STUDIA MATHEMATICA 152 (1) (2002)

# Separation properties for self-conformal sets

by

YUAN-LING YE (Hong Kong and Guangzhou)

**Abstract.** For a one-to-one self-conformal contractive system $\{w_j\}_{j=1}^m$ on $\mathbb{R}^d$ with attractor $K$ and conformality dimension $\alpha$, Peres *et al.* showed that the open set condition and strong open set condition are both equivalent to $0 < \mathcal{H}^\alpha(K) < \infty$. We give a simple proof of this result as well as discuss some further properties related to the separation condition.

**1. Introduction.** Let $U_0 \subset \mathbb{R}^d$ be a bounded open set. Let $w_j : U_0 \to U_0$ $(j = 1, \ldots, m)$ be contractive maps and suppose there exists a nonempty compact subset $X \subseteq U_0$ such that $w_j(X) \subseteq X$ for each $1 \le j \le m$. Then there exists a compact subset $K \subseteq X$ such that $K = \bigcup_{j=1}^m w_j(K)$. We say that $\{w_j\}_{j=1}^m$ satisfies the *open set condition* (OSC) if there exists a nonempty bounded open set $U \subseteq U_0$ such that

$$w_j(U) \subseteq U \quad \text{and} \quad w_i(U) \cap w_j(U) = \emptyset \quad \text{for } i \neq j.$$

Such a $U$ is called a *basic open set* for $\{w_j\}_{j=1}^m$. If moreover $U \cap K \neq \emptyset$, then $\{w_j\}_{j=1}^m$ is said to satisfy the *strong open set condition* (SOSC). In [S], Schief made use of an idea of Bandt [BG] and showed that for similitude, the two conditions are equivalent, and furthermore they are equivalent to $0 < \mathcal{H}^\alpha(K) < \infty$ where $\alpha$ is the similarity dimension of $K$.

Recently, Peres, Rams, Simon and Solomyak [P] extended Schief's theorem to self-conformal maps. A simple proof was also given by Lau, Rao and the author for the equivalence of the OSC and SOSC [L]. In a private communication, Peres asked if there is a short proof of the equivalence to $0<\mathcal{H}^\alpha(K)<\infty$. In this note we answer his question affirmatively. The main idea and some of the proofs are already in [L] and [FL]; we will modify them to fit our purpose. In [LX] Lau and Xu considered the boundary dimension of self-similar sets. We extend some of their results to self-conformal maps.

2000 *Mathematics Subject Classification*: Primary 28A78, 54E40; Secondary 54H15.

*Key words and phrases*: self-conformal set, conformality dimension, OSC, SOSC, basic open set, Hausdorff dimension.

The research was partially supported by a direct grant of CUHK and the NSF of Guangdong.

For one-to-one contractive self-conformal IFS $\{w_j\}_{j=1}^m$, we define the *conformality dimension* of the IFS to be the (positive) number $\alpha$ such that the *Ruelle operator* $T_\alpha : C(K) \to C(K)$ defined by

$$T_\alpha f(x) = \sum_{j=1}^{m} |w_j'(x)|^\alpha f(w_j(x))$$

has spectral radius 1 [FL]. We let $\mathcal{H}^\alpha$ be the $\alpha$-Hausdorff measure.

We prove Theorem 1.1 below by constructing a basic open set $U$ which satisfies both the SOSC and $\dim_{\mathrm{H}}(K \setminus U) < \alpha$. The key to the proof is Lemma 3.4. Furthermore we remark that in the previous considerations of self-conformality, it was additionally assumed that the open set $U$ in the OSC is connected (see e.g. [MU], [P]); we will see that this assumption is redundant (Lemma 2.1 and the remark there). Our basic results are:

THEOREM 1.1. *Let $\{w_j\}_{j=1}^m$ be a one-to-one self-conformal contractive IFS with $\{|w_j'(x)|\}_{j=1}^m$ satisfying* (2.1) *and the Dini condition. Then the following are equivalent*:

(i) $\{w_j\}_{j=1}^m$ *satisfies the OSC.*
(ii) $\{w_j\}_{j=1}^m$ *satisfies the SOSC.*
(iii) $0 < \mathcal{H}^\alpha(K) < \infty$.

THEOREM 1.2. *Let $\{w_j\}_{j=1}^m$ be as in Theorem* 1.1 *and satisfy the OSC. Then there exists a basic open set $U$ such that* $\dim_{\mathrm{H}}(K \setminus U) < \alpha$.

**2. Preliminaries.** Let $\{w_j\}_{j=1}^m$ be *self-conformal* on $U_0$ (i.e. for each $j$ and each $x \in U_0$, $w_j'(x)$ is a self-similar matrix and $|w_j'(\cdot)|$ is continuous). We assume that there exists a nonempty compact set $X$ such that $X \subseteq U_0$, and for each $1 \le j \le m$, $w_j(X) \subseteq X$, $w_j$ is one-to-one on $U_0$ and $|w_j'(\cdot)|$ is Dini continuous on $U_0$ with

$$(2.1)\qquad 0 < \inf_{x\in U_0} |w_j'(x)| \le \sup_{x\in U_0} |w_j'(x)| < 1 \quad \text{for each } 1 \le j \le m,$$

where $|w_j'(x)| := |\det w_j'(x)|^{1/d}$ is the operator norm of the matrix $w_j'(x)$ on $\mathbb{R}^d$. Enlarging $X$ to $X_0 \subseteq U_0$ by taking a $\delta$-neighborhood, we can show easily from the contractiveness of $w_j$'s that there exists $k'$ such that

$$\bigcup_{|J|=k} w_J(X_0) \subseteq X_0 \quad \text{for any } k \ge k'$$

where $J = j_1 \ldots j_k$, $1 \le j_i \le m$, $w_J = w_{j_1} \circ \ldots \circ w_{j_k}$. Hence we may assume without loss of generality that $\overline{X^\circ} = X$ and $B(K, \delta) \subseteq X^\circ$ for some $\delta > 0$ ($B(K, \delta)$ denotes the open $\delta$-neighborhood of $K$).

We set $\mathcal{J} = \{J = j_1 \dots j_n : 1 \le j_i \le m,\ n \in \mathbb{N}\}$, and for any $J \in \mathcal{J}$ define

$$K_J = w_J(K), \qquad r_J = \inf_{x \in U_0} |w_J'(x)|, \qquad R_J = \sup_{x \in U_0} |w_J'(x)|.$$

LEMMA 2.1. *Suppose $X$ and $\{w_j\}_{j=1}^m$ are defined as above.*

(i) *There exists a $c_1 > 1$ such that*

$$R_J \le c_1 r_J \quad \text{for any } J \in \mathcal{J}, \tag{2.2}$$

$$c_1^{-1} r_I r_J \le r_{IJ} \le c_1 r_I r_J \quad \text{for any } I, J \in \mathcal{J}. \tag{2.3}$$

(ii) *There exist $c_2 \ge c_1$ and $\delta > 0$ such that for $x, y, z \in X$ with $|x-y| \le \delta$,*

$$c_2^{-1} |w_J'(z)| \le \frac{|w_J(x) - w_J(y)|}{|x-y|} \le c_2 |w_J'(z)| \quad \text{for any } J \in \mathcal{J}. \tag{2.4}$$

(iii) *There exist $c_3 \ge c_2$ and $k_0$ such that for any $x, y \in X$,*

$$|w_J(x) - w_J(y)| \le c_3 r_J |x-y| \quad \text{for any } J \in \mathcal{J} \text{ with } |J| > k_0. \tag{2.5}$$

*Proof.* The proof of (i) and (ii) is in [FL, Lemma 2.3]. We include the proof of (ii) for completeness. For any $x \in X$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subseteq U_0$. Since $X$ is compact, there exists $\delta > 0$ (the Lebesgue number) such that for any $x, y \in X$, if $|x-y| \le \delta$, then $x, y \in B(x', \delta_{x'})$ for some $x' \in X$. For such $x, y \in X$, we have $w_J(x), w_J(y) \in B(y', \delta_{y'})$ for some $y' \in X$. Then the self-similar property of $w_J$ implies that

$$|w_J(x) - w_J(y)| \le R_J |x-y|. \tag{2.6}$$

On the other hand, let $u_J(\cdot)$ be the inverse of $w_J$ on $B(y', \delta_{y'}) \cap w_J(B(x', \delta_{x'}))$, i.e.,

$$u_J(x) := w_J^{-1}(x) \quad \text{for any } x \in B(y', \delta_{y'}) \cap w_J(B(x', \delta_{x'})).$$

Then

$$R_J^{-1} \le |u_J'(x)| \le r_J^{-1} \quad \text{for any } x \in B(y', \delta_{y'}) \cap w_J(B(x', \delta_{x'})).$$

By the self-similar property of $w_J(\cdot)$, we deduce that $B(y', \delta_{y'}) \cap w_J(B(x', \delta_{x'}))$ is convex connected, hence similarly to (2.6), we have

$$|u_J(w_J(x)) - u_J(w_J(y))| \le r_J^{-1} |w_J(x) - w_J(y)|.$$

Consequently, $r_J|x-y| \le |w_J(x) - w_J(y)| \le R_J|x-y|$. This together with (2.2) implies (ii).

(iii) follows directly from the choice of $\delta$ and (ii). ∎

To make use of the local connectedness of $X$, we take $0 < \varepsilon < 2^{-1} c_3^{-1} \delta$. Then $2c_3\varepsilon \le \delta$, and hence by the assumption on $X$, we have

$$B(K, c_3\varepsilon) \subseteq X. \tag{2.7}$$

For $J \in \mathcal{J}$, let

$$G_J = w_J(B(K, \varepsilon)).$$

Consequently, by (2.7) and (2.4), we have for any $x \in K$,

$$B(w_J(x), c_2^{-1}\varepsilon r_J) \subseteq w_J(B(x,\varepsilon)) \subseteq B(w_J(x), c_2\varepsilon r_J). \tag{2.8}$$

It follows that

$$\begin{aligned} B(K_J, c_2^{-1}\varepsilon r_J) &= \bigcup_{x\in K} B(w_J(x), c_2^{-1}\varepsilon r_J) \\ &\subseteq G_J = \bigcup_{x\in K} w_J(B(x,\varepsilon)) \subseteq \bigcup_{x\in K} B(w_J(x), c_2\varepsilon r_J) \\ &= B(K_J, c_2\varepsilon r_J). \end{aligned} \tag{2.9}$$

We remark that in [MU] and [P] the connectedness of $X$ was used to apply the mean value theorem so as to deduce (2.8) and (2.9); the above argument shows that the local connectedness of $X$ is sufficient. Hence we can study separation properties without assuming the connectedness of $U_0$ so long as we regard the relevant sets as unions of subsets whose diameters are less than $\delta$.

For $0 < b < 1$, we let

$$\Lambda_b = \{J = j_1 \ldots j_n : r_{j_1\ldots j_n} < b \le r_{j_1\ldots j_{n-1}}\}.$$

As in [L], our most crucial difference from [S] and [P] is the following inductive way of defining an index set $\Lambda(J), J \in \mathcal{J}$: Let $k_0$ be as in Lemma 2.1(iii). For $J$ with $|J| = k_0$, we define

$$\Lambda(J) = \{I \in \Lambda_{\operatorname{diam} G_J} : K_I \cap G_J \neq \emptyset\}.$$

Supposing $\Lambda(J)$ is defined, for any $1 \le j \le m$, we define $\Lambda(jJ) = \mathcal{A} \cup \mathcal{B}$ where

$$\mathcal{A} = \{jI : I \in \Lambda(J)\}, \qquad \mathcal{B} = \{I \in \Lambda_{\operatorname{diam} G_{jJ}} : i_1 \neq j \text{ and } K_I \cap G_{jJ} \neq \emptyset\}.$$

(Note that in [S], the $\Lambda(J)$ is defined as $\{I \in \Lambda_{\operatorname{diam} G_J} : K_I \cap G_J \neq \emptyset\}$.) It is easy to see from the construction that each $I \in \Lambda(J)$ is of type either $\mathcal{A}$ or $\mathcal{B}$, and $K_I \cap G_J \neq \emptyset$; also $K_I$ and $K_J$ are comparable in size by the following lemma.

LEMMA 2.2. *Suppose $\{w_j\}_{j=1}^m$ is as in Lemma 2.1. Then there exist $k_1$ and $c_4 > 0$ such that $c_4^{-1} \le r_J/r_I \le c_4$ for all $I \in \Lambda(J)$ and $J \in \mathcal{J}$ with $|J| \ge k_1$.*

*Proof.* The idea is in [L, Lemma 3.1]; we modify it to fit our purpose. Let $k_1 \ge k_0$ be such that

$$\min\{|I| : I \in \Lambda(J) \text{ and } |J| \ge k_1\} > k_0.$$

For any $I \in \Lambda(J)$ and $J \in \mathcal{J}$ with $|J| \ge k_1$, we consider two cases:

(i) If $i_1 \neq j_1$, by the construction of $\mathcal{B}$, we have $I \in \Lambda_{\operatorname{diam} G_J}$. Then

$$r_I \le \operatorname{diam} G_J \le r_{i_1\ldots i_{n-1}} \le c_1 r^{-1} r_I \tag{2.10}$$

where $r = \min_{1\le j\le m}\{r_j\}$. As $\varepsilon < 2^{-1}c_3^{-1}\delta < \delta$, it follows from (2.4) that $\operatorname{diam} G_J \ge c_2^{-1}\varepsilon r_J$. Hence

$$c_2^{-1}\varepsilon r_J \le \operatorname{diam} G_J \le c_1 r^{-1} r_I. \tag{2.11}$$

Also by (2.9), we have $\operatorname{diam} G_J \le 2c_2\varepsilon r_J + |K_J|$. Then by (2.10) and (2.5), it follows that

$$r_I \le \operatorname{diam} G_J \le c_3(2\varepsilon + |K|)r_J. \tag{2.12}$$

Hence (2.11) and (2.12) imply that there exists $a > 0$ such that

$$a^{-1} \le r_J/r_I \le a. \tag{2.13}$$

(ii) If $i_1 = j_1$, we write

$$J = j_1 \dots j_l\ j_{l+1} \dots j_n := j_1 \dots j_l J', \qquad I = j_1 \dots j_l\ i_{l+1} \dots i_m := j_1 \dots j_l I'$$

where $j_{l+1} \ne i_{l+1}$. Then by the construction of $\mathcal{A}$, we see inductively that $I' \in \Lambda(J')$ and by (2.13), $a^{-1} \le r_{J'}/r_{I'} \le a$. Together with Lemma 2.1(i), this implies that

$$(ac_1^2)^{-1} \le r_J/r_I \le ac_1^2.$$

If we let $c_4 = ac_1^2$, then the lemma follows from the conclusion of the two cases. ∎

We remark that for fixed $J_0 \in \mathcal{J}$, the construction of the set $\mathcal{A}$ implies trivially that

$$\Lambda(jJ_0) \supseteq \{jI : I \in \Lambda(J_0)\}, \qquad j = 1, \dots, m.$$

The key to proving the SOSC is to find $J_0$ such that equality holds (Lemma 3.4 below). In this case the set $\mathcal{B}$ is empty.

**3. The proof of the main results.** We need a few notations and lemmas. For any two subsets $E, F$ in $\mathbb{R}^d$, we define

$$D(E,F) = \inf\{|x-y| : x \in E,\ y \in F\};$$
$$d(E,F) = \inf\{\varepsilon : E \subseteq B(F,\varepsilon),\ F \subseteq B(E,\varepsilon)\}.$$

LEMMA 3.1 [FL, Lemma 2.8]. *Let $w$ be conformal and invertible, let $D$ be a Borel subset in the domain of $w$, and $0 < \mathcal{H}^\alpha(D) < \infty$. Then we have the following change of variable formula*:

$$\mathcal{H}^\alpha(w(D)) = \int_D |w'(x)|^\alpha \, d\mathcal{H}^\alpha(x).$$

LEMMA 3.2. *Let $\{w_j\}_{j=1}^m$ be as in Theorem 1.1. Suppose $0 < \mathcal{H}^\alpha(K) < \infty$. Then*

$$\mathcal{H}^\alpha(K_I \cap K_J) = 0 \quad \textit{for any incomparable } I, J \in \mathcal{J}.$$

*Proof.* Since $T_\alpha$ has spectral radius 1, by [FL, Theorem 1.1], there exists $0 < h \in C(K)$ such that $h(x) = \sum_{j=1}^m |w_j'(x)|^\alpha h(w_j x)$. Then

$$\sum_{j=1}^m \int_{K_j} h(x)\, d\mathcal{H}^\alpha(x) \geq \int_{\bigcup_{j=1}^m K_j} h(x)\, d\mathcal{H}^\alpha(x) = \int_K h(x)\, d\mathcal{H}^\alpha(x)$$
$$= \int_K \sum_{j=1}^m |w_j'(x)|^\alpha h(w_j x)\, d\mathcal{H}^\alpha(x) = \sum_{j=1}^m \int_{K_j} h(x)\, d\mathcal{H}^\alpha(x).$$

(The last equality follows from Lemma 3.1.) This implies that $\mathcal{H}^\alpha(K_i \cap K_j) = 0$ for any $i \neq j$. It follows immediately that $\mathcal{H}^\alpha(K_I \cap K_J) = 0$ for any incomparable $I, J \in \mathcal{J}$. ∎

Lemma 3.3. *Let $\{w_j\}_{j=1}^m$ be as in Lemma 3.2. Then there exists $\delta_0 > 0$ such that for any $L \in \mathcal{J}$,*

$$\|w_I(\cdot) - w_J(\cdot)\|_{C(K)} \geq \delta_0 r_L \quad \text{for any } I, J \in \Lambda(L) \text{ with } I \neq J.$$

*Proof.* Since $0 < \mathcal{H}^\alpha(K) < \infty$, there exists an open set $U$ such that $K \subseteq U \subset X$ and

$$0 < \mathcal{H}^\alpha(U) \leq \mathcal{H}^\alpha(K) + 1 < \infty.$$

Let $c_1$ and $\delta$ be as in Lemma 2.1 and let $0 < \eta < 2^{-1}c_1^{-\alpha}$. There exists an open covering $\{V_i\}_{i=1}^n$ of $K$ such that

$$K \subseteq V := \bigcup_{i=1}^n V_i \subseteq U, \quad \delta' := D(K, V^{\mathrm{c}}) < \delta, \tag{3.1}$$

$$0 < \mathcal{H}^\alpha(K) \leq \mathcal{H}^\alpha(V) \leq \sum_{i=1}^n |V_i|^\alpha < (1+\eta)\mathcal{H}^\alpha(K). \tag{3.2}$$

For any $I, J \in \Lambda(L)$, assume without loss of generality that $\mathcal{H}^\alpha(K_I) \leq \mathcal{H}^\alpha(K_J)$. Then for any given $\varepsilon$ satisfying $c_1^\alpha \eta < \varepsilon < 1$, we have

$$\varepsilon \mathcal{H}^\alpha(K_I) < \mathcal{H}^\alpha(K_J). \tag{3.3}$$

We claim that $d(K_I, K_J) \geq \delta' r_I$. Otherwise, by (3.1) and Lemma 2.1(ii), we have $D(K_I, w_I(V^{\mathrm{c}})) \geq \delta' r_I$, and then $K_J \subseteq w_I(V)$. Hence by (3.3) and Lemma 3.2, we have

$$(1+\varepsilon)\mathcal{H}^\alpha(K_I) < \mathcal{H}^\alpha(K_I) + \mathcal{H}^\alpha(K_J) = \mathcal{H}^\alpha(K_I \cup K_J) \leq \mathcal{H}^\alpha(w_I(V)).$$

This together with (3.2) implies that

$$\varepsilon r_I^\alpha \mathcal{H}^\alpha(K) \leq \varepsilon \mathcal{H}^\alpha(K_I) < \mathcal{H}^\alpha(w_I(V \setminus K)) \leq (c_1 r_I)^\alpha \mathcal{H}^\alpha(V \setminus K)$$
$$< (c_1 r_I)^\alpha \eta \mathcal{H}^\alpha(K).$$

(The first and third inequalities follow from Lemmas 3.1 and 2.1(i).) Then $\varepsilon < c_1^\alpha \eta$, which contradicts the choice of $\varepsilon$. The claim is proved, and the lemma follows. ∎

LEMMA 3.4. *Let* $\{w_j\}_{j=1}^m$ *be as in Lemma* 3.2. *Then* $\gamma := \sup_{|L|\ge k_1} \sharp\Lambda(L) < \infty$. *If* $J_0 \in \mathcal{J}$ *is such that* $|J_0| \ge k_1$ *and* $\sharp\Lambda(J_0) = \gamma$, *then*

$$\Lambda(IJ_0) = \{IJ : J \in \Lambda(J_0)\} \quad \textit{for all } I \in \mathcal{J}. \tag{3.4}$$

*Proof.* Let $c_3$, $c_4$ and $\delta_0$ be the constants given in Lemmas 2.1, 2.2 and 3.3 respectively. Let $\delta' = (3c_3c_4)^{-1}\delta_0$. We can find a finite set $Z \subset K$ whose $\delta'$-neighborhood contains $K$. For any $L \in \mathcal{J}$ with $|L| \ge k_1$ and for all different $I, J \in \Lambda(L)$, by Lemma 3.3, there exists $x \in K$ such that $|w_I(x) - w_J(x)| \ge \delta_0 r_L$. For that $x$ there exists $z \in Z$ such that $|x - z| < \delta'$; then by (2.5) and the choice of $k_1$ (see the proof of Lemma 2.2), we have

$$|w_I(x) - w_I(z)| \le \tfrac{1}{3}\delta_0 r_L \quad \text{and} \quad |w_J(x) - w_J(z)| \le \tfrac{1}{3}\delta_0 r_L.$$

It follows that for any different $I, J \in \Lambda(L)$, there exists some $z \in Z$ such that

$$|w_I(z) - w_J(z)| \ge \tfrac{1}{3}\delta_0 r_L. \tag{3.5}$$

For each $z \in Z$, set

$$P_z(L) = \{I \in \Lambda(L) : \exists J \in \Lambda(L) \text{ such that (3.5) holds}\}.$$

Hence (3.5) implies that

$$\Lambda(L) = \bigcup_{z\in Z} P_z(L).$$

To prove $\sup_{|L|\ge k_1} \sharp\Lambda(L) < \infty$, we observe that for each $z \in Z$, the sets

$$\bigl\{B\bigl(w_I(z), \tfrac{1}{6}\delta_0 r_L\bigr) : I \in P_z(L)\bigr\}$$

are disjoint by (3.5) and are contained in $B\bigl(G_L, \operatorname{diam} K_I + \frac{1}{6}\delta_0 r_L\bigr)$ by the definition of $\Lambda(L)$. By Lemma 2.1(iii) and Lemma 2.2, there exist $c > 0$ (independent of $L$) and $x \in K$ such that $B\bigl(G_L, \operatorname{diam} K_I + \frac{1}{6}\delta_0 r_L\bigr) \subseteq B(x, cr_L)$. By a simple volume argument, we deduce that there exists an $\ell$ (independent of $L$) such that $\max_{z\in Z} \sharp P_z(L) \le \ell$. Then

$$\sharp\Lambda(L) \le \sharp Z \cdot \max_{z\in Z} \sharp P_z(L) \le \ell \cdot \sharp Z.$$

We conclude that $\gamma = \sup_{|L|\ge k_1} \sharp\Lambda(L) < \infty$. Hence there exists $J_0$ such that $|J_0| \ge k_1$ and $\sharp\Lambda(J_0) = \gamma$.

To prove (3.4), we have remarked after the definition of $\Lambda(J)$ that

$$\Lambda(jJ_0) \supseteq \{jI : I \in \Lambda(J_0)\}, \quad j = 1, \ldots, m.$$

On the other hand, the choice of $J_0$ implies that $\sharp\{IJ : J \in \Lambda(J_0)\} = \gamma$. Thus the definition of $\gamma$ implies that $\sharp\Lambda(IJ_0) = \gamma$ also and (3.4) follows. ∎

*Proof of Theorem 1.1.* It is obvious that (ii) implies (i). That (i) implies (iii) is shown in [MU] and [FL]. We have to prove (iii)⇒(ii). The proof needs only a small modification of [S] and is the same as in [L]; we include it here for completeness. Let $J_0 \in \mathcal{J}$ be as in Lemma 3.4. For any fixed $1 \le l \le m$ and $J = j_1 \dots j_n \in \mathcal{J}$ with $j_1 \ne l$, we consider the family

$$\mathcal{K}_l = \{K_L : L \in \Lambda_{\operatorname{diam} G_{JJ_0}} \text{ with } l_1 = l\}$$

where $l_1$ is the first element of the multiple index $L$. Then $\mathcal{K}_l$ is a cover of $K_l$. Since $j_1 \ne l_1$, (3.4) implies that $L \notin \Lambda(JJ_0)$. Then by the construction of $\mathcal{B}$, $K_L \cap G_{JJ_0} = \emptyset$. Hence by (2.9), we have $D(K_L, K_{JJ_0}) \ge c_2^{-1}\varepsilon r_{JJ_0}$, which implies

$$D(K_l, K_{JJ_0}) \ge c_2^{-1}\varepsilon r_{JJ_0} \quad \text{for } l \ne j_1. \tag{3.6}$$

Now we let $G_J^* = w_J(B(K, 2^{-1}c_2^{-2}\varepsilon))$ and

$$U = \bigcup_{J \in \mathcal{J}} G_{JJ_0}^*.$$

We claim that the $U$ satisfies the condition of the SOSC. Indeed, $U$ is a bounded open set, $U \cap K \ne \emptyset$ and

$$w_j(U) = \bigcup_{J \in \mathcal{J}} w_j(G_{JJ_0}^*) = \bigcup_{J \in \mathcal{J}} G_{jJJ_0}^* \subseteq U.$$

Now we prove that

$$w_i(U) \cap w_j(U) = \emptyset \quad \text{for } i \ne j.$$

For otherwise, there are $I, J$ such that $G_{iIJ_0}^* \cap G_{jJJ_0}^* \ne \emptyset$. We assume $r_{iIJ_0} \ge r_{jJJ_0}$. Let $y$ be in the intersection; then there exist $y_1 \in K_{iIJ_0}$ and $y_2 \in K_{jJJ_0}$ such that

$$d(y, y_1) < c_2 \cdot \frac{1}{2c_2^2}\,\varepsilon \cdot r_{iIJ_0} \le \frac{c_2^{-1}\varepsilon}{2}\, r_{iIJ_0},$$

$$d(y, y_2) < c_2 \cdot \frac{1}{2c_2^2}\,\varepsilon \cdot r_{jJJ_0} \le \frac{c_2^{-1}\varepsilon}{2}\, r_{iIJ_0}.$$

Then $d(y_1, y_2) < c_2^{-1}\varepsilon r_{iIJ_0}$. Hence

$$D(K_{iIJ_0}, K_j) < c_2^{-1}\varepsilon r_{iIJ_0},$$

which contradicts (3.6). This completes the proof. ∎

LEMMA 3.5 [FL, Theorem 2.9]. *Let* $\{w_j\}_{j=1}^m$ *be as in Theorem* 1.1 *and satisfy the OSC. Let* $\nu = \mathcal{H}^\alpha|_K$. *Then* $\nu$ *is an invariant measure for* $T_\alpha$, *i.e.,* $T_\alpha^*\nu = \nu$.

*Proof of Theorem 1.2.* By assumption and Theorem 1.1, we have $0 < \mathcal{H}^\alpha(K) < \infty$. We recall the proof of Theorem 1.1 and let $U$ be as constructed there. To prove $\dim_{\mathrm{H}}(K\backslash U) < \alpha$, let $\mu = \mathcal{H}^\alpha(K)^{-1}\mathcal{H}^\alpha$. Then by Lemma 3.5, $\mu$ is an invariant probability measure of $T_\alpha$, i.e.,

$$\mu = \sum_{j=1}^m (|w_j'(x)|^\alpha \mu) \circ w_j^{-1}. \tag{3.7}$$

Let $k := |J_0|$. Then by Lemmas 2.1 and 3.1, we have

$$\mu(K_{J_0}) \geq c_1^{-\alpha} r_{J_0}^{\alpha}; \tag{3.8}$$

for any $L \in \mathcal{J}$,

$$\begin{aligned} \mu(K_{LJ_0}) &= \mathcal{H}^{\alpha}(K)^{-1} \int_K |w_L'(w_{J_0}x)|^{\alpha} |w_{J_0}'(x)|^{\alpha}\, d\mathcal{H}^{\alpha}(x) \\ &\geq r_L^{\alpha} r_{J_0}^{\alpha} \geq c_1^{-\alpha} r_{J_0}^{\alpha} \mu(K_L). \end{aligned} \tag{3.9}$$

For any integer $n$, let

$$U_n = \bigcup_{\ell=0}^{n-1} \bigcup_{|J|=k\ell} G_{JJ_0}^{*}.$$

Then $U_n \subseteq U$. Let

$$\begin{aligned} \mathcal{J}(n) &= \{j_1 \ldots j_{kn} : 1 \leq j_i \leq m\}, \\ \mathcal{L}_n &= \{L = l_1 \ldots l_{kn} \in \mathcal{J}(n) : l_{k\ell+1} \ldots l_{k\ell+k} \neq J_0 \ \forall 0 \leq \ell < n\}. \end{aligned}$$

For any $J$ with $0 \leq |J| = k\ell < kn$, we deduce from $K = \bigcup_{j=1}^{m} w_j(K)$ that

$$K_{JJ_0} = w_{JJ_0}(K) = \bigcup_{|J'|=k(n-1)-|J|} K_{JJ_0J'}. \tag{3.10}$$

Then

$$\begin{aligned} K \setminus U \subseteq K \setminus U_n &= K \setminus \bigcup_{\ell=0}^{n-1} \bigcup_{|J|=k\ell} G_{JJ_0}^{*} \subseteq K \setminus \bigcup_{\ell=0}^{n-1} \bigcup_{|J|=k\ell} K_{JJ_0} \\ &= \bigcap_{\ell=0}^{n-1} \bigcap_{|J|=k\ell} \bigcap_{|J'|=k(n-1)-|J|} K_{JJ_0J'}^{\mathrm{c}} \subseteq \bigcup_{L \in \mathcal{L}_n} K_L. \end{aligned} \tag{3.11}$$

We need to estimate the value of $\mathcal{H}^{\alpha}(\bigcup_{L \in \mathcal{L}_n} K_L)$. For this we will prove inductively that

$$\sum_{L \in \mathcal{L}_n} \mu(K_L) \leq (1 - c_1^{-\alpha} r_{J_0}^{\alpha})^n \quad \text{for any } n. \tag{3.12}$$

Indeed, by Lemma 3.2, we have $\mu(K_I \cap K_J) = 0$ for any $I, J \in \mathcal{L}_n$ with $I \neq J$. This together with (3.8) implies that

$$\sum_{L \in \mathcal{L}_1} \mu(K_L) = \mu\Big( \bigcup_{\substack{L \neq J_0 \\ |L|=k}} K_L \Big) = 1 - \mu(K_{J_0}) \leq 1 - c_1^{-\alpha} r_{J_0}^{\alpha}.$$

Assume that

$$\sum_{L \in \mathcal{L}_n} \mu(K_L) \leq (1 - c_1^{-\alpha} r_{J_0}^{\alpha})^n.$$

Since $\sum_{|J|=k}\mu(K_{LJ})=\mu(K_L)$, we have

$$\begin{aligned}\sum_{L\in\mathcal{L}_{n+1}}\mu(K_L) &= \sum_{L\in\mathcal{L}_n}\sum_{|J|=k}\mu(K_{LJ})-\sum_{L\in\mathcal{L}_n}\mu(K_{LJ_0})\\ &= \sum_{L\in\mathcal{L}_n}\mu(K_L)-\sum_{L\in\mathcal{L}_n}\mu(K_{LJ_0})\\ &\le \sum_{L\in\mathcal{L}_n}\mu(K_L)-c_1^{-\alpha}r_{J_0}^{\alpha}\sum_{L\in\mathcal{L}_n}\mu(K_L) \quad \text{by (3.9)}\\ &= (1-c_1^{-\alpha}r_{J_0}^{\alpha})\sum_{L\in\mathcal{L}_n}\mu(K_L)\le(1-c_1^{-\alpha}r_{J_0}^{\alpha})(1-c_1^{-\alpha}r_{J_0}^{\alpha})^n\\ &= (1-c_1^{-\alpha}r_{J_0}^{\alpha})^{n+1}.\end{aligned}$$

Let $\delta_n:=\max\{\operatorname{diam}K_L : L\in\mathcal{L}_n\}$ and $r=\min_{1\le j\le m}\{r_j\}$. Take

$$\beta:=\alpha-\frac{\log(1-c_1^{-\alpha}r_{J_0}^{\alpha})}{k\log r}.$$

Then $\beta<\alpha$. Set $c_4=(c_3\operatorname{diam}K)^{\beta}$. Then for large $n$, we have

$$\begin{aligned}&\mathcal{H}^{\beta}_{\delta_n}(K\setminus U)\le\mathcal{H}^{\beta}_{\delta_n}\Big(\bigcup_{L\in\mathcal{L}_n}K_L\Big)\le\sum_{L\in\mathcal{L}_n}(\operatorname{diam}K_L)^{\beta} \quad \text{by (3.11)}\\ &\quad\le\sum_{L\in\mathcal{L}_n}(c_3r_L\operatorname{diam}K)^{\beta}=c_4\sum_{L\in\mathcal{L}_n}r_L^{\beta}=c_4\sum_{L\in\mathcal{L}_n}r_L^{\beta-\alpha}r_L^{\alpha} \quad \text{by (2.5)}\\ &\quad\le c_4\Big(r^{nk(\beta-\alpha)}\sum_{L\in\mathcal{L}_n}\mu(K_L)\Big)\le c_4(r^{k(\beta-\alpha)}(1-c_1^{-\alpha}r_{J_0}^{\alpha}))^n\le c_4 \quad \text{by (3.12)}.\end{aligned}$$

Since $\lim_{n\to\infty}\delta_n=0$, we obtain $\mathcal{H}^{\beta}(K\setminus U)\le c_4<\infty$, hence $\dim_{\mathrm H}(K\setminus U)\le\beta$. ∎

COROLLARY 3.6. *Let* $\{w_j\}_{j=1}^m$ *be as in Theorem* 1.2. *Then*

$$\dim_{\mathrm H}(w_i(K)\cap w_j(K))<\alpha \quad \textit{for } i\ne j.$$

*Proof.* Let $J_0$ and $\beta$ be as in the proof of Theorem 1.2. Using (3.12), we can show similarly to [LX, Theorem 1.6] that $\dim_{\mathrm H}(w_i(K)\cap w_j(K))\le\beta<\alpha$. ∎

THEOREM 3.7. *Let* $\{w_j\}_{j=1}^m$ *be as in Theorem* 1.2. *If there is a basic open set* $U$ *such that* $U\setminus\bigcup_{j=1}^m w_j(\overline{U})\ne\emptyset$, *then* $\dim_{\mathrm H}K<d$.

*Proof.* Suppose that $\dim_{\mathrm H}K=d$. Since $\{w_j\}_{j=1}^m$ satisfies the OSC, we know from [FL, Theorem 2.7] that $T_d$ has spectral radius 1, i.e., $\alpha=d$. Since $U\setminus\bigcup_{j=1}^m w_j(\overline{U})$ is an open subset of $\mathbb{R}^d$, the proof will be finished if we can show that $\mathcal{H}^d(U \setminus \bigcup_{j=1}^m w_j(\overline{U})) = 0$. For this, let

$$V = U \setminus \bigcup_{j=1}^{m} w_j(\overline{U}). \tag{3.13}$$

We claim that

$$w_I(V) \cap w_J(V) = \emptyset \quad \forall I, J \in \mathcal{J},\ I \neq J. \tag{3.14}$$

In fact, for $I, J$ comparable, we have $J = II_0$. Since $U$ is a basic open set, we have

$$w_i(U) \subset U \quad \text{and} \quad w_i(U) \cap w_j(U) = \emptyset, \quad \forall i \neq j. \tag{3.15}$$

Therefore $w_{I_0}(V) \subset \bigcup_{j=1}^m w_j(U)$, and thus $w_{I_0}(V) \cap V = \emptyset$. Hence

$$w_I(V) \cap w_J(V) \subseteq w_I(V \cap w_{I_0}(V)) = \emptyset.$$

If $I, J$ are incomparable, let $I = i_1 \ldots i_p$, $J = j_1 \ldots j_q$ and $r = \min\{k : i_k \neq j_k\}$. Define $I_0 = i_1 \ldots i_{r-1}$. By (3.13) and (3.15), we have

$$w_I(V) \cap w_J(V) \subseteq w_{I_0}(w_{i_r}(U) \cap w_{j_r}(U)) = \emptyset.$$

This completes the proof of the claim.

By (3.14) and Lemma 3.1, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \int_V \sum_{|J|=n} |w_J'(x)|^d \, d\mathcal{H}^d(x) &= \sum_{n=1}^{\infty} \sum_{|J|=n} \mathcal{H}^d(w_J(V)) \\ &= \mathcal{H}^d\Big(\bigcup_{J \in \mathcal{J}} w_J(V)\Big) \leq \mathcal{H}^d(U) < \infty. \end{aligned} \tag{3.16}$$

On the other hand, for any fixed $y_0 \in K$ and any $x \in X$, by Lemma 2.1(i),

$$c_1^{-d} |w_J'(y_0)|^d \leq |w_J'(x)|^d.$$

Hence

$$c_1^{-d} \sum_{|J|=n} |w_J'(y_0)|^d \leq \sum_{|J|=n} |w_J'(x)|^d, \qquad x \in X.$$

Since $\alpha = d$, it follows from [FL, Theorem 1.1] that

$$\lim_n \sum_{|J|=n} |w_J'(\cdot)|^d = h(\cdot) \quad \text{uniformly on } K$$

where $0 < h \in C(K)$ is the 1-eigenfunction of the Ruelle operator $T_d$. Then

$$\begin{aligned} c_1^{-d} h(y_0) \mathcal{H}^d(V) &= c_1^{-d} \cdot \lim_n \int_V \sum_{|J|=n} |w_J'(y_0)|^d \, d\mathcal{H}^d(x) \\ &\leq \liminf_n \int_V \sum_{|J|=n} |w_J'(x)|^d \, d\mathcal{H}^d(x). \end{aligned}$$

By (3.16), the right side is 0, hence $\mathcal{H}^d(V) = \mathcal{H}^d(U \setminus \bigcup_{j=1}^m w_j(\overline{U})) = 0$. ∎

Corollary 3.8. *Let* $\{w_j\}_{j=1}^m$ *be as in Theorem* 1.1. *If* $\alpha = d$ *and* $\mathcal{H}^d(K) > 0$, *then* $K^\circ \neq \emptyset$ *and* $\dim_{\mathrm{H}} \partial K < d$.

*Proof.* Let $U$ be the basic open set constructed in the proof of Theorem 1.1. By assumption and Theorem 3.7, we have $\mathcal{H}^d(U \setminus \bigcup_{j=1}^m w_j(\overline{U})) = 0$. Then

$$\overline{U} = \bigcup_{j=1}^{m} w_j(\overline{U}).$$

By the uniqueness of the invariant set $K$, we have $K = \overline{U}$, and then $K^\circ \supseteq U \neq \emptyset$. In view of the proof of Theorem 1.2, we have

$$\dim_{\mathrm{H}} \partial K \leq \dim_{\mathrm{H}}(K \setminus U) < d. \ \blacksquare$$

**Acknowledgements.** I would like to express my deepest gratitude to Professor Ka-Sing Lau for numerous valuable discussions and for going through the manuscript. Thanks are also due to Dr. H. Rao for valuable discussions.

## References

[BG] C. Bandt and S. Graf, *Self-similar sets. VII. A characterization of self-similar fractals with positive Hausdorff measure*, Proc. Amer. Math. Soc. 114 (1992), 995–1001.

[FL] A. H. Fan and K. S. Lau, *Iterated function system and Ruelle operator*, J. Math. Anal. Appl. 231 (1999), 319–344.

[H] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. 30 (1981), 713–747.

[L] K. S. Lau, H. Rao and Y. L. Ye, *Corrigendum: "Iterated function system and Ruelle operator"*, J. Math. Anal. Appl. 262 (2001), 446–451.

[LX] K. S. Lau and Y. Xu, *On the boundary of attractors with non-void interior*, Proc. Amer. Math. Soc. 128 (2000), 1761–1768.

[MU] R. D. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc. 73 (1996), 106–154.

[P] Y. Peres, M. Rams, K. Simon and B. Solomyak, *Equivalence of positive Hausdorff measure and the open set condition for self-conformal sets*, Proc. Amer. Math. Soc. 129 (2001), 2689–2699.

[S] A. Schief, *Separation properties for self-similar sets*, ibid. 122 (1994), 111–115.

Department of Mathematics
The Chinese University of Hong Kong
Shatin, Hong Kong
E-mail: ylye@math.cuhk.edu.hk

*Present address*:
Department of Mathematics
South China Normal University
Guangzhou 510631, P.R. China
E-mail: ylye@scnu.edu.cn

*Received June 13, 2001* (4758)