y) = C(X,Y) \cap l_\infty(X,Y)$ of all bounded continuous functions from X to Y.

A function $f : X \to Y$ is called *bounded* if its image $f(X)$ is bounded in Y. The latter means that for every neighborhood U of the origin in Y there is a real number r with $f(X) \subset rU$.

The space $l_\infty(X,Y)$ will be considered as a locally convex space endowed with the topology of uniform convergence. If Y is a Banach space with norm $\|\cdot\|$, then the topology of $l_\infty(X,Y)$ is generated by the sup-norm $\|f\|_\infty = \sup_{x\in X} \|f(x)\|$.

If Y is the real line $\mathbb{R}$, then we omit the symbol $\mathbb{R}$ and write $l_\infty(X)$, $C(X)$, $C_A(X)$, and $C_\infty(X)$ instead of $l_\infty(X,\mathbb{R})$, $C(X,\mathbb{R})$, $C_A(X,\mathbb{R})$, and $C_\infty(X,\mathbb{R})$.

A classical result on conv-extenders belongs to J. Dugundji [Dug].

THEOREM 0.1 (Dugundji). *For every closed subspace A of a metrizable space X and every locally convex space Y there is a linear* conv*-extender $u : Y^A \to Y^X$ such that $u(C(A,Y)) \subset C(X,Y)$.*

In [Bor] C. Borges has shown that Dugundji's Theorem is still true for any closed subspace A of a stratifiable space X. On the other hand, R. W. Heath and D. J. Lutzer [HL] discovered that for the Michael line $\mathbb{R}_\mathbb{Q}$ and its closed subspace $\mathbb{Q}$ even a weaker form of the Dugundji Theorem is not true: no linear $\overline{\text{conv}}$-extender $C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$ exists. Afterwards it was found that even a monotone extender $C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$ does not exist (see [vD$_1$], [SV], [GHO]).

The Michael line $\mathbb{R}_\mathbb{Q}$ is a particular case of the following construction due to Bing [Bi] and Hanner [Han] (see [Eng, 5.1.22]). Given a subspace A of a topological space X let X_A denote the set X endowed with the *Hanner topology*

$$\tau_A = \{D \cup U : D \subset X \setminus A \text{ and } U \text{ is open in } X\},$$

which is discrete on $X \setminus A$ but coincides with the original topology at A. The space X_A is sometimes called the *Hannerization* of X with respect to A.

Observe that each function $f : X \to Y$ continuous at all points of the set A is (globally) continuous with respect to the Hanner topology τ_A. This is important because it allows us to reduce the study of extenders $C(A,Y) \to C_A(X,Y)$ to studying extenders of the form $C(A,Y) \to C(X_A,Y)$.

In spite of the fact that no linear $\overline{\text{conv}}$-extender $C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$ exists, a linear $\overline{\text{conv}}$-extender $C_\infty(\mathbb{Q}) \to C_\infty(\mathbb{R}_\mathbb{Q})$ for *bounded* continuous functions does exist. This is a particular case of the following result of R. W. Heath and D. J. Lutzer [HL].

THEOREM 0.2 (Heath–Lutzer). *For a closed subset A of a GO-space X there is a linear $\overline{\text{conv}}$-extender $u : C_\infty(A) \to C_\infty(X)$.*

We recall that a topological space X is called *a generalized ordered space* (briefly, a *GO-space*) if X is Hausdorff and for a suitable linear order $\le$ on X the space X has a base of the topology consisting of order-convex sets (see [Lu]). The Michael line $\mathbb{R}_\mathbb{Q}$ is a typical example of a GO-space.

In light of the Dugundji Theorem it was natural to ask about possible generalizations of the Heath–Lutzer Theorem to locally convex spaces (see Question (2) in [HL]). In this paper we give many different answers to this question. Moreover, we show that various properties of locally convex (ordered) spaces Y and pairs (X,A) can be characterized with the help of extenders (see Theorems 1.1, 1.4, 4.1, 5.1, 7.1, 9.1).

For the convenience of the reader we first survey the principal results of the paper and their interplay with known results, and also prove some easy immediate corollaries. Afterwards we present proofs of more difficult theorems.

When working with different topologies on a set Y, we shall write Y_τ to specify a chosen topology τ on Y.

1. Characterizing pairs (X,A) admitting various $\mathcal{C}$-extenders. In this section we search for conditions on a pair (X,A) guaranteeing the existence of a (linear) $\mathcal{C}$-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$ for a given locally convex space Y. We start with a [probably known] characterization of pairs (X,A) admitting a (linear) conv-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$ for every locally convex space Y.

For a Tikhonov space X let $P(\beta X)$ denote the space of probability measures on the Stone-Čech compactification of X. The space $P(\beta X)$ can be identified with the set of all positive norm-one linear functionals on the Banach lattice $C(\beta X) = C_\infty(X)$ of bounded continuous functions on X. The space $P(\beta X)$ is endowed with the weak-star topology induced from $C^*(\beta X)$. It is well-known that this topology is generated by the subbase consisting of the sets $\{\mu \in P(\beta X) : \mu(U) > a\}$ where $a \in \mathbb{R}$ and U runs over the topology of X. The *support* $\mathrm{supp}(\mu)$ of a measure $\mu \in P(\beta X)$ is the smallest closed subset $F \subset \beta X$ with $\mu(F) = 1$. Five subspaces of $P(\beta X)$ will be of interest:

$$\begin{aligned}
P_2(\beta X) &= \{\mu \in P(\beta X) : |\mathrm{supp}(\mu)| \le 2\},\\
P_\omega(X) &= \{\mu \in P(\beta X) : \mu(F) = 1 \text{ for some finite subset } F \subset X\},\\
P_R(X) &= \{\mu \in P(\beta X) : \mu(K) = 1 \text{ for some } \sigma\text{-compact subset } K \subset X\},\\
P_\tau(X) &= \{\mu \in P(\beta X) : \mu(K) = 0 \text{ for every compact subset } K \subset \beta X \setminus X\},\\
P_\sigma(X) &= \{\mu \in P(\beta X) : \mu(K) = 0 \text{ for every closed } G_\delta\text{-subset } K \subset \beta X\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } K \cap X = \emptyset\}.
\end{aligned}$$

Measures from the sets $P_R(X)$, $P_\tau(X)$, and $P_\sigma(X)$ are called *Radon*, *τ-additive*, and *σ-additive* measures on X, respectively. By [Fe, §1], measures from the set $P_\sigma(X)$ can be identified with σ-additive probability measures on X. This justifies the choice of notation. A more detailed information on the spaces $P_R(X)$ and $P_\tau(X)$ can be found in [Vr] and [Ba$_1$], [Ba$_2$].

Quite often, it happens that $P_\tau(X) = P_\sigma(X)$. In particular, this equality holds for all Lindelöf spaces X (or, more generally, for all paracompact spaces not containing a closed discrete subset of Ulam-measurable cardinality; see [BCF, §2]). On the other hand, $P_R(X) = P_\sigma(X)$ for all Polish spaces X (more generally, for universally measurable spaces; see [BCF, §2]).

A Tikhonov subspace A of a topological space X is called a *$P\beta$-valued retract* of X if there is a continuous map $r : X \to P(\beta A)$ such that $\mathrm{supp}(r(a)) = \{a\}$ for every $a \in A$. The latter means that $r(a)$ coincides with the Dirac measure δ_a at a. If $r(X) \subset P_2(\beta A)$ then we say that A is a *$P_2\beta$-valued retract* of X. If $r(X) \subset P_\omega(A)$, then A is called a *P_ω-valued retract* of X. By analogy we define *P_R-valued*, *P_τ-valued*, and *P_σ-valued retracts* of X.

We recall that $C_A(X, Y)$ stands for the space of functions from X to Y that are continuous at each point of the subset $A \subset X$.

THEOREM 1.1. *For a Tikhonov subspace A of a topological space X the following conditions are equivalent*:

(1) *For every linear space Y there is a linear* conv-*extender $u : Y^A \to Y^X$ such that $u(C_\infty(A, Y_\tau)) \subset C_\infty(X, Y_\tau)$ for any locally convex linear topology τ on Y.*
(2) *There is a* conv-*extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$ for $Y = C^*_\infty(A)$ endowed with the weak-star topology.*
(3) *A is a P_ω-valued retract of X.*

Proof. The implication (1)⇒(2) is trivial.

To prove that (2)⇒(3), fix a conv-extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$ where $Y = C^*_\infty(A)$ with the weak-star topology. Consider the bounded continuous map $\delta : A \to C^*_\infty(A)$ assigning to each point $a \in A$ the Dirac measure δ_a. Let $r = u(\delta) : X \to Y$ be the continuous extension of δ given by the conv-extender u. It follows that $r(X) \subset \mathrm{conv}(\delta(A)) = P_\omega(A)$, which means that $r : X \to P_\omega(A)$ is the required P_ω-valued retraction of X onto A.

(3)⇒(1). Fix a P_ω-valued retraction $r : X \to P_\omega(A)$. For every $x \in X$ the measure $\mu_x = r(x) \in P_\omega(X)$ can be uniquely written as the convex combination $\mu_x = \sum_{a \in S_x} \mu_x(a)\delta_a$ where $S_x = \{a \in A : \mu_x(a) > 0\}$ is the (finite) support of μ_x.

Now given a linear space Y, define a linear conv-extender $u : Y^A \to Y^X$ assigning to each function $f : A \to Y$ the function $\bar{f} : X \to Y$ defined by

$$\bar{f}(x) = \int_A f\, d\mu_x = \sum_{a \in S_x} \mu_x(a) \cdot f(a) \quad \text{for } x \in X.$$

It is a standard exercise to check that for every locally convex linear topology τ on Y, we get $u(C_\infty(A, Y_\tau)) \subset C_\infty(X, Y_\tau)$ (see also the proof of the corresponding implication in Theorem 1.4). ∎

For P_σ-valued retracts we have a slightly weaker result that will be applied in the proof of Theorem 7.1.

PROPOSITION 1.2. *If a Tikhonov subspace A of a topological space X is a P_σ-valued retract of X, then for every separable Banach space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$.*

Proof. Let $r : X \to P_\sigma(A)$ be a P_σ-valued retraction of X onto A. Given any bounded continuous function $f : A \to Y$, consider the closed convex hull K of $f(A)$ in Y. For the Polish space K we have the equality $P_R(K) = P_\sigma(K)$ and we can also consider the continuous map $b : P_R(K) \to K$ assigning to each measure $\mu \in P_R(K)$ its barycenter $b(\mu) \in K$ (see [Kh$_1$], [Kh$_2$], [Ba$_2$]). The continuous map $f : A \to K$ induces a continuous map $P_\sigma(f) : P_\sigma(A) \to P_\sigma(K) = P_R(K)$. Then the composition $b \circ P_\sigma(f) : P_\sigma(A) \to K \subset Y$ is continuous and so is the composition $\overline{f} = b \circ P_\sigma(f) \circ r : X \to Y$. Observe that for every $a \in A$ we get $\overline{f}(a) = b \circ P_\sigma(f) \circ r(a) = a$, which means that the operator $u : C_\infty(A, Y) \to C_\infty(X, Y)$, $u : f \mapsto \overline{f}$, is a $\overline{\mathrm{conv}}$-extender. The linearity of u follows from the observation that

$$b \circ P_\sigma(f)(\mu) = \int\limits_A f \, d\mu, \quad \mu \in P_\sigma(A),$$

and the linearity of the vector integral. ∎

QUESTION 1.3. *Is a Tikhonov subspace A of a topological space X a P_σ-valued retract in X if for each* (*separable*) *Banach space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$?*

Next, in terms of $P\beta$-valued retracts we characterize pairs (X, A) admitting a (linear) $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$ for each semireflexive locally convex space Y.

A locally convex space Y is called *semireflexive* if each bounded closed convex subset of Y is compact in the weak topology of Y. For a Banach space semireflexivity is equivalent to reflexivity (see [HHZ, Th. 65]). By the Banach–Steinhaus Uniform Boundedness Principle each dual Banach space Y^* endowed with the weak-star topology is semireflexive.

We define a linear topological space Y to be *countably semireflexive* if $\bigcap_{n\in\omega} C_n \neq \emptyset$ for any decreasing sequence $(C_n)_{n\in\omega}$ of non-empty bounded closed convex subsets of Y. It is clear that each semireflexive locally convex space is countably semireflexive. By the Shmul'yan Theorem 1.13.6 in [Me], the converse is true for normed spaces: *A normed space is (semi)reflexive if and only if it is countably semireflexive.*

A linear topology τ on a locally convex space Y will be called *admissible* if τ is stronger than the weak topology and for each neighborhood $U \in \tau$ of zero in Y there is a convex neighborhood $W \in \tau$ whose closure in Y lies in U. The space Y endowed with an admissible topology τ will be denoted by Y_τ.

THEOREM 1.4. *For a Tikhonov subspace A of a topological space X the following conditions are equivalent*:

(1) *For every semireflexive locally convex space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $u(C_\infty(A, Y_\tau)) \subset C_A(X, Y_\tau)$ for every admissible topology τ on Y.*

(2) *There is a* $\overline{\mathrm{conv}}$*-extender* $u : C_\infty(A,Y) \to C_A(X,Y)$ *for* $Y = C^*_\infty(A)$ *with the weak-star topology.*
(3) A *is a* $P\beta$*-valued retract of* X_A.

Since the norm topology is admissible for the weak-star topology on a dual Banach space, Theorem 1.4 implies

COROLLARY 1.5. *Assume that a Tikhonov subspace* A *of a topological space* X *is a* $P\beta$*-valued retract of* X. *Then for every dual Banach space* Y^* *there is a linear* $\overline{\mathrm{conv}}^*$*-extender* $u : l_\infty(A,Y^*) \to l_\infty(X,Y^*)$ *such that* $u(C_\infty(A,Y^*)) \subset C_A(X,Y^*)$.

In its turn, the above corollary will be applied to construct linear wcc-extenders for functions with values in Banach spaces Y that are norm-one complemented in their biduals Y^{**}. The class of such Banach spaces includes all dual Banach spaces [Me, 3.2.23] and also some non-dual spaces like L_1. The latter fact follows from Theorem 1.c.4 of [LT] asserting that each weakly sequentially complete Banach lattice (in particular, each Banach lattice of the form $L_1(\mu)$) is norm-one complemented in its bidual. On the other hand, c_0 is not complemented in $l_\infty = (c_0)^{**}$ (see [Me, 3.2.22]).

THEOREM 1.6. *Assume that a Banach space* Y *is norm-one complemented in* Y^{**} *and a Tikhonov subspace* $A \subset X$ *is a* $P\beta$*-valued retract of* X. *Then there is a linear* wcc*-extender* $u : l_\infty(A,Y) \to l_\infty(X,Y)$ *with* $\|u\| = 1$ *such that* $u(C_\infty(A,Y)) \subset C_A(X,Y)$.

Proof. Let $P : Y^{**} \to Y$ be a linear projector with $\|P\| = 1$. It induces a norm-one linear operator $P^X : l_\infty(X,Y^{**}) \to l_\infty(X,Y)$ assigning to each bounded function $f : X \to Y^{**}$ the function $P^X(f) = P \circ f$. The continuity of P implies that $P^X(C_\infty(X_A,Y^{**})) \subset C_\infty(X_A,Y)$.

By Corollary 1.5, there is a linear $\overline{\mathrm{conv}}^*$-extender $u : l_\infty(A,Y^{**}) \to l_\infty(X,Y^{**})$ with unit norm such that $u(C_\infty(A,Y^{**})) \subset C_\infty(X_A,Y^{**})$. Now consider the norm-one linear operator $v = P^X \circ u : l_\infty(A,Y) \to l_\infty(X,Y)$, which assigns to each $f \in l_\infty(A,Y) \subset l_\infty(A,Y^{**})$ the function $P^X \circ u(f) : X \to Y$. It follows that $v(C_\infty(A,Y)) \subset C_\infty(X_A,Y)$. If K is a weakly compact convex subset of Y, then K is weak-star closed in Y^{**}, and consequently $u(l_\infty(A,K)) \subset l_\infty(X,K)$. Since $P^X(f) = f$ for each $f \in l_\infty(X,Y)$, we conclude that $v(l_\infty(A,K)) \subset l_\infty(X,K)$, which means that v is a wcc-extender. ∎

QUESTION 1.7. *Is there a linear* (*continuous*) *extender* $u : C_\infty(\mathbb{Q},c_0) \to C_\infty(\mathbb{R}_\mathbb{Q},c_0)$?

2. Linear extenders on ordered spaces. In this section we shall construct nice linear extenders on linearly ordered topological spaces (briefly LOTS). Those are topological spaces X carrying the interval topology with respect to some linear order $\le$ on X. The interval topology is generated by the subbase consisting of left and right rays $(\leftarrow, a) = \{x \in X : x < a\}$ and $(a, \rightarrow) = \{x \in X : x > a\}$ for $a \in X$. A Hausdorff topology on $(X, \leq)$ having a base consisting of order-convex sets is called a *GO-topology*. It can be shown that the interval topology is the weakest GO-topology on $(X, \leq)$.

A set A with the discrete topology will be denoted by A_d.

The principal result of this section is the following

THEOREM 2.1. *Let A be a non-empty subset of a linearly ordered space $(X, \leq)$, and let $i : A_d \to A$ denote the identity map.*

(1) *There is a function $r : X \to P_2(\beta A_d)$ such that $r(a) = \delta_a$, $a \in A$, and for every GO-topology $g \ni X \setminus A$ on X, the map $P(\beta i) \circ r : X_g \to P_2(\beta A_g)$ is continuous.*
(2) *For any semireflexive locally convex space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $u(C_\infty(A_g, Y_\tau)) \subset C_\infty(X_g, Y_\tau)$ for every GO-topology $g \ni X \setminus A$ on X and every admissible topology τ on Y.*
(3) *For every dual Banach space Y^* there is a linear $\overline{\mathrm{conv}}^*$-extender $u : l_\infty(A, Y^*) \to l_\infty(X, Y^*)$ such that $u(C_\infty(A_g, Y^*)) \subset C_\infty(X_g, Y^*)$ for every GO-topology $g \ni X \setminus A$ on X.*
(4) *For any Banach space Y that is norm-one complemented in Y^{**} there is a linear* wcc*-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ with $\|u\| = 1$ such that $u(C_\infty(A_g, Y)) \subset C_\infty(X_g, Y)$ for every GO-topology $g \ni X \setminus A$ on X.*

Applying this theorem to GO-spaces, we obtain a less complicated corollary generalizing the Heath–Lutzer Theorem 0.2.

COROLLARY 2.2. *Let A be a closed subset of a GO-space X.*

(1) *A is a $P_2\beta$-valued retract of X.*
(2) *For any semireflexive locally convex space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $u(C_\infty(A, Y)) \subset C_\infty(X, Y)$.*
(3) *For every dual Banach space Y^* there is a linear $\overline{\mathrm{conv}}^*$-extender $u : l_\infty(A, Y^*) \to l_\infty(X, Y^*)$ such that $u(C_\infty(A, Y^*)) \subset C_\infty(X, Y^*)$.*
(4) *For any Banach space Y that is norm-one complemented in Y^{**} there is a linear* wcc*-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $\|u\| = 1$ and $u(C_\infty(A, Y)) \subset C_\infty(X, Y)$.*

3. The strong Choquet properties and games. In this section we shall introduce the so-called strong Choquet property of a subset A in a topological space X, which is necessary for the existence of a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_\infty(X_A, Y)$ for functions with values in non-reflexive Banach spaces Y. This will prove that the semireflexivity assumption cannot be removed from Theorems 1.4 and 2.1.

We shall need two modifications of the classical strong Choquet game introduced by G. Choquet to give a convenient game characterization of Polish spaces (see [Ch, Th. 8.7] and also [Ke, §8]).

Our modifications, called the *strong Choquet game* $G_s(A,X)$ and the *relative strong Choquet game* $G_r(A,X)$, are played by two players, I and II, for a subset A of a topological space X. The games $G_s(A,X)$ and $G_r(A,X)$ are played in the same manner and differ only by the definition of the outcome.

Player I starts the game selecting a point $a_0 \in A$ and a neighborhood U_0 of a_0 in X. Player II responds with a neighborhood $V_0 \subset U_0$ of a_0. Continuing in this fashion, at the nth inning player I selects a point $a_n \in V_{n-1} \cap A$ and a neighborhood $U_n \subset V_{n-1}$ of a_n while player II responds with a neighborhood $V_n \subset U_n$ of a_n. Thus the players construct a sequence of points $\{a_n\}_{n\in\omega} \subset A$ and two sequences of open subsets $(U_n)_{n\in\omega}$ and $(V_n)_{n\in\omega}$ of X such that $a_n \in V_n \subset U_n \subset V_{n-1}$ for all $n \in \mathbb{N}$. Player I is declared the winner in the game $G_s(A,X)$ (resp. $G_r(A,X)$) if $\bigcap_{n\in\omega} U_n = \emptyset$ (resp. $\emptyset \neq \bigcap_{n\in\omega} U_n \subset X \setminus A$). Otherwise, player II wins.

Definition 3.1. If player II has a winning strategy in the game $G_r(A,X)$ (resp. $G_s(A,X)$), then we shall say that the subset A is *strong Choquet in* X (resp. the space X is *strong Choquet at* A). A topological space X is *strong Choquet* if X is strong Choquet at X.

Let us observe that our definition of a strong Choquet space is equivalent to the classical definition from [Ke, 8.14]. This justifies our choice of the terminology.

According to Choquet's Theorem 8.18 in [Ke], a Tikhonov (metrizable separable) space X is strong Choquet if (and only if) X is Čech complete. The latter means that X is a G_δ-set in its Stone–Čech compactification βX.

The following theorem, which is one of the main results of this article, shows that the (countable) semireflexivity necessarily appears as soon as we consider linear $\overline{\mathrm{conv}}$-extenders.

Theorem 3.2. *If for a Tikhonov subspace A of a topological space X and a linear topological space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A,Y) \to C_A(X,Y)$, then either Y is countably semireflexive or the subset A is strong Choquet in X.*

In light of Theorem 3.2 it is important to study strong Choquet subsets in more detail. This is done in the following

Theorem 3.3. *Let A be a subspace of a topological space X.*

(1) *If X is strong Choquet, then X is strong Choquet at A.*
(2) *X is strong Choquet at A if and only if X_A is strong Choquet at A if and only if X_A is strong Choquet.*
(3) *If A is strong Choquet, then A is strong Choquet in X.*

(4) *The space A is strong Choquet if X is strong Choquet at A and A is strong Choquet in X.*

Now we give a simple condition guaranteeing that a space X is strong Choquet at a subset $A \subset X$.

DEFINITION 3.4. We shall say that a space X is *complete at A* if there is a countable family $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of covers of A by open subsets of X such that a decreasing sequence $(V_n)_{n\in\mathbb{N}}$ of open subsets of X has non-empty intersection $\bigcap_{n\in\mathbb{N}} V_n$ provided for every $n \in \mathbb{N}$ the following conditions are satisfied: (i) $V_n \cap A \neq \emptyset$, (ii) $\overline{V}_{n+1} \subset V_n$, and (iii) $\overline{V}_n \subset U$ for some $U \in \mathcal{U}_n$.

PROPOSITION 3.5. *Assume that a Tikhonov space X is complete at a subset $A \subset X$. Then*

(1) *The space X is strong Choquet at A.*
(2) *The space A is strong Choquet if and only if A is strong Choquet in X.*
(3) *The space A is strong Choquet if there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$ for a linear topological space Y that fails to be countably semireflexive.*

Proof. (1) We need to describe a winning strategy for player II in the game $G_s(A, X)$. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of A witnessing that X is complete at A. To win the game $G_s(A, X)$, player II at an nth inning should select a neighborhood V_n of the point a_n given by player I such that $a_n \in V_n \subset \overline{V}_n \subset U_n \cap U$ for some set $U \in \mathcal{U}_n$. Such a choice guarantees the victory of player II because $\bigcap_{n\in\omega} V_n \neq \emptyset$.

(2) The second item follows from the first one and Proposition 3.3(3, 4).

(3) The third item follows from the second one and Theorem 3.2. ∎

Next, we show that the notion of a total π-base considered in [SV] also leads to strong Choquet subsets. Following [SV, 1.3], we say that a family $\mathcal{B}$ of open subsets of a topological space X is a *total π-base* at a subset $A \subset X$ if

(1) each $B \in \mathcal{B}$ meets A;
(2) each open subset $U \subset X$ meeting A contains a set $B \in \mathcal{B}$;
(3) each decreasing sequence $B_1 \supset B_2 \supset B_3 \supset \cdots$ of elements of $\mathcal{B}$ has non-empty intersection.

PROPOSITION 3.6. *Assume that a topological space X has a total π-base at a subset $A \subset X$. If the space A is not Baire, then player I has a winning strategy in the game $G_r(A, X)$ and hence A fails to be strong Choquet in X.*

Proof. The space A is not Baire and hence contains an open non-empty subspace $W \subset A$ of the first Baire category. Write $W = \bigcup_{n\in\omega} W_n$ where $(W_n)_{n\in\omega}$ is an increasing sequence of nowhere dense subsets in W.

Now we describe a wining strategy of player I in the game $G_r(A,X)$. To start the game she selects an open set $U_0 \in \mathcal{B}$ and a point $x_0 \in A$ such that $x_0 \in U_0 \cap A \subset W \setminus W_0$. At the nth inning she receives an open neighborhood $V_{n-1} \subset U_{n-1}$ of x_{n-1} from player II and then chooses a set $U_n \in \mathcal{B}$ and a point $x_n \in U_n \cap A$ such that $U_n \subset V_{n-1} \setminus W_n$. The existence of U_n follows from the nowhere density of W_n in W and the definition of the total π-base $\mathcal{B}$.

This strategy of player I is winning because $\bigcap_{n\in\omega} U_n$ is not empty and avoids A. ∎

Applying this proposition and [SV] to the Michael line, we obtain

COROLLARY 3.7. *The subset $\mathbb{Q}$ is not strong Choquet in the Michael line $\mathbb{R}_{\mathbb{Q}}$.*

REMARK 3.8. Proposition 3.6 shows that various spaces X, besides GO-spaces, have no linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A,Y) \to C_A(X,Y)$ for a non-reflexive Banach space Y. For example, the set $X = 2^{\omega_1}$ with the countable box topology has a total π-base at the closed subset $A = \{(t_\alpha)_{\alpha<\omega_1} : |\{\alpha < \omega_1 : t_\alpha \neq 0\}| < \aleph_0\}$ which is of the first Baire category [SV] and hence fails to be strong Choquet in X. This implies that the linear $\overline{\mathrm{conv}}$-extender property for bounded vector-valued functions can fail in ω_μ-metrizable spaces X.

4. Characterizing reflexive Banach spaces with the help of linear $\overline{\mathrm{conv}}$-extenders. Since for normed spaces (countable) semireflexivity coincides with the usual reflexivity (see [Me, 1.13.6]), we can combine Theorems 1.4, 3.2 and Corollary 2.2 to obtain the following characterization of reflexivity in Banach spaces.

THEOREM 4.1. *For a normed space Y the following conditions are equivalent*:

(1) *Y is reflexive.*
(2) *For every GO-space X and a closed subspace $A \subset X$ there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A,Y) \to l_\infty(X,Y)$ such that $u(C_\infty(A,Y)) \subset C_\infty(X,Y)$.*
(3) *For every topological space X and a Tikhonov subspace $A \subset X$ that is a $P\beta$-valued retract of X there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A,Y) \to l_\infty(X,Y)$ such that $u(C_\infty(A,Y)) \subset C_A(X,Y)$.*
(4) *There is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A,Y) \to C_A(X,Y)$ for some topological space X and some Tikhonov subspace $A \subset X$ that is not strong Choquet in X.*
(5) *There is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(\mathbb{Q},Y) \to C(\mathbb{R}_{\mathbb{Q}},Y)$.*

Proof. The implication (1)⇒(2, 3) follows from Corollary 2.2 and Theorem 1.4; (2)⇒(5) is trivial and (3)⇒(5) follows from Corollary 2.2(1). The implication (5)⇒(4) follows from Corollary 3.7 while (4)⇒(1) follows from Theorem 3.2 and the reflexivity of countably semireflexive normed spaces guaranteed by the Shmul'yan Theorem 1.13.6 of [Me]. ∎

5. Characterizing finite-dimensional Banach spaces with the help of extenders. By Theorem 4.1, the reflexivity of a Banach space Y is equivalent to the existence of a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(\mathbb{Q}, Y) \to C_\infty(\mathbb{R}_\mathbb{Q}, Y)$. Now we shall construct a space Π containing a countable closed discrete subset $N \subset \Pi$ for which the existence of a linear extender $u : C_\infty(N, Y) \to C_\infty(\Pi, Y)$ characterizes finite-dimensional Banach spaces Y.

In the Stone–Čech compactification $\beta\mathbb{N}$ of the set $\mathbb{N}$ of positive integers, take any free ultrafilter $p \in \beta\mathbb{N} \setminus \mathbb{N}$ and consider the subspace $\mathbb{N} \cup \{p\}$ with a unique non-isolated point p.

Let $[0, \omega_1)$ stand for the space of all countable ordinals with the order topology. Let $N = \mathbb{N} \times \{\omega_1\}$ and $\Pi = \mathbb{N} \times [0, \omega_1] \cup \{p\} \times [0, \omega_1)$ be subspaces of the product $(\mathbb{N} \cup \{p\}) \times [0, \omega_1]$.

THEOREM 5.1. *For a normed space Y the following conditions are equivalent*:

(1) *There is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(N, Y) \to C_\infty(\Pi, Y)$.*
(2) *There is an extender $u : C_\infty(N, Y) \to C(\Pi, Y)$.*
(3) *Y is finite-dimensional.*

Proof. The implication (1)⇒(2) is trivial.

(2)⇒(3). If Y is infinite-dimensional, then we can find a homeomorphism $f : N \to Y$ onto a bounded closed discrete subset of Y. We claim that there is no continuous map $\bar f : \Pi \to Y$ with $\bar f|N = f$. Supposing that such a continuous map exists, we can use the metrizability of Y to find a countable ordinal α such that $\bar f(n, \alpha) = \bar f(n, \omega_1)$ for all $n \in \mathbb{N}$. Now the continuity of $\bar f$ at the point (p, α) would imply that $\bar f(p, \alpha)$ is a limit point of the set $\bar f(\mathbb{N} \times \{\alpha\}) = f(N)$, which contradicts the choice of $f(N)$ as a closed discrete subset of Y.

(3)⇒(1). If Y is finite-dimensional, then each bounded map $f : N \to Y$ can be extended to a continuous map $\beta f : \beta N \to Y$ defined on $\beta N = \beta\mathbb{N} \times \{\omega_1\}$. Now extend f to a continuous map $\bar f : \Pi \to Y$ letting $\bar f(x, \alpha) = \beta f(x)$ for $(x, \alpha) \in \Pi \subset \beta\mathbb{N} \times [0, \omega_1]$. Observe that $\bar f(\Pi) \subset \overline{f(N)}$. Consequently, the operator $u : C_\infty(N, Y) \to C_\infty(\Pi, Y)$, $f \mapsto \bar f$, is a linear $\overline{\mathrm{conv}}$-extender. ∎

REMARK 5.2. Since $\Pi = \mathbb{N} \times [0, \omega_1] \cup \{p\} \times [0, \omega_1)$ is the union of two orderable spaces, we see that Corollary 2.2 cannot be generalized to spaces X that are unions of two orderable spaces.

6. Monotone extenders for functions with values in pospaces. It turns out that the method of proof of Theorem 3.2 can be modified to yield the non-existence of monotone extenders for functions with values in pospaces. As a result we obtain a general theorem that generalizes many known results on the non-existence of extenders (see [vD$_1$], [HL], [SV], [GHO]).

By a *pospace* we understand a topological space Y endowed with a partial order $\le$. For $y \in Y$ and $B \subset Y$ let

$$\uparrow y = \{x \in Y : x \ge y\} \quad \text{and} \quad \uparrow B = \bigcup_{b \in B} \uparrow b$$

be the upper cones of y and B in Y.

Observe that a subset B has an upper bound in Y if and only if $\bigcap_{b\in B} \uparrow b \neq \emptyset$.

We shall say that a subset $B \subset Y$ is *almost upper bounded* in Y if $\bigcap_{b\in B} \uparrow G_b \neq \emptyset$ for any family $\{G_b\}_{b\in B}$ of G_δ-subsets of Y with $b \in G_b$, $b \in B$.

It is clear that each upper bounded set $B \subset Y$ is almost upper bounded while the converse is true if each point $b \in B$ has countable pseudocharacter in Y. In particular, each almost upper bounded subset in a metrizable space is upper bounded.

By an *ω-increasing ray* in a pospace Y we shall understand a continuous map $\gamma : [0,\infty) \to Y$ such that $\gamma(n) \le \gamma(t)$ for any integer $n \in \omega$ and real $t \ge n$.

THEOREM 6.1. *If for a subspace Y_0 of a pospace Y and a Tikhonov subspace A of a topological space X there is a monotone extender $u : C(A, Y_0) \to C_A(X, Y)$, then either A is strong Choquet in X or else for each ω-increasing ray $\gamma : [0,\infty) \to Y_0$ the set $\gamma(\omega)$ is almost upper bounded in Y.*

Applying this theorem to the real line $\mathbb{R}$, we obtain the following corollary generalizing Theorem 1.4 of [SV].

COROLLARY 6.2. *A Tikhonov subspace A of a topological space X is strong Choquet in X if there is a monotone extender $u : C(A) \to C_A(X)$.*

Applying Theorem 6.1 to the Banach lattice c_0 (endowed with the natural partial order), we obtain another non-existence result. Here we remark that each linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, c_0) \to C_\infty(X, c_0)$ is monotone.

COROLLARY 6.3. *A Tikhonov subspace A of a topological space X is strong Choquet in X if there is a monotone extender $u : C_\infty(A, c_0) \to C_A(X, c_0)$.*

Proof. Denote by $(e_n)_{n\in\mathbb{N}}$ the standard basis of c_0. Let $\gamma : [0,\infty) \to c_0$ be the increasing piecewise linear function such that $\gamma(n) = \sum_{i=1}^n e_i$ for all $n \in \mathbb{N}$. It is clear that the set $\gamma(\omega)$ is norm-bounded but has no upper bound in c_0. Applying Theorem 6.1 we conclude that A is strong Choquet in X. ∎

Surprisingly, we do not know if the same result is true for the Banach lattice c of all convergent sequences.

QUESTION 6.4. *Is there a monotone extender* $u : C_\infty(\mathbb{Q}, c) \to C(\mathbb{R}_\mathbb{Q}, c)$*?*

7. Characterizing Polish spaces with the help of extenders. In this section we unify all results proved in the preceding sections and obtain the following characterization of Polish spaces.

THEOREM 7.1. *For a metrizable separable space A the following conditions are equivalent*:

(1) *A is Polish.*
(2) *A is strong Choquet.*
(3) *A is a P_ω-valued retract in each normal space X containing A as a closed subspace.*
(4) *A is a P_σ-valued retract in some topological space $X \supset A$ that is strong Choquet at A.*
(5) *For every locally convex linear topological space Y and every normal space X containing A as a closed subspace there is a linear* conv-*extender* $u : C(A, Y) \to C(X, Y)$.
(6) *For some infinite-dimensional Banach space Y and some topological space $X \supset A$ that is strong Choquet at A there is a* conv-*extender* $u : C_\infty(A, Y) \to C_A(X, Y)$.
(7) *For some topological space $X \supset A$ that is strong Choquet at A and some separable non-reflexive Banach space Y there is a linear* $\overline{\text{conv}}$-*extender* $u : C_\infty(A, Y) \to C_A(X, Y)$.
(8) *For some topological space $X \supset A$ that is strong Choquet at A there is a monotone extender* $u : C(A) \to C_A(X)$.
(9) *For some topological space $X \supset A$ that is strong Choquet at A there is a monotone extender* $u : C_\infty(A, c_0) \to C_A(X, c_0)$.

Proof. We shall establish the implications (2)⇒(1)⇒(5)⇒(3, 6)⇒(4)⇒(7)⇒(2) and (5)⇒(7, 8, 9)⇒(2).

The implication (2)⇒(1) is due to G. Choquet (see [Ke, 8.18]).

(1)⇒(5). Assume that A is a Polish space. By [Ke, 4.17], A admits a closed embedding $e : A \to \mathbb{R}^\omega$. Given any normal subspace X containing A as a closed subset, we can apply the Tietze–Urysohn Theorem to find a continuous map $g : X \to \mathbb{R}^\omega$ extending e. By the Dugundji Theorem 0.1, for every locally convex space Y there is a linear conv-extender $v : C(e(A), Y) \to C(\mathbb{R}^\omega, Y)$. Now define a linear conv-extender $u : C(A, Y) \to C(X, Y)$ by the formula $u(f) = v(f \circ e^{-1}) \circ g : X \to Y$.

(5)⇒(3). Let X be a normal space containing A as a closed subspace. Let $Y = C^*_\infty(A)$ with the weak-star topology. By (5), there is a conv-extender u :

$C_\infty(A,Y) \to C_\infty(X,Y)$. Then the embedding $\delta : A \to P_\omega(A) \subset C^*_\infty(A)$ assigning δ_a to each $a \in A$ has a continuous extension $r = u(\delta) : X \to Y$ given by the conv-extender u. It follows that $r(X) \subset \mathrm{conv}(\delta(A)) = P_\omega(A)$, which means that $r : X \to P_\omega(A)$ is the required P_ω-valued retraction of X onto A.

The implications (3)⇒(4) and (5)⇒(6–9) will follow as soon as we find a normal space $X \supset A$ that is strong Choquet at A. For this take any metrizable compactification K of A and consider the space K_A. The compactness of K implies the completeness of K_A at A. By Proposition 3.5(1), the space K_A is strong Choquet at A. The normality of K_A follows from [Eng, 5.1.22].

To prove the implication (6)⇒(4), assume that for some infinite-dimensional Banach space Y and some topological space $X \supset A$ that is strong Choquet at A there exists a conv-extender $u : C_\infty(A,Y) \to C(X_A,Y)$. Let K be any metrizable compactification of the separable metrizable space A and let $P(K)$ be the space of probability measures on K. Let $\delta : A \to P(K)$ be the embedding assigning δ_x to each $x \in A$. Observe that $P_\omega(A)$ coincides with the convex hull of $\delta(A)$ in $P(K)$.

According to [BP, §III.2], there is a continuous affine embedding $e : P(K) \to Y$. Consider the map $g = e \circ \delta : A \to Y$ and its continuous extension $\bar g = u(g) : X_A \to Y$. Since u is a conv-extender, $\bar g(X) \subset \mathrm{conv}(g(A)) = e(P_\omega(A))$. It is clear that the map $r = e^{-1} \circ \bar g : X_K \to P_\omega(A)$ is continuous and $r|A = \delta$, which means that A is a P_ω-valued retract of X_A. By Theorem 3.3(2), the space X_A is strong Choquet at A.

The implication (4)⇒(7) follows from Proposition 1.2, and (7, 8, 9)⇒(2) from Theorem 3.2 and Corollaries 6.2, 6.3, respectively. ∎

8. Monotone extenders for functions with values in Banach lattices. In light of Corollary 6.3 it is natural to ask about the existence of linear monotone extenders $u : C_\infty(A,Y) \to C_\infty(X,Y)$ for functions taking their values in a (non-reflexive) Banach lattice Y. Many classical Banach spaces like c_0, l_p, L_p, $C(K)$ have the natural structure of a Banach lattice.

We recall that a *Banach lattice* is a real Banach space $(Y, \|\cdot\|)$ endowed with a partial order $\le$ satisfying the following four axioms (see [LT]):

- $x \le y$ implies $x + z \le y + z$ for all $x, y, z \in Y$;
- $a \cdot x \ge 0$ for any $x \ge 0$ in Y and any real number $a \ge 0$;
- any two points $x, y \in Y$ have the largest lower and smallest upper bounds $x \wedge y$ and $x \vee y$ in Y;
- $\|x\| \le \|y\|$ whenever $|x| \le |y|$ where the absolute value $|x|$ of $x \in Y$ is defined by $|x| = -x \vee x$.

Let us remark that the dual Banach space Y^* to a Banach lattice Y is a Banach lattice with respect to the partial order $\le$ defined by declaring $x^* \le y^*$ for $x^*, y^* \in Y^*$ iff $x^*(z) \le y^*(z)$ for all $z \ge 0$ in Y.

We shall say that a Banach lattice Y is *positively norm-one complemented in its bidual* Y^{**} if there is a linear monotone projector $P : Y^{**} \to Y$ with $\|P\| = 1$. The class of such Banach lattices includes all dual Banach lattices and also all weakly sequentially complete Banach lattices (like $L_1(\mu)$) (see [LT, 1.c.4]).

The following theorem is a monotone version of Theorem 1.6 and can be proved by analogy.

THEOREM 8.1. *For any $P\beta$-valued retract A of a topological space X and every Banach lattice Y that is positively norm-one complemented in Y^{**} there is a linear monotone* wcc-*extender* $u : l_\infty(A, Y) \to l_\infty(X, Y)$ *such that* $\|u\| = 1$ *and* $u(C_\infty(A, Y)) \subset C_A(X, Y)$.

The same concerns the following corollary that can be derived from Theorem 2.1(2).

COROLLARY 8.2. *For every subset A of a linearly ordered space $(X, \leq)$ and every Banach lattice Y that is positively norm-one complemented in Y^{**} there is a linear monotone* wcc-*extender* $u : l_\infty(A, Y) \to l_\infty(X, Y)$ *such that* $\|u\| = 1$ *and* $u(C_\infty(A_g, Y)) \subset C_\infty(X_g, Y)$ *for any GO-topology* $g \ni X \setminus A$ *on* X.

Now we are going to characterize the σ-complete Banach lattices Y admitting a linear monotone extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$ for any closed subset A of a GO-space X. The characterization Theorem 9.1 below relies on the notion of a $\perp$-extender defined as follows.

Two elements x, y of a Banach lattice Y are called *disjoint* if $|x| \wedge |y| = 0$. For any point $y \in Y$ the set

$$y^\perp = \{x \in Y : |x| \wedge |y| = 0\}$$

of elements disjoint from y is a closed linear subspace in Y called the *polar* of y (see [LT]). For a subset $B \subset Y$ the closed linear subspace

$$B^\perp = \bigcap_{b \in B} b^\perp$$

is called the *polar* of B, and $B^{\perp\perp} = (B^\perp)^\perp$ is the *bipolar* of B. It is clear that $B^{\perp\perp}$ is a closed linear subspace of Y, containing B.

An extender $u : C_\infty(A, Y) \to Y^X$ will be called a $\perp$-*extender* if u is a $\mathcal{C}$-extender for the collection $\mathcal{C} = \{B^\perp : B \subset Y\}$ of polar sets. It is easy to see that an extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$ is a $\perp$-extender if and only if for every bounded function $f : A \to Y$ the image $\overline{f}(X)$ of the extended function $\overline{f} = u(f) : X \to Y$ lies in the bipolar $f(A)^{\perp\perp}$. An extender which is simultaneously a $\perp$-extender and a wcc-extender will be called a $\perp$-wcc-*extender.*

Now we shall derive from Theorem 6.1 a necessary condition for the existence of a monotone extender.

THEOREM 8.3. *Assume that a Tikhonov subspace A of a topological space X is not strong Choquet in X. If for a Banach lattice Y there is a monotone extender $u : C_\infty(A, Y) \to C_A(X, Y)$ (which is a $\perp$-extender), then each countable norm-bounded upward directed subset $D \subset Y$ has an upper bound in Y (in $D^{\perp\perp}$).*

Proof. Let $D = \{y_n : n \in \mathbb{N}\} \subset Y$ be a countable norm-bounded upward directed subset. Since D is upward directed, by induction we can construct an increasing sequence $\{z_n\}_{n\in\omega} \subset D$ such that $z_n \geq y_i$ for all $i \leq n$. Then each upper bound for the set $E = \{z_n\}_{n\in\omega}$ is also an upper bound for D. Consider the piecewise linear map $\gamma : [0, \infty) \to Y$ defined by $\gamma(k) = z_k$ for all $k \in \omega$. It follows that γ is an ω-increasing ray with bounded range $B = \gamma([0, \infty))$. Consequently, $C(A, B) \subset C_\infty(A, Y)$ and the restriction $v = u|C(A, B) : C(A, B) \to C_A(X, Y)$ is a well-defined monotone extender. Applying Theorem 6.1, we conclude that the set $\gamma(\omega) = E$ is (almost) upper bounded in Y.

Now assume that $u : C_\infty(A, Y) \to C_A(X, Y)$ is a $\perp$-extender and thus $u(C_\infty(A, D^{\perp\perp})) \subset C_A(X, D^{\perp\perp})$.

Observe that $E^{\perp\perp} \supset E$ is a linear subspace of Y and thus $B = \gamma([0, \infty)) \subset \mathrm{conv}(\gamma(\omega)) \subset \mathrm{conv}(E) \subset E^{\perp\perp} \subset D^{\perp\perp}$. Then $C(A, B) \subset C_\infty(A, D^{\perp\perp})$ and we can consider the monotone extender $v = u|C(A, B) : C(A, B) \to C_A(X, D^{\perp\perp})$. By Theorem 6.1 the set $E = \gamma(\omega)$ is (almost) upper bounded in $D^{\perp\perp}$. ∎

9. Characterizing weakly sequentially complete Banach lattices. In this section we characterize weakly sequentially complete Banach lattices with the help of monotone extenders.

We recall that a Banach lattice Y is called

- *σ-complete* if each upper bounded increasing sequence $\{y_n\}_{n\in\omega} \subset Y$ has the smallest upper bound $\vee_{n\in\omega} y_n$;
- *order continuous* if each downward directed subset $D \subset Y$ with $\wedge D = 0$ contains zero in its closure.

For example, c_0 is σ-complete but not order continuous while c is not σ-complete. By [LT, 1.a.8] each order continuous Banach lattice is σ-complete.

THEOREM 9.1. *For a Banach lattice Y the following conditions are equivalent:*

(1) *Y is weakly sequentially complete.*
(2) *Y does not contain a copy of c_0.*
(3) *Norm-bounded increasing sequences in Y converge.*

(4) *Y is order continuous and there is a monotone extender* $u : C_\infty(A,Y) \to C_A(X,Y)$ *for some topological space X and a Tikhonov subspace $A \subset X$, which is not strongly Choquet in X.*

(5) *Y is σ-complete, does not contain a copy of l_∞, and there is a monotone extender $u : C_\infty(A,Y) \to C_A(X,Y)$ for some topological space X and a Tikhonov subspace $A \subset X$, which is not strongly Choquet in X.*

Moreover, if $\mathrm{dens}(Y) < \mathfrak{c}$, *then conditions* (1)–(5) *are equivalent to*:

(6) *For every subset A of a linearly ordered space $(X, \le)$ there is a linear monotone* $\perp$-wcc-*extender* $u : l_\infty(A,Y) \to l_\infty(X,Y)$ *such that* $\|u\| = 1$ *and* $u(C_\infty(A_g,Y)) \subset C_\infty(X_g,Y)$ *for every GO-topology $g \ni X \setminus A$ on X.*

(7) *There is a monotone $\perp$-extender $u : C_\infty(A,Y) \to C_A(X,Y)$ for some topological space X and a Tikhonov subspace $A \subset X$, which is not strongly Choquet in X.*

Proof. The equivalence of the first three conditions is well-known and can be found in [LT, 1.c.4].

(2)$\Rightarrow$(5). Assume that Y does not contain a copy of c_0. Then it does not contain a copy of l_∞ either. By [LT, 1.c.4, 1.a.8], Y is σ-complete, and by [LT, 1.c.4], it is positively norm-one complemented in its bidual space. By Corollary 8.2, there is a monotone extender $u : C_\infty(\mathbb{Q}, Y) \to C_\infty(\mathbb{R}_\mathbb{Q}, Y)$. By Corollary 3.7, $\mathbb{Q}$ is not strong Choquet in $\mathbb{R}_A$. Thus (5) follows.

(5)$\Rightarrow$(4). Assume that Y is σ-complete but contains no copy of l_∞. By Propositions 1.a.7 and 1.a.8 of [LT], Y is order continuous.

(4)$\Rightarrow$(3). By Theorem 8.3, norm-bounded increasing sequences in Y are upper-bounded and thus converge by the order continuity of Y.

Now assume that $\mathrm{dens}(Y) < \mathfrak{c}$.

(1)$\Rightarrow$(6). Assume that Y is weakly sequentially complete, and let A be a subset of a linearly ordered space $(X, \le)$.

By Theorem 2.1(2), there is a linear $\overline{\mathrm{conv}}^*$-extender $u : l_\infty(A, Y^{**}) \to l_\infty(X, Y^{**})$ such that $u(C_\infty(A_g, Y^{**})) \subset C_\infty(X_g, Y^{**})$ for every GO-topology $g \ni X \setminus A$ on $(X, \le)$. Since the positive cone $Y^{**}_+ = \{y^{**} \in Y^{**} : y^{**} \ge 0\}$ is convex and closed in the weak-star topology of Y^{**}, we conclude that $u(l_\infty(A, Y^{**}_+)) \subset l_\infty(X, Y^{**}_+)$, which implies that u is monotone.

By Theorem 1.c.4 of [LT], the weak sequential completeness of Y implies the existence of a monotone norm-one projector $P : Y^{**} \to Y$ whose kernel coincides with $Y^\perp \subset Y^{**}$. Consequently, Y^{**} can be identified with $Y \oplus Y^\perp$. By analogy with the proof of Theorem 1.6, we can consider the extender $v : l_\infty(A,Y) \to l_\infty(X,Y)$ assigning to each $f \in l_\infty(A,Y)$ the function $P \circ u(f) : X \to Y$ and prove that v is a wcc-extender with $v(C_\infty(A,Y)) \subset$

$C_\infty(X,Y)$. Being the composition of two monotone norm one operators, the extender v is monotone and has norm $\|v\|=1$.

It remains to prove that v is a $\perp$-extender. Given any $B\subset Y$ we should prove that $v(f)(X)\subset B^\perp\subset Y$ for every bounded function $f:A\to B^\perp$. Since $B^\perp=\bigcap_{b\in B}b^\perp$, it suffices to check that $v(f)(X)\subset b^\perp$ for every $b\in B$.

By Proposition 1.a.9 of [LT], $Y=Y_1\oplus Y_2$ where $Y_1=b^\perp$ and $Y_2=b^{\perp\perp}$. Consequently, $Y^{**}=Y_1^{**}\oplus Y_2^{**}$ and Y_1^{**} is weak-star closed in Y^{**}. The sublattices Y_1,Y_2 of Y are weakly sequentially complete and by Theorem 1.c.4 of [LT], their biduals decompose as $Y_i^{**}=Y_i\oplus Y_i^\perp$ where the polar set $Y_i^\perp$ is taken in Y_i^{**}. Consequently, $Y^{**}=Y_1\oplus Y_2\oplus Y_1^\perp\oplus Y_2^\perp=Y\oplus Y^\perp$ and $P(Y_1^{**})\subset Y_1$. Since u is a $\overline{\mathrm{conv}}^*$-extender, $u(f)(X)\subset\overline{\mathrm{conv}}^*(f(A))\subset\overline{\mathrm{conv}}^*(Y_1)\subset Y_1^{**}$ and then $v(f)(X)=P(u(f)(X))\subset P(Y_1^{**})\subset Y_1=b^\perp$.

The implication (6)⇒(7) follows from Corollary 3.7.

It remains to prove that (7)⇒(2). By Theorem 8.3, condition (7) implies that each norm-bounded countable upward directed subset $D\subset Y$ has an upper bound in $D^{\perp\perp}$. Suppose that Y contains a copy of c_0. By the proof of Theorem 1.a.5 in [LT] (see remark after Theorem 1.c.4 there), Y contains c_0 as a sublattice. Denote by $(e_n)_{n\in\mathbb{N}}$ the standard basis of c_0 and let $c=\inf_{n\in\mathbb{N}}\|e_n\|>0$ where $\|\cdot\|$ stands for the norm of Y.

For every $A\subset\mathbb{N}$ consider the countable upward directed subset $D_A=\{\sum_{i\in F}e_i : F\subset A \text{ is finite}\}$. By Theorem 8.3, this set has an upper bound $b_A\in D_A^{\perp\perp}\subset Y$.

Let us show that $\|b_A-b_B\|\ge c$ for any distinct $A,B\subset\mathbb{N}$. Without loss of generality, there is an $n\in B\setminus A$. Since $e_n\in D_A^\perp$ and $b_A\in D_A^{\perp\perp}$, we see that b_A and e_n are disjoint and hence $b_A\wedge e_n=|b_A|\wedge|e_n|=0$. Now we see that

$$(b_B-b_A)\ge(b_B-b_A)\wedge e_n=b_B\wedge e_n-b_A\wedge e_n=b_B\wedge e_n-0=e_n$$

and hence $\|b_B-b_A\|\ge\|e_n\|\ge c$. Consequently, $\{b_A:\emptyset\ne A\subset\mathbb{N}\}$ is a discrete subset of size continuum in Y, which is not possible because $\mathrm{dens}(Y)<\mathfrak{c}$. ∎

Remark 9.2. For a subset A of a topological space X and a Banach lattice Y consider the following three properties:

(1) There is a linear $\overline{\mathrm{conv}}$-extender $u:C_\infty(A,Y)\to C_\infty(X,Y)$.
(2) There is a norm-one linear extender $u:C_\infty(A,Y)\to C_\infty(X,Y)$.
(3) There is a monotone linear extender $u:C_\infty(A,Y)\to C_\infty(X,Y)$.

It is clear that (1)⇒(2, 3). In [vD$_1$] and [vD$_2$] E. K. van Douwen asked if for $Y=\mathbb{R}$ there are other implications among these conditions. The results of this paper show that (1) does not follow from (2, 3). Indeed, by Theorem 9.1, for $Y=l_1$ and the Michael line $X=\mathbb{R}_\mathbb{Q}$ there is a monotone norm-one linear extender $u : C_\infty(A, l_1) \to C_\infty(\mathbb{R}_\mathbb{Q}, l_1)$ for every closed subset $A \subset \mathbb{R}_\mathbb{Q}$. Yet, by Theorem 4.1, no linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(\mathbb{Q}, l_1) \to C_\infty(\mathbb{R}_\mathbb{Q}, l_1)$ exists.

10. Proof of Theorem 1.4. Let A be a Tikhonov subspace of a topological space X.

The implication (1)⇒(2) will follow as soon as we show that $Y = C^*_\infty(A)$ with the weak-star topology is semireflexive. But this follows from the Banach–Steinhaus Uniform Boundedness Principle (see [HHZ, Th. 58]).

To prove that (2)⇒(3), fix a $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$ where $Y = C^*_\infty(A)$ with the weak-star topology. Since each $f \in C_\infty(A)$ admits a unique continuous extension to βA, we can identify $C_\infty(A)$ with $C(\beta A)$ and Y with $C^*(\beta A)$. Consider the bounded continuous map $\delta : A \to C^*(\beta A)$ assigning δ_a to each $a \in A$. Let $r = u(\delta) : X_A \to Y = C^*(\beta A)$ be the continuous extension of δ given by the $\overline{\mathrm{conv}}$-extender u. It follows that $r(X) \subset \overline{\mathrm{conv}}(\delta(A)) = P(\beta A)$, which means that $r : X_A \to P(\beta A)$ is the required $P\beta$-valued retraction of X_A onto A.

(3)⇒(1). Fix a $P\beta$-valued retraction $r : X_A \to P(\beta A)$. Denote by A_d the space A endowed with the discrete topology. The identity map $i : A_d \to A$ is continuous and hence extends to a continuous surjective map $\beta i : \beta A_d \to \beta A$. This map induces a surjective continuous map $P(\beta i) : P(\beta A_d) \to P(\beta A)$. The surjectivity of $P(\beta i)$ allows us to select a (generally discontinuous) function $s : X \to P(\beta A_d)$ such that $P(\beta i) \circ s = r$ and $s(a) = \delta_a$ for all $a \in A$.

Now, given a locally convex semireflexive space Y, we are ready to define a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$. Given any bounded function $f : A \to Y$, consider the closed convex hull $K_\mathrm{w} \subset Y_\mathrm{w}$ of $f(A)$, endowed with weak topology. The semireflexivity of Y guarantees that K_w is compact. Let $\beta f_d : \beta A_d \to K_\mathrm{w}$ be the continuous extension of $f_d = f \circ i : A_d \to K_\mathrm{w}$.

For every $x \in X$ consider the probability measure $\mu_x = s(x)$ and the Pettis integral

$$u(f)(x) = \int_{\beta A_d} (\beta f_d)\, d\mu_x \in K_\mathrm{w} \subset Y_\mathrm{w}, \tag{1}$$

which is well-defined because K_w is weakly compact and convex (see [DU, § II.3]).

The linearity of the Pettis integral implies that $u : C_\infty(A, Y) \to Y^X$, $f \mapsto u(f)$, is a well-defined linear $\overline{\mathrm{conv}}$-extender.

It remains to check that for any admissible topology τ on Y the function $u(f) : X \to Y_\tau$ is continuous at A provided $f : A \to Y_\tau$ is continuous. Given any $a \in A$ and an open convex neighborhood $O \subset Y_\tau$ of zero we should find a neighborhood $W \subset X$ of a such that $u(f)(W) \subset f(a) + O$. Since τ is admissible, there is an open convex symmetric neighborhood $U \subset Y_\tau$ of zero whose closure $\overline{U}$ in Y lies in $\frac{1}{3}O$. By the Hahn–Banach Theorem, the set $\overline{U}$, being closed and convex, is weakly closed. Now the continuity of $\beta f : \beta A \to Y_{\mathrm{w}}$ implies that $F = (\beta f)^{-1}(f(a) + \overline{U})$ is closed in βA. On the other hand, the continuity of $f : A \to Y_\tau$ implies that $V = f^{-1}(f(a) + U)$ is open in A. It follows from the regularity of βA that $\overline{V} \subset F$ is a neighborhood of a in βA. Consequently, F is a neighborhood of a in βA and $F' = (\beta i)^{-1}(F) \subset \beta A_d$ is a neighborhood of a in βA_d.

Since $f : A \to Y_\tau$ is bounded, there is a number m so large that $f(A) \subset mU$. Then $K_{\mathrm{w}} = \overline{\mathrm{conv}}(f(A)) \subset m\overline{U}$ and $K_{\mathrm{w}} - f(a) \in m\overline{U} - m\overline{U} = 2m\overline{U}$. It follows that $\mathcal{V} = \{\mu \in P(\beta A) : \mu(F) > 1 - 1/m\}$ is a neighborhood of δ_a in $P(\beta A)$. Since the $P\beta$-valued retraction r is continuous at a, there is a neighborhood $W \subset X$ of a such that $r(W) \subset \mathcal{V}$. We claim that $u(f)(W) \subset f(a) + O$.

Take any $x \in W$ and consider the measures $r(x) \in \mathcal{V}$ and $\mu_x = s(x) \in P(\beta A_d)$. It follows from $P(\beta i)(\mu_x) = P(\beta i)(s(x)) = r(x)$ that $\mu_x(F') > 1 - 1/m$. Then

$$\begin{aligned} u(f)(x) - f(a) &= \int_{\beta A_d} (\beta f_d - f(a))\, d\mu_x \\ &= \int_{\beta A_d \setminus F'} (\beta f_d - f(a))\, d\mu_x + \int_{F'} (\beta f_d - f(a))\, d\mu_x \\ &\in \mu_x(\beta A_d \setminus F') \cdot (K_{\mathrm{w}} - f(a)) + \mu_x(F') \cdot \overline{U} \\ &\subset \frac{1}{m} 2m\overline{U} + \overline{U} = 3\overline{U} \subset O. \end{aligned}$$

11. Proof of Theorem 2.1. Let A be a non-empty subset of a linearly ordered space $(X, \le)$. By [Lu, 2.9], the linearly ordered topological space $(X, \le)$ has a linearly ordered compactification $(\overline{X}, \le)$.

(1) Let A_d be A with the discrete topology and $i : A_d \to A$ be the identity map. We shall construct a function $r : \overline{X} \to P_2(\beta A_d)$ such that $r(a) = \delta_a$ and for every GO-topology $g \ni X \setminus A$ on X the composition $P(\beta i) \circ r : X_g \to P_2(\beta A_g)$ is continuous.

Let $\bar{A}$ be the closure of A in $\overline{X}$ and $\beta i : \beta A_d \to \bar{A}$ be the Stone-Čech extension of i. We shall identify βA_d with the set of Dirac measures in $P_2(\beta A_d)$. For every $a \in \bar{A}$ select an ultrafilter $u_a \in \beta A_d$ such that $\beta i(u_a) = a$ and $u_a = a$ if $a \in A$.

Write $\overline{X} \setminus \bar{A}$ as the disjoint union $\bigcup \mathcal{C}$ of the family $\mathcal{C}$ of order-convex components of $\overline{X} \setminus \bar{A}$. Those are maximal order-convex subsets of $\overline{X} \setminus \bar{A}$. For each $C \in \mathcal{C}$ we define an order-convex set $\widetilde{C} \supset C$ and a continuous map $r_C : \widetilde{C} \to P_2(\beta A_d)$ as follows.

If $C = (\leftarrow, \min \bar{A})$, then we put $\widetilde{C} = (\leftarrow, \min \bar{A}]$ and $b_C = u_{\min \bar{A}}$, and we let $r_C : \widetilde{C} \to \{b_C\} \subset \beta A_d \subset P_2(\beta A)$ be the constant map.

If $C = (\max \bar{A}, \to)$, then we put $\widetilde{C} = [\max \bar{A}, \to)$ and $a_C = u_{\max \bar{A}}$, and we let $r_C : \widetilde{C} \to \{a_C\} \subset \beta A_d \subset P_2(\beta A)$ be the constant map.

In the remaining case, $C = (a, b)$ for some $a < b$ in $\bar{A}$. Let $a_C = u_a$ and $b_C = u_b$. Using the normality of the compact space $\overline{X}$, find a continuous function $\lambda_C : \overline{X} \to [0,1]$ such that $\lambda_C(a) = 0$ and $\lambda_C(b) = 1$. Finally, define $r_C : \widetilde{C} \to P_2(\beta A_d)$ by the formula

$$r_C(x) = (1 - \lambda_C(x)) \cdot a_C + \lambda_C(x) \cdot b_C, \quad x \in C.$$

Putting together the maps r_C, $C \in \mathcal{C}$, define a function $r : \overline{X} \to P_2(\beta A_d)$ by the formula

$$r(x) = \begin{cases} u_x & \text{if } x \in \bar{A}, \\ r_C(x) & \text{if } x \in C \in \mathcal{C}. \end{cases}$$

Observe that $r|\widetilde{C}$ is continuous for every $C \in \mathcal{C}$. It remains to prove that r has the continuity property required in item (1) of Theorem 2.1. We shall return to this problem after establishing item (2).

(2) Given a semireflexive locally convex space Y, we shall define a linear $\overline{\mathrm{conv}}$-extender $v : l_\infty(A, Y) \to l_\infty(\overline{X}, Y)$. Take any bounded function $f : A \to Y$ and set $f_d = f \circ i : A_d \to Y$. The semireflexivity of Y guarantees that the closed convex hull K_{w} of $f(A)$ is compact in the weak topology of Y. Then $f_d : A_d \to K_{\mathrm{w}}$ has a continuous extension $\beta f_d : \beta A_d \to K_{\mathrm{w}}$. For every $x \in \overline{X}$ we can integrate this function against the measure $\mu_x = r(x) \in P_2(\beta A_d)$ to obtain the value $\bar{f}(x) = \int_{\beta A_d} (\beta f_d)\, d\mu_x$ of the function $v(f) = \bar{f}$ at the point $x \in \overline{X}$.

It is clear that the so-defined operator $v : l_\infty(A, Y) \to l_\infty(\overline{X}, Y)$, $f \mapsto \bar{f} = v(f)$, is a linear $\overline{\mathrm{conv}}$-extender. It induces a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$, $f \mapsto u(f)|X$.

To finish the proof of Theorem 2.1(2), it remains to show $u(C_\infty(A_g, Y_\tau)) \subset C_\infty(X_g, Y_\tau)$ for every GO-topology $g \ni X \setminus A$ on $(X, \le)$ and every admissible topology τ on Y. Take any $f \in C_\infty(A_g, Y_\tau)$ and $\bar{f} = v(f) : \overline{X} \to Y$. It follows from the definition of $\bar{f}$ that $\bar{f}|\widetilde{C}$ is continuous for every $C \in \mathcal{C}$. So it remains to check the continuity of $u(f) = \bar{f}|X$ at each $x_0 \in \bar{A} \cap X$.

It suffices to prove that the restrictions of $\bar{f}$ to the closed subsets

$$X_g^- = (\leftarrow, x_0] \cap X_g \quad \text{and} \quad X_g^+ = [x_0, \to) \cap X_g$$

are continuous at x_0. We shall do that for X_g^+; the argument for X_g^- is analogous. If x_0 is isolated in X_g^+, then there is nothing to prove: $\bar{f}|X_g^+$ is trivially continuous at x_0.

So suppose that x_0 is not isolated in X_g^+. First we consider the case $x_0 \in \bar{A} \setminus A$. Since A is closed in X_g, there is an order-convex open set $U \subset X_g^+$ such that $x_0 \in U \subset X_g^+ \setminus A$. Since x_0 is not isolated in X_g^+, there is a point $x_1 \in U$. It follows that $(x_0, x_1) \cap A = \emptyset$ and hence $(x_0, x_1) \subset C$

and $[x_0, x_1) \subset \widetilde{C}$ for some $C \in \mathcal{C}$. Now the continuity of $\bar{f}|[x_0, x_1)$ follows from the continuity of $\bar{f}|\widetilde{C}$.

Next, suppose that $x_0 \in A$. Given any open neighborhood $O \subset Y_\tau$ of zero, we should find a neighborhood $W \subset X_g^+$ of x_0 such that $\bar{f}(W) \subset f(a) + O$. Since the topology τ is admissible, there is an open convex neighborhood $U \subset Y_\tau$ whose closure $\overline{U}$ in Y lies in O.

The continuity of $f : A \to Y_\tau$ yields an open order-convex subset $V \subset X_g^+$ such that $x_0 \subset V \cap A \subset f^{-1}(U)$. Since x_0 is not isolated in X_g^+, there is a point $x_1 > x_0$ in V. If $[x_0, x_1) \subset \widetilde{C}$ for some component $C \in \mathcal{C}$, then the continuity of $\bar{f}|X_g^+$ at x_0 follows from the continuity of $\bar{f}|\widetilde{C}$. In the other case, (x_0, x_1) contains a point $x_2 \in A$. Let F be the closure of $[x_0, x_2] \cap A$ in βA_d. The continuity of $\beta i : \beta A_d \to \bar{A}$ implies that $(\beta i)^{-1}([x_0, x_2] \cap \bar{A}) = F$. Consequently, $u_a \in F$ for every $a \in [x_0, x_2] \cap \bar{A}$. Then $a_C, b_C \in F$ for every component $C \subset [x_0, x_2]$.

On the other hand, the continuity of $\beta f_d : \beta A_d \to Y_{\mathrm{w}}$ and the inclusion $\beta f_d([x_0, x_2] \cap A) \subset f(x_0) + U$ imply $\beta f_d(F) \subset f(x_0) + \overline{U} \subset f(x_0) + O$. Now looking at the definition of $\bar{f} = u(f)$, we see that $\bar{f}([x_0, x_2]) \subset \mathrm{conv}\{\beta f_d(u_a) : a \in [x_0, x_2] \cap \bar{A}\} \subset \mathrm{conv}(\beta f_d(F)) \subset f(x_0) + \overline{U} \subset f(x_0) + O$. Then $W = [x_0, x_2) \cap X$ is the required neighborhood of x_0 in X_g^+ with $\bar{f}(W) \subset f(x_0) + \overline{U} \subset f(x_0) + O$. This completes the proof of Theorem 2.1(2).

($1'$) Now we shall finish the proof of item (1), establishing the continuity property of $r : \overline{X} \to P_2(\beta A_d)$. Let $g \ni X \setminus A$ be any GO-topology on $(X, \leq)$. Let $Y = C^*(\beta A_g)$ with the weak-star topology and let $u : l_\infty(A, Y) \to l_\infty(X, Y)$ be the linear $\overline{\mathrm{conv}}$-extender constructed in (2). It has the property that $u(C_\infty(A_g, Y)) \subset C_\infty(X_g, Y)$.

Consider the Dirac embedding $\delta : A_g \to P(\beta A_g) \subset C^*(\beta A_g) = Y$ and its continuous extension $\bar{\delta} = u(\delta) : X_g \to Y$ given by the extender u. The definition of u implies that $\bar{\delta}$ is equal to the composition $P(\beta i) \circ r$ of the maps $r : X \to P_2(\beta A_d)$ and $P(\beta i) : P(\beta A_d) \to P(\beta A_g)$. Consequently, $P(\beta i) \circ r : X_g \to P_2(\beta A_g)$ is continuous.

(3) The third item follows from the second one and the fact that the norm-topology of a dual Banach space Y^* is admissible for the weak-star topology on Y^*.

(4) The fourth item can be derived from the third one by the argument of the proof of Theorem 1.6.

12. Proof of Theorem 3.2. Assume that for a subspace A of a Tikhonov space X and a linear topological space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$. Assuming that Y is not countably semireflexive, we shall prove that the subset A is strong Choquet in X. We should describe a winning strategy for player II in the game $G_r(A, X)$.

Since Y is not countably semireflexive, there is a decreasing sequence $(K_n)_{n=1}^{\infty}$ of non-empty closed bounded convex subsets of Y with $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Let $y_0 = 0$ and $y_n \in K_n$ for every $n \in \mathbb{N}$. For every $m \in \mathbb{N}$ consider the finite-dimensional linear subspace L_m of Y spanned by $y_0, \ldots, y_m$. Since the union $L = \bigcup_{m\in\omega} L_m$ has countable pseudocharacter, we can select a decreasing sequence $(O_n)_{n\in\omega}$ of open neighborhoods of zero in Y such that $L \cap \bigcap_{n\in\omega} O_n = \{0\}$. Since u is a linear $\overline{\mathrm{conv}}$-extender, for every bounded continuous function $f : A \to L_n$ the extension $\bar{f} = u(f)$ has range $\bar{f}(X) \subset L_n$.

Now we (somewhat informally) describe a winning strategy of player II in the game $G_r(A, X)$. The point is that at his nth inning player II chooses a neighborhood $V_n \subset U_n$ of the point a_n given by the first player with the help of a continuous function $\lambda_n : A \to [0, 1]$ such that

$$a_n \in V_n \cap A \subset \lambda_n^{-1}(1) \subset \lambda_n^{-1}(0, 1] \subset U_n \subset V_{n-1} \tag{2}$$

and keeps the functions λ_i from the previous innings in his memory.

Therefore players I and II will consecutively choose the pairs

$$\begin{tikzcd}
\mathrm{I:} & (U_1,a_1) \arrow[dr] & & (U_2,a_2) \arrow[dr] & & (U_3,a_3) \arrow[dr] & & \cdots \\
\mathrm{II:} & & (V_1,\lambda_1) \arrow[ur] & & (V_2,\lambda_2) \arrow[ur] & & (V_3,\lambda_3) &
\end{tikzcd}$$

so that condition (2) is satisfied.

Now we explain how to select the function λ_n and the neighborhood V_n at the nth inning. After receiving the point $a_n \in A$ and the neighborhood $U_n \subset X$ of a_n from the first player, player II uses the Tikhonov property of A to find a continuous function $\lambda_n : A \to [0, 1]$ such that $\lambda_n(A \setminus U_n) \subset \{0\}$ and a_n lies in the interior W_n of $\lambda_n^{-1}(1)$ in A. Now for every $k \le n$ consider the bounded continuous function $f_k = \sum_{i=1}^{k} \lambda_i \cdot (y_i - y_{i-1}) : A \to L_n$ and its extension $\bar{f}_k = u(f_k) : X \to L_k \subset Y$ given by the linear $\overline{\mathrm{conv}}$-extender u. It follows from $a_n \in W_n \subset V_i \cap A \subset \lambda_i^{-1}(1)$, $i < n$, that $\bar{f}_k(a_n) = f_k(a_n) \in f_k(W_n) \subset \{y_k\}$ for all $k \le n$. Using the continuity of $\bar{f}_k$ at a_n, choose a neighborhood $V_n \subset U_n$ of a_n with $V_n \cap A \subset W_n$ such that $\bar{f}_k(V_n) \subset y_k + O_n$ for all $k \le n$. Finally, player II presents the set V_n to player I as his nth move.

We claim that player II wins the game $G_r(A, X)$ if he chooses the sets V_n according to the strategy described above. Otherwise player I wins, which means that $\emptyset \ne \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} V_n \subset X \setminus A$. It follows from the last inclusion that the formula

$$f_\infty(x) = \sum_{i=1}^{\infty} \lambda_i(x) \cdot (y_i - y_{i-1}) = \sum_{i=1}^{\infty} (\lambda_i(x) - \lambda_{i+1}(x)) \cdot y_i \in \mathrm{conv}\{y_i\}_{i=1}^{\infty} \subset K_1$$

determines a well-defined continuous bounded function $f_\infty : A \to Y$. Consider its extension $\bar{f}_\infty = u(f_\infty)$ and pick a point $c \in \bigcap_{n=1}^{\infty} V_n$.

We claim that $\bar{f}_k(c) = y_k$ for all $k \in \mathbb{N}$. Indeed, for every $n \geq k$, the choice of V_n guarantees that $\bar{f}_k(c) \in \bar{f}_k(V_n) \subset y_k + O_n$ and thus $\bar{f}_k(c) - y_k \in L_k \cap \bigcap_{n \geq k} O_n = \{0\}$.

For every $n \in \mathbb{N}$ consider the function

$$g_n = f_\infty - f_n + y_n = y_n + \sum_{i>n} \lambda_i \cdot (y_i - y_{i-1}) = (1 - \lambda_{n+1}) \cdot y_n + \sum_{i>n} (\lambda_i - \lambda_{i+1}) \cdot y_i$$

and observe that $g_n(A) \subset \operatorname{conv}\{y_i\}_{i \geq n} \subset K_n$. Since u is a linear $\overline{\operatorname{conv}}$-extender, we get $u(g_n)(c) \in K_n$ and by the linearity of u,

$$K_n \ni u(g_n)(c) = \bar{f}_\infty(c) - \bar{f}_n(c) + y_n = \bar{f}_\infty(c) + 0,$$

which implies that $\bigcap_{n=1}^\infty K_n$ contains the point $\bar{f}_\infty(c)$ and thus is not empty. This contradiction completes the proof of Theorem 3.2.

13. Proof of Theorem 6.1. Assume that Y_0 is a subspace of a po-space Y, and A is a Tikhonov subspace of a topological space X.

Assuming that there is a monotone extender $u : C(A, Y_0) \to C_A(X, Y)$, we should prove that A is strong Choquet in X or else for each ω-increasing ray $\gamma : [0, \infty) \to Y_0$ the set $\gamma(\omega)$ is almost upper bounded in Y. Suppose that the latter condition does not hold, i.e., there is an ω-increasing ray $\gamma : [0, \infty) \to Y_0$ such that $\gamma(\omega)$ is not almost upper bounded in Y. The latter means that for some G_δ-sets $G_n \subset Y$ with $\gamma(n) \in G_n$, $n \in \omega$, the intersection $\bigcap_{n \in \omega} {\uparrow} G_n$ is empty. For every $n \in \omega$ select a decreasing sequence $(O_m(y_n))_{m \geq n}$ of open neighborhoods of $y_n = \gamma(n)$ such that $\bigcap_{m \geq n} O_m(y_n) \subset G_n$.

Now we modify the winning strategy constructed in the proof of Theorem 3.2 and describe a winning strategy of player II in the game $G_r(A, X)$. The key idea is the same: in his nth inning player II chooses a neighborhood $V_n \subset U_n$ of the point a_n with the help of a continuous function $\lambda_n : A \to [0, 1]$ such that

$$a_n \in V_n \cap A \subset \lambda_n^{-1}(1) \subset \lambda_n^{-1}(0, 1] \subset U_n \subset V_{n-1} \tag{3}$$

and keeps the functions λ_i from the previous innings in his memory.

The choice of $\lambda_n : A \to [0, 1]$ is the same as in the proof of Theorem 3.2 while the choice of the neighborhood $V_n \subset \lambda^{-1}(1)$ of a_n is a bit different. For every $0 \leq k \leq n$ consider the continuous function $s_k = \sum_{i=1}^k \lambda_i : A \to [0, \infty)$ (for $k = 0$, we put $s_0 \equiv 0$). Then $f_k = \gamma \circ s_k : A \to Y_0$ is continuous and hence its extension $\bar{f}_k = u(f_k) : X \to Y$ given by the monotone extender u is continuous at A. It follows from $a_n \in W_n \subset V_i \cap A \subset \lambda_i^{-1}(1)$, $i < n$, that $\bar{f}_k(a_n) = f_k(a_n) = \gamma(k) = y_k$ for all $k \leq n$. Using the continuity of $\bar{f}_k$ at a_n, choose a neighborhood $V_n \subset U_n$ of a_n with $V_n \cap A \subset W_n$ such that $\bar{f}_k(V_n) \subset O_n(y_k)$ for all $k \leq n$. Finally, player II presents the set V_n as his nth move in the game $G_r(A, X)$.

We claim that player II wins $G_r(A,X)$ if he chooses the sets V_n as above. Otherwise player I wins, which means that $\emptyset \neq \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} V_n \subset X \setminus A$. It follows from the last inclusion that the formula $s_\infty(a) = \sum_{i=1}^{\infty} \lambda_i(a)$, $a \in A$, determines a well-defined continuous function $s_\infty : A \to [0,\infty)$. Then the function $f_\infty = \gamma \circ s_\infty : A \to Y_0$ is also continuous. Consider its extension $\bar{f}_\infty = u(f_\infty) : X \to Y$ and pick a point $c \in \bigcap_{n=1}^{\infty} V_n$.

We claim that $\bar{f}_k(c) \in G_k$ for all $k \in \mathbb{N}$. Indeed, for every $n \geq k$, the choice of V_n guarantees that $\bar{f}_k(c) \in \bar{f}_k(V_n) \subset O_n(y_k)$ and thus $\bar{f}_k(c) \in \bigcap_{n\geq k} O_n(y_k) \subset G_k$.

The ω-increasing property of the ray γ implies that $f_\infty \geq f_k$ for all $k \geq 0$. Now the monotonicity of the extender u guarantees that $\bar{f}_\infty(c) \geq \bar{f}_k(c) \in G_k$ and thus $\bar{f}_\infty(c) \in \bigcap_{n\in\omega} {\uparrow} G_n$, which contradicts the choice of the G_δ-sets G_n. This contradiction completes the proof of Theorem 6.1.

14. Proof of Theorem 3.3. (1, 2) The first two items easily follow from the definitions.

(3) We need to prove that a subset A of a topological space X is strong Choquet in X provided A is strong Choquet as a topological space.

The latter means that player II has a winning strategy in the game $G_s(A,A)$. We shall prove that this winning strategy induces a winning strategy in $G_r(A,X)$ and even in a more difficult (for player II) game $G'_r(A,X)$ which differs from $G_r(A,X)$ in the definition of the result of the game. In $G'_r(A,X)$ player II is declared the winner if $\bigcap_{n\in\omega} V_n$ meets A; otherwise player I wins. It is clear that if player II wins $G'_r(A,X)$, then he also wins $G_r(A,X)$.

To win $G'_r(A,X)$ player II simultaneously plays $G_s(A,A)$ for himself and for player I and transforms his moves in $G_s(A,A)$ suggested by the winning strategy into moves in $G'_r(A,X)$.

Namely, after receiving the nth move (U_n,a_n) of player II in the nth inning, player I declares that $(U_n \cap A, a_n)$ is her nth move in the auxiliary game $G_s(A,A)$. Then the winning strategy in $G_s(A,A)$ instructs player II to select a neighborhood $V_n \subset U_n \cap A$ of a_n in A. Player II enlarges V_n to an open subset $\widetilde{V}_n \subset U_n$ in X such that $\widetilde{V}_n \cap A = V_n$ and makes $\widetilde{V}_n$ his move in the nth inning of the game $G'_r(A,X)$.

In such a way players I and II choose sequences (U_n,a_n), $(U_n \cap A, a_n)$, (V_n) and $(\widetilde{V}_n)$. Since the sets V_n, $n \in \mathbb{N}$, are chosen according to the winning strategy of player II in $G_s(A,A)$, we get $\bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$. Then $A \cap \bigcap_{n\in\mathbb{N}} \widetilde{V}_n = \bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$, so we conclude that player II also wins $G'_r(A,X)$.

(4) Assume that a topological space X is strong Choquet at A and A is strong Choquet in X. We should prove that A is strong Choquet. In the

language of strategies this means that given winning strategies of player II in the games $G_s(A,X)$ and $G_r(A,X)$ we should describe a winning strategy for player II in $G_s(A,A)$.

Repeating the argument from the preceding item, we can prove that player II has a winning strategy in $G_s(A,A)$ if and only if he has a winning strategy in $G'_r(A,X)$ described above. So it suffices to describe a winning strategy for player II in $G'_r(A,X)$.

Fix winning strategies of player II in $G_s(A,X)$ and $G_r(A,X)$. To win $G'_r(A,X)$, player II plays simultaneously two auxiliary games $G_s(A,X)$ and $G_r(A,X)$ as follows. In the nth inning of $G'_r(A,X)$ he receives from the first player a point $a_n \in A$ and a neighborhood $U_n \subset X$ of a_n and declares that (U_n, a_n) is the nth move of player I in the auxiliary game $G_s(A,X)$. The winning strategy of player II in $G_s(A,X)$ instructs him to make the nth move by choosing a neighborhood $W_n \subset U_n$ of a_n.

Then player II declares that (W_n, a_n) is the nth move of the first player in $G_r(A,X)$ and selects a neighborhood $V_n \subset W_n$ of a_n according to the winning strategy in $G_r(A,X)$.

The neighborhood V_n is the nth move of the second player in $G'_r(A,X)$. Let us show that if player II plays according to this strategy, then he wins the game $G'_r(A,X)$. Playing three games simultaneously, players I and II construct the sequences (U_n, a_n), (W_n), and (V_n). The choice of the sets W_n according to the winning strategy in $G_s(A,X)$ guarantees that $\bigcap_{n=1}^{\infty} W_n$ is not empty. Taking into account that $W_n \subset U_n \subset V_n \subset W_{n-1}$ for all n, we conclude that also $\bigcap_{n=1}^{\infty} V_n$ is not empty. Since player II won $G_r(A,X)$, the intersection $\bigcap_{n=1}^{\infty} V_n$, being non-empty, must meet A. This means that player II has won the game $G'_r(A,X)$ too.

Acknowledgments. We would like to thank Professor Haruto Ohta for his valuable comments. In particular, Theorem 2.1 had been obtained for finite-dimensional topological vector spaces Y in the discussion between him and the third author.

The third author is supported by the Ministry of Education, Culture, Sports, Science and Technology of Japan, Grant-in-Aid for Young Scientists (B), No. 19740027.

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Received February 28, 2008
Revised version September 24, 2008 (6310)