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On the isotropic constant of marginals

by

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Abstract. We show that if $\mu_1, \dots, \mu_m$ are log-concave subgaussian or supergaussian probability measures in $\mathbb{R}^{n_i}$, $i \leq m$, then for every F in the Grassmannian $G_{N,n}$, where $N = n_1 + \dots + n_m$ and $n < N$, the isotropic constant of the marginal of the product of these measures, $\pi_F(\mu_1 \otimes \dots \otimes \mu_m)$, is bounded. This extends known results on bounds of the isotropic constant to a larger class of measures.

1. Introduction. A famous open problem in convex geometry is the hyperplane conjecture (**HC**) asking if there exists a constant $c > 0$ such that for every $n \geq 1$ and any symmetric convex body K of volume 1 in $\mathbb{R}^n$ there exists $\theta \in S^{n-1}$ such that

$$|K \cap \theta^{\perp}| \geq c. \tag{1.1}$$

The question was posed in this form by J. Bourgain in [Bou1]. A classical reference on the subject is the paper of V. D. Milman and A. Pajor [MP] (see also [G]). In this paper we will consider an equivalent formulation of the hyperplane conjecture, given by K. Ball [Ba1]. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ (i.e. the density f_μ of μ is of the form $f_\mu(x) = e^{-V(x)}$, where $V : \mathbb{R}^n \to [0, \infty]$ is a convex function). Then the question is whether

$$L_\mu := f_\mu(0)^{1/n} \leq C, \tag{1.2}$$

where $C > 0$ is an absolute constant. The best known bound is due to B. Klartag [K2] who proved that $L_\mu \leq Cn^{1/4}$ (see also [Bou2] and [KM]).

The validity of (**HC**) has been verified in many cases (see e.g. the references in [P3]). In this work we intend to verify the conjecture for a large class of measures that contains (or more precisely is generated by) the class of subgaussian and supergaussian *log*-concave measures. We say that the

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measure μ in $\mathbb{R}^n$ is *subgaussian* (with constant b) if for every $\theta \in S^{n-1}$,

$$\mu\Big(\{x : |\langle x,\theta\rangle| \geq t \int_{\mathbb{R}^n} |\langle x,\theta\rangle|\, d\mu(x)\}\Big) \leq 2e^{-t^2/b^2}$$

for all $t \geq 1$, and *supergaussian* (with constant a) if for every $\theta \in S^{n-1}$,

$$\mu\Big(\{x : |\langle x,\theta\rangle| \geq t \int_{\mathbb{R}^n} |\langle x,\theta\rangle|\, d\mu(x)\}\Big) \geq 2e^{-a^2t^2}$$

for all $1 \leq t \leq \sqrt{n}/a$ (see also §4 for more information). There are several examples of measures that have either the one or the other property (see §4) and in both cases the (**HC**) has been verified ([Bou3], [DP], [KM] and [P3]).

However, it is quite easy to construct examples of measures that have subgaussian and supergaussian directions (e.g. the product of a subgaussian and a supergaussian measure). Moreover, it is not true that a direction will be either supergaussian or subgaussian. Actually, the results of [GPV1] show that if μ is a general log-concave measure in $\mathbb{R}^n$ then a "typical" marginal measure $\pi_F(\mu)$ of the measure on the subspace F of dimension $k \gg \sqrt{n}$ has directions that are neither supergaussian nor subgaussian.

Before we describe the class of measures that we will treat in this paper let us mention another famous conjecture—at first sight unrelated to the hyperplane conjecture—which was proposed by Kannan, Lovász and Simonovits [KLS]. We will use the abbreviation (**KLS**). In equivalent form, (**KLS**) asks if for any isotropic log-concave probability measure μ on $\mathbb{R}^n$ and any smooth function $g : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{var}_\mu(g) := \mathbb{E}|g - \mathbb{E}(g)|^2 \leq C\mathbb{E}\|\nabla g\|_2^2, \tag{1.3}$$

where $C > 0$ is an absolute constant (see [M] for other equivalent formulations of the question). Recently, Eldan and Klartag [EK] (see also [BN], also [Ba2]) showed that if (**KLS**) has a positive answer then (**HC**) is also true. More precisely, they showed that a weaker version of the (**KLS**) conjecture (the so-called variance conjecture) is sufficient. We refer to [GM] for the best known bound and more information related to the latter problem.

The validity of (**HC**) has been verified in many cases (see e.g. the references in [P3]); on the contrary, (**KLS**) has been established in some very special cases only (1-dimensional log-concave probability measures [Bo], and indicators of B_p^n [So]). However, it is known that if μ_1, μ_2 are two probability measures satisfying (1.3) with the same constant, then so does their product $\mu_1 \otimes \mu_2$ (see e.g. [L, p. 98]). Moreover, if μ satisfies (1.3) with some constant D, then any marginal $\pi_F(\mu)$ of μ also satisfies (1.3) with the same constant. So, combining these two operators one can construct a rich family of isotropic log-concave probability measures which satisfy (**KLS**).

The behavior of the isotropic constant with respect to the two operators described above is different. It is well known that the isotropic constant of the product of two measures is bounded by the maximum of the corresponding isotropic constants (see, for example, [G, Lemma 1.6.6]). It is not known if given an isotropic log-concave probability measure μ on $\mathbb{R}^N$ and a subspace $F \in G_{N,n}$ one has

$$L_{\pi_F(\mu)} \leq cL_\mu \tag{1.5}$$

for some universal constant $c > 0$. Actually, (1.5) is another equivalent formulation of (**HC**)—see §5 for the details.

Our main result states that the isotropic constant is stable under these two operators if we start with the class of supergaussian and subgaussian measures:

THEOREM 1.1. *There exists an absolute constant $c > 0$ such that the following is true: Let $m \geq 1$, $a, b > 0$ and $\mu_1, \ldots, \mu_m$ be isotropic log-concave probability measures in $\mathbb{R}^{n_i}$ such that, for every $i \leq N$, μ_i is either supergaussian with constant a or subgaussian with constant b. Let $N := n_1 + \cdots + n_m$, $n < N$ and $F \in G_{N,n}$ be any n-dimensional subspace of $\mathbb{R}^N$. Then for any probability measure of the form $\mu := \pi_F(\bigotimes_{i=1}^m \mu_i)$ one has*

$$L_\mu \leq c \max\{a, b\}, \tag{1.6}$$

where $c > 0$ is an absolute constant.

The paper is organized as follows. In §2 we gather some background material. In §3 we investigate the properties of classes of log-concave measures that are close under the cartesian product and marginal operation (coherent classes of measures). In particular, we show that the largest isotropic constant of measures in such a class that is created by two different classes is bounded by the maximum isotropic constant of measures in these two classes (Proposition 3.5). In §4 we investigate the properties of the classes of supergaussian and subgaussian measures. In particular, we show that the isotropic constant of measures in the coherent class that contains the supergaussian measures is bounded (Theorem 4.6). Then we conclude the proof of the main theorem. We conclude in §5 with some applications of the main theorem and some final remarks.

2. Preliminaries

2.1. Basic notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $| \cdot |$. We write D_n for the Euclidean ball of volume 1 and σ for the rotationally invariant probability measure on S^{n-1}.

The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$. Let $1 \le k \le n$ and $F \in G_{n,k}$. We will denote by P_F the orthogonal projection from $\mathbb{R}^n$ onto F.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \le b \le c_2 a$. If $A \subseteq \mathbb{R}^n$ with $|A| > 0$, we write $\widetilde{A} := |A|^{-1/n} A$.

2.2. Probability measures. We denote by $\mathcal{P}_{[n]}$ the class of all probability measures in n-dimensional Euclidean spaces which are absolutely continuous with respect to the Lebesgue measure. We write $\mathcal{A}_n$ for the Borel σ-algebra in the corresponding n-dimensional Euclidean space. The density of $\mu \in \mathcal{P}_{[n]}$ is denoted by f_μ. We also write $\mathcal{P} := \bigcup_{n=1}^{\infty} \mathcal{P}_{[n]}$.

The subclass $\mathcal{SP}_{[n]}$ consists of all symmetric measures $\mu \in \mathcal{P}_{[n]}$; μ is called symmetric if f_μ is an even function on $\mathbb{R}^n$.

The subclass $\mathcal{CP}_{[n]}$ consists of all $\mu \in \mathcal{P}_{[n]}$ that have center of mass at the origin; so, $\mu \in \mathcal{CP}_{[n]}$ if

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle \, d\mu(x) = 0 \tag{2.1}$$

for all $\theta \in S^{n-1}$.

Let $\mu \in \mathcal{P}_{[n]}$. For every $1 \le k \le n-1$ and $F \in G_{n,k}$, we define the *F-marginal* $\pi_F(\mu)$ of μ as follows: for every $A \in \mathcal{A}_F$,

$$\pi_F(\mu)(A) := \mu(P_F^{-1}(A)). \tag{2.2}$$

It is clear that $\pi_F(\mu) \in \mathcal{P}_{[\dim F]}$. Note that, by the definition, for every Borel measurable function $f : \mathbb{R}^n \to [0, \infty)$ we have

$$\int_F f(x) \, d\pi_F(\mu)(x) = \int_{\mathbb{R}^n} f(P_F(x)) \, d\mu(x). \tag{2.3}$$

The density of $\pi_F(\mu)$ is the function

$$f_{\pi_F(\mu)}(x) = \pi_F(f_\mu)(x) = \int_{x+F^\perp} f_\mu(y) \, dy. \tag{2.4}$$

Let $\mu_1 \in \mathcal{P}_{[n_1]}$ and $\mu_2 \in \mathcal{P}_{[n_2]}$. We will write $\mu_1 \otimes \mu_2$ for the measure in $\mathcal{P}_{[n_1+n_2]}$ which satisfies

$$(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2) \tag{2.5}$$

for all $A_1 \in \mathcal{A}_{n_1}$ and $A_2 \in \mathcal{A}_{n_2}$. It is easily checked that $f_{\mu_1 \otimes \mu_2} = f_{\mu_1} f_{\mu_2}$.

Moreover, the marginal operator and the product operator "commute": Let $1 \le k_i < N_i$, $F_i \in G_{N_i,k_i}$ and $\mu_i \in \mathcal{P}_{N_i}$ for $i = 1, 2$. Then

$$\pi_{F_1}(\mu_1) \otimes \pi_{F_2}(\mu_2) = \pi_F(\mu_1 \otimes \mu_2), \tag{2.6}$$

where $F := F_1 \otimes F_2$.

Let $\mu \in \mathcal{P}_{[n]}$ and $\lambda > 0$. We define $\mu_{(\lambda)} \in \mathcal{P}_{[n]}$ as the measure that has density $f_{\mu_{(\lambda)}}(x) := \lambda^n f_\mu(\lambda x)$. Moreover, if $T \in \mathrm{SL}(n)$ we define $\mu \circ T \in \mathcal{P}_{[n]}$ as the measure with density $f_{\mu \circ T}(x) := f_\mu(T^{-1}x)$.

If $\mu_i \in \mathcal{P}$ we write $\mu_i \Rightarrow \mu$ for the weak convergence of μ_i to μ.

2.3. Log-concave measures. We denote by $\mathcal{L}_{[n]}$ the class of all log-concave probability measures on $\mathbb{R}^n$. A measure μ on $\mathbb{R}^n$ is called *log-concave* if for any compact sets A, B and any $\lambda \in (0,1)$,

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

A function $f : \mathbb{R}^n \to [0,\infty)$ is called *log-concave* if $\log f$ is concave.

It is known that if $\mu \in \mathcal{L}_{[n]}$ and $\mu(H) < 1$ for every hyperplane H, then $\mu \in \mathcal{P}_{[n]}$ and its density f_μ is log-concave (see [Bor]). As an application of the Prékopa inequality one can check that if f is log-concave then, for every $k \leq n-1$ and $F \in G_{n,k}$, $\pi_F(f)$ is also log-concave. As before, we write $\mathcal{CL}_{[n]}$ or $\mathcal{SL}_{[n]}$ for the classes of centered or symmetric non-degenerate $\mu \in \mathcal{L}_{[n]}$ respectively.

If $\mu_1, \mu_2 \in \mathcal{L}_{[n]}$ we define their convolution $\mu_1 * \mu_2$ as the measure with density $f_{\mu_1 * \mu_2}(x) := \int_{\mathbb{R}^n} f_{\mu_1}(y) f_{\mu_2}(x-y)\, dy$. It follows from the Prékopa inequality that $\mu_1 * \mu_2$ is well defined and belongs to $\mathcal{L}_{[n]}$. In the notation given above, one can check that

$$(\mu_1 * \mu_2)_{(\sqrt{2})} = \pi_F(\mu_1 \otimes \mu_2), \tag{2.7}$$

where $F := \{(x,y) \in \mathbb{R}^{2n} : x = y\}$.

2.4. Convex bodies. A *convex body* in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is *symmetric* if $x \in C$ implies that $-x \in C$. We say that C is *centered* if $\int_C \langle x, \theta\rangle\, dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y\rangle : y \in C\}$. Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn–Minkowski inequality implies that $\mathbf{1}_{\widetilde{K}} \in \mathcal{L}_{[n]}$.

We denote by $\mathcal{K}_{[n]}$ the class of convex bodies in $\mathbb{R}^n$ and by $\widetilde{\mathcal{K}}_{[n]}$ the subclass of bodies of volume 1. Also, $\mathcal{CK}_{[n]}$ is the class of centered convex bodies (bodies with center of mass at the origin) and $\mathcal{SK}_{[n]}$ is the class of origin symmetric convex bodies in $\mathbb{R}^n$.

2.5. L_q-centroid bodies. Let $\mu \in \mathcal{P}_{[n]}$. For every $q \geq 1$ and $\theta \in S^{n-1}$ we define

$$h_{Z_q(\mu)}(\theta) := \Big(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^q f_\mu(x)\, dx\Big)^{1/q},$$

where f_μ is the density of μ. If μ is log-concave then $h_{Z_q(\mu)}(\theta) < \infty$ for every $q \geq 1$ and every $\theta \in S^{n-1}$. We define the *L_q-centroid body* $Z_q(\mu)$ of μ to be the centrally symmetric convex set with support function $h_{Z_q(\mu)}$. One can check that for any $T \in \mathrm{SL}(n)$ and $\lambda > 0$,

$$Z_p((\mu \circ T)_{(\lambda)}) = \frac{1}{\lambda} T(Z_p(\mu)). \tag{2.8}$$

Note that (2.3) implies that

$$P_F(Z_p(\mu)) = Z_p(\pi_F(\mu)). \tag{2.9}$$

L_q-centroid bodies were introduced, with a different normalization, in [LZ] (see also [LYZ] where an L_q affine isoperimetric inequality was proved). Here we follow the normalization (and notation) that appeared in [P1]. The original definition concerned the class of measures μ_K where μ_K is the uniform measure on a body $K \in \widetilde{\mathcal{K}}_{[n]}$. In this case, we also write $Z_q(K)$ instead of $Z_q(\mu_K)$.

If K is a compact set in $\mathbb{R}^n$ and $|K| = 1$, it is easy to check that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all $1 \le p \le q \le \infty$, where $Z_\infty(K) = \mathrm{conv}(\{K, -K\})$. Note that if $T \in \mathrm{SL}(n)$ then $Z_p(T(K)) = T(Z_p(K))$. Moreover, it was proved in [P2, Theorem 4.4] that, for all $1 \le n < N$, $K \in \mathcal{CK}_{[N]}$ and $F \in G_{N,n}$,

$$|P_F(Z_n(K))|^{1/n} |K \cap F^\perp|^{1/n} \simeq 1. \tag{2.10}$$

For additional information on L_q-centroid bodies, we refer to [P1] and [P2].

2.6. Isotropic probability measures. Let μ be a centered measure in $\mathcal{P}_{[n]}$. We say that μ is *isotropic* if $Z_2(\mu) = B_2^n$. Note that if $\mu \in \mathcal{CL}_{[n]}$, then there exist $T \in \mathrm{SL}(n)$ and $\lambda > 0$ such that $(\mu \circ T)_{(\lambda)}$ is isotropic. We write μ_{iso} for an "isotropic image" of μ. Note that μ_{iso} is unique up to orthogonal transformations. If $\mu \in \mathcal{CL}_{[n]}$ then we define the *isotropic constant* of μ by $L_\mu := f_{\mu_{\mathrm{iso}}}(0)^{1/n}$. We denote by $\mathcal{IL}$ the class of isotropic log-concave measures.

It is known (see [P2, Proposition 3.7]) that if $\mu \in \mathcal{IL}_{[n]}$, then

$$1/L_\mu \simeq |Z_n(\mu)|^{1/n}. \tag{2.11}$$

A centered convex body K is called *isotropic* if $Z_2(K)$ is a multiple of the Euclidean ball. We define the *isotropic constant* of K in $\mathcal{CK}_{[n]}$ of volume 1 by

$$L_K := \left(\frac{|Z_2(K)|}{|B_2^n|} \right)^{1/n}. \tag{2.12}$$

So, K in $\mathcal{CK}_{[n]}$ of volume 1 is isotropic if and only if $Z_2(K) = L_K B_2^n$. Let $K \in \mathcal{CK}_{[n]}$ and $a > 0$. We write $\mu_{K,a} := a^n \mathbf{1}_{K/a}$. Note that K is isotropic if and only if $\mu_{K,L_K} = L_K^n \mathbf{1}_{K/L_K}$ is isotropic as a measure.

We refer to [MP], [G] for additional information on isotropic convex bodies and to the books [S], [MS] and [Pi] for basic facts from the Brunn–Minkowski theory and the asymptotic theory of finite-dimensional normed spaces.

3. Coherent classes of measures. We start with the definition of coherent classes of measures (see [DP]). Recall that $\mathcal{P} := \bigcup_{n=1}^{\infty} \mathcal{P}_{[n]}$.

Definition 3.1. Let $\mathcal{C} \subseteq \mathcal{P}$ be a class of probability measures. Then $\mathcal{C}$ is called *coherent* if

1. For all n_1, n_2 and $\mu_1 \in \mathcal{C}_{[n_1]}$, $\mu_2 \in \mathcal{C}_{[n_2]}$ one has $\mu_1 \otimes \mu_2 \in \mathcal{C}_{[n_1+n_2]}$.
2. For all n, $1 \le k \le n-1$, $F \in G_{n,k}$ and $\mu \in \mathcal{C}_{[n]}$ one has $\pi_F(\mu) \in \mathcal{C}_{[k]}$.
3. If $\mu_i \in \mathcal{C}_{[n]}$, $i = 1, 2, \ldots$, and $\mu_i \Rightarrow \mu$, then $\mu \in \mathcal{C}$.

We will say that $\mathcal{C}$ is τ-*coherent* if condition 3 is replaced with the following:

4. If $\mu \in \mathcal{C}$, $\lambda > 0$ and $T \in \mathrm{SL}(n)$, then $(\mu \circ T)_{(\lambda)} \in \mathcal{C}$.

We also agree that the null class is coherent. Note that if $\mathcal{U}_1$ and $\mathcal{U}_2$ are τ-coherent then $\mathcal{U}_1 \cap \mathcal{U}_2$ is also τ-coherent. Denote $\mathcal{C}_{[n]} := \mathcal{C} \cap \mathcal{P}_{[n]}$. Observe that, by definition, a coherent class is stable under isometric image. Known results show that the classes $S\mathcal{P}$, $C\mathcal{P}$ and $\mathcal{L}$ are τ-coherent. Also, $\mathcal{I}$ is coherent (see [DP]).

Let $\mathcal{A} \subseteq \mathcal{P}$ be a family of probability measures. We define

$$\overline{\mathcal{A}} := \bigcap \{\mathcal{U} \subseteq \mathcal{P} : \mathcal{U} \text{ coherent and } \mathcal{A} \subseteq \mathcal{U}\}. \tag{3.1}$$

It is clear that if $\mathcal{A}_1 \subseteq \mathcal{A}_2$ then $\overline{\mathcal{A}_1} \subseteq \overline{\mathcal{A}_2}$.

Note that the class $\mathcal{K} := \bigcup_{n=1}^{\infty} \{\mu \in \mathcal{P}_{[n]} : \mu = \mathbf{1}_{\widetilde{K}},\, K \in \mathcal{K}_{[n]}\}$ is *not* coherent since a marginal of μ_K is not (in general) the uniform measure of a convex body.

Let $\mathcal{A} \subseteq \mathcal{CL}$. Then we define

$$L_{\mathcal{A}} := \sup\{L_\mu : \mu \in \mathcal{A}\}. \tag{3.2}$$

We will need the following fact (which is a corollary of [W, Proposition 2.11]).

Proposition 3.2. *Let $\mathcal{A} \subseteq \mathcal{IL}$ be a family of probability measures and set $\mathcal{C} := \overline{\mathcal{A}}$. Then, for every $n \ge 1$, for every $\mu \in \mathcal{C}_{[n]}$ and $\varepsilon > 0$, there exist $k \in \mathbb{N}$, $\mu_i \in \mathcal{A}_{n_i}$, $i \le k$, with $\sum_{i=1}^{k} n_i = N$ and $F \in G_{N,n}$ such that*

$$|L_\mu - L_\nu| \le \varepsilon, \quad \textit{where} \quad \nu := \pi_F\Big(\bigotimes_{i=1}^{k} \mu_i\Big). \tag{3.3}$$

Proof. Let $\mathcal{U} \subseteq \mathcal{IL}$ be the smallest class which is closed under products and marginals and contains $\mathcal{A}$. Then it is proved in [W, Proposition 2.11] that $\mathcal{C}$ is the closure of $\mathcal{U}$ with respect to the Lévy metric. So, in order to finish the proof, it is enough to observe that L_μ is continuous with respect to the Lévy metric and use (2.6). ∎

The next proposition follows from the definition of a τ-coherent class and (2.6).

Proposition 3.3. *Let $\mathcal{C} \subseteq \mathcal{CL}$ be a τ-coherent class and let $\mu_1, \mu_2 \in \mathcal{C}_{[n]}$. Then $\mu_1 * \mu_2 \in \mathcal{C}_{[n]}$.*

The next proposition allows us to work only with symmetric isotropic log-concave measures. The proof follows an argument of B. Klartag [K1].

Proposition 3.4. *Let $\mathcal{C} \subseteq \mathcal{CL}$ be a τ-coherent class of measures and set $\mathcal{SC} := \mathcal{SL} \cap \mathcal{C}$. Then*

$$L_{\mathcal{C}} \le e\sqrt{2}\, L_{\mathcal{SC}}.$$

Proof. Let $\mu \in \mathcal{C}_{[n]}$ and let $\bar{\mu}$ the measure with density $f_{\bar{\mu}}(x) = f_\mu(-x)$. Since $\mathcal{C}$ is τ-coherent, $\bar{\mu} \in \mathcal{C}$. Also, by Proposition 3.3, $\mu^s := \mu * \bar{\mu} \in \mathcal{C}$ and it is straightforward to check that μ^s is also symmetric. Note that $L_\mu = L_{\bar{\mu}}$. In order to finish the proof it is enough to show that for all $\mu_1, \mu_2 \in \mathcal{CL}$,

$$L_{\mu_1 * \mu_2} \le e\sqrt{2} \min\{L_{\mu_1}, L_{\mu_2}\}. \tag{3.4}$$

We may assume that μ_1, μ_2 are isotropic. Then one can check that

$$(\mu_1 * \mu_2)_{(\sqrt{2})} = (\mu_1)_{(\sqrt{2})} * (\mu_1)_{(\sqrt{2})}$$

is also isotropic. So,

$$\begin{aligned} L^n_{\mu_1*\mu_2} &:= f_{(\mu_1*\mu_2)_{(\sqrt{2})}}(0) = \int_{\mathbb{R}^n} (f_{\mu_1})_{(\sqrt{2})}(y)(f_{\mu_2})_{(\sqrt{2})}(-y)\, dy \\ &\le \|(f_{\mu_2})_{(\sqrt{2})}\|_\infty \le e^n (f_{\mu_2})_{(\sqrt{2})}(0) = (e\sqrt{2})^n f_{\mu_2}(0) \le (e\sqrt{2} L_{\mu_2})^n, \end{aligned}$$

where we have also used a theorem of M. Fradelizi [F] stating that, for any centered log-concave density f in $\mathbb{R}^n$, one has $\|f\|_\infty \le e^n f(0)$. We work in the same way to get $L^n_{\mu_1*\mu_2} \le (e\sqrt{2}\, L_{\mu_1})^n$. This proves (3.4). ∎

Let $\mu_i \in \mathcal{IL}_{[n_1]}$, $i \le m$, and $N := \sum_{i=1}^m n_i$. Then

$$L_{\mu_1 \otimes \cdots \otimes \mu_m} = \prod_{i=1}^m L_{\mu_i}^{n_i/N} \le \max\{L_{\mu_i} : i \le m\}. \tag{3.5}$$

Indeed, from the definition it follows that

$$L^N_{\mu_1 \otimes \cdots \otimes \mu_m} = f_{\mu_1 \otimes \cdots \otimes \mu_m}(0) = \prod_{i=1}^m f_{\mu_i}(0) = \prod_{i=1}^m L_{\mu_i}^{n_i}.$$

We will also need the following

Proposition 3.5. *Let $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{IL}$ be two coherent classes of measures. Then*

$$L_{\overline{\{\mathcal{A}_1, \mathcal{A}_2\}}} \le c \max\{L_{\mathcal{A}_1}, L_{\mathcal{A}_2}\}.$$

We need to introduce some additional notation before we give the proof. Let $\mu \in \mathcal{IL}_{[n]}$ and $p > -n$. Then we define

$$I_p(\mu) := \Big(\int\limits_{\mathbb{R}^n} \|x\|_2^p \, d\mu(x) \Big)^{1/p}.$$

Moreover, if $\delta > 1$, we define the quantity (see [DP])

$$q_{-c}(\mu, \delta) := \max\Big\{ p \geq 1 : I_{-p}(\mu) \geq \frac{1}{\delta}\sqrt{n} \Big\}.$$

We will use the following (see [DP, Proposition 5.1 and the proof of Theorem 6.1]):

Proposition 3.6. *Let $\mathcal{A} \subseteq \mathcal{IL}$ be a coherent class of measures. Then for any $n \geq 1$ there exists $\mu \in \mathcal{A}_{[n]}$ such that for any $\delta > 1$,*

1. $L_\mu \geq c_1 L_{\mathcal{A}_{[n]}}$,
2. $L_\mu \leq c_2 \delta \sqrt{\frac{n}{q_{-c}(\mu, \delta)} \log e \frac{n}{q_{-c}(\mu, \delta)}}$,

where $c_1, c_2 > 0$ are absolute constants.

We also need the following (see [P2, Proposition 4.8]):

Proposition 3.7. *Let $\mu \in \mathcal{IL}_{[n]}$. Then for every $p \geq -n/2$,*

$$I_p(\mu) \geq \frac{c\sqrt{n}}{L_\mu},$$

where $c > 0$ is an absolute constant.

Proof of Proposition 3.5. Let $L := \max\{L_{\mathcal{A}_1}, L_{\mathcal{A}_2}\}$. We assume that $L < \infty$, or else we have nothing to prove. Let $n \geq 1$ and let

$$\mu \in \overline{\{\mathcal{A}_1, \mathcal{A}_2\}}_{[n]} =: \mathcal{A}_{[n]}$$

be as in Proposition 3.6. As in Proposition 3.2 we may assume that there exist $N > n$, $F \in G_{N,n}$ and measures $\mu^{(1)}, \dots, \mu^{(k)} \in \mathcal{A}_1$, $\mu^{(k+1)}, \dots, \mu^{(m)} \in \mathcal{A}_2$ such that

$$\mu = \pi_F(\mu^{(1)} \otimes \cdots \otimes \mu^{(k)} \otimes \mu^{(k+1)} \otimes \cdots \otimes \mu^{(m)}). \tag{3.6}$$

However, since $\mathcal{A}_1, \mathcal{A}_2$ are coherent, we see that $\mu_1 := \mu^{(1)} \otimes \cdots \otimes \mu^{(k)} \in \mathcal{A}_1$ and $\mu_2 := \mu^{(k+1)} \otimes \cdots \otimes \mu^{(m)} \in \mathcal{A}_2$. So (3.6) becomes

$$\mu = \pi_F(\mu_1 \otimes \mu_2). \tag{3.7}$$

We may now assume that μ_1 is N_1-dimensional and μ_2 is N_2-dimensional. Let $F_1 := P_F \mathbb{R}^{N_1}$ and $F_2 := P_F \mathbb{R}^{N_2}$ and $n_1 := \dim F_1$ and $n_2 := \dim F_2$. Since $\operatorname{span}\{F_1, F_2\} \supseteq F$ we deduce that $n \leq n_1 + n_2$. In particular at least one of n_1, n_2 is greater than $n/2$. Assume that $n_1 \geq n/2$. Then

$$\begin{aligned}
I_{-n/4}^{-n/4}(\mu) &= \int_F \frac{1}{\|x\|_2^{n/4}}\, d\pi_F(\mu_1 \otimes \mu_2)(x) \\
&= \int_{\mathbb{R}^{N_1}} \int_{\mathbb{R}^{N_2}} \frac{1}{\|P_F(x_1,x_2)\|_2^{n/4}}\, d\mu_2(x_2)\, d\mu_1(x_1) \\
&= \int_{\mathbb{R}^{N_1}} \int_{\mathbb{R}^{N_2}} \frac{1}{(\|P_{F_1}x_1\|_2^2 + \|P_{F_2}x_2\|_2^2)^{n/8}}\, d\mu_2(x_2)\, d\mu_1(x_1) \\
&\leq \int_{\mathbb{R}^{N_1}} \int_{\mathbb{R}^{N_2}} \frac{1}{\|P_{F_1}x_1\|_2^{n/4}}\, d\mu_2(x_2)\, d\mu_1(x_1) = \int_{\mathbb{R}^{N_1}} \frac{1}{\|P_{F_1}x_1\|_2^{n/4}}\, d\mu_1(x_1) \\
&= I_{-n/4}^{-n/4}(\pi_{F_1}(\mu_1)) \leq \left(\frac{L_{\pi_{F_1}(\mu_1)}}{c\sqrt{n_1}}\right)^{n/4} \leq \left(\frac{L_{\mathcal{A}_1}}{c'\sqrt{n}}\right)^{n/4},
\end{aligned}$$

where we have also used Proposition 3.7. Working similarly in the case when $n_2 \geq n/2$ we conclude that

$$I_{-n/4}(\mu) \geq c \min\left\{\frac{\sqrt{n}}{L_{\mathcal{A}_1}}, \frac{\sqrt{n}}{L_{\mathcal{A}_2}}\right\} \geq \frac{\sqrt{n}}{c'L}.$$

In other words,

$$q_{-c}(\mu, cL) \geq \frac{n}{4}. \tag{3.8}$$

Then by Proposition 3.6 we get

$$L_{\mathcal{A}} \leq cL_\mu \leq c'L. \ \blacksquare$$

4. Supergaussian and subgaussian measures. Let $\mu \in \mathcal{IL}_{[n]}$ and $\theta \in S^{n-1}$. The *subgaussian constant of μ in the direction of θ* is defined by

$$\widetilde{\psi}_{2,\mu}(\theta) := \sup_{\lambda>0} \frac{1}{\lambda}\Big(\log \int_{\mathbb{R}^n} e^{\lambda\langle x,\theta\rangle}\, d\mu(x)\Big)^{1/2}. \tag{4.1}$$

We define the *subgaussian constant* of μ by

$$\beta_{2,\mu} := \sup_{\theta\in S^{n-1}} \widetilde{\psi}_{2,\mu}(\theta). \tag{4.2}$$

The usual definition of the subgaussian constant is different:

$$\psi_{2,\mu}(\theta) := \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} e^{|\langle x,\theta\rangle|^2/\lambda^2}\, d\mu(x) \leq 2\Big\}. \tag{4.3}$$

Our modification is justified by the next proposition (see [DP, Propositions 4.5 and 4.9]).

PROPOSITION 4.1. *Let $\mu \in \mathcal{SIL}_{[n]}$. Then, for every $\theta \in S^{n-1}$,*

$$\psi_{2,\mu}(\theta) \simeq \widetilde{\psi}_{2,\mu}(\theta). \tag{4.4}$$

Moreover, if for some $b > 0$ and for all $n \geq 1$ we define

$$\mathcal{SBG}(b)_{[n]} := \{\mu \in \mathcal{IL}_{[n]} : \beta_{2,\mu} \leq b\} \quad \text{and} \quad \mathcal{SBG}(b) := \bigcup_{n=1}^{\infty} \mathcal{SBG}(b)_{[n]}, \tag{4.5}$$

then $\mathcal{SBG}(b)$ is a coherent class.

Let γ_n be the standard Gaussian distribution. Then there exists a universal constant c_γ such that $\gamma_n \in \mathcal{SBG}(c_\gamma)_{[n]}$.

The fact that if μ is subgaussian with constant b then L_μ is bounded by a constant $c(b)$ depending only on b was first established by J. Bourgain in [Bou3]. His estimate $c(b) \leq cb \log b$ has been slightly improved in [DP]. The best known estimate is due to B. Klartag and E. Milman [KM]:

THEOREM 4.2. *There exists $c > 0$ such that for any $b \geq c_\gamma$,*

$$L_{\mathcal{SBG}(b)} \leq cb. \tag{4.6}$$

The assumption that $b \geq c_\gamma$ is only to guarantee that the class $\mathcal{SBG}(b)$ is not empty. We will need the following consequence of Theorem 4.2.

PROPOSITION 4.3. *Let K be an isotropic convex body in $\mathbb{R}^N$ which is subgaussian with constant b. Then, for any $F \in G_{N,n}$,*

$$|K \cap F^\perp|^{1/n} \leq cb, \tag{4.7}$$

where $c > 0$ is an absolute constant.

Proof. Note that μ_{K,L_K} is isotropic (as a measure) and $\pi_F(\mu_{K,L_K})$ is also subgaussian with constant b. So, using Theorem 4.2, we have

$$(cb)^n \geq \pi_{\mu_F(K,L_K)}(0) = L_K^N \left| \frac{K}{L_K} \cap F^\perp \right| = L_K^n |K \cap F^\perp| \geq c_0^n |K \cap F^\perp|. \ \blacksquare$$

The uniform measure on the Euclidean ball and the Gaussian measure are clearly in the class $\mathcal{SBG}$. Moreover, the uniform measure on the unit cube is also subgaussian. In general for $p \in [1, \infty]$ we write $\mu_{p,n}$ for the uniform measure in $B_p^n := \{x \in \mathbb{R}^n : |x_1|^p + \cdots + |x_n|^p \leq 1\}$. For $1 \leq p \leq q \leq \infty$ we write

$$\mathcal{L}_{[p,q]}^{(n)} := \{\mu_{r,n} : p \leq r \leq q\} \quad \text{and} \quad \mathcal{L}_{[p,q]} := \overline{\bigcup_{n=1}^{\infty} \mathcal{L}_{[p,q]}^{(n)}}.$$

F. Barthe and A. Koldobsky [BK, §6.2] (see also [BGMN]) proved that there exists $c > 0$ such that for every $n \geq 1$ and $p \in [2, \infty]$, $\mu_{p,n} \in \mathcal{SBG}(c)$. The next result follows from their result and Proposition 4.1.

THEOREM 4.4. *There exists $c > 0$ such that*

$$\mathcal{L}_{[2,\infty]} \subseteq \mathcal{SBG}(c).$$

One can check that the previous result cannot be extended to the range $p \in [1,2)$. In particular $\mu_{1,n}$ is subgaussian only with a constant of order $\sqrt{n}$. But as we will see, the measures $\mu_{p,n}$ for $p \in [1,2]$ are supergaussian.

Let $\mu \in \mathcal{IL}_{[n]}$. We say that μ is *supergaussian with constant* $a \geq L_{D_1} = 1/(2\sqrt{3})$ if, for all $p \geq 1$,

$$Z_p(\mu) \supseteq Z_p(\mathbf{1}_{D_n,a}). \tag{4.8}$$

An equivalent way to describe (4.8) is to say that, for every $1 \leq p \leq n$ and $\theta \in S^{n-1}$,

$$h_{Z_p(\mu)}(\theta) \geq \frac{\sqrt{p}}{ca}.$$

It is not difficult to show the following (see [P3, Proposition 5.1]).

PROPOSITION 4.5. *Let $\mu \in \mathcal{IL}$ be supergaussian with constant $a \geq 1/(2\sqrt{3})$. Then*

$$L_\mu \leq ca,$$

where $c > 0$ is an absolute constant.

The measure $\mu_{\infty,n}$ is supergaussian only with a constant of order $\sqrt{n}$. It is not difficult to see that any log-concave measure is supergaussian with a constant of order $\sqrt{n}$ (see [P3] Proposition 3.2).

It follows from the definition and (2.8) that if μ is supergaussian with constant a, then $\pi_F(\mu)$ is also supergaussian with constant a. However, the class of supergaussian measures (with constant less than a) is not a coherent class, because the product of two supergaussian measures fails (in general) to be supergaussian. (Consider the example of the uniform measure in the cube.) Assuming $a \geq L_{D_1}$ we see that the class of supergaussian measures with constant less than a is non-empty. So, for $a \geq L_{D_1}$ we define

$$\mathcal{SPG}(a)_{[n]} := \overline{\{\mu \in \mathcal{IL}_{[n]} : \mu \text{ is supergaussian with constant } a\}}$$

$$\mathcal{SPG}(a) := \bigcup_{n=1}^{\infty} \mathcal{SPG}(a)_{[n]}.$$

Let us emphasize that the class $\mathcal{SPG}$ contains probability measures that are not necessarily "supergaussian".

We will prove the following.

THEOREM 4.6. *There exists $c > 0$ such that for all $a \geq L_{D_1}$,*

$$L_{\mathcal{SPG}(a)} \leq ca.$$

Proof. We write $\mu_{a,n} := \mathbf{1}_{D_n,a}$. Let $\mu \in \mathcal{IL}_{[n]}$ be symmetric and supergaussian with constant a. Then, for every $t \in \mathbb{R}$, for every even integer $p \geq 2$ and for all $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |\langle x, y\rangle + t|^p \, d\mu(x) \geq \int_{\mathbb{R}^n} |\langle x, y\rangle + t|^p \, d\mu_{a,n}(x). \tag{4.9}$$

Indeed, since $\mu, \mu_{a,n}$ are symmetric,

$$\begin{aligned}\int\limits_{\mathbb{R}^n} |\langle x, y\rangle + t|^p \, d\mu(x) &= \sum_{i=0}^{p} \binom{p}{i} \int\limits_{\mathbb{R}^n} t^i \langle x, y\rangle^{p-i} \, d\mu(x) \\ &= \sum_{i=0,\, i \,\text{even}}^{p} \binom{p}{i} \int\limits_{\mathbb{R}^n} t^i \langle x, y\rangle^{p-i} \, d\mu(x) \\ &= \sum_{k=0}^{p/2} \binom{p}{2k} |t|^{2k} \int\limits_{\mathbb{R}^n} |\langle x, y\rangle|^{p-2k} \, d\mu(x) \\ &\geq \sum_{k=0}^{p/2} \binom{p}{2k} |t|^{2k} \int\limits_{\mathbb{R}^n} |\langle x, y\rangle|^{p-2k} \, d\mu_{a,n}(x) \\ &= \int\limits_{\mathbb{R}^n} |\langle x, y\rangle + t|^p \, d\mu_{a,n}(x).\end{aligned}$$

Let $n_1, \ldots, n_k \in \mathbb{N}$ and let $N := \sum_{i=1}^k n_i$. Let $\mu_1 \in \mathcal{SIL}_{n_1}, \ldots, \mu_k \in \mathcal{SIL}_{n_k}$ be supergaussian measures with constant a. Let

$$\mu_N := \mu_1 \otimes \cdots \otimes \mu_k \quad \text{and} \quad \bar{\mu}_a := \mu_{n_1,a} \otimes \cdots \otimes \mu_{n_k,a}.$$

Then, for every even integer $p \geq 2$ and every $\bar{y} := (y_1, \ldots, y_k) \in \mathbb{R}^N$, applying k times (4.9) and using Fubini's theorem, we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^N} |\langle \bar{x}, \bar{y}\rangle|^p \, d\mu_N(\bar{x}) &= \int\limits_{\mathbb{R}^{n_1}} \cdots \int\limits_{\mathbb{R}^{n_k}} \Big|\sum_{i=1}^k \langle x_i, y_i\rangle\Big|^p \, d\mu_k(x_k) \cdots d\mu_1(x_1) \\ &\geq \int\limits_{\mathbb{R}^{n_1}} \cdots \int\limits_{\mathbb{R}^{n_k}} \Big|\sum_{i=1}^k \langle x_i, y_i\rangle\Big|^p \, d\mu_{n_k,a}(x_k) \cdots d\mu_{n_1,a}(x_1) \\ &= \int\limits_{\mathbb{R}^N} |\langle \bar{x}, \bar{y}\rangle|^p \, d\bar{\mu}_a(\bar{x}).\end{aligned}$$

So, we have shown that

$$Z_p(\mu_N) \supseteq Z_p(\bar{\mu}_a). \tag{4.10}$$

Let $n \geq 1$ and $F \in G_{N,n}$. Let $D := D_{n_1} \times \cdots \times D_{n_k}$. Then D is isotropic (in the convex body sense) and subgaussian with some absolute constant $c > 0$. So, by Proposition 4.3, we have

$$|D \cap F^{\perp}|^{1/n} \simeq 1. \tag{4.11}$$

Moreover, for any $p > 0$,

$$Z_p(\bar{\mu}_a) = \frac{1}{a} Z_p(D). \tag{4.12}$$

Now, using (2.9), (4.10) and (4.12), we get

$$Z_p(\pi_F(\mu_N)) = P_F(Z_p(\mu_N)) \supseteq P_F(Z_p(\bar{\mu}_a)) = \frac{1}{a} P_F(Z_p(D)). \tag{4.13}$$

So, for $p = n$, using (2.11), (4.13), (2.10) and (4.11) we see that

$$\frac{1}{L_{\pi_F(\mu_N)}} \simeq |Z_n(\pi_F(\mu_N))|^{1/n} \geq \frac{c}{a}|P_F(Z_n(D))|^{1/n} \geq \frac{c'}{a}\frac{1}{|D \cap F^{\perp}|^{1/n}} \geq \frac{c''}{a}.$$

In other words,

$$L_{\pi_F(\mu_N)} \leq c''' a. \tag{4.14}$$

The result follows from Propositions 3.2 and 3.4. ■

Theorems 4.2, 4.6 and Proposition 3.5 imply the following reformulation of Theorem 1.1.

THEOREM 4.7. *There exist $c, c_0 > 0$ such that, for any $a, b \geq c_0$,*

$$L_{\overline{\{\mathcal{SPG}(a), \mathcal{SBG}(b)\}}} \leq c \max\{a, b\}.$$

5. Final remarks. The class $\mathcal{SPG}$ is quite rich, as the following two propositions show.

PROPOSITION 5.1. *There exists $c > 0$ such that*

$$\mathcal{IL}_{[1,2]} \subseteq \mathcal{SPG}(c).$$

Proof. Given $a > 0$, let $\gamma_{n,1/a}$ be the centered Gaussian measure in $\mathbb{R}^n$ with variance $1/a$. Let $\mathcal{B}(a)_{[n]}$ the class of all measures satisfying $Z_p(\mu) \supseteq Z_p(\gamma_{n,1/a})$ for all $p \geq 1$. Then, if $\mu_i \in \mathcal{B}_{n_i}$, $1 \leq i \leq k$, and if $N := \sum_{i=1}^{k} n_i$, working as in the proof of (4.10) we obtain

$$Z_p\Big(\bigotimes_{i=1}^{k} \mu_i\Big) \supseteq Z_p(\gamma_{N,1/a}) \supseteq Z_p(\mathbf{1}_{D_n,ca}) \tag{5.1}$$

for all $p \geq 1$, where $c > 0$ is an absolute constant. The first inclusion combined with (2.8) shows that $\mathcal{B}(a) := \bigcup_{n=1}^{\infty} \mathcal{B}(a)_{[n]}$ is a coherent class, and the second inclusion implies that

$$\mathcal{B}(a) \subseteq \mathcal{SPG}(ca) \tag{5.2}$$

for some $c > 0$. Let $p \in [1, 2]$ and write $\mu_{p,n}$ for the isotropic log-concave probability measure with density $f_{\mu_{p,n}}(x) = a_{p,n} e^{-\|x\|_p^p}$. It is a straightforward computation to check that $\mu_{p,1} \in \mathcal{B}(c_1)$ for some $c_1 > 0$. So (5.1) implies that $\mu_{n,p} \in \mathcal{B}(c_1)$.

Let K be a symmetric convex body in $\mathbb{R}^n$, let $\|\cdot\|_K$ be the corresponding norm to K and let $r > 0$. We define a probability density $g_{K,r}$ on $\mathbb{R}^n$ by

$$g_{K,r}(x) := \frac{1}{|K|\Gamma(\frac{n+r}{r})} e^{-\|x\|_K^r}.$$

Then (see [GPV2, Lemma 4.3]), for any $q > 0$,

$$Z_q(g_{K,r}) = \left(\frac{\Gamma(\frac{n+q+r}{r})}{\Gamma(\frac{n+r}{r})}\right)^{1/q} Z_q(\widetilde{K}). \tag{5.3}$$

Since $g_{B_p^n,p} = \mu_{p,n}$, it is not hard to check that, for all $q \le n$,

$$Z_q(\widetilde{B}_p^n) \simeq Z_q(\mu_{p,n}). \tag{5.4}$$

This shows that, for all $q \le n$,

$$Z_q(\mu_{B,p,n}) \supseteq c' Z_q(\widetilde{B_p}) \supseteq c'' Z_q(\mu_{p,n}) \supseteq c''' \sqrt{q}\, B_2^n. \ \blacksquare$$

We have already mentioned that S. Sodin [So] proved that the measures $\mu_{p,n}$ for $p \in [1,\infty]$ (see also [SV]) satisfy **KLS** with a universal constant $c > 0$. So any measure of the form

$$\pi_F(\mu_{p_1,n_1} \otimes \cdots \otimes \mu_{p_m,n_m}), \tag{5.5}$$

where $p_1, \ldots, p_m \in [1,\infty]$, $n_1 + \cdots + n_m = N$ and $F \in G_{N,n}$, satisfies (**KLS**) with the same universal constant $c > 0$. Therefore, by a result of K. Ball and V. Nguyen [BN], a measure of the form (5.5) has bounded isotropic constant. As a by-product of our method we can can give an alternative proof of this fact. Indeed by Theorems 1.1, 4.4 and Proposition 5.1 we have:

Proposition 5.2. *There exists a universal constant $c > 0$ such that*

$$L_{\mathcal{L}_{[1,\infty]}} \le c.$$

It was mentioned in the introduction that the main difficulty we had to overcome in this work was that it is not known whether there exists an absolute constant $a > 0$ such that

$$L_{\pi_F(\mu)} \le a L_\mu \tag{5.6}$$

for all $\mu \in \mathcal{IL}_{[n]}$ and $F \in G_{n,k}$. In fact, (5.6) is just another equivalent formulation of (**HC**). Indeed, if (**HC**) is true then clearly (5.6) is also true. The other direction follows from the next proposition.

Proposition 5.3. *Let $\mathcal{C} \subseteq \mathcal{IL}$ be a non-empty coherent class. Assume that there exists $a > 0$ such that, for any $\mu \in \mathcal{C}$,* (5.6) *holds. Then*

$$L_{\mathcal{C}} \le a. \tag{5.7}$$

Proof. Let $\mu \in \mathcal{C}_{[n]}$ satisfy $L_\mu = L_{\mathcal{C}}$. Since $\mathcal{C}$ is non-empty, we have $\gamma_N \in \mathcal{C}$ for all $N \ge 1$. We define $\mu_1 := \mu \otimes \gamma_N$. Note that if $F = \mathbb{R}^n$ then $\pi_F(\mu_1) = \mu$. Moreover, if N is large enough we have

$$\begin{aligned} L_{\mu_1} &= f_{\mu_1}(0)^{1/(n+N)} = (f_\mu(0) f_{\gamma_N}(0))^{1/(n+N)} \\ &\le (\sqrt{n})^{n/(n+N)} \left(\frac{1}{\sqrt{2\pi}}\right)^{N/(n+N)} \le 1. \end{aligned} \tag{5.8}$$

Applying (5.6) for $\pi_F(\mu_1) = \mu$, we get (5.7). $\blacksquare$

As we have shown, the class $\overline{\{\mathcal{SBG}(c), \mathcal{SPG}(c)\}}$ is quite rich. A natural question is whether every log-concave measure lies in this class or at least is "close" to a measure of this class. Here we choose a distance on measures that is stable with respect to the notions we are working with (supergaussian and subgaussian constant) and such that the isotropic constant is stable with respect to this distance. Let $\mu_1, \mu_2 \in \mathcal{IL}_{[n]}$. We define

$$d(\mu_1, \mu_2) := \inf\left\{\gamma_1\gamma_2 : \frac{1}{\gamma_1} Z_p(\mu_1) \subseteq Z_p(\mu_2) \subseteq \gamma_2 Z_p(\mu_1),\ \forall p \in [1, n]\right\}.$$

Note that the isotropy assumption implies $\gamma_1, \gamma_2 \geq 1$. So, as is easy to check, it follows from (2.11) that if $\mu_1, \mu_2 \in \mathcal{IL}_{[n]}$ and $d(\mu_1, \mu_2) \leq \gamma$ then $1/(c\gamma) \leq L_{\mu_1}/L_{\mu_2} \leq c\gamma$, where $c > 0$ is an absolute constant.

We are not aware of an example of an isotropic log-concave measure that is not in the class $\overline{\{\mathcal{SBG}(c), \mathcal{SPG}(c)\}}$. So we would like to conclude with the following question. Note that an affirmative answer would also imply that the hyperplane conjecture is true.

Question. Is it true that there exists a constant $c > 0$ such that for every $n \geq 1$ and an isotropic log-concave measure μ in $\mathbb{R}^n$ there exists $\mu_1 \in \overline{\{\mathcal{SBG}(c), \mathcal{SPG}(c)\}}$ with $d(\mu, \mu_1) \leq c$?

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References

[Ba1] K. M. Ball, *Logarithmically concave functions and sections of convex sets in* $\mathbb{R}^n$, Studia Math. 88 (1988), 69–84.

[Ba2] K. M. Ball, *Convex geometry: the information-theoretic viewpoint*, Lectures at the Institut Henri Poincaré, Paris, 2006.

[BN] K. M. Ball and V. H. Nguyen, *Entropy jumps for isotropic log-concave random vectors and spectral gap*, Studia Math., to appear.

[BK] F. Barthe and A. Koldobsky, *Extremal slabs in the cube and the Laplace transform*, Adv. Math. 174 (2003), 89–114.

[BGMN] F. Barthe, O. Guédon, S. Mendelson and A. Naor, *A probabilistic approach to the geometry of the* ℓ_p^n*-ball*, Ann. Probab. 33 (2005), 480–513.

[BH] S. G. Bobkov and C. Houdré, *Isoperimetric constants for product probability measures*, Ann. Probab. 25 (1997), 184–205.

[Bo] S. Bobkov, *Spectral gap and concentration for some spherically symmetric probability measures*, in: Geometric Aspects of Functional Analysis, V. D. Milman and G. Schechtman (eds.), Lecture Notes in Math. 1807, Springer, 2003, 37–43.

[Bor] C. Borell, *Convex set functions in d-space*, Period. Math. Hungar. 6 (1975), 111–136.

[Bou1] J. Bourgain, *On high dimensional maximal functions associated to convex bodies*, Amer. J. Math. 108 (1986), 1467–1476.

[Bou2] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, in: Geometric Aspects of Functional Analysis, J. Lindenstrauss and V. D. Milman (eds.), Lecture Notes in Math. 1469, Springer, 1991, 127–137.

[Bou3] J. Bourgain, *On the isotropy constant problem for ψ_2-bodies*, in: Geometric Aspects of Functional Analysis, Lecture Notes in Math. 1807, Springer, 2003, 114–121.

[DP] N. Dafnis and G. Paouris, *Small ball probability estimates, ψ_2-behavior and the hyperplane conjecture*, J. Funct. Anal. 258 (2010), 1933–1964.

[EK] R. Eldan and B. Klartag, *Approximately gaussian marginals and the hyperplane conjecture*, in: Concentration, Functional Inequalities and Isoperimetry, Contemp. Math. 545, Amer. Math. Soc., 2011, 55–68.

[F] M. Fradelizi, *Sections of convex bodies through their centroid*, Arch. Math. (Basel) 69 (1997), 515–522.

[G] A. Giannopoulos, *Isotropic convex bodies*, Warsaw Univ. notes, 2003; http://users.uoa.gr/~apgiannop/.

[GPV1] A. Giannopoulos, G. Paouris and P. Valettas, *ψ_α-Estimates for marginals of log-concave probability measures*, Proc. Amer. Math. Soc. 140 (2012), 1297–1308.

[GPV2] A. Giannopoulos, G. Paouris and P. Valettas, *On the existence of subgaussian directions for log-concave measures*, in: Concentration, Functional Inequalities and Isoperimetry, Contemp. Math. 545, Amer. Math. Soc., 2011, 103–122.

[Gr] L. Gross, *Logarithmic Sobolev inequalities and contractivity properties of semigroups Dirichlet forms*, in: Lecture Notes in Math. 1563, Springer, 1993, 54–88.

[GM] O. Guédon and E. Milman, *Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures*, Geom. Funct. Anal. 21 (2011), 1043–1068.

[KLS] R. Kannan, L. Lovász and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom. 13 (1995), 541–559.

[K1] B. Klartag, *An isomorphic version of the slicing problem*, J. Funct. Anal. 218 (2005), 372–394.

[K2] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. 16 (2006), 1274–1290.

[KM] B. Klartag and E. Milman, *Centroid bodies and the logarithmic Laplace transform—a unified approach*, J. Funct. Anal. 262 (2012), 10–34.

[L] M. Ledoux, *The Concentration of Measure Phenomenon*, Math. Surveys Monogr. 89, Amer. Math. Soc., 2001.

[LYZ] E. Lutwak, D. Yang and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. 56 (2000), 111–132.

[LZ] E. Lutwak and G. Zhang, *Blaschke–Santaló inequalities*, J. Differential Geom. 47 (1997), 1–16.

[M] E. Milman, *On the role of convexity in isoperimetry, spectral gap and concentration*, Invent. Math. 177 (2009), 1–43.

[MP] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, in: Geometric Aspects of Functional Analysis, Lecture Notes in Math. 1376, Springer, 1989, 64–104.

[MS] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Math. 1200, Springer, 1986.

[P1] G. Paouris, *Concentration of mass on convex bodies*, Geom. Funct. Anal. 16 (2006), 1021–1049.

[P2] G. Paouris, *Small ball probability estimates for log-concave measures*, Trans. Amer. Math. Soc. 364 (2012), 287–308.

[P3] G. Paouris, *On the existence of supergaussian directions on convex bodies*, Mathematika 58 (2012), 389–408.

[Pi] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Math. 94, Cambridge Univ. Press, 1989.

[S] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, Encyclopedia Math. Appl. 44, Cambridge Univ. Press, Cambridge, 1993.

[So] S. Sodin, *An isoperimetric inequality on the ℓ_p balls*, Ann. Inst. H. Poincaré Probab. Statist. 44 (2008), 362–373.

[SV] P. Stavrakakis and P. Valettas, *On the geometry of log-concave probability measures with bounded log-Sobolev constant*, http://users.uoa.gr/~apgiannop/stavrakakis.html.

[W] J. O. Wojtaszczyk, *No return to convexity*, Studia Math. 199 (2010), 227–239.

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