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# Hölder functions in Bergman type spaces

by

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**Abstract.** It seems impossible to extend the boundary value theory of Hardy spaces to Bergman spaces since there is no boundary value for a function in a Bergman space in general. In this article we provide a new idea to show what is the correct version of Bergman spaces by demonstrating the extension to Bergman spaces of a result of Hardy–Littlewood in Hardy spaces, which characterizes the Hölder class of boundary values for a function from Hardy spaces in the unit disc in terms of the growth of its derivative. To this end, a class of Hölder functions in Bergman spaces is introduced in terms of the modulus of continuity and we establish its characterization in terms of radial derivatives. The classical result of Hardy–Littlewood in the Hardy space can be thought of as the limit case, matching the fact that the Hardy space is a limit of Bergman spaces.

**1. Introduction.** Let $H(U)$ denote the set of holomorphic functions in the unit disc $U$ in the complex plane $\mathbb{C}$. The Hardy space $H^p(U)$ consists of holomorphic functions $f$ in $U$ such that

$$\sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right\}^{1/p}<\infty.$$

Hardy–Littlewood observed a very close relation between the mean growth of the derivative $f'(z)$ and the smoothness of the boundary function $f(e^{i\theta})$.

Theorem 1.1 ([HL2]). *Let $f\in H^p(U)$, $1\le p<\infty$, and $0<\alpha\le 1$. Then*

$$f(e^{i\theta})\in\Lambda^p_\alpha \iff M_p(r,f')=O((1-r)^{\alpha-1}),\qquad r\to 1^-.$$

*Here, $\Lambda^p_\alpha$ consists of all $\phi\in L^p[0,2\pi)$ such that*

$$\omega_p(t,\phi):=\sup_{0\le\theta\le t}\|\phi(e^{i\theta}z)-\phi(z)\|_{L^p[0,2\pi)}=O(t^\alpha),\qquad t\to 0^+.$$

Theorem 1.1 can be interpreted as an equivalence characterization of Hölder functions in Hardy space in terms of the growth of derivatives. A function $f$ in the Hardy space $H^p(U)$ is said to be Hölder if its boundary

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function $f(e^{i\theta})$ is Hölder in $L^p[0,2\pi)$, i.e., $f(e^{i\theta}) \in \Lambda^p_\alpha$. We remark that the modulus of continuity $\omega_p(t,\phi)$ is a precise way to measure the smoothness of a boundary function $f(e^{i\theta})$.

The purpose of this article is to extend the Hardy–Littlewood theorem to Bergman type spaces in bounded symmetric domains. We shall establish an equivalent characterization of Hölder functions in Bergman type spaces on bounded symmetric domains in terms of radial derivatives.

In contrast to Hardy space, functions in Bergman type spaces have no boundary values in general. However, we can define Hölder functions in Bergman type spaces in terms of the modulus of continuity. Thus the modulus of continuity of a function itself shall be considered in Bergman type spaces instead of the boundary value of the function in Hardy space. We refer to [RC, RW, WR] for the properties of the modulus of continuity of holomorphic functions in higher dimensions. We shall see that the result for the Hardy space in Theorem 1.1 can be thought of as the limit case of results for weighted Bergman spaces (see Theorem 5.1).

To state our main result, we need some notation.

Let $\Omega$ be a bounded symmetric domain in $\mathbb{C}^n$ with the standard Harish-Chandra realization. Let $S$ be the Shilov boundary of $\Omega$ with Lebesgue measure $\sigma$ normalized so that $\sigma(S) = 1$.

A positive continuous function $\varphi$ on $[0,1)$ is *normal* if there exist $0 < a < b$ and $0 \le r_0 < 1$ such that

(i) $(1-r)^{-a}\varphi(r)$ is non-increasing on $[r_0,1)$ and
$$\lim_{r\to 1^-}(1-r)^{-a}\varphi(r) = 0;$$

(ii) $(1-r)^{-b}\varphi(r)$ is non-decreasing on $[r_0,1)$ and
$$\lim_{r\to 1^-}(1-r)^{-b}\varphi(r) = \infty.$$

Throughout, $a$, $b$, and $r_0$ are taken to be fixed for the normal function $\varphi$.

Let $H(\Omega)$ be the set of holomorphic functions in $\Omega$. For any $f \in H(\Omega)$, we denote
$$f_r(z) = f(rz), \qquad f_{e^{ih}}(z) = f(e^{ih}z),$$
and denote the radial derivative of $f$ by
$$\mathscr{R}f(z) = \sum_{i=1}^n z_i \frac{\partial f}{\partial z_i}(z).$$

For $0 < p, q \le \infty$, the Bergman type space $H_{p,q,\varphi}$ consists of all $f \in H(\Omega)$ such that
$$\|f\|_{H_{p,q,\varphi}} = \Big(\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r) M_q^p(r,f)\,dr\Big)^{1/p} < \infty,$$

where

$$M_q(r,f) = \Big(\int_S |f(r\zeta)|^q \, d\sigma(\zeta)\Big)^{1/q}.$$

As usual, the case of $p = \infty$ or $q = \infty$ is in the limit sense.

DEFINITION 1.2. The *modulus of continuity* of $f$ is defined as

$$\omega(\delta, f, H_{p,q,\varphi}) = \sup_{0\le h\le\delta} \|\Delta_h f\|_{H_{p,q,\varphi}},$$

where

$$\Delta_h f(z) = f_{e^{ih}}(z) - f(z).$$

Let $0 < \alpha \le 1$. A function $f \in H_{p,q,\varphi}$ is said to belong to the *$\alpha$-Hölder class $\Lambda^\alpha(H_{p,q,\varphi})$* if

$$\omega(\rho, f, H_{p,q,\varphi}) = O(\rho^\alpha), \quad \rho \to 0^+.$$

Now we can state our main result of which Theorem 1.1 can be considered as a limit case.

THEOREM 1.3. *Let $f \in H_{p,q,\varphi}$, $0 < p, q \le \infty$, and $0 < \alpha \le 1$. Then*

$$f \in \Lambda^\alpha(H_{p,q,\varphi}) \;\Leftrightarrow\; \|(\mathscr{R}f)_\rho\|_{H_{p,q,\varphi}} = O((1-\rho)^{\alpha-1}),\ \rho \to 1^-.$$

**2. Bergman type spaces and radial derivatives.** In this section, we shall give some preliminary results concerning normal functions, Bergman type spaces and radial derivatives.

As usual, $f \lesssim g$ means $f \le Cg$ with some positive constant $C$ independent of $f$ and $g$. Similarly, $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$.

LEMMA 2.1 ([RK]). *For any $0 \le t \le r < 1$ and $s > 0$, we have $\varphi(r) \simeq \varphi(r^s)$ and*

$$\frac{\varphi(r)}{(1-r)^a} \lesssim \frac{\varphi(t)}{(1-t)^a}, \qquad \frac{\varphi(r)}{(1-r)^b} \gtrsim \frac{\varphi(t)}{(1-t)^b}.$$

With this lemma, one can always take the index $r_0$ to be zero in applications.

Associated to the norm of Bergman type spaces, we introduce a new quantity

$$\|f\|_{H_{p,q,\varphi}[\rho_2,\rho_1)} = \Big(\int_{\rho_2}^{\rho_1} \Phi(r) M_q^p(r,f)\, dr\Big)^{1/p}, \tag{2.1}$$

where $0 \le \rho_2 < \rho_1 < 1$ and

$$\Phi(r) = \Phi_p(r) := r^{2n-1}(1-r)^{-1}\varphi^p(r).$$

Clearly, $\|f\|_{H_{p,q,\varphi}} = \|f\|_{H_{p,q,\varphi}[0,1)}$.

LEMMA 2.2. *Let* $0 < p, q \leq \infty$ *and* $f \in H_{p,q,\varphi}$. *Then*

$$\|f\|_{H_{p,q,\varphi}[0,1/4)} \lesssim \|f\|_{H_{p,q,\varphi}[1/4,1/2)}. \tag{2.2}$$

*Proof.* We claim that, for $0 < p < \infty$,

$$\Phi(r) \lesssim \Phi(r + 1/4), \quad r \in [0, 1/4]. \tag{2.3}$$

Indeed, by the monotonicity of the functions $r^{2n-1}$ and $((1-r)^{-b}\varphi(r))^p$, it is sufficient to consider $\Phi(r) = (1-r)^{pb-1}$. In this case, both functions in (2.2) are equivalent to a non-zero constant.

Therefore, from (2.3) follows (2.2) as well as the monotonicity of $M_q(r, f)$ whenever $0 < q < \infty$. The case $q = \infty$ follows directly from the maximum principle, and the case $p = \infty$ from the limit process. ■

LEMMA 2.3. *Suppose that* $f \in H(\Omega)$, $0 < pb \leq 1$, $0 < q \leq \infty$, *and* $0 < r \leq R < 1$. *Then*

$$\|f\|_{H_{p,q,\varphi}[0,r)} \lesssim \|f\|_{H_{p,q,\varphi}[R/2,(R+r)/2)}.$$

*Proof.* Since $(1-r)^{pb-1}$ is increasing for any $0 < pb \leq 1$, as shown in (2.3) we have

$$\Phi(r) \lesssim \Phi(R), \quad \forall\, 0 < r \leq R < 1.$$

Making a change of variable $\rho = (R+t)/2$, we have

$$\begin{aligned}\|f\|^p_{H_{p,q,\varphi}[R/2,(R+r)/2)} &= \int_{R/2}^{(R+r)/2} \Phi(\rho) M_q^p(\rho, f)\, d\rho \\ &= \frac{1}{2}\int_0^r \Phi\left(\frac{R+t}{2}\right) M_q^p\left(\frac{R+t}{2}, f\right) dt \\ &\gtrsim \int_0^r \Phi(t) M_q^p(t, f)\, dt = \|f\|^p_{H_{p,q,\varphi}[0,r)}. \ ■\end{aligned}$$

LEMMA 2.4. *If* $0 \leq \rho \leq r < 1$, *then*

$$\Phi(\rho) \lesssim \Phi\left(\rho + \frac{1-r}{2}\right).$$

*Proof.* This follows directly from the fact that

$$1 - \left(\rho + \frac{1-r}{2}\right) \leq 1 - \rho \leq 2\left(1 - \left(\rho + \frac{1-r}{2}\right)\right). \ ■$$

LEMMA 2.5. *Let* $1/4 \leq \rho < r \leq 1$, $\theta \in [-\pi, \pi]$. *Then*

(i) $\left|\left(\rho + \frac{1-r}{2}\right)e^{i\theta} - \rho\right|^2 \geq \frac{1}{4\pi^2}((1-r)^2 + \theta^2)$;

(ii) $|1 - re^{i\theta}|^2 \geq \frac{1}{\pi^2}((1-r)^2 + \theta^2)$.

*Proof.* Denote $\rho_1 = \rho + (1-r)/2$. Notice that

$$|\rho_1 e^{i\theta} - \rho|^2 = (\rho_1 - \rho)^2 + 4\rho_1\rho \sin^2 \frac{\theta}{2},$$

which is an even function of $\theta$. We can thus assume that $\theta \in [0, \pi]$, so that (i) follows directly from the inequality

$$\sin\theta \geq \frac{2}{\pi}\theta, \qquad \forall\, 0 \leq \theta \leq \frac{\pi}{2}.$$

Statement (ii) can be proved similarly. ∎

LEMMA 2.6 ([G]). *Suppose that* $0 < p \leq 1$, $0 < r < 1$, *and* $g$ *is holomorphic in the closed unit disc* $\overline{U}$. *Then*

$$\Big(\int\limits_{[-\pi,\pi)} |g(re^{i\theta})|\, d\theta\Big)^p \lesssim (1-r)^{p-1} \int\limits_{[-\pi,\pi)} |g(e^{i\theta})|^p\, d\theta.$$

LEMMA 2.7. *Let* $\lambda > 0$, $\delta > 0$, *and* $s = \min\{1, p, q\}$. *If* $f \in H_{p,q,\varphi}$, *then*

$$\omega^s(\lambda\delta, f, H_{p,q,\varphi}) \leq (\lambda + 1)\omega^s(\delta, f, H_{p,q,\varphi}).$$

*Proof.* See Lemma 3.3 in [WR] for the special case $\varphi(r) = (1-r)^\alpha$ with $\alpha > 0$. Its proof is also suitable for general normal functions $\varphi$. ∎

In [H], Hua constructed a set of holomorphic homogeneous polynomials

$$\Big\{\varphi_{j,v} : j \in \mathbb{N} \cup \{0\},\ v = 1, \ldots, m_j = \frac{(n+j-1)!}{j!(n-1)!}\Big\},$$

which is complete and orthogonal on $\Omega$ and orthogonal on $S$. Every $f \in H(\Omega)$ has a series expansion (see [HM])

$$f(z) = \sum_{j=0}^{\infty}\sum_{v=1}^{m_j} a_{j,v}\varphi_{j,v}(z),$$

where the convergence is uniform on compact subsets of $\Omega$. The coefficients are given by the formula

$$a_{j,v} = \lim_{r\to 1^-} \int\limits_S f(r\xi)\overline{\varphi_{j,v}(\xi)}\, d\sigma(\xi).$$

For any $\beta > 0$ and $s \geq 0$, we define the *fractional radial derivative* $\mathscr{R}^{\beta,s}$ as (see [RK])

$$(\mathscr{R}^{\beta,s} f)(z) = \sum_{j=0}^{\infty}\sum_{v=1}^{m_j} (j+s)^\beta a_{j,v}\varphi_{j,v}(z).$$

LEMMA 2.8. *Let* $0 < p, q \leq \infty$, $\beta > 0$, $s > 0$, $t \geq 0$, *and let* $\varphi$ *be normal. Then, for any* $f \in H(\Omega)$,

$$\int\limits_0^1 r^t(1-r)^{-1}\varphi^p(r) M_q^p(r, f)\, dr \simeq \int\limits_0^1 r^t (1-r)^{p\beta-1}\varphi^p(r) M_q^p(r, \mathscr{R}^{\beta,s} f)\, dr.$$

*When $p=\infty$, the inequality is understood to be its limit case:*

$$M_q(r,f)=O(\varphi^{-1}(r)) \Leftrightarrow M_q(r,\mathscr{R}^{\beta,s}f)=O((1-r)^{-\beta}\varphi^{-1}(r))$$

*as $r\to 1^-$.*

*Proof.* When $t=0$, the result is proved in [RK]. The general case follows from the equivalence

$$\int_0^1 r^t(1-r)^{-1}\varphi^p(r)M_q^p(r,f)\,dr \simeq \int_0^1 (1-r)^{-1}\varphi^p(r)M_q^p(r,f)\,dr.$$

One direction of this is clear and the other follows from the change of variable $u=r^{t+1}$. ∎

LEMMA 2.9. *Let $0<p,q\le\infty$, $s>0$, and $\beta>0$. Let $\varphi$ be normal and denote $\psi(r)=(1-r)^\beta\varphi(r)$. If $f\in H(\Omega)$, then*

$$\omega(\delta,f,H_{p,q,\varphi})\simeq\omega(\delta,\mathscr{R}^{\beta,s}f,H_{p,q,\psi}). \tag{2.4}$$

*Proof.* By definition,

$$\Delta_h(\mathscr{R}^{\beta,s}f(z))=\mathscr{R}^{\beta,s}f(e^{ih}z)-\mathscr{R}^{\beta,s}f(z)=\mathscr{R}^{\beta,s}(\Delta_hf(z)).$$

Lemma 2.8 thus implies that

$$\begin{aligned}\omega(\delta,f,H_{p,q,\varphi}) &= \sup_{0\le h\le\delta}\Big(\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)M_q^p(r,\Delta_hf)\,dr\Big)^{1/p}\\ &\simeq \sup_{0\le h\le\delta}\Big(\int_0^1 r^{2n-1}(1-r)^{p\beta-1}\varphi^p(r)M_q^p(r,\mathscr{R}^{\beta,s}(\Delta_hf))\,dr\Big)^{1/p}\\ &= \sup_{0\le h\le\delta}\Big(\int_0^1 r^{2n-1}(1-r)^{p\beta-1}\varphi^p(r)M_q^p(r,\Delta_h(\mathscr{R}^{\beta,s}f))\,dr\Big)^{1/p}\\ &= \omega(\delta,\mathscr{R}^{\beta,s}f,H_{p,q,\psi}).\ ∎\end{aligned}$$

The maximal theorem of Hardy–Littlewood in Hardy spaces will be used.

LEMMA 2.10 ([D]). *Let $f\in H^p(U)$, $0<p\le\infty$, and let*

$$F(\theta)=\sup_{0<r<1}|f(re^{i\theta})|.$$

*Then $F\in L^p[0,2\pi]$ and*

$$\|F\|_{L^p[0,2\pi]}\le C(p)\|f\|_{H^p(U)}.$$

**3. Hardy–Littlewood direct theorem.** The Hardy–Littlewood direct theorem gives an estimate of the radial derivative $\mathscr{R}f$ in terms of the modulus of continuity of $f$.

THEOREM 3.1. *Let* $0 < p, q \le \infty$ *and* $f \in H_{p,q,\varphi}$. *Then*

$$\|\mathscr{R}f\|_{H_{p,q,\varphi}[0,\rho)} \lesssim \frac{\omega(1-\rho, f, H_{p,q,\varphi})}{1-\rho}, \qquad \forall \rho \in (0,1).$$

*Proof.* Let $f \in H(\Omega)$ and denote $f_\zeta(\lambda) = f(\lambda\zeta)$ with $\lambda \in U$ and $\zeta \in S$. Then $f_\zeta \in H(U)$. Let $1/4 \le \rho < r < 1$ and take $\rho_1 = \rho + (1-r)/2$ and $\rho_2 = \rho + (1-r)/4$. Then

$$1/4 \le \rho < \rho_2 < \rho_1 < 1.$$

For any fixed $\gamma > 0$, we consider the holomorphic function

$$g(\lambda) := \frac{f_\zeta(\rho_1) - f_\zeta(\lambda)}{(\rho_1 - r\lambda)^\gamma}, \qquad \forall\, |\lambda| \le \rho_1. \tag{3.1}$$

From the Cauchy formula,

$$\frac{f_\zeta(\rho_1) - f_\zeta(\lambda)}{(\rho_1 - r\lambda)^\gamma} = \frac{1}{2\pi i} \int\limits_{|w|=\rho_2} \frac{f_\zeta(\rho_1) - f_\zeta(w)}{(w-\lambda)(\rho_1 - rw)^\gamma}\, dw, \qquad \forall\, |\lambda| < \rho_2.$$

In particular, by taking $\lambda = \rho$, we have

$$\begin{aligned} \frac{f_\zeta(\rho_1) - f_\zeta(\rho)}{(\rho_1 - r\rho)^\gamma} &= \frac{1}{2\pi i} \int\limits_{|w|=\rho_2} \frac{f_\zeta(\rho_1) - f_\zeta(w)}{(w-\rho)(\rho_1 - rw)^\gamma}\, dw \\ &= \frac{\rho_2}{2\pi} \int\limits_{-\pi}^{\pi} \frac{(f_\zeta(\rho_1) - f_\zeta(\rho_2 e^{i\theta}))e^{i\theta}}{(\rho_2 e^{i\theta} - \rho)(\rho_1 - r\rho_2 e^{i\theta})^\gamma}\, d\theta, \end{aligned}$$

so that

$$|f_\zeta(\rho_1) - f_\zeta(\rho)| \le \frac{\rho_2(\rho_1 - r\rho)^\gamma}{2\pi(\rho_2 - \rho)} \int\limits_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1) - f_\zeta(\rho_2 e^{i\theta})|}{|\rho_1 - r\rho_2 e^{i\theta}|^\gamma}\, d\theta. \tag{3.2}$$

Since $g(\lambda)$ in (3.1) is holomorphic in $|\lambda| \le \rho_1$, we have

$$h(\lambda) := g(\rho_1\lambda) = \frac{f_\zeta(\rho_1) - f_\zeta(\rho_1\lambda)}{(\rho_1 - r\rho_1\lambda)^\gamma}, \qquad \lambda \in \overline{U},$$

is holomorphic, so that Lemma 2.6 implies that

$$\Big( \int\limits_{[-\pi,\pi)} |h(te^{i\theta})|\, d\theta \Big)^\eta \lesssim (1-t)^{\eta-1} \int\limits_{[-\pi,\pi)} |h(e^{i\theta})|^\eta\, d\theta$$

for any $0 < t, \eta \le 1$. Setting $t = \rho_2/\rho_1$, we then have

$$\Big( \int\limits_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1) - f_\zeta(\rho_2 e^{i\theta})|}{|\rho_1 - r\rho_2 e^{i\theta}|^\gamma}\, d\theta \Big)^\eta \lesssim \Big( \frac{\rho_1}{\rho_1 - \rho_2} \Big)^{1-\eta} \int\limits_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1) - f_\zeta(\rho_1 e^{i\theta})|^\eta}{|\rho_1 - r\rho_1 e^{i\theta}|^{\gamma\eta}}\, d\theta.$$

Inserting this into (3.2), we obtain

$$|f_\zeta(\rho_1) - f_\zeta(\rho)|^\eta \le \left(\frac{\rho_2}{2\pi}\right)^\eta \frac{(\rho_1 - r\rho)^{\gamma\eta}}{(\rho_2-\rho)^\eta}\left(\frac{\rho_1}{\rho_1-\rho_2}\right)^{1-\eta} \int_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1 e^{i\theta}) - f_\zeta(\rho_1)|^\eta}{|\rho_1 - r\rho_1 e^{i\theta}|^{\gamma\eta}}\,d\theta.$$

We now pick the parameter $\gamma = 4/\eta$. Since $1/4 < \rho_2 < \rho_1 < 1$, $\rho_1 - \rho_2 = \rho_2 - \rho = (1-r)/4$, and

$$\rho_1 - r\rho = (1-r)\left(\frac{1}{2} + \rho\right) \le \frac{3}{2}(1-r),$$

we have

$$|f_\zeta(\rho_1) - f_\zeta(\rho)|^\eta \lesssim (1-r)^3 \int_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1 e^{i\theta}) - f_\zeta(\rho_1)|^\eta}{|1 - re^{i\theta}|^4}\,d\theta \tag{3.3}$$

for any $1/4 \le \rho < r < 1$ and $\rho_1 = \rho + (1-r)/2$.

Now, we apply the Cauchy formula to the holomorphic function

$$g_1(\lambda) := \frac{f_\zeta(\lambda)}{(\rho_1 - r\lambda)^\gamma}, \qquad \forall\, |\lambda| \le \rho_1;$$

then

$$\frac{f_\zeta(\lambda)}{(\rho_1 - r\lambda)^\gamma} = \frac{1}{2\pi i}\int_{|w|=\rho_2} \frac{f_\zeta(w)}{(w-\lambda)(\rho_1 - rw)^\gamma}\,dw, \quad \forall\, |\lambda| < \rho_2. \tag{3.4}$$

On taking the derivative on both sides of (3.4) and noticing that

$$f'_\zeta(\lambda) = \frac{\mathscr{R}f(\lambda\zeta)}{\lambda},$$

we get

$$\frac{\mathscr{R}f(\lambda\zeta)}{\lambda(\rho_1 - r\lambda)^\gamma} + \frac{\gamma r f_\zeta(\lambda)}{(\rho_1 - r\lambda)^{\gamma+1}} = \frac{1}{2\pi i}\int_{|w|=\rho_2} \frac{f_\zeta(w)}{(w-\lambda)^2(\rho_1 - rw)^\gamma}\,dw$$

for any $|\lambda| < \rho_2$. Taking $\lambda = \rho$, we have

$$\frac{\mathscr{R}f(\rho\zeta)}{\rho(\rho_1 - r\rho)^\gamma} + \frac{\gamma r f_\zeta(\rho)}{(\rho_1 - r\rho)^{\gamma+1}} = \frac{\rho_2}{2\pi}\int_{-\pi}^{\pi} \frac{f_\zeta(\rho_2 e^{i\theta})e^{i\theta}}{(\rho_2 e^{i\theta} - \rho)^2(\rho_1 - r\rho_2 e^{i\theta})^\gamma}\,d\theta. \tag{3.5}$$

In particular, by setting $f \equiv 1$,

$$\frac{\gamma r}{(\rho_1 - r\rho)^{\gamma+1}} = \frac{\rho_2}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}}{(\rho_2 e^{i\theta} - \rho)^2(\rho_1 - r\rho_2 e^{i\theta})^\gamma}\,d\theta. \tag{3.6}$$

We subtract (3.5) from a suitable multiple of (3.6) to obtain

$$\mathscr{R}f(\rho\zeta) = \frac{\rho\rho_2(\rho_1 - r\rho)^\gamma}{2\pi}\int_{-\pi}^{\pi} \frac{(f_\zeta(\rho_2 e^{i\theta}) - f_\zeta(\rho))e^{i\theta}}{(\rho_2 e^{i\theta} - \rho)^2(\rho_1 - r\rho_2 e^{i\theta})^\gamma}\,d\theta,$$

so that

$$|\mathscr{R}f(\rho\zeta)| \le \frac{\rho\rho_2(\rho_1 - r\rho)^\gamma}{2\pi(\rho_2-\rho)^2} \int_{-\pi}^{\pi} \frac{|f_\zeta(\rho_2 e^{i\theta}) - f_\zeta(\rho)|}{|\rho_1 - r\rho_2 e^{i\theta}|^\gamma}\, d\theta.$$

Again from Lemma 2.6, for any $0 < \eta \le 1$, we have

$$|\mathscr{R}f(\rho\zeta)|^\eta \lesssim \left(\frac{\rho\rho_2(\rho_1 - r\rho)^\gamma}{2\pi(\rho_2-\rho)^2}\right)^\eta \left(\frac{\rho_1}{\rho_1-\rho_2}\right)^{1-\eta} \int_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1 e^{i\theta}) - f_\zeta(\rho)|^\eta}{|\rho_1 - r\rho_1 e^{i\theta}|^{\gamma\eta}}\, d\theta.$$

Recalling that $1/4 \le \rho < \rho_2 < \rho_1 < 1$,

$$\rho_2 - \rho \simeq 1-r, \quad \rho_1 - \rho_2 \simeq 1-r, \quad \rho_1 - r\rho \le \frac{3}{2}(1-r),$$

we thus obtain

$$|\mathscr{R}f(\rho\zeta)|^\eta \lesssim (1-r)^{3-\eta} \int_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1 e^{i\theta}) - f_\zeta(\rho)|^\eta}{|1 - re^{i\theta}|^4}\, d\theta \tag{3.7}$$

for any $1/4 \le \rho < r < 1$ and $\rho_1 = \rho + (1-r)/2$.

From Lemma 2.5, we have

$$\int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^4} \lesssim \int_{-\pi}^{\pi} \frac{d\theta}{((1-r)^2+\theta^2)^2}$$

$$= \frac{2}{(1-r)^3} \int_0^{\pi} \frac{d\left(\frac{\theta}{1-r}\right)}{\left(1+\left(\frac{\theta}{1-r}\right)^2\right)^2} \lesssim \frac{1}{(1-r)^3}.$$

Since $|f_\zeta(\rho_1 e^{i\theta}) - f_\zeta(\rho)|^\eta \le |f_\zeta(\rho_1 e^{i\theta}) - f_\zeta(\rho_1)|^\eta + |f_\zeta(\rho_1) - f_\zeta(\rho)|^\eta$, it follows from (3.7) and (3.3) that

$$|\mathscr{R}f(\rho\zeta)|^\eta \lesssim (1-r)^{3-\eta} \int_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1 e^{i\theta}) - f_\zeta(\rho_1)|^\eta}{|1 - re^{i\theta}|^4}\, d\theta.$$

By replacing $\eta$ with $s = \min\{1, p, q\}$, we have

$$|\mathscr{R}f(\rho\zeta)|^s \lesssim (1-r)^{3-s} \int_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1 e^{i\theta}) - f_\zeta(\rho_1)|^s}{|1 - re^{i\theta}|^4}\, d\theta$$

for any $1/4 \le \rho < r < 1$ and $\rho_1 = \rho + (1-r)/2$.

We now take the $q/s$-power, integrate over $S$, and apply Minkowski's inequality to get

$$M_q^s(\rho, \mathscr{R}f) \lesssim (1-r)^{3-s} \left( \int_S \left( \int_{-\pi}^{\pi} \frac{|f_\zeta(\rho_1 e^{i\theta}) - f_\zeta(\rho_1)|^s}{|1-re^{i\theta}|^4}\, d\theta \right)^{q/s} d\sigma(\zeta) \right)^{s/q}$$

$$\le (1-r)^{3-s} \int_{-\pi}^{\pi} \left( \int_S |f(\rho_1 e^{i\theta}\zeta) - f(\rho_1\zeta)|^q \, d\sigma(\zeta) \right)^{s/q} \frac{d\theta}{|1-re^{i\theta}|^4}.$$

Again we take the $p/s$-power, integrate over $[1/4, r)$, and apply Minkowski's inequality as well as Lemma 2.4 to deduce that

$$\Big(\int_{1/4}^{r} \Phi(\rho) M_q^p(\rho, \mathscr{R}f)\, d\rho\Big)^{s/p}$$
$$\lesssim (1-r)^{3-s}\Big(\int_{1/4}^{r} \Phi(\rho)\Big(\int_{-\pi}^{\pi} \frac{M_q^s(\rho_1, f_{e^{i\theta}} - f)\, d\theta}{|1-re^{i\theta}|^4}\Big)^{p/s} d\rho\Big)^{s/p}$$
$$\lesssim (1-r)^{3-s}\int_{-\pi}^{\pi}\Big(\int_0^1 \Phi(\rho_1) M_q^p(\rho_1, f_{e^{i\theta}} - f)\, d\rho\Big)^{s/p} \frac{d\theta}{|1-re^{i\theta}|^4}.$$

Therefore, from Lemma 2.7,

$$\|\mathscr{R}f\|^s_{H_{p,q,\varphi}[1/4,r)} \lesssim (1-r)^{3-s}\int_{-\pi}^{\pi} \frac{\omega^s(|\theta|, f, H_{p,q,\varphi})}{|1-re^{i\theta}|^4}\, d\theta$$
$$\lesssim (1-r)^{3-s}\omega^s(1-r, f, H_{p,q,\varphi})\int_0^{\pi} \frac{\frac{\theta}{1-r}+1}{|1-re^{i\theta}|^4}\, d\theta$$
$$\lesssim \frac{\omega^s(1-r, f, H_{p,q,\varphi})}{(1-r)^s},$$

i.e.

$$\|\mathscr{R}f\|_{H_{p,q,\varphi}[1/4,r)} \lesssim \frac{\omega(1-r, f, H_{p,q,\varphi})}{1-r}, \qquad \forall\, 1/4 < r < 1. \tag{3.8}$$

(i) *Case* $pb > 1$. When $r \ge 1/2$, from Lemma 2.2 and (3.8) we have

$$\|\mathscr{R}f\|_{H_{p,q,\varphi}[0,r)} \lesssim \|\mathscr{R}f\|_{H_{p,q,\varphi}[0,1/4)} + \|\mathscr{R}f\|_{H_{p,q,\varphi}[1/4,r)} \tag{3.9}$$
$$\lesssim \|\mathscr{R}f\|_{H_{p,q,\varphi}[1/4,r)} \lesssim \frac{\omega(1-r, f, H_{p,q,\varphi})}{1-r}.$$

When $r < 1/2$, we apply the fact that $1-r \simeq 1$, (3.9) with $r = 1/2$, and Lemma 2.7 to get

$$\|\mathscr{R}f\|_{H_{p,q,\varphi}[0,r)} \le \|\mathscr{R}f\|_{H_{p,q,\varphi}[0,1/2)} \lesssim \frac{\omega(1/2, f, H_{p,q,\varphi})}{1/2}$$
$$\lesssim \Big(\frac{1}{2(1-r)}+1\Big)^{1/s}\omega(1-r, f, H_{p,q,\varphi})$$
$$\lesssim \frac{\omega(1-r, f, H_{p,q,\varphi})}{1-r}.$$

(ii) *Case* $0 < pb \le 1$. It follows from Lemmas 2.3 and 2.7, and (3.8), that

$$\|\mathscr{R}f\|_{H_{p,q,\varphi}[0,r)} \lesssim \|\mathscr{R}f\|_{H_{p,q,\varphi}[\frac{1}{2},\frac{1+r}{2})} \lesssim \frac{\omega\big(1-\frac{1+r}{2},f,H_{p,q,\varphi}\big)}{1-\frac{1+r}{2}}$$
$$\lesssim \frac{\omega(1-r,f,H_{p,q,\varphi})}{1-r}. \ \blacksquare$$

THEOREM 3.2. *Let* $0<p,q\le\infty$ *and* $f\in H_{p,q,\varphi}$. *Then*

$$\|(\mathscr{R}f)_r\|_{H_{p,q,\varphi}} \lesssim \frac{\omega(1-r,f,H_{p,q,\varphi})}{1-r}, \qquad \forall\, 0<r<1.$$

*Proof.* We split the proof into several cases. We denote

$$N_{p,q,\varphi}(r,f)=\|f_r\|_{H_{p,q,\varphi}}.$$

(i) *Case* $p\ge 1/a$. From Lemma 2.1,

$$\begin{aligned} N_{p,q,\varphi}(r,f) &= \Big(\int_0^1 \rho^{2n-1}\frac{\varphi^p(\rho)}{1-\rho}\Big(\int_S |f(r\rho\zeta)|^q\,d\sigma(\zeta)\Big)^{p/q}d\rho\Big)^{1/p}\\ &= \Big(\int_0^r \frac{1}{r}\Big(\frac{t}{r}\Big)^{2n-1}\frac{\varphi^p\big(\frac{t}{r}\big)}{1-\frac{t}{r}}\Big(\int_S |f(t\zeta)|^q\,d\sigma(\zeta)\Big)^{p/q}dt\Big)^{1/p}\\ &\lesssim \Big(\frac{1}{r^{2n}}\Big)^{1/p}\Big(\int_0^r t^{2n-1}\frac{\varphi^p(t)}{1-t}\Big(\int_S |f(t\zeta)|^q\,d\sigma(\zeta)\Big)^{p/q}dt\Big)^{1/p}\\ &\lesssim \Big(\frac{1}{r^{2n}}\Big)^{1/p}\|f(z)\|_{H_{p,q,\varphi}[0,r)}. \end{aligned}$$

Thus, from Theorem 3.1,

$$N_{p,q,\varphi}(r,\mathscr{R}f) \lesssim \Big(\frac{1}{r^{2n}}\Big)^{1/p}\frac{\omega(1-r,f,H_{p,q,\varphi})}{1-r}. \tag{3.10}$$

(A) When $r\ge 1/2$, from (3.10) we obtain

$$N_{p,q,\varphi}(r,\mathscr{R}f) \lesssim \frac{\omega(1-r,f,H_{p,q,\varphi})}{1-r}. \tag{3.11}$$

(B) When $r<1/2$, we have $1-r\simeq 1$ so that (3.11) and Lemma 2.7 imply that

$$\begin{aligned} N_{p,q,\varphi}(r,\mathscr{R}f) &\le N_{p,q,\varphi}(1/2,\mathscr{R}f) \lesssim \omega(1/2,f,H_{p,q,\varphi})\\ &\le \Big(\frac{1}{2(1-r)}+1\Big)^{1/s}\omega(1-r,f,H_{p,q,\varphi})\\ &\lesssim \frac{\omega(1-r,f,H_{p,q,\varphi})}{1-r}. \end{aligned}$$

(ii) *Case* $0<p<1/a$. Take $\beta$ such that $\beta\ge 1/p-a>0$. Let $s>0$ and $\psi(\rho)=(1-\rho)^\beta\varphi(\rho)$. From Lemmas 2.8 and 2.9,

$$\begin{aligned}N_{p,q,\varphi}(r,f) &= \Big(\int_0^1 \rho^{2n-1}(1-\rho)^{-1}\varphi^p(\rho)\Big(\int_S |f(r\rho\zeta)|^q\,d\sigma(\zeta)\Big)^{p/q}\,d\rho\Big)^{1/p}\\ &\simeq \Big(\int_0^1 \rho^{2n-1}(1-\rho)^{p\beta-1}\varphi^p(\rho)\Big(\int_S |\mathscr{R}^{\beta,s}f(r\rho\zeta)|^q\,d\sigma(\zeta)\Big)^{p/q}\,d\rho\Big)^{1/p}\\ &\lesssim \frac{1}{r^{2n/p}}\|\mathscr{R}^{\beta,s}f\|_{H_{p,q,\psi}[0,r)}.\end{aligned}$$

As in the proof of case (i), this implies that

$$N_{p,q,\varphi}(r,f) \lesssim \frac{\omega(1-r,\mathscr{R}^{\beta,s}f,H_{p,q,\psi})}{1-r},$$

or, by (2.4),

$$N_{p,q,\varphi}(r,f) \lesssim \frac{\omega(1-r,f,H_{p,q,\varphi})}{1-r}.$$

This finishes the proof. ∎

**4. Hardy–Littlewood inverse theorem.** The Hardy–Littlewood inverse theorem provides an estimate of the modulus of continuity of $f$ in terms of the growth of the radial derivative.

LEMMA 4.1. *Let $f \in H(\Omega)$ and $0 < h < r < 1$.*

(i) *If $q \ge 1$, then*

$$\Big(\int_S |f(r\zeta)-f((r-h)\zeta)|^q\,d\sigma(\zeta)\Big)^{1/q} \lesssim \int_0^h \Big(\int_S \Big|\frac{\mathscr{R}f((r-t)\zeta)}{r-t}\Big|^q\,d\sigma(\zeta)\Big)^{1/q}\,dt.$$

(ii) *If $0 < q < 1$, then*

$$\int_S |f(r\zeta)-f((r-h)\zeta)|^q\,d\sigma(\zeta) \lesssim \int_{h/2}^h (h-t)^{q-1}\int_S \Big|\frac{\mathscr{R}f((r-h+t)\zeta)}{r-h}\Big|^q\,d\sigma(\zeta)\,dt.$$

*Proof.* (i) Assume that $q \ge 1$. Since $f_\zeta(\lambda) = f(\lambda\zeta)$ is holomorphic, we have

$$\begin{aligned}f(r\zeta)-f((r-h)\zeta) &= -\int_0^h \frac{\partial}{\partial t}(f_\zeta(r-t))\,dt\\ &= \int_0^h \frac{\mathscr{R}f((r-t)\zeta)}{r-t}\,dt,\end{aligned}$$

so that the Minkowski inequality yields

$$\Big(\int_S |f(r\zeta)-f((r-h)\zeta)|^q\,d\sigma(\zeta)\Big)^{1/q} \le \bigg(\int_S\bigg(\int_0^h\bigg|\frac{\mathscr{R}f((r-t)\zeta)}{r-t}\bigg|\,dt\bigg)^q d\sigma(\zeta)\bigg)^{1/q}$$
$$\le \int_0^h\bigg(\int_S\bigg|\frac{\mathscr{R}f((r-t)\zeta)}{r-t}\bigg|^q d\sigma(\zeta)\bigg)^{1/q} dt.$$

(ii) For $0<q<1$, let $\{h_k\}_{k=0}^\infty$ be a partition of the interval $[0,h)$, i.e.,

$$0=h_0<h_1<\cdots<h_k<\cdots\to h \quad (k\to\infty).$$

Then

$$|f(r\zeta)-f((r-h)\zeta)|^q \le \sum_{k=0}^\infty |f_\zeta(r-h+h_k)-f_\zeta(r-h+h_{k+1})|^q$$
$$\le \sum_{k=0}^\infty\Big(\int_{h_k}^{h_{k+1}}|f'_\zeta(r-h+t)|\,dt\Big)^q$$
$$\le \sum_{k=0}^\infty (h_{k+1}-h_k)^q \sup_{h_k<t<h_{k+1}} |f'_\zeta(r-h+t)|^q.$$

To estimate its integral over $S$, we resort to the identity (see [R])

$$\int_S f\,d\sigma = \int_S d\sigma\,\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}\zeta)\,d\theta \tag{4.1}$$

and the maximal theorem of Hardy–Littlewood as given in Lemma 2.10. We have

$$\int_S \frac{1}{2\pi}\int_0^{2\pi}\sup_{h_k<t<h_{k+1}}\bigg|\frac{\partial f_\zeta}{\partial t}((r-h+t)e^{i\theta})\bigg|^q d\theta\,d\sigma(\zeta)$$
$$\lesssim \int_S \frac{1}{2\pi}\int_0^{2\pi}\bigg|\frac{\partial f_\zeta}{\partial t}((r-h+h_{k+1})e^{i\theta})\bigg|^q d\theta\,d\sigma(\zeta)$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\int_S\bigg|\frac{\mathscr{R}f((r-h+h_{k+1})e^{i\theta}\zeta)}{r-h+h_{k+1}}\bigg|^q d\sigma(\zeta)\,d\theta$$
$$\le \int_S\bigg|\frac{\mathscr{R}f((r-h+h_{k+1})\zeta)}{r-h}\bigg|^q d\sigma(\zeta).$$

As a result, for $0<q<1$,

$$\int_S |f(r\zeta)-f((r-h)\zeta)|^q\,d\sigma(\zeta) \lesssim \sum_{k=0}^\infty (h_{k+1}-h_k)^q\int_S\bigg|\frac{\mathscr{R}f((r-h+h_{k+1})\zeta)}{r-h}\bigg|^q d\sigma(\zeta). \tag{4.2}$$

Now we set

$$h_k = \left(1 - \frac{1}{2^k}\right)h, \qquad k = 0, 1, 2, \ldots.$$

Then

$$(h_{k+1} - h_k)^q = 2(h_{k+2} - h_{k+1})(h - h_{k+1})^{q-1}.$$

Notice that $(h-t)^{q-1}$ $(0 < q < 1)$ and

$$\int_S |\mathscr{R} f((r-h+t)\zeta)|^q \, d\sigma(\zeta)$$

are non-decreasing functions of $t$ in $[0, h]$. This allows us to estimate further the left side of (4.2) and obtain

$$\begin{aligned}
&\int_S |f(r\zeta) - f((r-h)\zeta)|^q \, d\sigma(\zeta) \\
&\qquad \lesssim \sum_{k=0}^{\infty} (h_{k+2} - h_{k+1})(h - h_{k+1})^{q-1} \int_S \left|\frac{\mathscr{R} f((r-h+h_{k+1})\zeta)}{r-h}\right|^q d\sigma(\zeta) \\
&\qquad \le \sum_{k=0}^{\infty} \int_{h_{k+1}}^{h_{k+2}} (h-t)^{q-1} \, dt \int_S \left|\frac{\mathscr{R} f((r-h+h_{k+1})\zeta)}{r-h}\right|^q d\sigma(\zeta) \\
&\qquad \le \int_{h/2}^{h} (h-t)^{q-1} \int_S \left|\frac{\mathscr{R} f((r-h+t)\zeta)}{r-h}\right|^q d\sigma(\zeta) \, dt.
\end{aligned}$$

This completes the proof. ∎

LEMMA 4.2. *Let $f \in H(\Omega)$, $q > 0$, and $0 < r < 1$. Then*

$$\int_S |f(re^{ih}\zeta) - f(r\zeta)|^q \, d\sigma(\zeta) \lesssim |h|^q \int_S |\mathscr{R} f(r\zeta)|^q \, d\sigma(\zeta).$$

*Proof.* Notice that

$$f(re^{ih}\zeta) - f(r\zeta) = \int_0^h \frac{\partial}{\partial\theta}(f_\zeta(re^{i\theta})) \, d\theta = i \int_0^h (\mathscr{R} f)(re^{i\theta}\zeta) \, d\theta.$$

When $q \ge 1$, the Minkowski inequality shows that

$$\begin{aligned}
\int_S |f(re^{ih}\zeta) - f(r\zeta)|^q \, d\sigma(\zeta) &\le \left(\int_0^h \left(\int_S |\mathscr{R} f(re^{i\theta}\zeta)|^q \, d\sigma(\zeta)\right)^{1/q} d\theta\right)^q \\
&= |h|^q \int_S |\mathscr{R} f(r\zeta)|^q \, d\sigma(\zeta),
\end{aligned}$$

where the last step used the rotational invariance of the measure $d\sigma$.

When $0 < q < 1$, it is known that (see [S])

$$\int_0^{2\pi} |F(e^{i(\theta+h)}) - F(e^{i\theta})|^q \, d\theta \lesssim |h|^q \int_0^{2\pi} |F'(e^{i\theta})|^q \, d\theta \tag{4.3}$$

for any $F \in H(\overline{U})$. From (4.3) and (4.1), we have

$$\begin{aligned}\int_S |f(re^{ih}\zeta) - f(r\zeta)|^q \, d\sigma(\zeta) &= \int_S \frac{1}{2\pi} \int_0^{2\pi} |f_{r\zeta}(e^{i(h+\theta)}) - f_{r\zeta}(e^{i\theta})|^q \, d\theta \, d\sigma(\zeta) \\ &\lesssim \int_S |h|^q \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\partial}{\partial\theta}(f_{r\zeta}(e^{i\theta}))\right|^q d\theta \, d\sigma(\zeta) \\ &= \int_S |h|^q \frac{1}{2\pi} \int_0^{2\pi} |i\mathscr{R} f(re^{i\theta}\zeta)|^q \, d\theta \, d\sigma(\zeta) \\ &= |h|^q \int_S |\mathscr{R} f(r\zeta)|^q \, d\sigma(\zeta). \ \blacksquare\end{aligned}$$

DEFINITION 4.3 (see [L]). A function $\Omega(t)$ on $\mathbb{R}^+$ is called a *modulus of continuity type function* if it has the following properties:

(a) $\Omega(t) \to \Omega(0) = 0$ as $t \to 0$;
(b) $\Omega(t)$ is non-negative and non-decreasing on $\mathbb{R}^+$;
(c) $\Omega(t)$ is subadditive: $\Omega(t_1 + t_2) \le \Omega(t_1) + \Omega(t_2)$ for any $t_1, t_2 \ge 0$.

THEOREM 4.4. *Let $0 < p, q \le \infty$, $\mu = \min\{1, p, q\}$, and $f \in H_{p,q,\varphi}$. Assume that $\Omega(t)$ is a modulus of continuity type function such that*

$$\int_0^1 \frac{\Omega^\mu(t)}{t} \, dt < \infty.$$

*If*

$$\|(\mathscr{R} f)_\rho\|_{H_{p,q,\varphi}} \lesssim \frac{\Omega(1-\rho)}{1-\rho}, \qquad \forall\, 0 < \rho < 1,$$

*then*

$$\omega(\tau, f, H_{p,q,\varphi}) \lesssim \left(\int_0^\tau \frac{\Omega^\mu(t)}{t} \, dt\right)^{1/\mu}, \qquad \forall\, 0 < \tau < 1.$$

*Proof.* From Lemma 2.2,

$$\|\Delta_h f\|_{H_{p,q,\varphi}[0,1)} \lesssim \|\Delta_h f\|_{H_{p,q,\varphi}[1/4,1)}.$$

Therefore, the proof reduces to establishing that

$$\|\Delta_h f\|_{H_{p,q,\varphi}[1/4,1)} \lesssim \left(\int_0^\tau \frac{\Omega^\mu(t)}{t} \, dt\right)^{1/\mu}, \qquad \forall\, 0 < h < \tau < 1. \tag{4.4}$$

Case I: $0 < q < 1$, $p \geq q$. In this case, we have $\mu = \min\{1, p, q\} = q$.

(A) *Case* $p \geq 1/a$.

(i) We first prove (4.4) for $\tau \in [0, 1/8]$. Now we take

$$0 < h < \tau \leq 1/8, \qquad 1/4 < r < 1. \tag{4.5}$$

We can write

$$\begin{aligned} \Delta_h f(r\zeta) &= |f(r\zeta e^{ih}) - f(r\zeta)| \\ &\leq |f(r\zeta) - f((r-h)\zeta)| + |f((r-h)\zeta) - f((r-h)e^{ih}\zeta)| \\ &\quad + |f(re^{ih}\zeta) - f((r-h)e^{ih}\zeta)| \\ &=: \Delta_1 + \Delta_2 + \Delta_3. \end{aligned}$$

Then

$$\|\Delta_h f\|_{H_{p,q,\varphi}[1/4,1)} \lesssim \sum_{k=1}^{3} \|\Delta_k\|_{H_{p,q,\varphi}[1/4,1)}. \tag{4.6}$$

First, we shall estimate $\|\Delta_1\|_{H_{p,q,\varphi}[1/4,1)}$. Since $r - h > 1/8$, it follows from (4.5) and Lemma 4.1 that

$$\int_S |\Delta_1|^q \, d\sigma(\zeta) \lesssim \int_{h/2}^{h} (h-t)^{q-1} \int_S |\mathscr{R} f((r-h+t)\zeta)|^q \, d\sigma(\zeta)\, dt.$$

Therefore, from the Minkowski inequality ($p \geq q$), we have

$$\begin{aligned} \|\Delta_1\|^q_{H_{p,q,\varphi}[1/4,1)} &\lesssim \Big( \int_{1/4}^{1} \Phi(r) \Big( \int_{h/2}^{h} (h-t)^{q-1} M_q^q(r-h+t, \mathscr{R} f)\, dt \Big)^{p/q} dr \Big)^{q/p} \\ &\leq \int_{h/2}^{h} (h-t)^{q-1} \Big( \int_{1/4}^{1} \Phi(r) M_q^p(r-h+t, \mathscr{R} f)\, dr \Big)^{q/p} dt. \end{aligned}$$

For any $t \in (h/2, h)$ with $h$ as well as $r$ as in (4.5), we make a change of variable

$$r - h + t = (1 - h + t)\rho.$$

Then

$$r = \rho + (1-\rho)(h-t) > \rho, \qquad \rho > r - h + t > 1/16.$$

Since $p \geq 1/a$ and $r > \rho > 1/16$, we then have

$$\Phi(r) \leq (1-r)^{pa-1} \Big( \frac{\varphi(r)}{(1-r)^a} \Big)^p \lesssim \Phi(\rho). \tag{4.7}$$

The preceding integral can be further estimated by

$$\int_{h/2}^{h}(h-t)^{q-1}\Big(\int_0^1 \Phi(\rho)M_q^p((1-h+t)\rho,\mathscr{R}f)\,d\rho\Big)^{q/p}dt$$
$$=\int_{h/2}^{h}(h-t)^{q-1}N_{p,q,\varphi}^q(1-h+t,\mathscr{R}f)\,dt$$
$$\lesssim\int_{h/2}^{h}(h-t)^{q-1}\frac{\Omega^q(h-t)}{(h-t)^q}\,dt$$
$$\le\int_0^h\frac{\Omega^q(t)}{t}\,dt.$$

That is,

$$\|\Delta_1\|_{H_{p,q,\varphi}[1/4,1)}\lesssim\Big(\int_0^h\frac{\Omega^q(t)}{t}\,dt\Big)^{1/q}. \tag{4.8}$$

With the same approach, we also have

$$\|\Delta_3\|_{H_{p,q,\varphi}[1/4,1)}\lesssim\Big(\int_0^h\frac{\Omega^q(t)}{t}\,dt\Big)^{1/q}. \tag{4.9}$$

Finally, we turn to estimating $\|\Delta_2\|_{H_{p,q,\varphi}[1/4,1)}$. By Lemma 4.2, we have

$$\|\Delta_2\|_{H_{p,q,\varphi}[1/4,1)}$$
$$=\Big(\int_{1/4}^{1}\Phi(r)\Big(\int_S|f((r-h)\zeta)-f((r-h)e^{ih}\zeta)|^q\,d\sigma(\zeta)\Big)^{p/q}dr\Big)^{1/p}$$
$$\lesssim\Big(\int_{1/4}^{1}\Phi(r)|h|^pM_q^p(r-h,\mathscr{R}f)\,dr\Big)^{1/p}.$$

Make a change of variable $r-h=(1-h)\rho$. Then

$$r>\rho>r-h>1/16, \tag{4.10}$$

which implies that $\Phi(r)\lesssim\Phi(\rho)$ and so

$$\|\Delta_2\|_{H_{p,q,\varphi}[1/4,1)}\le|h|N_{p,q,\varphi}(1-h,\mathscr{R}f)\le\Omega(h).$$

Since $\Omega^q(h)$ is a modulus of continuity type function, it follows that for any $0<t<h$ (see [DL, L])

$$\Omega^q(h)\le\Big(\frac{h}{t}+1\Big)\Omega^q(t)$$

so that

$$\frac{\Omega^q(h)}{h} \leq \frac{2\Omega^q(t)}{t}.$$

Therefore,

$$\|\Delta_2\|_{H_{p,q,\varphi}[1/4,1)} \leq \Omega(h) \lesssim \Big(\int_0^h \frac{\Omega^q(t)}{t}\,dt\Big)^{1/q}.$$

Combining all the results above, we find that (4.4) holds true for any $\tau \in [0, 1/8]$.

(ii) Let $\tau \in (1/8, 1)$. Due to the properties of the modulus of continuity and Lemma 2.7, we have

$$\begin{aligned}\omega(\tau, f, H_{p,q,\varphi}) &\lesssim \omega(1, f, H_{p,q,\varphi}) \leq (8+1)^{1/q}\omega(1/8, f, H_{p,q,\varphi}) \\ &\lesssim \Big(\int_0^{1/8} \frac{\Omega^q(t)}{t}\Big)^{1/q} \lesssim \Big(\int_0^{\tau} \frac{\Omega^q(t)}{t}\,dt\Big)^{1/q}.\end{aligned}$$

(B) *Case* $p < 1/a$. Take $s > 0$, $\beta \geq 1/p - a > 0$, and $\psi(r) = (1-r)^{\beta}\varphi(r)$. Lemma 2.8 then implies that

$$\begin{aligned}N_{p,q,\psi}(\rho, \mathscr{R}^{\beta,s} f) &= \Big(\int_0^1 r^{2n-1}(1-r)^{p\beta-1}\varphi^p(r) M_q^p(\rho r, \mathscr{R}^{\beta,s} f)\,dr\Big)^{1/p} \\ &\simeq \Big(\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r) M_q^p(\rho r, f)\,dr\Big)^{1/p} \\ &= N_{p,q,\varphi}(\rho, f) \lesssim \frac{\Omega(1-\rho)}{1-\rho}.\end{aligned}$$

Since $p > 1/(\beta + a)$, we can apply the result of Case (A) to the normal function $\psi$. Therefore,

$$\omega(\tau, \mathscr{R}^{\beta,s} f, H_{p,q,\psi}) \lesssim \Big(\int_0^{\tau} \frac{\Omega^q(t)}{t}\,dt\Big)^{1/q},$$

so that, by Lemma 2.9,

$$\omega(\tau, f, H_{p,q,\varphi}) \lesssim \Big(\int_0^{\tau} \frac{\Omega^q(t)}{t}\,dt\Big)^{1/q}.$$

Case II: $0 < q < 1$, $p < q$. In this case, $\mu = \min\{1, p, q\} = p$. We shall adopt the same approach as in Case I and only prove the analog of (4.8), since others are similar.

We take $h_k = (1 - (1/2)^k)h$ as in Lemma 4.1(ii). Applying (4.2) and the Minkowski inequality, we have

$$\begin{aligned}\|\Delta_1\|^p_{H_{p,q,\varphi}[1/4,1)} &\lesssim \int_{1/4}^{1} \Phi(r)\Big(\sum_{k=0}^{\infty}(h_{k+1}-h_k)^q M_q^q(r-h+h_{k+1},\mathscr{R}f)\Big)^{p/q}\,dr \\ &\le \sum_{k=0}^{\infty}(h_{k+1}-h_k)^p \int_{1/4}^{1} \Phi(r) M_q^p(r-h+h_{k+1},\mathscr{R}f)\,dr \\ &\le \sum_{k=0}^{\infty}\int_{h_{k+1}}^{h_{k+2}} (h-t)^{p-1} \int_{1/4}^{1} \Phi(r) M_q^p(r-h+h_{k+1},\mathscr{R}f)\,dr \\ &\le \int_{h/2}^{h} (h-t)^{p-1} \int_{1/4}^{1} \Phi(r) M_q^p(r-h+t,\mathscr{R}f)\,dr.\end{aligned}$$

We take a change of variable $r-h+t=(1-h+t)\rho$. With the same trick as in Case I, without loss of generality, we can assume $p\ge 1/a$, so that (4.7) holds. Therefore,

$$\begin{aligned}\|\Delta_1\|_{H_{p,q,\varphi}[1/4,1)} &\lesssim \Big(\int_{h/2}^{h}(h-t)^{p-1} N^p_{p,q,\varphi}(1-h+t,\mathscr{R}f)\,dt\Big)^{1/p} \\ &\le \Big(\int_{h/2}^{h}(h-t)^{p-1}\frac{\Omega^p(h-t)}{(h-t)^p}\,dt\Big)^{1/p} \\ &= \Big(\int_0^h \frac{\Omega^p(t)}{t}\,dt\Big)^{1/p}.\end{aligned}$$

Case III: $0<p<1$, $1\le q\le\infty$. In this case, $\mu=\min\{1,p,q\}=p$. As in Case II, we only need to prove the analogue of (4.8). From the same approach as in Case I, it is sufficient to consider assumption (4.5). With this assumption, we have $r-h>1/8$ and the inequality in Lemma 4.1(i) can be modified as

$$\begin{aligned}(4.11)\qquad \Big(\int_S |f(r\zeta)-f((r-h)\zeta)|^q\,d\sigma(\zeta)\Big)^{1/q} &\le \int_0^h \Big(\int_S |\mathscr{R}f((r-t)\zeta)|^q\,d\sigma(\zeta)\Big)^{1/q} dt \\ &= \int_0^h \Big(\int_S |\mathscr{R}f((r-h+t)\zeta)|^q\,d\sigma(\zeta)\Big)^{1/q} dt.\end{aligned}$$

With (4.11) in place of the inequality in Lemma 4.1(ii), the desired result follows with the same proof as in Case II.

Case IV: $1 \le p \le \infty$, $1 \le q \le \infty$. In this case, $\mu = \min\{1, p, q\} = 1$. With the same reasoning, we need only prove the analogue of (4.8) under assumption (4.5).

From Lemma 4.1 and the Minkowski inequality, we obtain

$$\begin{aligned}\|\Delta_1\|_{H_{p,q,\varphi}[1/4,1)} &= \Big(\int_{1/4}^{1} \Phi(r)\Big(\int_S |f(r\zeta) - f((r-h)\zeta)|^q \, d\sigma(\zeta)\Big)^{p/q} dr\Big)^{1/p} \\ &\lesssim \Big(\int_{1/4}^{1} \Phi(r)\Big(\int_0^h \Big(\int_S \Big|\frac{\mathscr{R}f((r-t)\zeta)}{r-t}\Big|^q d\sigma(\zeta)\Big)^{1/q} dt\Big)^p dr\Big)^{1/p} \\ &\le \int_0^h \Big(\int_{1/4}^{1} \Phi(r) M_q^p(r-t, \mathscr{R}f)\, dr\Big)^{1/p} dt.\end{aligned}$$

Making a change of variable $r-t = (1-t)\rho$, we find that (4.10) holds, which yields

$$\|\Delta_1\|_{H_{p,q,\varphi}[1/4,1)} \lesssim \int_0^h N_{p,q,\varphi}(1-t, \mathscr{R}f)\, dt \lesssim \int_0^h \frac{\Omega(t)}{t}\, dt.$$

The above proof can be modified to apply to the case of $p = \infty$ or $q = \infty$. This completes the proof. ∎

**5. Hardy–Littlewood theorem.** As a direct corollary of Theorems 3.2 and 4.4 with $\Omega(t) = t^\alpha$, we obtain a slightly stronger result than that in Theorem 1.3.

Theorem 5.1. *Let $0 < \alpha \le 1$, $0 < p, q \le \infty$, and $f \in H_{p,q,\varphi}$. Then*

$$\omega(\rho, f, H_{p,q,\varphi}) \le C\rho^\alpha, \qquad \forall \rho \in (0,1),$$

*if and only if*

$$\|(\mathscr{R}f)_\rho\|_{H_{p,q,\varphi}} \le C(1-\rho)^{\alpha-1}, \qquad \forall \rho \in (0,1),$$

*where the constants $C = C(p,q,\varphi)$ are arbitrary and independent of $f$.*

Because of its own interest, we finally record Theorem 5.1 in the special case of the unit ball.

Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^n$ and $L_a^p(\mathbb{B}, dV_\beta)$ be the weighted Bergman space, which consists of all holomorphic functions $f$ on $\mathbb{B}$ such that

$$\|f\|_{L_a^p(\mathbb{B},dV_\beta)} = \Big(\int_{\mathbb{B}} |f(z)|^p \, dV_\beta(z)\Big)^{1/p} < \infty,$$

where $0 < p \le \infty$, $\beta > -1$, and $dV_\beta = (1-|z|^2)^\beta dV(z)$ with $dV(z)$ being the normalized Lebesgue measure in the unit ball.

The modulus of continuity of $f$ is defined as

$$\omega(\delta, f, L_a^p(\mathbb{B}, dV_\beta)) = \sup_{0\le h\le\delta} \Big(\int_{\mathbb{B}} |f(e^{ih}z) - f(z)|^p \, dV_\beta(z)\Big)^{1/p}.$$

Let $0 < \alpha \le 1$. A function $f \in L_a^p(\mathbb{B}, dV_\beta)$ is said to belong to the $\alpha$-Hölder class $\Lambda^\alpha(L_a^p(\mathbb{B}, dV_\beta))$ if

$$\omega(\rho, f, L_a^p(\mathbb{B}, dV_\beta)) \le C\rho^\alpha, \qquad \forall \rho \in (0,1),$$

where $C$ is a constant depending on $f$.

We denote the radial derivative of $f$ as

$$\mathscr{R}f(z) = \sum_{i=1}^n z_i \frac{\partial f}{\partial z_i}(z)$$

and write $(\mathscr{R}f)_\rho(z) = \mathscr{R}f(\rho z)$.

THEOREM 5.2. *Let* $0 < \alpha \le 1$, $\beta > -1$, *and* $f(z) \in L_a^p(\mathbb{B}, dV_\beta)$. *Then*

$$f \in \Lambda^\alpha(L_a^p(\mathbb{B}, dV_\beta)) \;\Leftrightarrow\; \|(\mathscr{R}f)_\rho\|_{L_a^p(\mathbb{B},\, dV_\beta)} = O((1-\rho)^{\alpha-1}),\ \rho \to 1^-.$$

It is well known that in some sense the Hardy space $H^p(U)$ can be thought of as the limit of weighted Bergman spaces $L_a^p(U, dV_\beta)$ as $\beta \to -1^+$. In this way, Theorem 1.1 is exactly the limit case of Theorem 5.1 with $\beta \to -1^+$.

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