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Nonlocal Poincaré inequalities on Lie groups with polynomial volume growth and Riemannian manifolds

by

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Abstract. Let G be a real connected Lie group with polynomial volume growth endowed with its Haar measure dx. Given a C^2 positive bounded integrable function M on G, we give a sufficient condition for an L^2 Poincaré inequality with respect to the measure $M(x)\,dx$ to hold on G. We then establish a nonlocal Poincaré inequality on G with respect to $M(x)\,dx$. We also give analogous Poincaré inequalities on Riemannian manifolds and deal with the case of Hardy inequalities.

1. Introduction. Let G be a unimodular connected Lie group endowed with a measure $M(x)\,dx$ where $M \in L^1(G)$ and dx stands for the Haar measure on G. By "unimodular", we mean that the Haar measure is left- and right-invariant. We always assume that M is bounded and $M = e^{-v}$ where v is a C^2 function on G. If we denote by $\mathcal{G}$ the Lie algebra of G, we consider a family

$$\mathbb{X} = \{X_1, \ldots, X_k\}$$

of left-invariant vector fields on G satisfying the Hörmander condition, i.e. $\mathcal{G}$ is the Lie algebra generated by the X_i's. A standard metric on G , called the Carnot–Carathéodory metric, is naturally associated with $\mathbb{X}$ and is defined as follows. Let $\ell : [0,1] \to G$ be an absolutely continuous path. We say that ℓ is *admissible* if there exist measurable functions $a_1, \ldots, a_k : [0,1] \to \mathbb{C}$ such that, for almost every $t \in [0,1]$,

$$\ell'(t) = \sum_{i=1}^{k} a_i(t) X_i(\ell(t)).$$

If ℓ is admissible, its length is defined by

$$|\ell| = \int_0^1 \Big(\sum_{i=1}^{k} |a_i(t)|^2 \, dt\Big)^{1/2}.$$

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For all $x, y \in G$, define $d(x,y)$ as the infimum of the lengths of all admissible paths joining x to y (such a curve exists by the Hörmander condition). This distance is left-invariant. For short, we denote by $|x|$ the distance between e, the neutral element of the group, and x, so that the distance from x to y is equal to $|y^{-1}x|$.

For all $r > 0$, denote by $B(x,r)$ the open ball in G with respect to the Carnot–Carathéodory distance and by $V(r)$ the Haar measure of any ball. There exists $d \in \mathbb{N}^*$ (called the *local dimension* of $(G, \mathbb{X})$) and $0 < c < C$ such that, for all $r \in (0,1)$,

$$cr^d \le V(r) \le Cr^d$$

(see [NSW]). When $r > 1$, two situations may occur (see [G]):

- There exist $c, C, D > 0$ such that, for all $r > 1$,
$$cr^D \le V(r) \le Cr^D$$
where D is called the *dimension at infinity* of the group (note that, unlike d, D does not depend on $\mathbb{X}$). The group is then said to have *polynomial volume growth*.
- There exist $c_1, c_2, C_1, C_2 > 0$ such that, for all $r > 1$,
$$c_1 e^{c_2 r} \le V(r) \le C_1 e^{C_2 r}.$$
The group is then said to have *exponential volume growth*.

When G has polynomial volume growth, it is plain that there exists $C > 0$ such that, for all $r > 0$,

$$V(2r) \le CV(r), \tag{1.1}$$

which implies that there exist $C > 0$ and $\kappa > 0$ such that, for all $r > 0$ and all $\theta > 1$,

$$V(\theta r) \le C\theta^\kappa V(r). \tag{1.2}$$

Denote by $H^1(G, d\mu_M)$ the Sobolev space of functions $f \in L^2(G, d\mu_M)$ such that $X_i f \in L^2(G, d\mu_M)$ for all $1 \le i \le k$. We are interested in L^2 Poincaré inequalities for the measure $d\mu_M$. In order to state sufficient conditions for such an inequality to hold, we introduce the operator

$$L_M f = -M^{-1} \sum_{i=1}^k X_i\{MX_i f\}$$

for all f such that

$$f \in \mathcal{D}(L_M) := \left\{ g \in H^1(G, d\mu_M) : \frac{1}{\sqrt{M}} X_i\{MX_i g\} \in L^2(G, dx),\ \forall 1 \le i \le k \right\}.$$

One therefore has, for all $f \in \mathcal{D}(L_M)$ and $g \in H^1(G, d\mu_M)$,

$$\int_G L_M f(x) g(x)\, d\mu_M(x) = \sum_{i=1}^{k} \int_G X_i f(x) \cdot X_i g(x)\, d\mu_M(x).$$

In particular, the operator L_M is symmetric on $L^2(G, d\mu_M)$.

Following [BBCG], say that a C^2 function $W : G \to \mathbb{R}$ is a *Lyapunov function* if $W(x) \geq 1$ for all $x \in G$ and there exist constants $\theta > 0$, $b \geq 0$ and $R > 0$ such that, for all $x \in G$,

$$-L_M W(x) \leq -\theta W(x) + b\mathbf{1}_{B(e,R)}(x), \tag{1.3}$$

where, for all $A \subset G$, $\mathbf{1}_A$ denotes the characteristic function of A. We first claim:

Theorem 1.1. *Assume that G is unimodular and that there exists a Lyapunov function W on G. Then $d\mu_M$ satisfies the following L^2 Poincaré inequality: there exists $C > 0$ such that, for every function $f \in H^1(G, d\mu_M)$ with $\int_G f(x)\, d\mu_M(x) = 0$,*

$$\int_G |f(x)|^2\, d\mu_M(x) \leq C \sum_{i=1}^{k} \int_G |X_i f(x)|^2\, d\mu_M(x). \tag{1.4}$$

Let us give, as a corollary, a sufficient condition on v for (1.4) to hold:

Corollary 1.2. *Assume that G is unimodular and there exist constants $a \in (0,1)$, $c > 0$ and $R > 0$ such that, for all $x \in G$ with $|x| > R$,*

$$a \sum_{i=1}^{k} |X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq c. \tag{1.5}$$

Then (1.4) *holds.*

Notice that, if (1.5) holds with $a \in (0, 1/2)$, then the Poincaré inequality (1.4) admits the following improvement:

Proposition 1.3. *Assume that G is unimodular and there exist constants $c > 0$, $R > 0$ and $\varepsilon \in (0,1)$ such that, for all $x \in G$,*

$$\frac{1-\varepsilon}{2} \sum_{i=1}^{k} |X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq c \quad \text{whenever } |x| > R. \tag{1.6}$$

Then there exists $C > 0$ such that, for every function $f \in H^1(G, d\mu_M)$ with $\int_G f(x)\, d\mu_M(x) = 0$,

$$\int_G |f(x)|^2 \Big(1 + \sum_{i=1}^{k} |X_i v(x)|^2\Big)\, d\mu_M(x) \leq C \sum_{i=1}^{k} \int_G |X_i f(x)|^2\, d\mu_M(x). \tag{1.7}$$

Observe that conditions (1.5) and (1.6) are satisfied for instance, when $M = \exp(-|x|^2/2)$ is a Gaussian measure, but also when $M(x) = e^{-|x|}$, and more generally when $M(x) = e^{-|x|^\alpha}$ with $\alpha \geq 1$.

Finally, we obtain Poincaré inequalities for $d\mu_M$ involving a nonlocal term.

Main Theorem 1.4. *Let G be a unimodular Lie group with polynomial growth. Let $d\mu_M = M dx$ be a measure absolutely continuous with respect to the Haar measure on G where $M = e^{-v} \in L^1(G)$ is assumed to be bounded and $v \in C^2(G)$.*

(i) *Assume that there exist constants $a \in (0,1)$, $c > 0$ and $R > 0$ such that, for all $x \in G$ with $|x| > R$, (1.5) holds. Let $\alpha \in (0,2)$. Then there exists $\lambda_\alpha(M) > 0$ such that, for every function $f \in \mathcal{D}(G)$ satisfying $\int_G f(x)\, d\mu_M(x) = 0$,*

$$\int_G |f(x)|^2\, d\mu_M(x) \leq \lambda_\alpha(M) \iint_{G\times G} \frac{|f(x)-f(y)|^2}{V(|y^{-1}x|)|y^{-1}x|^\alpha}\, dx\, d\mu_M(y). \tag{1.8}$$

(ii) *Assume that there exist constants $c > 0$, $R > 0$ and $\varepsilon \in (0,1)$ such that (1.6) holds. Let $\alpha \in (0,2)$. Then there exists $\lambda_\alpha(M) > 0$ such that, for every function $f \in \mathcal{D}(G)$ satisfying $\int_G f(x)\, d\mu_M(x) = 0$,*

$$\begin{aligned}\int_G |f(x)|^2 \Big(1 + \sum_{i=1}^k |X_i v(x)|^2\Big)^{\alpha/2} d\mu_M(x) \qquad\qquad \\ \leq \lambda_\alpha(M) \iint_{G\times G} \frac{|f(x)-f(y)|^2}{V(|y^{-1}x|)|y^{-1}x|^\alpha}\, dx\, d\mu_M(y).\end{aligned} \tag{1.9}$$

Note that (1.8) (resp. (1.9)) is an extension of (1.4) (resp. (1.7)) in terms of fractional nonlocal quantities. The proof follows the same lines as in [MRS] but we concentrate here on a more geometric context.

Before describing our method, let us give some motivation for obtaining fractional Poincaré inequalities. Fractional diffusions naturally appear in many models, ranging from plasma turbulence [DCL] or geostrophic flows [CV] in fluid dynamics, grazing collisions in kinetic theory (cf. the Boltzmann equation for long-range interactions [VI, M, MS, GS]), all the way to stockmarket modeling based on Lévy processes [DOP]. They also appear naturally in mathematics: in probability they appear in the important class of infinitely divisible Markov processes (cf. the Lévy–Khinchin representation [FE]); in analysis they naturally appear in the study of singular integral operators (e.g. for the Boltzmann equation, cf. references above) as well as in the so-called "Dirichlet-to-Neumann" boundary value problem and in the Signorini (obstacle) problem [SIG] (see for instance among other references [SIL] and [CF]). The search for a Poincaré inequality for the nonlocal fractional energy associated with fractional diffusion is therefore a natural and interesting question since this inequality governs the spectral gap of the underlying operator and the speed of (fractional) diffusion towards an equilibrium.

In order to prove Theorem 1.4, we need to introduce fractional powers of L_M. This is the object of the following developments. Since the operator L_M is symmetric and nonnegative on $L^2(G, d\mu_M)$, we can define the usual power L_M^β for any $\beta \in (0,1)$ by means of spectral theory.

Section 2 is devoted to the proof of Theorem 1.1 and Corollary 1.2. Then, in Section 3, we check L^2 "off-diagonal" estimates for the resolvent of L_M and use them to establish Theorem 1.4.

Analogous results can be obtained on Riemannian manifolds (under certain assumptions) and we refer the reader to Section 4 for a complete description. Finally the last section deals with Hardy inequalities.

2. A proof of the Poincaré inequality for $d\mu_M$. We follow closely the approach of [BBCG]. Recall first that the following L^2 local Poincaré inequality holds on G for the measure dx: for all $R > 0$, there exists $C_R > 0$ such that, for all $x \in G$, all $r \in (0, R)$, every ball $B := B(x, r)$ and every function $f \in C^\infty(B)$,

$$\int_B |f(x) - f_B|^2\, dx \le C_R r^2 \sum_{i=1}^k \int_B |X_i f(x)|^2\, dx, \tag{2.1}$$

where $f_B := V(r)^{-1} \int_B f(x)\, dx$. In the Euclidean context, Poincaré inequalities for vector fields satisfying Hörmander conditions were obtained by Jerison [J]. A proof of (2.1) in the case of unimodular Lie groups can be found in [SA], but the idea goes back to [VA]. A nice survey of this topic can be found in [HK]. Notice that no global growth assumption on the volume of balls is required for (2.1) to hold.

The proof of (1.4) relies on the following inequality:

Lemma 2.1. *Assume that W is a Lyapunov function. For every function $f \in H^1(G, d\mu_M)$ on G,*

$$\int_G \frac{L_M W}{W}(x) f(x)^2\, d\mu_M(x) \le \sum_{i=1}^k \int_G |X_i f(x)|^2\, d\mu_M(x). \tag{2.2}$$

Proof. Assume first that f is compactly supported on G. Using the definition of L_M, one has

$$\begin{aligned}\int_G \frac{L_M W}{W}(x) f(x)^2\, d\mu_M(x) &= \sum_{i=1}^k \int_G X_i\left(\frac{f^2}{W}\right)(x)\cdot X_i W(x)\, d\mu_M(x)\\ &= 2\sum_{i=1}^k \int_G \frac{f}{W}(x) X_i f(x)\cdot X_i W(x)\, d\mu_M(x)\\ &\quad - \sum_{i=1}^k \int_G \frac{f^2}{W^2}(x)|X_i W(x)|^2\, d\mu_M(x)\\ &= \sum_{i=1}^k \int_G |X_i f(x)|^2\, d\mu_M(x)\\ &\quad - \sum_{i=1}^k \int_G \left|X_i f - \frac{f}{W} X_i W\right|^2(x)\, d\mu_M(x)\\ &\le \sum_{i=1}^k \int_G |X_i f(x)|^2\, d\mu_M(x).\end{aligned}$$

Notice that all the above integrals are finite because of the support condition on f. Now, if f is as in Lemma 2.1, consider a nondecreasing sequence of smooth compactly supported functions χ_n satisfying

$$\mathbf{1}_{B(e,nR)} \le \chi_n \le 1 \quad \text{and} \quad |X_i \chi_n| \le 1 \quad \text{for all } 1 \le i \le k.$$

Applying (2.2) to $f\chi_n$ and letting n go to $+\infty$ yields the desired conclusion, by use of the monotone convergence theorem on the left-hand side and the dominated convergence theorem on the right-hand side. ∎

Let us now establish (1.4). Let g be a smooth function on G and let $f := g - c$ on G where c is a constant to be chosen. By assumption (1.3),

$$\begin{aligned}\text{(2.3)}\qquad &\int_G f(x)^2\, d\mu_M(x)\\ &\le \int_G f(x)^2 \frac{L_M W}{\theta W}(x)\, d\mu_M(x) + \int_{B(e,R)} f(x)^2 \frac{b}{\theta W}(x)\, d\mu_M(x).\end{aligned}$$

Lemma 2.1 shows that (2.2) holds. Let us now turn to the second term on the right-hand side of (2.3). Fix c such that $\int_{B(e,R)} f(x)\, d\mu_M(x) = 0$. By (2.1) applied to f on $B(e,R)$ and the fact that M is bounded from above and below on $B(e,R)$, one has

$$\int_{B(e,R)} f(x)^2\, d\mu_M(x) \le CR^2 \sum_{i=1}^k \int_{B(e,R)} |X_i f(x)|^2\, d\mu_M(x)$$

where the constant C depends on R and M. Therefore, as $W \geq 1$ on G,

$$\int_{B(e,R)} f(x)^2 \frac{b}{\theta W}(x)\, d\mu_M(x) \leq CR^2 \sum_{i=1}^{k} \int_{B(e,R)} |X_i f(x)|^2\, d\mu_M(x) \tag{2.4}$$

where the constant C depends on R, M, θ and b. Gathering (2.3), (2.2) and (2.4) yields

$$\int_G (g(x)-c)^2\, d\mu_M(x) \leq C \sum_{i=1}^{k} \int_G |X_i g(x)|^2\, d\mu_M(x),$$

which easily implies (1.4) for the function g (and the same dependence for the constant C). ■

Proof of Corollary 1.2. According to Theorem 1.1, it is enough to find a Lyapunov function W. Define

$$W(x) := e^{\gamma(v(x)-\inf_G v)}$$

(remember that M, and therefore v, are bounded) where $\gamma > 0$ will be chosen later. Since

$$-L_M W(x) = \gamma\Big(\sum_{i=1}^{k} X_i^2 v(x) - (1-\gamma)\sum_{i=1}^{k} |X_i v(x)|^2\Big) W(x),$$

W is a Lyapunov function for $\gamma := 1-a$ because of the assumption on v. Indeed, one can take $\theta = c\gamma$ and $b = \max_{B(e,R)}\{-L_M W + \theta W\}$ (recall that M is a C^2 function). ■

Let us now prove Proposition 1.3. Observe first that, since v is C^2 on G and (1.6) holds, there exists $\alpha \in \mathbb{R}$ such that, for all $x \in G$,

$$\frac{1-\varepsilon}{2} \sum_{i=1}^{k} |X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq \alpha. \tag{2.5}$$

Let f be as in the statement of Proposition 1.3 and let $g := f M^{1/2}$. Since, for all $1 \leq i \leq k$,

$$X_i f = M^{-1/2} X_i g - \frac{1}{2} g M^{-3/2} X_i M,$$

inequality (2.5) yields two positive constants β, γ such that

$$\begin{aligned} &\sum_{i=1}^{k} \int_G |X_i f(x)|^2(x)\, d\mu_M(x) \\ &\quad = \sum_{i=1}^{k} \int_G \Big(|X_i g(x)|^2 + \frac{1}{4} g(x)^2 |X_i v(x)|^2 + g(x) X_i g(x) X_i v(x)\Big)\, dx \end{aligned} \tag{2.6}$$

$$= \sum_{i=1}^{k} \int_G \Big(|X_i g(x)|^2 + \frac{1}{4} g(x)^2 |X_i v(x)|^2 + \frac{1}{2} X_i(g^2)(x) X_i v(x) \Big)\, dx$$
$$\geq \sum_{i=1}^{k} \int_G g(x)^2 \Big(\frac{1}{4} |X_i v(x)|^2 - \frac{1}{2} X_i^2 v(x) \Big)\, dx$$
$$\geq \sum_{i=1}^{k} \int_G f(x)^2 (\beta |X_i v(x)|^2 - \gamma)\, d\mu_M(x).$$

The conjunction of (1.4), which holds because of (1.6), and (2.6) yields the desired conclusion. ■

3. Proof of Theorem 1.4. We divide the proof into several steps.

3.1. Rewriting the Poincaré inequalities. The first step in the proof of 1.4 consists in rewriting the "initial" Poincaré inequality in terms of operators. Let us first consider item (i). By the definition of L_M, inequality (1.4) means, in terms of operators in $L^2(G, d\mu_M)$, that, for some $\lambda > 0$,

$$L_M \geq \lambda I, \tag{3.1}$$

where I is the identity operator. Using a functional calculus argument (see [D, p. 110]), one deduces from (3.1) that, for any $\alpha \in (0,2)$,

$$L_M^{\alpha/2} \geq \lambda^{\alpha/2} I,$$

which implies, thanks to the fact $L_M^{\alpha/2} = (L_M^{\alpha/4})^2$ and the symmetry of $L_M^{\alpha/4}$ on $L^2(G, d\mu_M)$, that

$$\int_G |f(x)|^2\, d\mu_M(x) \leq C \int_G |L_M^{\alpha/4} f(x)|^2 \, d\mu_M(x) = C \|L_M^{\alpha/4} f\|^2_{L^2(G, d\mu_M)}.$$

As far as item (ii) is concerned, the conclusion of Proposition 1.3 means that

$$L_M \geq \lambda \mu \tag{3.2}$$

for some $\lambda > 0$, where μ is the multiplication operator by $1 + \sum_{i=1}^k |X_i v|^2$. Arguing similarly, one deduces from (3.2) that, for any $\alpha \in (0,2)$,

$$L_M^{\alpha/2} \geq \lambda^{\alpha/2} \mu^{\alpha/2},$$

which implies now that

$$\int_G |f(x)|^2 \Big(1 + \sum_{i=1}^{k} |X_i v(x)|^2 \Big)^{\alpha/2} d\mu_M(x) \leq C \int_G |L_M^{\alpha/4} f(x)|^2\, d\mu_M(x) = C \|L_M^{\alpha/4} f\|^2_{L^2(G, d\mu_M)}.$$

Therefore, the conclusions of Theorem 1.4 will follow by estimating the quantity $\|L_M^{\alpha/4} f\|^2_{L^2(G, d\mu_M)}$.

3.2. Off-diagonal L^2 estimates for the resolvent of L_M. The crucial estimates to derive the desired inequality are some L^2 "off-diagonal" estimates for the resolvent of L_M, in the spirit of [G]. This is the object of the following lemma.

LEMMA 3.1. *There exists C with the following property: for all closed disjoint subsets $E, F \subset G$ with $d(E,F) =: d > 0$, every function $f \in L^2(G, d\mu_M)$ supported in E and all $t > 0$,*

$$\|(I+tL_M)^{-1}f\|_{L^2(F,d\mu_M)} + \|tL_M(I+tL_M)^{-1}f\|_{L^2(F,d\mu_M)} \leq 8e^{-Cd/\sqrt{t}}\|f\|_{L^2(E,d\mu_M)}.$$

Proof. We argue as in [AHLMT, Lemma 1.1]. From the fact that L_M is self-adjoint on $L^2(G, d\mu_M)$ we have

$$\|(L_M-\mu)^{-1}\|_{L^2(G,d\mu_M)\to L^2(G,d\mu_M)} \leq \frac{1}{\operatorname{dist}(\mu, \Sigma(L_M))}$$

where $\Sigma(L_M)$ denotes the spectrum of L_M, and $\mu \notin \Sigma(L_M)$. Then we deduce that $(I+tL_M)^{-1}$ is bounded with norm less than 1 for all $t > 0$, and it is clearly enough to consider the case $0 < \sqrt{t} < d$.

In the following computations, we will make explicit the dependence of the measure $d\mu_M$ on M for clarity. Define $u_t = (I+tL_M)^{-1}f$, so that, for every function $v \in H^1(G, d\mu_M)$,

$$\int_G u_t(x)v(x)M(x)\,dx + t\sum_{i=1}^k \int_G X_iu_t(x)\cdot X_iv(x)M(x)\,dx = \int_G f(x)v(x)M(x)\,dx. \tag{3.3}$$

Fix now a nonnegative function $\eta \in \mathcal{D}(G)$ vanishing on E (by $\mathcal{D}(G)$ we denote the space of C^∞ functions on G with compact support). Since f is supported in E, applying (3.3) with $v = \eta^2 u_t$ (remember that $u_t \in H^1(G, d\mu_M)$) yields

$$\int_G \eta(x)^2|u_t(x)|^2M(x)\,dx + t\sum_{i=1}^k \int_G X_iu_t(x)\cdot X_i(\eta^2u_t)M(x)\,dx = 0,$$

which implies

$$\begin{aligned}
&\int_G \eta(x)^2|u_t(x)|^2M(x)\,dx + t\int_G \eta(x)^2\sum_{i=1}^k |X_iu_t(x)|^2M(x)\,dx \\
&\quad = -2t\sum_{i=1}^k \int_G \eta(x)u_t(x)X_i\eta(x)\cdot X_iu_t(x)M(x)\,dx \\
&\quad \leq t\int_G |u_t(x)|^2\sum_{i=1}^k |X_i\eta(x)|^2M(x)\,dx + t\int_G \eta(x)^2\sum_{i=1}^k |X_iu_t(x)|^2M(x)\,dx,
\end{aligned}$$

hence

$$\int_G \eta(x)^2|u_t(x)|^2 M(x)\,dx \le t\int_G |u_t(x)|^2 \sum_{i=1}^k |X_i\eta(x)|^2 M(x)\,dx. \tag{3.4}$$

Let ζ be a nonnegative smooth function on G such that $\zeta = 0$ on E, and let $\lambda > 0$ be so chosen that $\eta := e^{\lambda\,\zeta} - 1 \ge 0$ and η vanishes on E. Choosing this particular η in (3.4) with $\lambda > 0$ gives

$$\begin{aligned}\int_G |e^{\lambda\zeta(x)} - 1|^2|u_t(x)|^2 M(x)\,dx \\ \le \lambda^2 t\int_G |u_t(x)|^2 \sum_{i=1}^k |X_i\zeta(x)|^2\, e^{2\lambda\zeta(x)} M(x)\,dx.\end{aligned}$$

Taking $\lambda = 1/(2\sqrt{t}\|\sum_{i=1}^k |X_i\zeta|^2\|_\infty^{1/2})$, one obtains

$$\int_G |e^{\lambda\,\zeta(x)} - 1|^2|u_t(x)|^2 M(x)\,dx \le \frac{1}{4}\int_G |u_t(x)|^2 e^{2\lambda\zeta(x)} M(x)\,dx.$$

Since the norm of $(I+tL_M)^{-1}$ is bounded by 1 uniformly in $t > 0$, this gives

$$\begin{aligned}\|e^{\lambda\zeta}\, u_t\|_{L^2(G,d\mu_M)} &\le \|(e^{\lambda\zeta} - 1)\, u_t\|_{L^2(G,d\mu_M)} + \|u_t\|_{L^2(G,d\mu_M)} \\ &\le \frac{1}{2}\|e^{\lambda\zeta}\, u_t\|_{L^2(G,d\mu_M)} + \|f\|_{L^2(G,d\mu_M)},\end{aligned}$$

therefore

$$\int_G |e^{\lambda\zeta(x)}|^2|u_t(x)|^2 M(x)\,dx \le 4\int_G |f(x)|^2 M(x)\,dx.$$

We now choose ζ such that $\zeta = 0$ on E as before and additionally that $\zeta = 1$ on F. It can furthermore be chosen with $\max_{i=1,\dots,k}\|X_i\zeta\|_\infty \le C/d$, which yields the desired conclusion for the L^2 norm of $(I+tL_M)^{-1}f$ with a factor 4 on the right-hand side. Since $tL_M(I+tL_M)^{-1}f = f - (I+tL_M)^{-1}f$, the desired inequality with a factor 8 readily follows. ∎

3.3. Control of $\|L_M^{\alpha/4}f\|_{L^2(G,d\mu_M)}$ and conclusion of the proof of Theorem 1.4. This is now the heart of the proof to reach the conclusion of Theorem 1.4. The following first lemma is a standard quadratic estimate on powers of subelliptic operators. It is based on spectral theory.

LEMMA 3.2. *Let $\alpha \in (0,2)$. There exists $C > 0$ such that, for all $f \in \mathcal{D}(L_M)$,*

$$\|L_M^{\alpha/4}f\|^2_{L^2(G,d\mu_M)} \le C\int_0^\infty t^{-1-\alpha/2}\|tL_M(I+tL_M)^{-1}f\|^2_{L^2(G,d\mu_M)}\,dt. \tag{3.5}$$

We now come to the desired estimate.

LEMMA 3.3. *Let $\alpha \in (0,2)$. There exists $C > 0$ such that, for all $f \in \mathcal{D}(G)$,*

$$\int_0^\infty t^{-1-\alpha/2} \|tL_M(I+tL_M)^{-1}f\|^2_{L^2(G,d\mu_M)}\,dt \\ \leq C \iint_{G\times G} \frac{|f(x)-f(y)|^2}{V(|y^{-1}x|)|y^{-1}x|^\alpha} M(x)\,dx\,dy.$$

Proof. Fix $t \in (0,+\infty)$. Following Lemma 3.2, we give an upper bound of

$$\|tL_M(I+tL_M)^{-1}f\|^2_{L^2(G,d\mu_M)}$$

involving first order differences for f. Using (1.1), one can pick a countable family x_j^t, $j \in \mathbb{N}$, such that the balls $B(x_j^t, \sqrt{t})$ are pairwise disjoint and

$$G = \bigcup_{j\in\mathbb{N}} B(x_j^t, 2\sqrt{t}). \tag{3.6}$$

By Lemma 6.1 in Appendix A, there exists a constant $\widetilde{C} > 0$ such that for all $\theta > 1$ and all $x \in G$, there are at most $\widetilde{C}\theta^{2\kappa}$ indices j such that $|x^{-1}x_j^t| \leq \theta\sqrt{t}$ where κ is given by (1.2).

For fixed j, one has

$$tL_M(I+tL_M)^{-1}f = tL_M(I+tL_M)^{-1}g^{j,t}$$

where, for all $x \in G$,

$$g^{j,t}(x) := f(x) - m^{j,t}$$

and $m^{j,t}$ is defined by

$$m^{j,t} := \frac{1}{V(2\sqrt{t})} \int_{B(x_j^t,2\sqrt{t})} f(y)\,dy.$$

Note that, here, the mean value of f is computed with respect to the Haar measure on G. Since (3.6) holds, one clearly has

$$\begin{aligned}\|tL_M(I+tL_M)^{-1}f\|^2_{L^2(G,d\mu_M)} &\leq \sum_{j\in\mathbb{N}} \|tL_M(I+tL_M)^{-1}f\|^2_{L^2(B(x_j^t,2\sqrt{t}),d\mu_M)} \\ &= \sum_{j\in\mathbb{N}} \|tL_M(I+tL_M)^{-1}g^{j,t}\|^2_{L^2(B(x_j^t,2\sqrt{t}),d\mu_M)},\end{aligned}$$

and we are left with the task of estimating

$$\|tL_M(I+tL_M)^{-1}g^{j,t}\|^2_{L^2(B(x_j^t,2\sqrt{t}),d\mu_M)}.$$

For that purpose, set

$$C_0^{j,t} = B(x_j^t, 4\sqrt{t}) \quad \text{and} \quad C_k^{j,t} = B(x_j^t, 2^{k+2}\sqrt{t}) \setminus B(x_j^t, 2^{k+1}\sqrt{t}), \quad \forall k \geq 1,$$

and $g_k^{j,t} := g^{j,t}\,\mathbf{1}_{C_k^{j,t}}$, $k \geq 0$. Since $g^{j,t} = \sum_{k\geq 0} g_k^{j,t}$ one has

$$\text{(3.7)}\quad \|tL_M(I+tL_M)^{-1}g^{j,t}\|_{L^2(B(x_j^t,2\sqrt{t}),d\mu_M)} \leq \sum_{k\geq 0} \|tL_M(I+tL_M)^{-1}g_k^{j,t}\|_{L^2(B(x_j^t,2\sqrt{t}),d\mu_M)}$$

and, using Lemma 3.1, one obtains (for some constants $C,c>0$)

$$\text{(3.8)}\quad \|tL_M(I+tL_M)^{-1}g^{j,t}\|_{L^2(B(x_j^t,2\sqrt{t}),d\mu_M)} \leq C\Big(\|g_0^{j,t}\|_{L^2(C_0^{j,t},d\mu_M)} + \sum_{k\geq 1} e^{-c2^k}\|g_k^{j,t}\|_{L^2(C_k^{j,t},d\mu_M)}\Big).$$

By Cauchy–Schwarz's inequality, we deduce (for another constant $C'>0$)

$$\text{(3.9)}\quad \|tL_M(I+tL_M)^{-1}g^{j,t}\|^2_{L^2(B(x_j^t,2\sqrt{t}),d\mu_M)} \leq C'\Big(\|g_0^{j,t}\|^2_{L^2(C_0^{j,t},d\mu_M)} + \sum_{k\geq 1} e^{-c2^k}\|g_k^{j,t}\|^2_{L^2(C_k^{j,t},d\mu_M)}\Big).$$

As a consequence, we have

$$\text{(3.10)}\quad \int_0^\infty t^{-1-\alpha/2}\|tL_M(I+tL_M)^{-1}f\|^2_{L^2(G,d\mu_M)}\,dt \leq C'\int_0^\infty t^{-1-\alpha/2}\sum_{j\geq 0}\|g_0^{j,t}\|^2_{L^2(C_0^{j,t},d\mu_M)}\,dt + C'\int_0^\infty t^{-1-\alpha/2}\sum_{k\geq 1}e^{-c2^k}\sum_{j\geq 0}\|g_k^{j,t}\|^2_{L^2(C_k^{j,t},d\mu_M)}\,dt.$$

The proof of the following lemma is postponed to Appendix B:

Lemma 3.4. *There exists $\bar{C}>0$ such that, for all $t>0$ and all $j\in\mathbb{N}$:*

(A) *For the first term:*

$$\|g_0^{j,t}\|^2_{L^2(C_0^{j,t},M)} \leq \frac{\bar{C}}{V(\sqrt{t})}\int_{x\in B(x_j^t,4\sqrt{t})}\int_{y\in B(x_j^t,4\sqrt{t})} |f(x)-f(y)|^2\,d\mu_M(x)\,dy.$$

(B) *For all $k\geq 1$,*

$$\|g_k^{j,t}\|^2_{L^2(C_k^{j,t},d\mu_M)} \leq \frac{\bar{C}}{V(\sqrt{t})}\int_{x\in B(x_j^t,2^{k+2}\sqrt{t})}\int_{y\in B(x_j^t,2^{k+2}\sqrt{t})} |f(x)-f(y)|^2\,d\mu_M(x)\,dy.$$

We now finish the proof of the theorem. Using Lemma 3.4(A), summing up on $j\geq 0$ and integrating over $(0,\infty)$, we get

$$\begin{aligned}
\int_0^\infty t^{-1-\alpha/2} \sum_{j\geq 0} \|g_0^{j,t}\|^2_{L^2(C_0^{j,t},d\mu_M)}\,dt &= \sum_{j\geq 0}\int_0^\infty t^{-1-\alpha/2}\|g_0^{j,t}\|^2_{L^2(C_0^{j,t},d\mu_M)}\,dt\\
&\leq \bar{C}\sum_{j\geq 0}\int_0^\infty \frac{t^{-1-\alpha/2}}{V(\sqrt{t})}\Big(\int_{B(x_j^t,4\sqrt{t})}\int_{B(x_j^t,4\sqrt{t})} |f(x)-f(y)|^2\,d\mu_M(x)\,dy\Big)\,dt\\
&\leq \bar{C}\sum_{j\geq 0}\iint_{(x,y)\in G\times G} |f(x)-f(y)|^2 M(x)\\
&\quad\times\Big(\int_{t\geq \max\{|x^{-1}x_j^t|^2/16;\,|y^{-1}x_j^t|^2/16\}} \frac{t^{-1-\alpha/2}}{V(\sqrt{t})}\,dt\Big)\,dx\,dy.
\end{aligned}$$

The Fubini theorem now shows

$$\begin{aligned}
&\sum_{j\geq 0}\int_{t\geq \max\{|x^{-1}x_j^t|^2/16;\,|y^{-1}x_j^t|^2/16\}} \frac{t^{-1-\alpha/2}}{V(\sqrt{t})}\,dt\\
&\qquad\qquad = \int_0^\infty \frac{t^{-1-\alpha/2}}{V(\sqrt{t})}\sum_{j\geq 0}\mathbf{1}_{(\max\{|x^{-1}x_j^t|^2/16;\,|y^{-1}x_j^t|^2/16\},+\infty)}(t)\,dt.
\end{aligned}$$

Observe that, by Lemma 6.1, there is a constant $N\in\mathbb{N}$ such that, for all $t>0$, there are at most N indices j such that $|x^{-1}x_j^t|^2<16t$ and $|y^{-1}x_j^t|^2<16t$, and for those j, one has $|x^{-1}y|<8\sqrt{t}$. Therefore

$$\sum_{j\geq 0}\mathbf{1}_{(\max\{|x^{-1}x_j^t|^2/16;\,|y^{-1}x_j^t|^2/16\},+\infty)}(t)\leq N\,\mathbf{1}_{(|x^{-1}y|^2/64,+\infty)}(t),$$

so that, by (1.1),

$$\begin{aligned}
(3.11)\qquad &\int_0^\infty t^{-1-\alpha/2}\sum_j \|g_0^{j,t}\|^2_{L^2(C_0^{j,t},d\mu_M)}\,dt\\
&\qquad\leq \bar{C}N\iint_{G\times G}|f(x)-f(y)|^2M(x)\Big(\int_{|x^{-1}y|^2/64}^\infty \frac{t^{-1-\alpha/2}}{V(\sqrt{t})}\,dt\Big)\,dx\,dy\\
&\qquad\qquad\leq \bar{C}N\iint_{G\times G}\frac{|f(x)-f(y)|^2}{V(|x^{-1}y|)|x^{-1}y|^\alpha}\,d\mu_M(x)\,dy.
\end{aligned}$$

Using Lemma 3.4(B), we obtain, for all $j\geq 0$ and all $k\geq 1$,

$$\begin{aligned}
&\int_0^\infty t^{-1-\alpha/2}\sum_{j\geq 0}\|g_k^{j,t}\|_2^2\,dt\\
&\quad\leq \bar{C}\sum_{j\geq 0}\int_0^\infty \frac{t^{-1-\alpha/2}}{V(\sqrt{t})}\Big(\iint_{B(x_j^t,2^{k+2}\sqrt{t})\times B(x_j^t,2^{k+2}\sqrt{t})}|f(x)-f(y)|^2M(x)\,dx\,dy\Big)\,dt
\end{aligned}$$

$$\leq \bar{C} \sum_{j\geq 0} \iint_{x,y\in G} |f(x)-f(y)|^2 M(x)$$
$$\times \left(\int_0^\infty \frac{t^{-1-\alpha/2}}{V(\sqrt{t})} \mathbf{1}_{(\max\{|x^{-1}x_j^t|^2/4^{k+2},|y^{-1}x_j^t|^2/4^{k+2}\},+\infty)}(t)\,dt \right) dx\,dy.$$

But, given $t>0$, $x,y\in G$, by Lemma 6.1 again, there exist at most $\widetilde{C}\,2^{2k\kappa}$ indices j such that

$$|x^{-1}x_j^t| \leq 2^{k+2}\sqrt{t} \quad \text{and} \quad |y^{-1}x_j^t| \leq 2^{k+2}\sqrt{t},$$

and for those j, $|x^{-1}y| \leq 2^{k+3}\sqrt{t}$. As a consequence of these observations and (1.2),

$$\int_0^\infty \frac{t^{-1-\alpha/2}}{V(\sqrt{t})} \sum_{j\geq 0} \mathbf{1}_{(\max\{|x^{-1}x_j^t|^2/4^{k+2},|x^{-1}x_j^t|^2/4^{k+2}\},+\infty)}(t)\,dt \tag{3.12}$$
$$\leq \widetilde{C}\,2^{2k\kappa} \int_{t\geq |x^{-1}y|^2/4^{k+3}} \frac{t^{-1-\alpha/2}}{V(\sqrt{t})}\,dt$$
$$\leq \widetilde{C}\,2^{3k\kappa} \int_{t\geq |x^{-1}y|^2/4^{k+3}} \frac{t^{-1-\alpha/2}}{V(2^k\sqrt{t})}\,dt \leq \widetilde{C}' \frac{2^{k(3\kappa+\alpha)}}{V(|x^{-1}y|)|x^{-1}y|^\alpha},$$

for some other constant $\widetilde{C}'>0$, and therefore

$$\int_0^\infty \frac{t^{-1-\alpha/2}}{V(\sqrt{t})} \sum_j \|g_k^{j,t}\|^2_{L^2(C_0^{j,t},d\mu_M)}\,dt$$
$$\leq \bar{C}\widetilde{C}'\,2^{k(3\kappa+\alpha)} \iint_{G\times G} \frac{|f(x)-f(y)|^2}{V(|x^{-1}y|)|x^{-1}y|^\alpha} M(x)\,dx\,dy.$$

We can now conclude the proof of Lemma 3.3, using Lemma 3.2, (3.8), (3.11) and (3.12). We have proved, by reconsidering (3.10),

$$\int_0^\infty t^{-1-\alpha/2} \|tL_M(I+tL_M)^{-1}f\|^2_{L^2(G,d\mu_M)}\,dt \tag{3.13}$$
$$\leq C'\bar{C}N \iint_{G\times G} \frac{|f(x)-f(y)|^2}{V(|x^{-1}y|)|x-y|^\alpha} M(x)\,dx\,dy$$
$$+ \sum_{k\geq 1} C'\bar{C}\widetilde{C}'\,2^{k(3\kappa+\alpha)} e^{-c2^k} \iint_{G\times G} \frac{|f(x)-f(y)|^2}{V(|x^{-1}y|)|x^{-1}y|^\alpha} M(x)\,dx\,dy$$

and we deduce that

$$\int_0^\infty t^{-1-\alpha/2}\|tL_M(I+tL_M)^{-1}f\|^2_{L^2(G,d\mu_M)}\,dt \\ \leq C\iint_{G\times G}\frac{|f(x)-f(y)|^2}{V(|x^{-1}y|)|x^{-1}y|^\alpha}\,d\mu_M(x)\,dy$$

for some constant C as claimed in the statement. ∎

REMARK 3.5. In the Euclidean context, Strichartz [STR] proved that, when $0<\alpha<2$, for all $p\in(1,+\infty)$,

$$\|(-\Delta)^{\alpha/4}f\|_{L^p(\mathbb{R}^n)}\leq C_{\alpha,p}\|S_\alpha f\|_{L^p(\mathbb{R}^n)} \tag{3.14}$$

where

$$S_\alpha f(x)=\Big(\int_0^\infty\Big(\int_B|f(x+ry)-f(x)|\,dy\Big)^2\frac{dr}{r^{1+\alpha}}\Big)^{1/2},$$

and also [STE]

$$\|(-\Delta)^{\alpha/4}f\|_{L^p(\mathbb{R}^n)}\leq C_{\alpha,p}\|D_\alpha f\|_{L^p(\mathbb{R}^n)} \tag{3.15}$$

where

$$D_\alpha f(x)=\Big(\int_{\mathbb{R}^n}\frac{|f(x+y)-f(x)|^2}{|y|^{n+\alpha}}\,dy\Big)^{1/2}.$$

In [CRT], these inequalities were extended to the setting of a unimodular Lie group endowed with a sublaplacian Δ, relying on semigroup techniques and Littlewood–Paley–Stein functionals. In particular, in [CRT], the authors use *pointwise* estimates of the kernel of the semigroup generated by Δ. In the present paper, we deal with the operator L_M for which these pointwise estimates are not available, but it turns out that L^2 off-diagonal estimates are enough for our purpose. Note that we do not obtain L^p inequalities here.

4. The case of Riemannian manifolds. Let $\mathcal{M}$ be a Riemannian manifold, denote by n its dimension, by $d\mu$ its Riemannian measure and by Δ the Laplace–Beltrami operator. For all $x\in\mathcal{M}$ and all $r>0$, let $B(x,r)$ be the open geodesic ball centered at x with radius r, and $V(x,r)$ its measure.

In order to apply our method, we will need to be able to control from below the volume of any geodesic ball $B(x,r)$ by a quantity of the type r^p. The goal of the next paragraph is to give sufficient assumptions on M such that this control occurs.

The first one is a Faber–Krahn inequality on $\mathcal{M}$. For any bounded open subset $\Omega\subset\mathcal{M}$, denote by $\lambda_1^D(\Omega)$ the principal eigenvalue of $-\Delta$ on Ω under the Dirichlet boundary condition. If $p\geq n$, consider the following

Faber–Krahn inequality: there exists $C > 0$ such that

$$\lambda_1^D(\Omega) \geq C\mu(\Omega)^{2/p} \quad \text{for all bounded subsets } \Omega \subset M. \tag{4.1}$$

Let $\Lambda_p > 0$ be the greatest constant C for which (4.1) is satisfied. In other words,

$$\Lambda_p = \inf \frac{\lambda_1^D(\Omega)}{\mu(\Omega)^{2/p}},$$

where the infimum is taken over all bounded subsets $\Omega \subset \mathcal{M}$. The Faber–Krahn inequality (4.1) is satisfied in particular when an isoperimetric inequality holds on $\mathcal{M}$, namely there exist $C > 0$ and $p \geq n$ such that, for any bounded smooth subset $\Omega \subset \mathcal{M}$,

$$\sigma(\partial\Omega) \geq C\mu(\Omega)^{1-1/p}, \tag{4.2}$$

where $\sigma(\partial\Omega)$ denotes the surface measure of $\partial\Omega$. If $\mathcal{M}$ has nonnegative Ricci curvature, then (4.2) with $p = n$ and (4.1) with $p = n$ are equivalent. More generally, if $\mathcal{M}$ has Ricci curvature bounded from below by a constant, then (4.1) with $p > 2n$ implies (4.2) with $p/2$ in place of p ([CA1, Proposition 3.1], see also [CO] where the injectivity radius of $\mathcal{M}$ is furthermore assumed to be bounded). Note that there exists a Riemannian manifold satisfying (4.1) for some $p \geq n$ but for which (4.2) does not hold for any $p \geq n$ ([CA1, Proposition 3.4]).

It is a well-known fact that (4.1) implies a lower bound for the volume of geodesic balls in $\mathcal{M}$. Namely ([CA1, Proposition 2.4]), if (4.1) holds, then, for all $x \in \mathcal{M}$ and all $r > 0$,

$$V(x,r) \geq \left(\frac{\Lambda_p}{2^{p+2}}\right)^{p/2} r^p. \tag{4.3}$$

We will also need another assumption on the volume growth of balls in $\mathcal{M}$, already encountered in the present work in the case of Lie groups. Say that $\mathcal{M}$ has the *doubling property* if there exists $C > 0$ such that, for all $x \in \mathcal{M}$ and all $r > 0$,

$$V(x,2r) \leq CV(x,r). \tag{D}$$

There is a wide class of manifolds on which (D) holds. First, as already said in the introduction (see (1.1)), it is true on Lie groups with polynomial volume growth (in particular on nilpotent Lie groups). Next, (D) is true if $\mathcal{M}$ has nonnegative Ricci curvature thanks to the Bishop comparison theorem (see [BC]). Recall also that (D) remains valid if $\mathcal{M}$ is quasi-isometric to a manifold with nonnegative Ricci curvature, or is a cocompact covering manifold whose deck transformation group has polynomial growth, [CSC]. Unlike the doubling property, the nonnegativity of the Ricci curvature is not stable under quasi-isometry.

The last assumption we need on $\mathcal{M}$ is a local L^2 Poincaré inequality on balls *for the Riemannian measure.* Namely, if $R > 0$, say that $\mathcal{M}$ satisfies (P_R) if there exists $C_R > 0$ such that, for all $x \in \mathcal{M}$, all $r \in (0, R)$ and every function $f \in C^\infty(B(x,r))$,

$$(P_R) \qquad \int_{B(x,r)} |f(x) - f_{B(x,r)}|^2 \, d\mu(x) \le C_R r^2 \int_{B(x,r)} |\nabla f(x)|^2 \, d\mu(x).$$

Note that (2.1) shows that, on a unimodular Lie group G equipped with vector fields as in the introduction, such a Poincaré inequality always holds. Recall that (P_R) always holds for all $R > 0$ for instance when M has nonnegative Ricci curvature ([B]).

Under these assumptions, the proof developed above in the context of groups can be adopted *verbatim* to give the following result.

MAIN THEOREM 4.1. *Let $\mathcal{M}$ be a complete noncompact Riemannian manifold. Assume that* (4.1) *holds, $\mathcal{M}$ has the doubling property and* (P_R) *holds for some $R > 0$. Let v be a C^2 function on $\mathcal{M}$ and $M = e^{-v}$.*

(i) *Assume that there exists $x_0 \in \mathcal{M}$ and constants $a \in (0,1)$ and $c > 0$ such that, for all $x \in G$ with $d(x, x_0) > R$,*

$$a|\nabla v(x)|^2 - \Delta v(x) \ge c. \tag{4.4}$$

Then there exists $C > 0$ such that, for every $f \in H^1(\mathcal{M}, M\,d\mu)$ with $\int_{\mathcal{M}} f(x)M(x)\,d\mu = 0$, and for all $\alpha \in (0,2)$,

$$\int_{\mathcal{M}} f(x)^2 M(x)\,d\mu(x) \le C \iint_{\mathcal{M}\times\mathcal{M}} \frac{|f(y) - f(x)|^2}{d(x,y)^{p+\alpha}} M(x)\,d\mu(x)\,d\mu(y). \tag{4.5}$$

(ii) *Assume there exist $x_0 \in \mathcal{M}$ and constants $c > 0$ and $\varepsilon \in (0,1)$ such that, for all $x \in \mathcal{M}$,*

$$\frac{1-\varepsilon}{2}|\nabla v(x)|^2 - \Delta v(x) \ge c \quad \text{whenever } d(x, x_0) > R. \tag{4.6}$$

Then there exists $C > 0$ such that, for every $f \in H^1(\mathcal{M}, M\,d\mu)$ with $\int_{\mathcal{M}} f(x)M(x)\,d\mu = 0$, and for all $\alpha \in (0,2)$,

$$\begin{aligned} \int_{\mathcal{M}} f(x)^2 (1 + |\nabla v|^2)^{\alpha/2} M(x)\,d\mu(x) \qquad\qquad \\ \le C \iint_{\mathcal{M}\times\mathcal{M}} \frac{|f(y) - f(x)|^2}{d(x,y)^{p+\alpha}} M(x)\,d\mu(x)\,d\mu(y). \end{aligned} \tag{4.7}$$

5. Hardy inequalities. One can also use the previous method to obtain a nonlocal version of Hardy inequalities. The simplest Hardy inequality on $\mathbb{R}^n$

asserts that, if $n \geq 3$,

$$\int_{\mathbb{R}^n} \frac{u(x)^2}{|x|^2}\,dx \lesssim \int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx = \|u\|_{\dot{H}^1(\mathbb{R}^n)}, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n). \tag{5.1}$$

A nonlocal version of (5.1) can be given, where the $\dot{H}^1$ norm on the right hand side is replaced by an $\dot{H}^s$ norm for $0 < s < n/2$ (see [BCG]):

$$\int_{\mathbb{R}^n} \frac{u(x)^2}{|x|^{2s}}\,dx \lesssim \|u\|^2_{\dot{H}^s(\mathbb{R}^n)}, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n). \tag{5.2}$$

When $0 < s < 1$, it is well-known (see for instance [A]) that $\|u\|_{\dot{H}^s(\mathbb{R}^n)}$ can be represented by means of an integral quantity involving first order differences of u, and (5.2) can therefore be rewritten as

$$\int_{\mathbb{R}^n} \frac{u(x)^2}{|x|^{2s}}\,dx \lesssim \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy. \tag{5.3}$$

These Hardy inequalities (i.e. the local and the nonlocal version) were transposed to the framework of the Heisenberg group in [BCG, BCX]. More precisely, in the Heisenberg group $\mathcal{H}^d$ ($d \geq 1$), the following Hardy inequality was established in [BCX]:

$$\int_{\mathcal{H}^d} \frac{u(x)^2}{\rho(x)^2}\,dx \lesssim \|\nabla_{\mathcal{H}} u\|_2^2, \qquad \forall u \in \mathcal{D}(\mathcal{H}^d), \tag{5.4}$$

where $\rho(x)$ denotes the distance of x to the origin and $\nabla_{\mathcal{H}}$ stands for the gradient associated to the vector fields $Z_1, \dots, Z_{2d}$ (see [BCX] and the notations therein). The nonlocal version of (5.4), which was proven in [BCG] (where it was derived from precise inequalities involving Besov norms) says that, for $0 < s < d+1$,

$$\int_{\mathcal{H}^d} \frac{u(x)^2}{\rho(x)^{2s}}\,dx \lesssim \|u\|^2_{\dot{H}^s}, \qquad \forall u \in \mathcal{D}(\mathcal{H}^d). \tag{5.5}$$

When $0 < s < 1$, an integral representation for the fractional Sobolev homogeneous norm was proven in [CRT] (note that an analogous representation holds in any connected Lie group with polynomial volume growth, and even in any unimodular Lie group if one works with the inhomogeneous version of this norm), so that (5.5) can be rewritten as

$$\int_{\mathcal{H}^d} \frac{u(x)^2}{\rho(x)^{2s}}\,dx \lesssim \iint_{\mathcal{H}^d\times\mathcal{H}^d} \frac{|u(x)-u(y)|^2}{\rho(y^{-1}x)^{2d+2+2s}}\,dx\,dy. \tag{5.6}$$

Hardy inequalities in *local* versions on more general Lie groups, namely Carnot groups, were obtained in [KO]. The Lie group G is called a *Carnot* *group* if G is simply connected and the Lie algebra of G admits a stratification, i.e. there exist linear subspaces $V_1, \dots, V_k$ of $\mathcal{G}$ such that

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_k \quad \text{with} \quad [V_1, V_i] = V_{i+1}$$

for $i = 1, \dots, k-1$ and $[V_1, V_k] = 0$. By $[V_1, V_i]$ we mean the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ where $X \in V_1$ and $Y \in V_i$. Recall that the class of Carnot groups is a strict subclass of nilpotent groups. Moreover, if G is a Carnot group, there exists $n \in \mathbb{N}$, called the *homogeneous dimension* of G, such that, for all $r > 0$,

$$V(r) \sim r^n \tag{5.7}$$

(see [FS]). The Heisenberg group $\mathcal{H}^d$ is a Carnot group with $n = 2d + 2$.

Let G be a Carnot group, denote by δ the Dirac distribution supported at the origin and let u be a solution of

$$-\Delta_G u = \delta.$$

Define $N(x) = u(x)^{1/(2-n)}$ for $x \neq 0$ and $N(0) = 0$. The function N is a homogeneous norm on N by [FO]. Kombe [KO] proved the following Hardy inequality on G: for $\alpha > 2-n$, there exists $C > 0$ such that, for all functions $u \in \mathcal{D}(G \setminus \{0\})$,

$$\left(\frac{n+\alpha-2}{2}\right)^2 \int_G u(x)^2 \frac{|\nabla_G N(x)|^2}{|N(x)|^2} N(x)^\alpha \, dx \leq C \int_G |\nabla_G u(x)|^2 N(x)^\alpha \, dx. \tag{5.8}$$

Using the same method as before, we obtain the following nonlocal version of (5.8):

Main Theorem 5.1. *Let G be a Carnot group with homogeneneous dimension $n \geq 3$. Then for all $\alpha > 2-n$ and all $s \in (0, 2)$,*

$$\int_G u(x)^2 \left(\frac{|\nabla_G N(x)|}{|N(x)|}\right)^s N(x)^\alpha \, dx \lesssim \iint_{G \times G} \frac{|u(x) - u(y)|^2}{|y^{-1}x|^{n+s}} N(x)^\alpha \, dx \, dy, \qquad \forall u \in \mathcal{D}(G \setminus \{0\}). \tag{5.9}$$

As far as Riemannian manifolds are concerned, a general principle was developed in [CA2] to derive Hardy inequalities. Let us recall here an example of such an inequality. Let $\mathcal{M}$ be a complete noncompact Riemannian manifold as in Section 4. Below, we use the same notation, as well as d for exterior differentiation. Assume that $\rho : \mathcal{M} \to [0, +\infty)$ satisfies

$$|d\rho| \leq 1 \quad \text{on } \mathcal{M}, \tag{5.10}$$

and

$$\Delta\rho \geq C/\rho \quad \text{in the distribution sense,} \tag{5.11}$$

where $C > 0$. Then, for all $\alpha > 1 - C$, and all $u \in \mathcal{D}(\mathcal{M} \setminus \rho^{-1}(0))$,

$$\left(\frac{C+\alpha-1}{2}\right)^2 \int\limits_{\mathcal{M}} \left(\frac{u}{\rho}\right)^2 (x)\rho(x)^\alpha\, dx \leq \int\limits_{\mathcal{M}} |du(x)|^2 \rho(x)^\alpha\, dx. \tag{5.12}$$

Moreover, if the codimension of $\rho^{-1}(0)$ is greater than $2 - \alpha$, (5.12) holds for every function $u \in \mathcal{D}(\mathcal{M})$ (see [CA2, Théorème 1.4 and Remarque 1.5], see also [KOZ]).

Remark 5.2. Observe that, in the Euclidean context, (5.8) and (5.12) amount to the same inequality. Indeed, on the one hand, when $G = \mathbb{R}^n$, one has $N(x) = |x|$, and (5.8) exactly means that

$$\left(\frac{n+\alpha-2}{2}\right)^2 \int\limits_{\mathbb{R}^n} u(x)^2 |x|^{\alpha-2}\, dx \leq \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2 |x|^\alpha\, dx$$

as soon as $\alpha > 2 - n$. On the other hand, when $M = \mathbb{R}^n$, assumption (5.10) is satisfied with $\rho(x) = |x|$ and $C = n - 1$, so that (5.12) means that

$$\left(\frac{n+\alpha-2}{2}\right)^2 \int\limits_{\mathbb{R}^n} u(x)^2 |x|^{\alpha-2}\, dx \leq \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2 |x|^\alpha\, dx$$

whenever $\alpha > 2 - n$.

Always using the same method, we obtain:

Main Theorem 5.3. *Let $\mathcal{M}$ be a complete noncompact Riemannian manifold. Assume that* (4.1) *holds and $\mathcal{M}$ has the doubling property. Assume also that $C > 0$ and $\rho : \mathcal{M} \to [0, +\infty)$ are such that* (5.10) *and* (5.11) *hold. Then, if $\alpha > 1 - C$ and $\rho^{-1}(0)$ has codimension greater than $2 - \alpha$, one has, for all $s \in (0, 2)$,*

$$\begin{aligned} &\int\limits_{\mathcal{M}} u(x)^2 \rho(x)^{\alpha-s}\, dx \\ &\quad \lesssim \iint\limits_{\mathcal{M}\times\mathcal{M}} \frac{|u(y)-u(x)|^2}{d(x,y)^{p+s}} \rho(x)^\alpha\, dx\, dy, \qquad \forall u \in \mathcal{D}(\mathcal{M} \setminus \rho^{-1}(0)). \end{aligned} \tag{5.13}$$

6. Appendix A: Technical lemma.

We prove the following lemma.

Lemma 6.1. *Let G and the x_j^t be as in the proof of Lemma* 3.3. *Then there exists a constant $\widetilde{C} > 0$ with the following property: for all $\theta > 1$ and all $x \in G$, there are at most $\widetilde{C}\theta^{2\kappa}$ indices j such that $|x^{-1}x_j^t| \leq \theta\sqrt{t}$.*

Proof. The argument is very simple (see [KA]) and we give it for the sake of completeness. Let $x \in G$ and denote

$$I(x) := \{j \in \mathbb{N} : |x^{-1}x_j^t| \leq \theta\sqrt{t}\}.$$

Since, for all $j \in I(x)$,

$$B(x_j^t, \sqrt{t}) \subset B(x, (1+\theta)\sqrt{t}), \qquad B(x, \sqrt{t}) \subset B(x_j^t, (1+\theta)\sqrt{t}),$$

by (1.2) and since the balls $B(x_j^t, \sqrt{t})$ are pairwise disjoint one has

$$\begin{aligned} |I(x)| V(x, \sqrt{t}) &\le \sum_{j \in I(x)} V(x_j^t, (1+\theta)\sqrt{t}) \le C(1+\theta)^\kappa \sum_{j \in I(x)} V(x_j^t, \sqrt{t}) \\ &\le C(1+\theta)^\kappa V(x, (1+\theta)\sqrt{t}) \le C(1+\theta)^{2\kappa} V(x, \sqrt{t}), \end{aligned}$$

and we obtain the desired conclusion. ∎

7. Appendix B: Estimates for g_j^t. We prove Lemma 3.4. For all $x \in G$,

$$\begin{aligned} g_0^{j,t}(x) &= f(x) - \frac{1}{V(2\sqrt{t})} \int_{B(x_j^t, 2\sqrt{t})} f(y)\, dy \\ &= \frac{1}{V(2\sqrt{t})} \int_{B(x_j^t, 2\sqrt{t})} (f(x) - f(y))\, dy. \end{aligned}$$

By the Cauchy–Schwarz inequality and (1.1), it follows that

$$|g_0^{j,t}(x)|^2 \le \frac{C}{V(\sqrt{t})} \int_{B(x_j^t, 4\sqrt{t})} |f(x) - f(y)|^2\, dy.$$

Therefore,

$$\|g_0^{j,t}\|^2_{L^2(C_0^{j,t}, M)} \le \frac{C}{V(\sqrt{t})} \int_{B(x_j^t, 4\sqrt{t})} \int_{B(x_j^t, 4\sqrt{t})} |f(x) - f(y)|^2\, d\mu_M(x)\, dy,$$

which shows assertion (A). We argue similarly for (B) and obtain

$$\begin{aligned} &\|g_k^{j,t}\|^2_{L^2(C_k^{j,t}, M)} \\ &\quad \le \frac{C}{V(2^k\sqrt{t})} \int_{x \in B(x_j^t, 2^{k+2}\sqrt{t})} \int_{y \in B(x_j^t, 2^{k+2}\sqrt{t})} |f(x) - f(y)|^2\, d\mu_M(x)\, dy, \end{aligned}$$

which ends the proof.

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