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Brézis–Gallouët–Wainger type inequality for Besov–Morrey spaces

by

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Abstract. The aim of the present paper is to obtain an inequality of Brézis–Gallouët–Wainger type for Besov–Morrey spaces. We investigate these spaces in a self-contained manner. Also, we verify that our result is sharp.

1. Introduction. In the present paper we shall obtain an inequality of Brézis–Gallouët–Wainger type for Besov–Morrey spaces. Let us begin by describing Morrey spaces.

Let $0 < q \le p < \infty$. Then the *Morrey* (*quasi-*)*norm* is given by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_B |B|^{1/p-1/q}\Big(\int_B |f(x)|^q\,dx\Big)^{1/q}, \tag{1.1}$$

where B runs over all the open balls in $\mathbb{R}^n$. Note that Morrey spaces include the L^p spaces as a special case when $0 < p = q < \infty$ and that $\mathcal{M}^p_q$ is monotone with respect to q. An easy calculation yields the following example.

Proposition 1.1. *$|x|^{-n/p} \in \mathcal{M}^p_q \setminus L^p$ with $0 < q < p < \infty$.*

As Proposition 1.1 shows, Morrey spaces can deal directly with functions having singularity $|x|^{-n/p}$. From this fact it seems that the parameter p in the Morrey space $\mathcal{M}^p_q$ reflects the global regularity.

With this in mind, let us describe the Besov–Morrey norm. About a decade ago, Besov–Morrey spaces were investigated in connection with the Navier–Stokes equations by Kozono and Yamazaki (see [5]). Later, several people studied Besov–Morrey spaces and their variants (see [6, 7, 11, 10, 12]). Let $0 < r \le \infty$ and $0 < q \le p < \infty$. We pick a smooth function $\psi \in \mathcal{S}$ so that $\chi_{B_1} \le \psi \le \chi_{B_2}$. Here and below we shall denote by B_r the open ball centered at the origin and of radius $r > 0$. We write $\varphi_j \equiv \psi(2^{-j}\cdot) - \psi(2^{-j+1}\cdot)$ and $\psi_j \equiv \psi(2^{-j}\cdot)$ for $j \in \mathbb{Z}$. We denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and

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its inverse respectively. Given $f \in \mathcal{S}'$ and $\tau \in \mathcal{S}$, we set $\tau(D)f \equiv \mathcal{F}^{-1}\tau * f$. Then define the *Besov–Morrey norm* by

$$\|f\|_{\mathcal{N}^s_{pqr}} \equiv \|\psi(D)f\|_{\mathcal{M}^p_q} + \Big(\sum_{j=1}^{\infty} 2^{jsr}\|\varphi_j(D)f\|^r_{\mathcal{M}^p_q}\Big)^{1/r} \tag{1.2}$$

for $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$. An important observation made in [12] is that the definition of the Besov–Morrey norm (1.2) is independent of the choice of ψ and φ: more precisely, different choices of admissible ψ and φ will yield equivalent norms. In [11] we defined the *homogeneous Besov–Morrey norm* by

$$\|f\|_{\dot{\mathcal{N}}^s_{pqr}} \equiv \Big(\sum_{j=-\infty}^{\infty} 2^{jsr}\|\varphi_j(D)f\|^r_{\mathcal{M}^p_q}\Big)^{1/r} \tag{1.3}$$

for $f \in \mathcal{S}'/\mathcal{P}$, where $\mathcal{P} \subset \mathcal{S}'$ denotes the set of all polynomials, and $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$. Recall that the homogeneous and nonhomogeneous Besov norms are given by

$$\|f\|_{\dot{B}^{s,p,q}} \equiv \Big(\sum_{j=-\infty}^{\infty} 2^{jsq}\|\varphi_j(D)f\|^q_p\Big)^{1/q}, \tag{1.4}$$

$$\|f\|_{B^{s,p,q}} \equiv \|\psi(D)f\|_p + \Big(\sum_{j=1}^{\infty} 2^{jsq}\|\varphi_j(D)f\|^q_p\Big)^{1/q}, \tag{1.5}$$

respectively, where $0 < p, q \le \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$. Hence we see that these Besov–Morrey norms generalize the corresponding Besov norms. We refer to [5, 6, 7, 11, 12] for more details on Besov–Morrey spaces.

In [10], by using the homogeneous and nonhomogeneous Hölder norms $\dot{\mathcal{C}}^{s-n/p}$ and $\mathcal{C}^{s-n/p}$, we have established that

$$\dot{\mathcal{N}}^s_{pqr} \hookrightarrow \dot{\mathcal{C}}^{s-n/p}, \quad \mathcal{N}^s_{pqr} \hookrightarrow \mathcal{C}^{s-n/p} \tag{1.6}$$

for $s > n/p$, which is a new formulation of the Morrey lemma obtained originally in 1938 [9]. The following theorem, which is the main theorem in the present paper, quantifies (1.6) more precisely. Here and below we shall write $\log^\alpha x = (\log x)^\alpha$ for $\alpha \in \mathbb{R}$ and $x > 0$.

THEOREM 1.1. *Let $1 \le \sigma \le \infty$, $s > \alpha > 0$ and $0 < q \le n/(s-\alpha)$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{B^{0,\infty,1}} \le C\|u\|_{\dot{B}^{0,\infty,\sigma}} \log^{1-1/\sigma}\left(2 + \frac{\|u\|_{\mathcal{N}^s_{n/(s-\alpha),q,\infty}}}{\|u\|_{\dot{B}^{0,\infty,\sigma}}}\right). \tag{1.7}$$

It is worth noting that we only need a very weak assumption on the growth of f at infinity, say, $f \in \mathcal{N}^s_{n/(s-\alpha),q,\infty}$ with $0 < q \ll n/(s-\alpha)$, in order for $f \in \dot{B}^{0,\infty,\sigma}$ to belong to $f \in B^{0,\infty,1}$. In Section 4 we shall show that (1.7) is sharp (see Propositions 4.1 and 4.2).

Now let us look back briefly on the foregoing results. The inequality (1.8) below, which Theorem 1.1 extends, dates back to the results by Brézis and Gallouët and by Brézis and Wainger [1, 2]. They established the following inequality:

PROPOSITION 1.2. *Let $1 < p < \infty$, $1 \le r \le \infty$. Assume in addition that s is an integer with $s > n/r$. Then there exists $\lambda > 0$ such that*

$$\|u\|_{L^\infty(\mathbb{R}^n)}^{p/(p-1)} \le \lambda(1 + \log(1 + \|u\|_{W^{s,r}(\mathbb{R}^n)})) \tag{1.8}$$

for all $u \in W^{n/p,p}(\mathbb{R}^n) \cap W^{s,r}(\mathbb{R}^n)$ under the normalization

$$\|u\|_{W^{n/p,p}(\mathbb{R}^n)} = 1.$$

Brézis and Gallouët [2] proved Proposition 1.2 for $s = r = n = p = 2$. Brézis and Wainger [1] generalized it to the present form.

Kozono, Ogawa and Taniuchi [4] extended the Brézis–Gallouët–Wainger inequality considerably:

PROPOSITION 1.3. *Let $1 \le p, \rho, \sigma \le \infty$, $1 \le q < \infty$ and $s > n/q$. Then*

$$\|u\|_{L^\infty(\mathbb{R}^n)}^{\rho/(\rho-1)} \le C \log(2 + \|u\|_{B^{s,q,\sigma}(\mathbb{R}^n)}) \quad \textit{whenever} \quad \|u\|_{\dot{B}^{n/p,p,\rho}(\mathbb{R}^n)} = 1.$$

We learn from Theorem 1.1 that the assumption $1 \le q$ is not necessary in Proposition 1.3 and that we only have to postulate a weak restriction on the growth at infinity.

We are not concerned with the constants C in Theorem 1.1 and Proposition 1.3, since the Besov norms in question depend on the specific choice of ψ. However, as a special case of Theorem 1.1, Morii, Sato and Wadade obtained the following beautiful inequality. To formulate it, we let $\Lambda_1 \equiv \omega_{n-1}^{-1/(n-1)}$ and $\Lambda_2 \equiv \Lambda_1/n$. Define $\dot{C}^\alpha(B_1)$ as the set of all continuous functions f supported on $\overline{B_1}$ for which the seminorm $\|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}$ is finite.

PROPOSITION 1.4 ([8]). *Let $0 < \alpha < 1$. Then there exists a large constant $C > 0$ such that*

$$\left(\frac{\|f\|_\infty}{\|\nabla f\|_n}\right)^{n/(n-1)} \le \frac{\Lambda_1}{\alpha} \log\left(1 + \frac{\|f\|_{\dot{C}^\alpha(B_1)}}{\|\nabla f\|_n}\right) + \frac{\Lambda_2}{\alpha} \log\left(1 + \log\left(1 + \frac{\|f\|_{\dot{C}^\alpha(B_1)}}{\|\nabla f\|_n}\right)\right) + C \tag{1.9}$$

for all $f \in \dot{C}^\alpha(B_1)$. Conversely, if $\lambda_2 < \Lambda_2/\alpha$, then for any constant $C > 0$ we can find $f \in C_c^\infty(B_1)$ such that

$$\left(\frac{\|f\|_\infty}{\|\nabla f\|_n}\right)^{n/(n-1)} > \frac{\Lambda_1}{\alpha}\log\left(1+\frac{\|f\|_{\dot{C}^\alpha(B_1)}}{\|\nabla f\|_n}\right) + \lambda_2 \log\left(1+\log\left(1+\frac{\|f\|_{\dot{C}^\alpha(B_1)}}{\|\nabla f\|_n}\right)\right)+C. \tag{1.10}$$

To conclude this section, we describe the organization of the paper. Some preliminary facts are collected in Section 2. All the results in Section 2 are known and the references are given. Section 3 is devoted to the proof of Theorem 1.1. We show that Theorem 1.1 is sharp in Section 4.

2. Some estimates for band-limited distributions. This section collects some preliminary estimates for band-limited distributions. The proofs are supplied except for Theorems 2.1 and 2.2.

THEOREM 2.1 ([3, 12]). *Let $1 < q \le p < \infty$. Then*

$$\|Mf\|_{\mathcal{M}^p_q} \le C\|f\|_{\mathcal{M}^p_q} \tag{2.1}$$

for all $f \in \mathcal{M}^p_q$, where M denotes the Hardy–Littlewood maximal operator.

THEOREM 2.2 ([13, Theorem 1.3.1, Section 1.4.1]). *Let $f \in \mathcal{S}'$ have frequency support in B_1. Then for all $\eta > 0$, there exists a constant $C > 0$ independent of f such that*

$$\sup_{y\in\mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{n/\eta}} \le CM[|f|^\eta](x)^{1/\eta} \tag{2.2}$$

for every $x \in \mathbb{R}^n$.

The next estimate is an immediate corollary of Theorems 2.1 and 2.2.

COROLLARY 2.1 ([11, Corollary 2.3]). *Let $0 < q \le p < \infty$. Then*

$$\|f\|_\infty \le C\|f\|_{\mathcal{M}^p_q} \tag{2.3}$$

for all $f \in \mathcal{M}^p_q \cap \mathcal{S}'$ with $\operatorname{supp}(\mathcal{F}f) \subset B_1$.

Proof. Let $0 < \eta < q$. By Theorem 2.2, we have

$$\|f\|_\infty = \sup_B(\sup_{x\in B}|f(x)|) \le C\sup_B\Big(\int_B M[|f|^\eta](x)^{q/\eta}\,dx\Big)^{1/q}, \tag{2.4}$$

where B runs over all balls in $\mathbb{R}^n$ of radius 1. In view of the definition of the Morrey norm (1.1), we have

$$\begin{aligned}\sup_B\Big(\int_B M[|f|^\eta](x)^{q/\eta}\,dx\Big)^{1/q} &\le C\|M[|f|^\eta]\|^{1/\eta}_{\mathcal{M}^{p/\eta}_{q/\eta}} \\ &= C\||f|^\eta\|^{1/\eta}_{\mathcal{M}^{p/\eta}_{q/\eta}} \le C\|f\|_{\mathcal{M}^p_q}.\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we obtain (2.3). ∎

We transform (2.3) into a form which we shall use in the present paper.

LEMMA 2.1 ([11, Corollary 2.9]). *Let $0 < q \le p < \infty$ and $R > 0$. Then*

$$\|f\|_\infty \le C R^{n/p} \|f\|_{\mathcal{M}^p_q} \tag{2.6}$$

for all $f \in \mathcal{M}^p_q \cap \mathcal{S}'$ with $\operatorname{supp}(\mathcal{F}f) \subset B_R$, *where C is the constant of* (2.3).

Proof. This is just a matter of dilation of the estimate (2.3). Apply Corollary 2.1 to $f(R^{-1}\,\cdot)$. ∎

3. Proof of Theorem 1.1. Our proof consists of establishing the following two inequalities:

$$\|\psi(D)u\|_\infty \le C\|u\|_{\dot{B}^{0,\infty,\sigma}} \log^{1-1/\sigma}\left(2 + \frac{\|u\|_{\mathcal{N}^s_{n/(s-\alpha),q,\infty}}}{\|u\|_{\dot{B}^{0,\infty,\sigma}}}\right), \tag{3.1}$$

$$\|(1-\psi(D))u\|_{B^{0,\infty,1}} \le C\|u\|_{\dot{B}^{0,\infty,\sigma}} \log^{1-1/\sigma}\left(2 + \frac{\|u\|_{\mathcal{N}^s_{n/(s-\alpha),q,\infty}}}{\|u\|_{\dot{B}^{0,\infty,\sigma}}}\right). \tag{3.2}$$

Actually, instead of (3.2), we shall prove more:

$$\|(1-\psi(D))u\|_{B^{0,\infty,1}} \le C\|u\|_{\dot{B}^{0,\infty,\sigma}} \log^{1-1/\sigma}\left(2 + \frac{\|u\|_{\dot{\mathcal{C}}^\alpha}}{\|u\|_{\dot{B}^{0,\infty,\sigma}}}\right). \tag{3.3}$$

Let us begin by proving (3.3). Taking into account the frequency support of the functions and using the triangle inequality, we obtain

$$\begin{aligned}\|(1-\psi(D))u\|_{B^{0,\infty,1}} &= C\Big\|\sum_{j=0}^{\infty}(1-\psi(D))\varphi_j(D)u\Big\|_{B^{0,\infty,1}} \\ &\le C\sum_{j=0}^{\infty}\|(1-\psi(D))\varphi_j(D)u\|_{B^{0,\infty,1}} \\ &= C\sum_{j=0}^{\infty}\|(\varphi_{j-1}(D)+\varphi_j(D)+\varphi_{j+1}(D))\varphi_j(D)u\|_\infty.\end{aligned} \tag{3.4}$$

By the Young inequality we have

$$\begin{aligned}&\|(\varphi_{j-1}(D)+\varphi_j(D)+\varphi_{j+1}(D))\varphi_j(D)u\|_\infty \\ &\qquad\le C\|\mathcal{F}^{-1}[\varphi_{j-1}]+\mathcal{F}^{-1}[\varphi_j]+\mathcal{F}^{-1}[\varphi_{j+1}]\|_1\cdot\|\varphi_j(D)u\|_\infty.\end{aligned} \tag{3.5}$$

A change of variables yields

$$\|\mathcal{F}^{-1}[\varphi_{j-1}]+\mathcal{F}^{-1}[\varphi_j]+\mathcal{F}^{-1}[\varphi_{j+1}]\|_1 \le 3\|\mathcal{F}^{-1}[\varphi]\|_1. \tag{3.6}$$

If we combine (3.4)–(3.6), we obtain

$$\|(1-\psi(D))u\|_{B^{0,\infty,1}} \le C\sum_{j=0}^{\infty}\|\varphi_j(D)u\|_\infty = C\sum_{j=0}^{J}\|\varphi_j(D)u\|_\infty + C\sum_{j=J+1}^{\infty}\|\varphi_j(D)u\|_\infty, \tag{3.7}$$

where J is a constant to be fixed later. We estimate the first term of the right-hand side of (3.7) by the $\dot{B}^{0,\infty,q}$-norm:

$$\sum_{j=0}^{J}\|\varphi_j(D)u\|_\infty \le CJ^{1-1/\sigma}\Big(\sum_{j=0}^{J}\|\varphi_j(D)u\|_\infty^\sigma\Big)^{1/\sigma} \le CJ^{1-1/\sigma}\|u\|_{\dot{B}^{0,\infty,\sigma}}, \tag{3.8}$$

while the second term is estimated by the Hölder norm:

$$\sum_{j=J+1}^{\infty}\|\varphi_j(D)u\|_\infty \le \sum_{j=J+1}^{\infty}2^{-j\alpha}\|u\|_{\dot{\mathcal{C}}^\alpha} = C\,2^{-J\alpha}\|u\|_{\dot{\mathcal{C}}^\alpha}. \tag{3.9}$$

From (3.7)–(3.9) we deduce

$$\|(1-\psi(D))u\|_{B^{0,\infty,1}} \le CJ^{1-1/\sigma}\|u\|_{\dot{B}^{0,\infty,\sigma}} + C\,2^{-J\alpha}\|u\|_{\dot{\mathcal{C}}^\alpha}. \tag{3.10}$$

This is an estimate we are looking for.

Assuming, for the time being, that

$$\|u\|_{\dot{\mathcal{C}}^\alpha} \ge 2^{\alpha+1}\|u\|_{\dot{B}^{0,\infty,\sigma}}, \tag{3.11}$$

we take the smallest $J\in\mathbb{N}$ such that

$$J^{1-1/\sigma}\|u\|_{\dot{B}^{0,\infty,\sigma}} \ge 2^{-J\alpha}\|u\|_{\dot{\mathcal{C}}^\alpha}. \tag{3.12}$$

Note that (3.11) implies that $J\ge 2$, which in turn yields

$$(J-1)^{1-1/\sigma}\|u\|_{\dot{B}^{0,\infty,\sigma}} < 2^{-(J-1)\alpha}\|u\|_{\dot{\mathcal{C}}^\alpha} \tag{3.13}$$

in view of the minimality. From (3.12) and (3.13) we deduce

$$2^{-J\alpha}\|u\|_{\dot{\mathcal{C}}^\alpha} \le J^{1-1/\sigma}\|u\|_{\dot{B}^{0,\infty,\sigma}} \le C\,2^{-J\alpha}\|u\|_{\dot{\mathcal{C}}^\alpha}. \tag{3.14}$$

In particular, we have

$$\|u\|_{\dot{B}^{0,\infty,\sigma}} \le C\,2^{-J\alpha}\|u\|_{\dot{\mathcal{C}}^\alpha}, \quad \text{or equivalently,} \quad J \le C\log_2\Big(2+\frac{\|u\|_{\dot{\mathcal{C}}^\alpha}}{\|u\|_{\dot{B}^{0,\infty,\sigma}}}\Big). \tag{3.15}$$

Hence it follows from (3.14) and (3.15) that

$$\begin{aligned}
\|(1-\psi(D))u\|_{B^{0,\infty,1}} &\le CJ^{1-1/\sigma}\|u\|_{\dot{B}^{0,\infty,\sigma}} \\
&\le C\|u\|_{\dot{B}^{0,\infty,\sigma}} \log^{1-1/\sigma}\biggl(2+\frac{\|u\|_{\dot{\mathcal{C}}^\alpha}}{\|u\|_{\dot{B}^{0,\infty,\sigma}}}\biggr),
\end{aligned} \tag{3.16}$$

provided (3.11) holds.

If (3.11) fails, then use (3.10) with $J=1$. Then we have

$$\begin{aligned}
\|(1-\psi(D))u\|_{B^{0,\infty,1}} &\le C\|u\|_{\dot{B}^{0,\infty,\sigma}} \\
&\le C\|u\|_{\dot{B}^{0,\infty,\sigma}} \log^{1-1/\sigma}\biggl(2+\frac{\|u\|_{\dot{\mathcal{C}}^\alpha}}{\|u\|_{\dot{B}^{0,\infty,\sigma}}}\biggr).
\end{aligned} \tag{3.17}$$

From (3.16) and (3.17) we deduce the inequality (3.3).

Now let us establish (3.1). An argument which we used to deduce (3.7) and (3.8) yields

$$\begin{aligned}
\Bigl\|\sum_{j=-J}^{-1} \varphi_j(D)u\Bigr\|_\infty &\le J^{1-1/\sigma}\Bigl(\sum_{j=0}^{J}\|\varphi_j(D)u\|_\infty^\sigma\Bigr)^{1/\sigma} \\
&\le J^{1-1/\sigma}\|u\|_{\dot{B}^{0,\infty,\sigma}}.
\end{aligned} \tag{3.18}$$

By using Theorems 2.1 and 2.2 we have

$$\begin{aligned}
\|\psi_{-J-1}(D)f\|_\infty &= \|\psi_{-J-1}(D)\psi(D)f\|_\infty \\
&\le C\,2^{-J(s-\alpha)}\|u\|_{\mathcal{N}^s_{n/(s-\alpha),q,\infty}}.
\end{aligned} \tag{3.19}$$

(3.18) and (3.19) yield

$$\|\psi(D)f\|_\infty \le C\,2^{-J(s-\alpha)}\|u\|_{\mathcal{N}^s_{n/(s-\alpha),q,\infty}}. \tag{3.20}$$

Once (3.20) was established, going through the same technique used in (3.10)–(3.18), we obtain (3.1).

4. Sharpness of Theorem 1.1. Now we shall establish that (1.7) is sharp. Motivated by the example in [8], we shall construct a counterexample which is quite close to a function having a log-singularity. We shall distort the function $\log|x|$ to construct counterexamples whose Besov norm and L^∞ norm are easy to calculate, while in [8] it was necessary to quantify the log-singularity in order to obtain the inequality there.

PROPOSITION 4.1. *Let $0<\alpha<1$ and $1<\sigma\le\infty$. Then for any $C>0$, we can find $\kappa\in\mathcal{S}$ such that*

$$\|\kappa\|_\infty > C\|\kappa\|_{\dot{B}^{0,\infty,\sigma}} \log^{1-1/\sigma}\biggl(2+\frac{\|\kappa\|_{\dot{\mathcal{C}}^\alpha}}{\|\kappa\|_{\dot{B}^{0,\infty,\sigma}}}\biggr). \tag{4.1}$$

Proof. To calculate the Besov norms, we shall specify the Littlewood–Paley decomposition more precisely and quantitatively. Let us take $\kappa\in$

$C^\infty(\mathbb{R})$ so that $\chi_{(-\infty,1)} \le \kappa \le \chi_{(-\infty,11/10)}$. We define $\varphi_j(x) = \kappa(2^{-j}|x|) - \kappa(2^{-j+1}|x|)$ and $\psi_j(x) = \kappa(2^{-j}|x|)$ for $j \in \mathbb{Z}$. Note that

$$\chi_{B_{2^{-j}} \setminus B_{2^{-j}\cdot 11/20}} \le \varphi_j \le \chi_{B_{2^{-j}\cdot 11/10} \setminus B_{2^{-j}\cdot 1/2}}. \tag{4.2}$$

Recall that the Besov norm $\|f\|_{\dot{B}^{0,\infty,\sigma}}$ is given by

$$\|f\|_{\dot{B}^{0,\infty,\sigma}} = \Big(\sum_{j=-\infty}^{\infty} \|\varphi_j(D)f\|_\infty^\sigma \Big)^{1/\sigma}. \tag{4.3}$$

Take an auxiliary radial function $\tau \in \mathcal{S}$ so that

$$\chi_{B_{2^{-j}\cdot 4/5} \setminus B_{2^{-j}\cdot 7/10}} \le \tau \le \chi_{B_{2^{-j}\cdot 9/10} \setminus B_{2^{-j}\cdot 3/5}}. \tag{4.4}$$

We set $\tau_j(\xi) \equiv \tau(2^{-j}\xi)$ for $j \in \mathbb{N}$. We define

$$\kappa_J(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Big(\sum_{j=-J}^{-1} \tau(2^{-j}\xi) \Big) \exp(i\xi \cdot x) \, \frac{d\xi}{|\xi|^n}. \tag{4.5}$$

Then we have

$$\begin{aligned} \|\kappa_J\|_\infty = \kappa_J(0) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Big(\sum_{j=-J}^{-1} \tau(2^{-j}\xi) \Big) \frac{d\xi}{|\xi|^n} \\ &= (2\pi)^{-n/2} J \int_{\mathbb{R}^n} \tau(\xi) \, \frac{d\xi}{|\xi|^n} \end{aligned} \tag{4.6}$$

from (4.5).

To estimate the $\mathcal{C}^\alpha$ norm, we use the following: From (4.5) we have

$$\partial^\beta \kappa_J(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Big(\sum_{j=-J}^{-1} \tau(2^{-j}\xi) \Big) (i\xi)^\beta \exp(i\xi \cdot x) \, \frac{d\xi}{|\xi|^n}$$

for all $\beta \in (\mathbb{N} \cup \{0\})^n \setminus \{(0,\dots,0)\}$, which implies that there exists a constant C independent of J such that

$$\|\partial^\beta \kappa_J\|_\infty \le C < \infty. \tag{4.7}$$

Set

$$\mathcal{B}^M = \{ f \in C^M : \partial^\alpha f \in L^\infty \text{ for all } |\alpha| \le M \},$$

which is normed by

$$\|f\|_{\mathcal{B}^M} = \sum_{|\beta| \le M} \|\partial^\beta f\|_\infty.$$

From (4.6) and (4.7) we have

$$\|\kappa_J\|_{\mathcal{B}^M} = \sum_{|\alpha| \le M} \|\partial^\alpha \kappa_J\|_\infty \le CJ. \tag{4.8}$$

Since $\mathcal{B}^{\lfloor\alpha\rfloor+1} \hookrightarrow \mathcal{C}^\alpha$, (4.8) yields

$$\|\kappa_J\|_{\mathcal{C}^\alpha} \le C\|\kappa_J\|_{\mathcal{B}^M} \le CJ. \tag{4.9}$$

Finally, we shall estimate the Besov norm $\|\kappa_J\|_{\dot{B}^{0,\infty,\sigma}}$. With the norm specified by (4.2), we conclude from (4.3) that

$$\|\kappa_J\|_{\dot{B}^{0,\infty,\sigma}} = C\Big(\sum_{j=-J}^{-1}\Big\|\mathcal{F}^{-1}\Big[\frac{\tau_j}{|\xi|^n}\Big]\Big\|_\infty^\sigma\Big)^{1/\sigma} = C\Big\|\mathcal{F}^{-1}\Big[\frac{\tau}{|\xi|^n}\Big]\Big\|_\infty J^{1/\sigma}. \tag{4.10}$$

It is easy to see that

$$\lim_{J\to\infty}\frac{J}{J^{1/\sigma}\log^{1-1/\sigma}J} = \lim_{J\to\infty}\Big(\frac{J}{\log J}\Big)^{1-1/\sigma} = \infty, \tag{4.11}$$

provided $1 < \sigma \le \infty$. If $J \gg 1$, we see that (4.1) holds from (4.6) and (4.9)–(4.11). ∎

Proposition 4.2. *Let $s > \alpha > 0$, $0 < q \le \infty$ and $1 < \sigma \le \infty$. Assume that $u < 1 - 1/\sigma$. Then for any $C > 0$, we can find $\mu \in \mathcal{S}$ such that*

$$\|\mu\|_\infty > C\|\mu\|_{\dot{B}^{0,\infty,\sigma}}\log^u\Big(2 + \frac{\|\mu\|_{B^{s,n/(s-\alpha),q}}}{\|\mu\|_{\dot{B}^{0,\infty,\sigma}}}\Big). \tag{4.12}$$

Proof. Maintain the same notation as in the proof of Proposition 4.1. Set

$$\mu_J = \sum_{j=1}^{J}\mu_j. \tag{4.13}$$

In analogy with (4.6) and (4.10), we can deduce

$$\|\mu_J\|_\infty = (2\pi)^{-n/2}J\int_{\mathbb{R}^n}\tau(\xi)\,\frac{d\xi}{|\xi|^n}, \tag{4.14}$$

$$\|\mu_J\|_{\dot{B}^{0,\infty,\sigma}} = \|\mathcal{F}^{-1}[\tau_0]\|_\infty J^{1/\sigma}. \tag{4.15}$$

What is different from the proof of Proposition 4.1 is the estimate of the Besov norm $\|\mu_J\|_{B^{s,n/(s-\alpha),q}}$. However, a similar strategy works. Indeed, we can calculate with ease

$$\begin{aligned}\|\mu_J\|_{B^{s,n/(s-\alpha),q}} &= C\Big(\sum_{j=1}^{J}2^{jsq}\|\mathcal{F}^{-1}[\tau_0](2^j\cdot)\|_{n/(s-\alpha)}^q\Big)^{1/q}\\ &= C\Big(\sum_{j=1}^{J}2^{j\alpha q}\Big)^{1/q} \le C\,2^{J\alpha}.\end{aligned} \tag{4.16}$$

Since $u < 1 - 1/\sigma$, we see that (4.12) holds with $\mu = \mu_J$, provided $J \gg 1$. ∎

Proposition 4.3. *Let $s > \alpha > 0$, $0 < q \le \infty$ and $1 < \sigma \le \infty$. Then for any $C > 0$, we can find $\zeta \in \mathcal{S}$ such that*

$$\|\zeta\|_\infty > C\|\zeta\|_{\dot{B}^{0,\infty,\sigma}} \log^{1-1/\sigma}\left(2 + \frac{\|\zeta\|_{\dot{\mathcal{N}}^s_{n/(s-\alpha)\,qr}}}{\|\zeta\|_{\dot{B}^{0,\infty,\sigma}}}\right). \tag{4.17}$$

Proof. Let $C > 0$ be fixed. First, we shall prove

$$\|\zeta\|_\infty > C\|\zeta\|_{\dot{B}^{0,\infty,\sigma}} \tag{4.18}$$

for some $\zeta \in \mathcal{S}$, which is well-known. For convenience, we supply a short proof. Use the notation of Proposition 4.1. Set $\zeta_M = \sum_{j=1}^M j^{-2/(1+\sigma)}\mathcal{F}^{-1}[\tau_j]$. Then $\zeta_M(0) \to \infty$, while $\sup_{M\in\mathbb{N}} \|\zeta_M\|_{\dot{B}^{0,\infty,\sigma}} < \infty$. Hence, we obtain (4.18).

With this in mind we NOW prove Proposition 4.3. Given $\zeta \in \mathcal{S}$, we have

$$\|\zeta_j\|_\infty = \|\zeta\|_\infty, \quad \|\zeta_j\|_{\dot{B}^{0,\infty,\sigma}} = \|\zeta\|_{\dot{B}^{0,\infty,\sigma}},$$
$$\|\zeta_j\|_{\dot{\mathcal{N}}^s_{n/(s-\alpha)qr}} = 2^{-j\alpha}\|\zeta\|_{\dot{\mathcal{N}}^s_{n/(s-\alpha)qr}},$$

where we have set $\zeta_j = \zeta(2^{-j}\cdot)$. Therefore, if we choose $j \gg 1$ so that

$$\frac{\|\zeta_j\|_{\dot{\mathcal{N}}^s_{n/(s-\alpha)\,qr}}}{\|\zeta\|_{\dot{B}^{0,\infty,\sigma}}} < e - 2,$$

then with the help of (4.18), we obtain (4.17). ∎

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