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Grauert's theorem for subanalytic open sets in real analytic manifolds

by

Daniel Barlet (Vandœuvre-lès-Nancy) and
Teresa Monteiro Fernandes (Lisboa)

Abstract. By an open neighbourhood in $\mathbb{C}^n$ of an open subset Ω of $\mathbb{R}^n$ we mean an open subset Ω' of $\mathbb{C}^n$ such that $\mathbb{R}^n \cap \Omega' = \Omega$. A well known result of H. Grauert implies that any open subset of $\mathbb{R}^n$ admits a fundamental system of Stein open neighbourhoods in $\mathbb{C}^n$. Another way to state this property is to say that each open subset of $\mathbb{R}^n$ is Stein. We shall prove a similar result in the subanalytic category: every subanalytic open subset in a paracompact real analytic manifold M admits a fundamental system of subanalytic Stein open neighbourhoods in any complexification of M.

1. Introduction. A classical result of H. Grauert states that an open set in a real analytic manifold $M_{\mathbb{R}}$ is locally the trace on $M_{\mathbb{R}}$ of a Stein open set in any given complexification $M_{\mathbb{C}}$ of $M_{\mathbb{R}}$.

The analogous result in the semi-analytic setting is easy to obtain because when f is a real analytic function, say near 0 in $\mathbb{R}^n$, the set $\{f > 0\}$ is near 0 the trace on $\mathbb{R}^n$ of the Stein open set $\{\Re(f) > 0\}$ intersected with a small open ball in $\mathbb{C}^n$.

We solve the subanalytic case of this problem using the following rather deep result (Theorem 2.1 below):

- each compact subanalytic set in $\mathbb{R}^n$ is the zero set of a $\mathscr{C}^2$ subanalytic function on $\mathbb{R}^n$.

The construction of the subanalytic Stein open subset we are looking for is then an easy consequence of H. Grauert's idea.

Let us mention without technical details that applications of our result arise naturally in the theory of sheaves on subanalytic sites, as developed by L. Prelli [13] (cf. [10] for the foundations of the theory of ind-sheaves). It entails, for instance, that the subanalytic sheaf of tempered analytic functions on a real analytic manifold is concentrated in degree zero as in the classical case.

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We conclude this article by computing one very simple example which is not semi-analytic in order to show that the subanalytic case is much more involved and also to explain to non-specialists of subanalytic geometry (like ourselves) the ideas and tools hidden behind this construction.

2. Main results and proofs. We refer to [1], [3], [11] and [15] for the basic material on subanalytic geometry.

The following result due to Bierstone, Milman and Pawłucki comes from a 1995 private letter to W. Schmid and K. Vilonen (cf. [14]). We refer to [4, C.11] for a proof in the more general setting of o-minimal structures.

THEOREM 2.1. *Let A be a compact subanalytic set in $\mathbb{R}^n$ and let $p \in \mathbb{N}$. Then there exists a $\mathscr{C}^p$ subanalytic function f on $\mathbb{R}^n$ such that $A = f^{-1}(0)$.*

REMARK 2.2. Let U be an open ball in $\mathbb{R}^n$ and Z a relatively compact subanalytic open set in U. Then there exists a $\mathscr{C}^2$ subanalytic function $g : \mathbb{R}^n \to \mathbb{R}^+$ with compact support in U such that

$$Z = \{x \in \mathbb{R}^n ; g(x) > 0\}.$$

To see this, apply the previous theorem to $\bar{U} \setminus Z$ and define g to be f on U and 0 on $\mathbb{R}^n \setminus U$. As U is subanalytic and f is identically zero around ∂U, this function g has the required properties.

Moreover, we can divide g by any given positive constant without changing the set Z, so for each $\varepsilon > 0$ we may assume that the Levi form of g is uniformly bounded on $\mathbb{R}^n$ by $\varepsilon \|z\|^2$.

COROLLARY 2.3. *Let Ω be a subanalytic open set in a paracompact real analytic manifold $M_{\mathbb{R}}$. Then, for any complexification $M_{\mathbb{C}}$ of $M_{\mathbb{R}}$, and for any given smooth hermitian metric on the complex tangent bundle of $M_{\mathbb{C}}$ there exists a subanalytic non-negative function f on $M_{\mathbb{C}}$ of class $\mathscr{C}^2$ such that*

$$\{f > 0\} \cap M_{\mathbb{R}} = \Omega$$

and such that the Levi form of f is bounded by the given hermitian metric. Moreover, f can be chosen so that $\operatorname{supp} f$ *is contained in any given open set in $M_{\mathbb{C}}$ containing the closed set $\bar{\Omega}$.*

Proof. For $\epsilon > 0$, denote by B_ϵ an open ball of $\mathbb{R}^n$ of radius ϵ and by $B_\epsilon^{\mathbb{C}}$ the corresponding ball in $\mathbb{C}^n$.

For each $p \in \bar{\Omega}$ (the closure of Ω) there exist two relatively compact open subanalytic neighbourhoods $V \subset\subset U$ of p in $M_{\mathbb{C}}$ and a complex analytic isomorphism φ from an open neighbourhood W of $\bar{U}$ to an open ball $B_\epsilon^{\mathbb{C}}$ such that $\varphi(\bar{V})$ is the closed ball $\bar{B}_{\epsilon/2}^{\mathbb{C}}$, and φ is real on $W \cap M_{\mathbb{R}}$. In particular, $\bar{V} \cap M_{\mathbb{R}} \subset U$ is a compact subanalytic subset, and $\bar{U}$ is a compact subanalytic subset of W.

As $M_{\mathbb{R}}$ is paracompact, we get a locally finite countable cover $(U_i)_{i\in\mathbb{N}^*}$ of $\bar{\Omega}$ such that the conditions above are satisfied. On each U_i, by the remark following Theorem 2.1, we may choose a $\mathscr{C}^2$ non-negative subanalytic function f_i on $M_{\mathbb{C}}$ with compact support in U_i whose non-zero set is exactly $V_i \cap \Omega$, and such that its Levi form is bounded by $h/2^i$ for any given hermitian metric h on $M_{\mathbb{C}}$. Then define $f := \sum_{i=1}^{\infty} f_i$. As this sum is locally finite, it clearly satisfies our requirements.

The last assertion follows by applying this construction in any open neighbourhood W of $\bar{\Omega}$ in $M_{\mathbb{C}}$ regarded as a complexification of $W \cap M_{\mathbb{R}}$. ∎

THEOREM 2.4. *Let Ω be a subanalytic open set in a paracompact real analytic manifold $M_{\mathbb{R}}$. Then, given a complexification $M_{\mathbb{C}}$ of $M_{\mathbb{R}}$, there exists a subanalytic Stein open subset $\Omega_{\mathbb{C}}$ of $M_{\mathbb{C}}$ such that*

$$\Omega = \Omega_{\mathbb{C}} \cap M_{\mathbb{R}}. \tag{2.1}$$

Proof. Let n be the dimension of $M_{\mathbb{R}}$. By Grauert's Theorem 3 [5, p. 470], there exist a natural number $N \in \mathbb{N}$ and a real analytic regular proper embedding φ of $M_{\mathbb{R}}$ in the euclidean space $\mathbb{R}^N$. By complexification, one defines a holomorphic map $\varphi_{\mathbb{C}}$ in a neighbourhood V of $M_{\mathbb{R}}$ in $M_{\mathbb{C}}$ taking values in $\mathbb{C}^N$, such that $\varphi_{\mathbb{C}}|_{M_{\mathbb{R}}} = \varphi$ and the rank of $\varphi_{\mathbb{C}}$ is everywhere equal to n.

Note that the Levi form of the real analytic function

$$g(z_1, \ldots, z_N) = \sum_{j=1}^{N} (\Im z_j)^2$$

is half the square norm in $\mathbb{C}^N$, hence g is strictly plurisubharmonic on $\mathbb{C}^N$. By the maximality of the rank of $\varphi_{\mathbb{C}}$, the function $\varphi_{\mathbb{C}}^*(g)$ is also strictly plurisubharmonic on V and subanalytic (in fact analytic).

Fix now a smooth hermitian metric (1) h on $T_{\mathbb{C}}V$ such that the Levi form of $\varphi_{\mathbb{C}}^*(g)$ is larger than $2h$ at each point.

By Corollary 2.3, there exists a subanalytic $\mathscr{C}^2$ non-negative function f with support in V such that

$$\{f > 0\} \cap M_{\mathbb{R}} = \Omega$$

and the Levi form of f is bounded by h. So the Levi form of the $\mathscr{C}^2$ subanalytic function

$$\psi := \varphi_{\mathbb{C}}^*(g) - f$$

is positive definite at each point of V. It follows that the open set

$$\Omega_{\mathbb{C}} = \{\psi < 0\} \cap V$$

(1) For instance 1/2 of the Levi form of $\varphi_{\mathbb{C}}^*(g)$ may be chosen as Kähler form on V.

is (strongly 1-complete) Stein by Grauert's famous result and subanalytic in $M_{\mathbb{C}}$ by construction.

Moreover, as $\varphi_{\mathbb{C}}^*(g) = 0$ in $M_{\mathbb{R}}$, it follows that $\Omega_{\mathbb{C}} \cap M_{\mathbb{R}} = \Omega$. ■

3. Example: A strange four-leaved trefoil. Our aim is now to give an explicit construction of the function f in Theorem 2.1 in the case of one of the simplest examples which is not semi-analytic. For that purpose we shall only use the Łojasiewicz inequalities and Theorem 3.2 below, which are basic tools in subanalytic geometry. We think that this analysis will convince the reader of the strength and usefulness of Theorem 2.1 and that this tool is far from being elementary.

We shall need the following refinement of subanalyticity.

3.1. Strong subanalyticity. For a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ to be subanalytic simply means that its graph is a subanalytic set in $\mathbb{R}^n \times \mathbb{R}$, but in the non-continuous case we shall use a stronger assumption, in order to control the behaviour of the graph near points where f is not locally bounded. We restrict ourselves to the context we need.

DEFINITION 3.1. Let $\Omega \subset\subset \mathbb{R}^n$ a relatively compact subanalytic open set, and let $f : \Omega \to \mathbb{R}$ be a continuous function. We shall say that f is *strongly subanalytic* if the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ defined by extending f by 0 on $\mathbb{R}^n \setminus \Omega$ has a subanalytic graph in $\mathbb{R}^n \times \mathbb{P}_1$, where $\mathbb{P}_1$ is the 1-dimensional projective space $\mathbb{R} \cup \{\infty\}$.

It is easy to see that strong subanalyticity implies that the growth of f near a boundary point in $\partial\Omega$ has to be bounded by some power of the function $d(x, \partial\Omega)$ thanks to the Łojasiewicz inequalities ([1]).

If $\tilde{f}$ is continuous this condition reduces to the usual subanalyticity of the graph of $\tilde{f}$ in $\mathbb{R}^n \times \mathbb{R}$.

We shall also need the following theorem (cf. [11, Theorem (2.4)]).

THEOREM 3.2. *Let $\Omega \subset\subset \mathbb{R}^n$ be a relatively compact subanalytic open set, and let $f : \Omega \to \mathbb{R}$ be a strongly subanalytic $\mathscr{C}^1$ function. Then any partial derivative of f in Ω is also strongly subanalytic.*

Since, in Definition 3.1, the continuity of $\tilde{f}$ just means that $f(x)$ goes to 0 when $x \in \Omega$ goes to the boundary $\partial\Omega$, using the Łojasiewicz inequalities we easily obtain the following corollary:

COROLLARY 3.3. *In the situation of the previous theorem, assume that $\tilde{f}$ is continuous. Then there exists an integer N_1 such that $\tilde{f}^{N_1}$ is $\mathscr{C}^1$ on $\mathbb{R}^n$ and subanalytic.*

Now applying again the ideas of the previous corollary we finally obtain:

COROLLARY 3.4. *In the situation of the previous corollary there exists an integer N_2 such that $\tilde{f}^{N_2}$ is $\mathscr{C}^2$ on $\mathbb{R}^n$ and subanalytic.*

REMARK 3.5. In view of the preceding results, the remaining non-trivial step to prove the existence of a subanalytic $\mathscr{C}^2$ function which vanishes exactly on $\mathbb{R}^n \setminus \Omega$ as stated in Theorem 2.1, is to show the existence of a $\mathscr{C}^2$ strictly positive (strongly) subanalytic function f on Ω which vanishes at the boundary. The natural candidate is, of course, the function $x \mapsto d(x, \partial\Omega)$. But then all conditions are satisfied except smoothness. And non-smoothness points may go to the boundary. If one tries to use the "desingularization theorem" of H. Hironaka to solve this problem, a new difficulty arises because the jacobian of the modification may vanish inside Ω and not only at some points of $\partial\Omega$.

3.2. Example. Let us consider the analytic map $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(x, y, z) = \big(y(e^x - 1) + x^2 + y^2 + z^2 - \varepsilon^2, y(e^{x\sqrt{2}} - 1), y(e^{x\sqrt{3}} - 1)\big).$$

Denote by Ω the interior of the image $\tilde{\Omega}$ of the compact ball $\bar{B}_3(0, \varepsilon)$. We start by showing that the image under F of the sphere S_ε (the boundary of $\bar{B}(0, \varepsilon)$) is a subanalytic compact subset of $\mathbb{R}^3$ which is not semi-analytic in the neighbourhood of $(0, 0, 0)$. This example is extracted from [8, Example I.6].

LEMMA 3.6. *The compact set $F(S_\varepsilon)$ is not semi-analytic in the neighbourhood of the origin.*

Proof. Since this compact set has an empty interior, if it were semi-analytic in a neighbourhood of the origin, there would exist an analytic function $f : U \to \mathbb{R}$ on a ball U centred at 0, not identically zero, such that $f^{-1}(0)$ contains $U \cap F(S_\varepsilon)$. Let

$$f = \sum_{m \ge m_0} P_m$$

be the Taylor series of f at the origin, which we may assume to be convergent in U small enough. We shall assume that the homogeneous polynomial P_{m_0} is not identically zero. Hence, considering $(x, y, z) \in S_\varepsilon$ close enough to $(0, 0, \varepsilon)$, the definition of F entails the equality

$$0 \equiv \sum_{m \ge m_0} y^m P_m((e^x - 1), (e^{x\sqrt{2}} - 1), (e^{x\sqrt{3}} - 1))$$

for $(x, y) \in \mathbb{R}^2$ close enough to $(0, 0)$. We conclude that $P_{m_0}(e^x - 1, e^{x\sqrt{2}} - 1, e^{x\sqrt{3}} - 1)$ is identically zero for x in a neighbourhood of 0. Hence this analytic function vanishes identically on $\mathbb{R}$. Its behaviour at infinity easily entails ([2]) that we must have $P_{m_0} \equiv 0$, which gives a contradiction. ∎

We shall now describe the open set Ω. Let us remark that the jacobian of F is given by

$$J(F)(x,y,z) = 2yz\big((\sqrt{2}-\sqrt{3})e^{x(\sqrt{2}+\sqrt{3})} - \sqrt{2}\,e^{x\sqrt{2}} + \sqrt{3}\,e^{x\sqrt{3}}\big)$$

and for ε small enough, it does not vanish on $\{xyz \neq 0\}$ within the ball $\bar{B}_3(0,\varepsilon)$. Indeed, the brackets give an analytic function of a single variable x; hence it has an isolated zero at $x = 0$. The image of $\{xy = 0\} \cap \bar{B}_3(0,\varepsilon)$ under F is $[-\varepsilon^2, 0] \times \{(0,0)\}$, which is contained in ([3]) the boundary of $\tilde{\Omega}$. The image of $\{z = 0\}$ is more complicated to describe.

Now consider the analytic morphism $G : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G(x,y) := \big(y(e^{x\sqrt{2}} - 1), y(e^{x\sqrt{3}} - 1)\big).$$

Denote by Γ the image under G of the ball $\bar{B}_2(0,\varepsilon)$ of $\mathbb{R}^2$. If $(v,w) \in \Gamma \setminus \{(0,0)\}$ then the fibre $G^{-1}(v,w)$ reduces to a single point (for ε small enough). In fact we must have $vw \neq 0$ and

$$\frac{(e^{x\sqrt{2}} - 1)}{(e^{x\sqrt{3}} - 1)} = \frac{v}{w} = \frac{\sqrt{2}}{\sqrt{3}} h(x)$$

whenever $h \in \mathbb{C}\{x\}$ converges for $|x| < 2\pi/\sqrt{3}$ and satisfies $h(0) = 1$ and $h'(0) = (\sqrt{2} - \sqrt{3})/2$; these equations determine a unique $x \in [-\varepsilon, \varepsilon]$, for $\varepsilon \ll 1$, and hence a unique y. Note that for x in a neighbourhood of 0, v/w is close to $\sqrt{2}/\sqrt{3}$. Therefore Γ approaches $(0,0)$ only along that direction.

The fibre over $(0,0)$ of G is the curve $\{xy = 0\} \cap \bar{B}_2(0,\varepsilon)$.

Observe that the points in the sphere $\{x^2 + y^2 = \varepsilon^2\}$ are mapped to the boundary of Γ. Indeed, those on $\{xy = 0\}$ are mapped to the origin. On the other hand, for those points not mapped to the origin, the jacobian of G does not vanish and the boundary of $\bar{B}_2(0,\varepsilon)$ is mapped to the boundary of Γ in a neighbourhood of that point.

Hence, any point of the interior Γ' of Γ is the image under G of some point in $B_2(0,\varepsilon) \setminus \{xy = 0\}$.

We shall denote by $\varphi : \Gamma \setminus \{(0,0)\} \to \mathbb{R}$ the subanalytic function ([4]) given by $\varphi(v,w) = \|G^{-1}(v,w)\|^2$, in other words, the composition of G^{-1} with the square of the euclidean norm in $\mathbb{R}^2$.

([2]) This is equivalent to the algebraic independence of the functions $e^x - 1$, $e^{x\sqrt{2}} - 1$, $e^{x\sqrt{3}} - 1$.

([3]) See the description of Γ near $(0,0)$ given below

([4]) The graph of $G^{-1} : \Gamma \setminus \{(0,0)\} \to \bar{B}_2(0, \varepsilon \setminus \{xy = 0\}$ is the same as the graph of $G : \bar{B}_2(0,\varepsilon) \setminus \{xy = 0\} \to \Gamma \setminus \{(0,0)\}$.

We shall denote by $\psi : \varGamma \setminus \{(0,0)\} \to \mathbb{R}$ the subanalytic function defined by setting $\psi(v,w) = y(e^x - 1)$ where $G^{-1}(v,w) = (x,y)$, and we set

$$\begin{aligned} \varDelta^+ &:= \big\{(\psi(v,w), v, w)\,;\, (v,w) \in \varGamma \setminus \{(0,0)\}\big\},\\ \varDelta^- &:= \big\{(\psi(v,w) + \varphi(v,w) - \varepsilon^2, v, w)\,;\, (v,w) \in \varGamma \setminus \{(0,0)\}\big\},\\ \varDelta^0 &:= [-\varepsilon^2, 0] \times \{(0,0)\}. \end{aligned}$$

Note that

$$\varDelta^+ \cap \varDelta^- = \big\{(u,v,w) \in \mathbb{R} \times (\varGamma \setminus \{(0,0)\})\,;\, u = \psi(v,w) \text{ and } \varphi(v,w) = \varepsilon^2\big\}$$

is the graph of the restriction of ψ to $\partial\varGamma \setminus \{(0,0)\}$.

We now have the following description of $\tilde{\varOmega}$ and of its interior $\varOmega$.

Lemma 3.7. *One has $\partial\tilde{\varOmega} = \varDelta^+ \cup \varDelta^- \cup \varDelta^0$. The interior $\varOmega$ is the open set*

$$\varOmega = \big\{(u,v,w) \in \mathbb{R} \times \varGamma'\,;\, \psi(v,w) + \varphi(v,w) - \varepsilon^2 < u < \psi(v,w)\big\}$$

where $\varGamma'$ denotes the interior of $\varGamma$.

Proof. Let $(u,v,w) \in \tilde{\varOmega}$. If $vw = 0$ then $xy = 0$ and $v = w = 0$, and $u = x^2 + y^2 + z^2 - \varepsilon^2$ belongs to $[-\varepsilon^2, 0]$ which is contained in $\varDelta^0$. Since the projection of $\varOmega$ on $\mathbb{R}^2$ is an open set contained in $\varGamma$, hence in $\varGamma'$, the point (u,v,w) does not belong to $\varOmega$.

Let us now assume $uv \neq 0$. There is a point $(x,y,z) \in \bar{B}_3(0,\varepsilon)$ such that $F(x,y,z) = (u,v,w)$ with $xy \neq 0$. Then $(x,y) \in \bar{B}_2(0,\varepsilon) \setminus \{xy = 0\}$ and $G(x,y) = (v,w)$ is not $(0,0)$. Since $\varphi(v,w) = x^2 + y^2$ we have

$$u = \psi(v,w) + \varphi(v,w) + z^2 - \varepsilon^2$$

where $z \in [-\varepsilon, \varepsilon]$ is, up to sign, determined by this equation. We conclude that the inequalities

$$\psi(v,w) + \varphi(v,w) - \varepsilon^2 \leq u \leq \psi(v,w) \tag{3.1}$$

hold on $\tilde{\varOmega}$. We have to check that $\partial\tilde{\varOmega}\setminus\varDelta^0$ is exactly described by the equality

$$(u - \psi(v,w) - \varphi(v,w) + \varepsilon^2)(\psi(v,w) - u) = 0. \tag{3.2}$$

Since the projection on $\mathbb{R}^2$ is open, if $(v,w) \notin \varGamma'$ then it must lie at the boundary of $\varOmega$. It suffices to prove that for $(v,w) \in \varGamma'$ the equality above implies that (v,w) is at the boundary. This is clear because near any $(u,v,w) \in \varOmega$ one can find $\delta > 0$ such that $]u-\delta, u+\delta[\times \{(v,w)\}$ is contained in $\varOmega$, which is not possible by the inequalities (3.1) at a point where the equality (3.2) is satisfied.

Hence it is sufficient to prove that $\tilde{\varOmega} \setminus \varDelta^0$ is the set of points (u,v,w) in $\mathbb{R} \times (\varGamma \setminus \{(0,0)\})$ satisfying (3.1). Indeed, any choice of $(v,w) \in \varGamma \setminus \{(0,0)\}$ gives a unique point $(x,y) \in B_2(0,\varepsilon)$ such that $G(x,y) = (v,w)$ and (3.1) entails that we can find $z \in \mathbb{R}$ such that $z^2 = u - \psi(v,w) - \varphi(v,w) + \varepsilon^2$ and $\varphi(v,w) + z^2 \leq \varepsilon^2$. Note that if $u = \psi(v,w) + \varphi(v,w) - \varepsilon^2$ we have $z = 0$.

Therefore, the boundary Δ^- corresponds to the image of $\bar{B}_3(0,\varepsilon)\cap\{z=0\}\setminus\Delta^0$. Similarly the equality $u=\psi(v,w)$ corresponds to the image of the sphere $\{x^2+y^2+z^2=\varepsilon^2\}$ with Δ^0 removed. ∎

Now define $f:\mathbb{R}^3\to\mathbb{R}^+$ by

$$f(u,v,w)=\begin{cases}(\psi(v,w)-u)(u-\psi(v,w)-\varphi(v,w)+\varepsilon^2) & \text{for } (u,v,w)\in\Omega,\\ 0 & \text{for } (u,v,w)\notin\Omega.\end{cases}$$

Note that f is strictly positive on Ω by Lemma 3.7, and that it is analytic on the complement of $\partial\Omega$, since the functions φ and ψ are analytic on Γ'. Moreover f is bounded.

Let us now define $\tilde{f}(u,v,w)=f(u,v,w)v^2w^2$.

LEMMA 3.8. *The function $\tilde{f}:\mathbb{R}^3\to\mathbb{R}^+$ is subanalytic and continuous, it satisfies*

$$\Omega=\{(u,v,w)\in\mathbb{R}^3\,;\,\tilde{f}(u,v,w)>0\}$$

and it is $\mathscr{C}^\infty$ on $\mathbb{R}^3\setminus\partial\Omega$.

Proof. First we prove that f is subanalytic ([5]). Since its graph is the union of the graph of its restriction to Ω and the set $(\mathbb{R}^3\setminus\Omega)\times\{0\}$ which is subanalytic, Ω being an open subanalytic set of $\mathbb{R}^3$, it is sufficient to prove that the graph of the restriction of f to Ω is subanalytic.

Consider the polynomial morphism $h:\mathbb{R}^3\to\mathbb{R}$ given by

$$h(x,y,z)=(\varepsilon^2-(x^2+y^2+z^2))z^2$$

and denote by X, X_1, X_2 the graphs of the restrictions of h respectively to $\bar{B}_3(0,\varepsilon)$, $\partial B_3(0,\varepsilon)$, $\bar{B}_3(0,\varepsilon)\cap\{xy=0\}$, and by Y, Y_1, Y_2 the respective images of these graphs under the morphism $F\times\mathrm{id}:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3\times\mathbb{R}$.

Let us prove that the graph of the restriction of f to Ω is equal to $Y\setminus(Y_1\cup Y_2)$. Indeed, for $(u,v,w)\in\Omega$, if $(x,y,z)\in\bar{B}_3(0,\varepsilon)$ satisfies $F(x,y,z)=(u,v,w)$, we get $\varphi(v,w)=x^2+y^2$, $\psi(v,w)=y(e^x-1)$ and $u=\psi(v,w)+\varphi(v,w)+z^2-\varepsilon^2$.

One sees that $f(u,v,w)=(\varepsilon^2-(x^2+y^2+z^2))z^2$. To finish, it is enough to note that the points of $F(\bar{B}_3(0,\varepsilon)\cap\{xy=0\})$ and of $F(\partial B_3(0,\varepsilon))$ are never in Ω. Hence $\tilde{f}$ is subanalytic.

Let us show that it is continuous along $\partial\Omega$, since it is $\mathscr{C}^\infty$ on $\mathbb{R}^3\setminus\partial\Omega$. Let $(u_0,v_0,w_0)\in\partial\Omega$. First assume that $(u_0,v_0,w_0)\in\Delta^+$. Then $u_0=\psi(v_0,w_0)$, in other words, we get the image under F of a point $(x,y,z)\in\partial B_3(0,\varepsilon)\setminus\{xy=0\}$. Hence the limit of $u-\psi(v,w)$ when $(u,v,w)\in\Omega$ tends to (u_0,v_0,w_0) is zero. As the functions ψ and φ are bounded on Ω, the limit of $\tilde{f}$ is zero at such a point.

([5]) As pointed out by the referee, this fact is a consequence of basic stability properties of subanalytic functions. We give a direct proof for non-specialists.

If $(u_0, v_0, w_0) \in \Delta^-$, then we have the image of a point in

$$(\bar{B}_3(0,\varepsilon) \cap \{z = 0\}) \setminus \{xy = 0\}.$$

Since the function ψ is bounded on Γ the limit of f at such a point is zero, and so is the case for $\tilde{f}$.

If $(u_0, v_0, w_0) \in \Delta^0$ then $v_0 = w_0 = 0$ and the function f is bounded, hence $\tilde{f}$ tends to 0 at such a point.

Let us finally show that Ω is the set where $\tilde{f}$ is strictly positive. It is sufficient to check that $vw \neq 0$ on Ω. But $vw = 0$ entails $xy = 0$ and so $v = w = 0$ and $u = x^2 + y^2 + z^2 - \varepsilon^2$, in other words, $(u, v, w) \in [-\varepsilon^2, 0] \times (0, 0) = \Delta^0$. Hence such a (v, w) belongs to $\partial\Omega$.

We have now constructed a subanalytic function $\tilde{f}$ on $\mathbb{R}^3$ which is continuous and strictly positive exactly on $\Omega \subset\subset \mathbb{R}^3$. By Corollary 3.4 there exists a positive integer N such that $\tilde{f}^N$ is of class $\mathscr{C}^2$. Then one gets a Stein open subanalytic set of $\mathbb{C}^3$ which intersects $\mathbb{R}^3$ exactly in Ω as in the general proof of Theorem 2.4. ∎

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Daniel Barlet
Institut Élie Cartan
Université de Nancy
Laboratoire de Mathématiques
B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France
E-mail: Daniel.Barlet@iecn.u-nancy.fr

Teresa Monteiro Fernandes
Centro de Matemática e Aplicações Fundamentais
Departamento de Matemática
da Faculdade de Ciências
da Universidade de Lisboa
Edifício C6, P.2, Campo Grande
1749-16 Lisboa, Portugal
E-mail: tmf@ptmat.fc.ul.pt

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