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Maps preserving zero products

by

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Abstract. A linear map T from a Banach algebra A into another B preserves zero products if $T(a)T(b) = 0$ whenever $a, b \in A$ are such that $ab = 0$. This paper is mainly concerned with the question of whether every continuous linear surjective map $T\colon A \to B$ that preserves zero products is a weighted homomorphism. We show that this is indeed the case for a large class of Banach algebras which includes group algebras.

Our method involves continuous bilinear maps $\phi\colon A \times A \to X$ (for some Banach space X) with the property that $\phi(a, b) = 0$ whenever $a, b \in A$ are such that $ab = 0$. We prove that such a map necessarily satisfies $\phi(a\mu, b) = \phi(a, \mu b)$ for all $a, b \in A$ and for all μ from the closure with respect to the strong operator topology of the subalgebra of $\mathcal{M}(A)$ (the multiplier algebra of A) generated by doubly power-bounded elements of $\mathcal{M}(A)$. This method is also shown to be useful for characterizing derivations through the zero products.

Introduction. S. Banach [4] was the first to describe isometries on $L^p([0,1])$ with $p \neq 2$. Although Banach did not give the full proof for this case (this was provided by J. Lamperti [26]), he made the key observation that isometries on $L^p([0,1])$ must take functions with disjoint support into functions with disjoint support. This property arises in a variety of situations and has been considered by several authors. For example, in the theory of Banach lattices there is an extensive literature about linear maps $T\colon X \to Y$, where X and Y are Banach lattices, with the property that $|T(x)| \wedge |T(y)| = 0$ whenever $x, y \in X$ are such that $|x| \wedge |y| = 0$. Such maps are usually called *disjointness preserving operators* or *d-homomorphisms*. The reader interested in this setting is referred to the monograph [1]. The concept of a disjointness preserving operator was exported to function algebras by E. Beckenstein and L. Narici (see [5] for general information).

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Let A and B be function algebras. Linear operators $T\colon A \to B$ with the property that $T(a)T(b) = 0$ whenever $a, b \in A$ are such that $ab = 0$ have been studied over the years and by many authors. They are usually considered under the name *Lamperti operators* or *separating maps*. Typical examples are provided by weighted composition operators, and it turns out that in certain important cases these are the canonical continuous separating maps. The concept of separating maps can be extended in various ways to general (possibly noncommutative) Banach algebras, and in fact even to pure algebras. The most common and natural way is to consider literally the same condition, that is,

$$a, b \in A,\ ab = 0 \ \Rightarrow\ T(a)T(b) = 0, \tag{0.1}$$

where T is a linear map between algebras A and B. Usually such operators are called *zero product preserving maps* and the standard goal is to characterize them as weighted homomorphisms. For recent results on such and related maps defined on operator algebras we refer the reader to [2, 3, 9–13, 20, 21, 25, 30, 35], and on the other hand to [17] where group algebras corresponding to locally compact abelian groups are considered.

Our method for investigating a continuous linear map $T\colon A \to B$ preserving zero products consists in introducing a continuous bilinear map $\phi\colon A \times A \to B$ by $\phi(a, b) = T(a)T(b)$ $(a, b \in A)$ which obviously satisfies the condition $\phi(a, b) = 0$ whenever $a, b \in A$ are such that $ab = 0$. Motivated by this fact we study bilinear maps $\phi\colon A \times A \to X$ into some Banach space X that preserve zero products in the sense that

$$a, b \in A,\ ab = 0 \ \Rightarrow\ \phi(a, b) = 0. \tag{0.2}$$

The study of (0.2) was initiated in our preceding paper [2]. However, [2, Lemma 2.1] considers (0.2) only in a very special setting and the conclusion is much less general than the one that will be obtained in this paper.

The paper is organized as follows. In the first section we introduce the notation and terminology, and then the class of faithful Banach algebras A satisfying the following condition:

$(\mathbb{A})$ $A \subset \overline{\mathcal{D}(A)}^{\mathrm{so}}$.

Here $\mathcal{D}(A)$ stands for the subalgebra of the multiplier algebra of A generated by the doubly power-bounded elements and $\overline{(\cdot)}^{\mathrm{so}}$ denotes the closure with respect to the strong operator topology. We show that the class of Banach algebras having the property $(\mathbb{A})$ is really large, it includes: group algebras, C^*-algebras, and Banach algebras generated by idempotents.

Section 2 is concerned with continuous bilinear maps preserving zero products in the sense of (0.2). At the beginning we introduce another class of Banach algebras, namely those that have the following property:

($\mathbb{B}$) for any Banach space X and every continuous bilinear map $\phi\colon A\times A \to X$, (0.2) implies that $\phi(ab,c)=\phi(a,bc)$ for all $a,b,c\in A$.

Under mild technical assumptions, the condition $\phi(ab,c)=\phi(a,bc)$ further implies that ϕ is of the form

$$\phi(a,b)=\Phi(ab)$$

for some continuous linear map $\Phi : A\to X$, which is clearly the best possible description of a map satisfying (0.2). For example, if A is unital, then just take $b=\mathbf{1}$ and note that Φ can be defined according to $\Phi(a)=\phi(a,\mathbf{1})$. However, except in a few situations, throughout the paper we try to avoid the assumption that our algebras are unital, in order to cover some important examples of Banach algebras such as group algebras. On the other hand, we will often assume the existence of an approximate identity.

We show that the property ($\mathbb{B}$) is stable under the usual methods of constructing Banach algebras. However, it does not seem obvious why Banach algebras with the property ($\mathbb{B}$) should exist at all. A crucial result of the paper is Theorem 2.11 which tells us that for every Banach algebra A we have:

(0.3) $\qquad A$ has the property ($\mathbb{A}$) $\Rightarrow$ A has the property ($\mathbb{B}$).

More precisely, Theorem 2.11 states that (0.2) implies that $\phi(a\mu,b)=\phi(a,\mu b)$ for all $a,b\in A$ and all $\mu\in\overline{\mathcal{D}(A)}^{\mathrm{so}}$, from which (0.3) clearly follows. As is evident from this statement, our approach to zero product preservers is based on some special invertible elements (typical examples are point mass measures in the context of group algebras and unitary elements in C^*-algebras). This seems to be a novelty. The traditional approach is based on some other types of elements, particularly on idempotents. We also remark that the proof rests heavily on a technical result (Lemma 2.10) about $\ell^1(\mathbb{Z})$ using classical Fourier analysis.

Of course (0.3) gives a convincing justification for dealing with the class of Banach algebras with the property ($\mathbb{B}$). The part of the paper after Theorem 2.11 is mostly concerned with this class.

In the third section we consider the central theme of the paper, that is, continuous linear maps $T\colon A\to B$ between Banach algebras A and B that satisfy $T(a)T(b)=0$ whenever $a,b\in A$ are such that $ab=0$. Roughly speaking, the goal is to show that T is close to a homomorphism. The heart of the paper are Theorems 3.4 and 3.8 which essentially say that under rather mild assumptions on Banach algebras A and B, a continuous linear map $T\colon A\to B$ that preserves zero products is a weighted homomorphism, whenever A has the property ($\mathbb{B}$). As a matter of fact, we show that if G_1 and G_2 are locally compact groups and $T\colon L^1(G_1)\to L^1(G_2)$ is a continuous surjective linear map preserving zero products, then there are an invertible

measure μ in the centre of the measure algebra $M(G_2)$ and a continuous epimorphism $\Phi\colon L^1(G_1) \to L^1(G_2)$ with $T = \mu\Phi$.

Finally, in Section 4 we show that the study of bilinear maps preserving zero products contained in the second section is also useful for characterizing derivations through zero products. Over the last years a considerable attention has been paid to characterizations of such kind [2, 8, 9, 23, 37]. In the recent papers [2, 8], it was observed that characterizing derivations through the zero product analysis can be a powerful tool in studying local derivations, which is another popular research area.

1. The seminal property. The purpose of this section is to introduce the notation and terminology, and then the class of Banach algebras with the property $(\mathbb{A})$. We are actually mostly interested in another class of Banach algebras, namely those with the property $(\mathbb{B})$. Nevertheless, the property $(\mathbb{A})$ is of extreme importance for this paper because of its close connection with the property $(\mathbb{B})$.

1.1. *Some notation and terminology.* All Banach spaces and Banach algebras which we consider throughout this paper are assumed to be complex.

Let X be a linear space. The linear span of a nonempty subset $S \subset X$ is denoted by $\operatorname{lin} S$. The algebra of linear operators from X into itself is denoted by $\mathcal{L}(X)$. Let $\mathcal{B}(X)$ denote the Banach algebra of all continuous linear operators on a given nonzero complex Banach space X, and let X^* denote the topological dual space of X. We write $\langle\cdot,\cdot\rangle$ for the dual pairing of X and X^*.

Let A be an algebra. Then:

(i) A is *left faithful* if $\{a \in A\colon aA = \{0\}\} = \{0\}$;
(ii) A is *right faithful* if $\{a \in A\colon Aa = \{0\}\} = \{0\}$;
(iii) A is *faithful* if $\{a \in A\colon aA = Aa = \{0\}\} = \{0\}$.

We write A^{op} for the *opposite algebra* to A. For a nonempty subset $S \subset A$ and $n \in \mathbb{N}$, set

$$S^n = \operatorname{lin}\{a_1 \cdots a_n\colon a_1, \ldots, a_n \in S\}.$$

Further, $\operatorname{alg} S$ denotes the subalgebra of A generated by S.

Let A be a Banach algebra. Then:

(i) A *factors weakly* if $A = A^2$;
(ii) A is *essential* if $A = \overline{A^2}$.

A *left* [*right*] *approximate identity* for A is a net $(\rho_i)_{i\in I}$ in A such that $\lim \rho_i a = a$ [$\lim a\rho_i = a$] for each $a \in A$. An *approximate identity* for A is a net $(\rho_i)_{i\in I}$ which is both a left and a right approximate identity. A [left or right] approximate identity $(\rho_i)_{i\in I}$ is *bounded* if $\sup_{i\in I} \|\rho_i\| < \infty$. It should be pointed out that if a Banach algebra has a bounded left approximate

identity and a bounded right approximate identity, then it has a bounded approximate identity [14, Proposition 2.9.3]. In general, the existence of both a left approximate identity and a right approximate identity in a Banach algebra does not imply the existence of an approximate identity [15, §2]. Obviously, if A has a left approximate identity, then A is right faithful and essential. According to Cohen's factorization theorem [14, Theorem 2.9.24], if A has a bounded left approximate identity, then A factors weakly, in fact $A = \{ab\colon a, b \in A\}$.

Since approximate identities will play an important role in this paper, it seems appropriate to mention some relevant examples of Banach algebras containing them. For a more complete account we refer the reader to [14, 15], where the corresponding hereditary properties are also studied. We shall mention just a few concrete examples and a few classes of algebras in which we are mostly interested in this paper (see remarks concerning the conditions ($\mathbb{A}$) and ($\mathbb{B}$) below).

- Let G be a locally compact group with left Haar measure λ. As is customary, we write $L^1(G)$ for the Banach algebra of those (equivalence classes of) Borel functions on G which are integrable with respect to λ. The group algebra $L^1(G)$ has a bounded approximate identity of bound 1 [14, Theorem 3.3.23].
- Every C^*-algebra has a bounded approximate identity of bound 1 [14, Theorem 3.2.21].
- The Schatten class $S_p(H)$ has an approximate identity for every Hilbert space H and $1 \le p < \infty$.
- If X is a Banach space with the approximation property, then $\mathcal{A}(X)$, the algebra of all approximable operators, has a left approximate identity [32, Proposition 3.1.1]. Furthermore, $\mathcal{A}(X)$ has a bounded left approximate identity if and only if the Banach space X has the bounded approximation property [32, Corollary 3.1.3]. The usual examples of separable Banach spaces have the bounded approximation property and, as a matter of fact, they even have a basis. The space $L^p(\mu)$ for any measure space (Ω, Σ, μ) and $1 \le p \le \infty$ and the space $C_0(\Omega)$ for any locally compact Hausdorff space have the bounded approximation property (see [32, Definition 3.1.8 and Examples 3.1.12]).

1.2. *Introducing the property* ($\mathbb{A}$). Let A be an algebra. A *multiplier* on A is a pair (L, R), where $L, R\colon A \to A$ are linear maps such that, for all $a, b \in A$, the following identities hold:

$$L(ab) = L(a)b, \qquad R(ab) = aR(b), \quad \text{and} \quad aL(b) = R(a)b.$$

The set of all multipliers on A is denoted by $\mathcal{M}(A)$. It is a unital subalgebra of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$ and it is called the *multiplier algebra* of A. The map

$$a \mapsto (L_a, R_a), \qquad A \to \mathcal{M}(A),$$

where $L_a(b) = ab = R_b(a)$ for all $a, b \in A$, is a homomorphism, and it is a monomorphism if and only if A is faithful. In this case we shall always identify A with its image via this embedding and it is easily checked that A is a two-sided ideal of $\mathcal{M}(A)$. Moreover, A is unital if and only if $A = \mathcal{M}(A)$. As usual, the *unitisation* $A_{\mathbf{1}}$ of A is the subalgebra of $\mathcal{M}(A)$ generated by A and $\mathbf{1}$.

We now turn to the case when A is a faithful Banach algebra. Then every multiplier on A consists of continuous linear operators on A so that $\mathcal{M}(A)$ becomes a unital closed subalgebra of $\mathcal{B}(A) \times \mathcal{B}(A)^{\mathrm{op}}$ and the embedding of A into $\mathcal{M}(A)$ is continuous [14, Proposition 2.5.12]. An invertible element $\mu \in \mathcal{M}(A)$ is said to be *doubly power-bounded* if $\sup_{k\in\mathbb{Z}} \|\mu^k\| < \infty$. Faithful Banach algebras generated by the doubly power-bounded elements in the multiplier algebra will play the crucial role throughout this paper. Let $\mathcal{D}(A)$ stand for the subalgebra of $\mathcal{M}(A)$ generated by the doubly power-bounded elements in $\mathcal{M}(A)$. The *strong operator topology* on $\mathcal{M}(A)$, denoted by "so", is determined by the family of seminorms $\{p_a \colon a \in A\}$, where

$$p_a(L, R) = \max\{\|L(a)\|, \|R(a)\|\} \qquad ((L, R) \in \mathcal{M}(A)).$$

Therefore, a net $(\mu_i)_{i\in I}$ in $\mathcal{M}(A)$ converges to $\mu \in \mathcal{M}(A)$ with respect to the strong operator topology if and only if $\lim \mu_i a = \mu a$ and $\lim a\mu_i = a\mu$ with respect to the norm for each $a \in A$.

Definition 1.1. We say that a Banach algebra A *has the property* $(\mathbb{A})$ if it is faithful and satisfies

$$A \subset \overline{\mathcal{D}(A)}^{\mathrm{so}}.$$

Further, we say that a faithful Banach algebra A *has the property* $(\mathbb{A}_{\mathbf{1}})$ if

$$A_{\mathbf{1}} = \overline{\mathrm{alg}\{u \in A_{\mathbf{1}} \colon u \text{ is doubly power-bounded}\}}.$$

Of course, $(\mathbb{A}_{\mathbf{1}})$ is stronger than $(\mathbb{A})$.

Remark 1.2. There are several important classes of elements in the multiplier algebra $\mathcal{M}(A)$ of a Banach algebra A which lie in the closed algebra generated by doubly power-bounded elements.

(i) Let A be a unital Banach algebra. For $a \in A$, we define the *numerical range* of a by

$$V(a) = \{\langle a, f\rangle \colon f \in A^*,\ \langle \mathbf{1}, f\rangle = \|f\| = 1\};$$

this is a compact convex subset of the complex plane containing the spectrum of a. An element $a \in A$ is called *hermitian* if $V(a) \subset \mathbb{R}$, which is equivalent to the property that $\|\exp(ita)\| = 1$ for each $t \in \mathbb{R}$ (see [6, §10]). It is well known that hermitian elements of a

unital C^*-algebra are precisely the self-adjoint elements [6, Proposition 20 in §12].

It should be pointed out that if A is a Banach algebra and $\mu \in \mathcal{M}(A)$ is hermitian, then

$$\mu = \frac{1}{i} \lim_{t \to 0} \frac{\exp(it\mu) - \mathbf{1}}{t} \in \overline{\mathcal{D}(A)}.$$

(ii) Throughout, the set of idempotents of a given algebra A is denoted by $\mathcal{I}(A)$. Let A be a Banach algebra and let $e \in \mathcal{I}(\mathcal{M}(A))$. Then $(\mathbf{1} - 2e)^2 = \mathbf{1}$, which obviously implies that $\mathbf{1} - 2e$ is doubly power-bounded. Moreover, $e = \frac{1}{2}\mathbf{1} - \frac{1}{2}(\mathbf{1} - 2e)$ and hence $e \in \mathcal{D}(A)$.

1.3. *Examples of Banach algebras with the property* $(\mathbb{A})$. The goal of this short section is to collect some basic examples of Banach algebras which have the property $(\mathbb{A})$.

1. *Group algebras.* Let G be a locally compact group. We write $M(G)$ for the Banach algebra of all complex, regular Borel measures on G. Of course, $L^1(G)$ can be thought of as the two-sided closed ideal of $M(G)$ consisting of all measures in $M(G)$ which are absolutely continuous with respect to λ. In fact, by Wendel's theorem [14, Theorem 3.3.40], $M(G)$ is nothing other than the multiplier algebra of $L^1(G)$. For every $t \in G$, the unit point mass measure δ_t at t is clearly doubly power-bounded. Moreover, [14, Proposition 3.3.41] shows that $\overline{\operatorname{lin}\{\delta_t \colon t \in G\}}^{\mathrm{so}} = M(G)$. Hence $L^1(G)$ has the property $(\mathbb{A})$.

2. *C^*-algebras.* Let A be a C^*-algebra. Then $A_{\mathbf{1}}$ becomes a unital C^*-algebra (see [14, Proposition 3.2.39]). It is well known that every element in $A_{\mathbf{1}}$ is a linear combination of four unitary elements [34, Proposition I.4.9]; recall that an element $u \in A_{\mathbf{1}}$ is *unitary* if $uu^* = u^*u = \mathbf{1}$. Since unitary elements are clearly doubly power-bounded, $A_{\mathbf{1}}$ is generated by the doubly power-bounded elements and so A has the property $(\mathbb{A}_{\mathbf{1}})$.

3. *Banach algebras generated by idempotents.* In view of Remark 1.2(ii), if A is a Banach algebra with the property that $A = \overline{\operatorname{alg} \mathcal{I}(A)}$, then A has the property $(\mathbb{A}_{\mathbf{1}})$. Some examples of such Banach algebras are the following.

(1) The Banach algebras $\mathcal{A}(X)$ and $\mathcal{N}(X)$ of all *approximable operators* and all *nuclear operators*, respectively, on any Banach space X.
(2) The classical *Schatten class* $S_p(H)$ for any Hilbert space H and $1 \leq p \leq \infty$.
(3) The Banach algebras $\ell^p(I)$ with $1 \leq p < \infty$ and $c_0(I)$ for any nonempty set I.
(4) The Banach algebras $L^p(G)$ with $1 \leq p < \infty$ and $C(G)$ (with convolution product) for any compact group G. Indeed, on account of [19, Theorem 27.21], the linear space $\mathcal{T}(G)$ of all *trigonometric polynomials* on G (see [19, Definition 27.7]) is the linear span of the

idempotents, and [16, Theorem 5.11] shows that $\mathcal{T}(G)$ is dense in $L^p(G)$ for $1 \le p < \infty$ and $C(G)$.

(5) Topologically simple Banach algebras containing a nontrivial idempotent. Indeed, a nontrivial idempotent e (i.e., $e \neq 0$ and $e \neq \mathbf{1}$) in such an algebra A clearly cannot be contained in the centre of A. This implies that $\operatorname{alg}\mathcal{I}(A)$ contains a nonzero ideal of A [8, Lemma 2.1], from which $A = \overline{\operatorname{alg}\mathcal{I}(A)}$ follows.

2. Bilinear maps preserving zero products

Definition 2.1. Let A be a Banach algebra and let $\phi : A \times A \to X$, where X is a Banach space, be a continuous bilinear map. We say that ϕ *preserves zero products* if

$$a, b \in A,\ ab = 0 \ \Rightarrow\ \phi(a, b) = 0.$$

2.1. *Introducing the property* $(\mathbb{B})$

Definition 2.2. A Banach algebra A *has the property* $(\mathbb{B})$ if every continuous bilinear map $\phi\colon A \times A \to X$, where X is an arbitrary Banach space, preserving zero products satisfies

$$\phi(ab, c) = \phi(a, bc) \qquad (a, b, c \in A).$$

The basic result in this section is Theorem 2.11 which shows that the class of Banach algebras with the property $(\mathbb{B})$ is large since it contains the class of algebras with the property $(\mathbb{A})$.

A typical example of a map ϕ from the above definition is the one given by $\phi(a, b) = \Phi(ab)$ where $\Phi : A \to X$ is a linear operator. Under mild assumptions this is also the only possible example.

Lemma 2.3. *Let A be a Banach algebra with the property $(\mathbb{B})$. If A has a left approximate identity, then every continuous bilinear map $\phi\colon A \times A \to X$ preserving zero products is of the form*

$$\phi(a, b) = \Phi(ab) \qquad (a, b \in A)$$

for some linear operator $\Phi : A^2 \to X$. Moreover, if A has a bounded left approximate identity, then Φ is continuous.

Proof. Let $(\rho_i)_{i\in I}$ be a left approximate identity of A. Since A has the property $(\mathbb{B})$, for all $a, b \in A$ we have

$$\phi(a, b) = \lim_{i\in I} \phi(\rho_i a, b) = \lim_{i\in I} \phi(\rho_i, ab). \tag{2.1}$$

Therefore the net $(\phi(\rho_i, a))_{i\in I}$ is convergent for each $a \in A^2$ and so we may define a linear operator

$$\Phi\colon A^2 \to X, \qquad \Phi(a) = \lim_{i\in I} \phi(\rho_i, a) \qquad (a \in A^2).$$

According to the definition of Φ and in view of (2.1), we have $\phi(a,b) = \Phi(ab)$ for all $a, b \in A$.

If $(\rho_i)_{i\in I}$ is bounded, then, for every $a \in A$,

$$\|\phi(\rho_i, a)\| \le \|\phi\| \sup_{i\in I} \|\rho_i\| \, \|a\| \qquad (i \in I)$$

and so Φ is continuous with $\|\Phi\| \le \|\phi\| \sup_{i\in I}\|\rho_i\|$. ∎

COROLLARY 2.4. *Let A be a commutative Banach algebra with the property* $(\mathbb{B})$. *If A has a left approximate identity, then every continuous bilinear map $\phi\colon A \times A \to X$ preserving zero products is symmetric, that is, $\phi(a,b) = \phi(b,a)$ for all $a, b \in A$.*

Proof. By Lemma 2.3 we have $\phi(a,b) = \Phi(ab) = \Phi(ba) = \phi(b,a)$. ∎

2.2. *The heredity of the property* $(\mathbb{B})$

PROPOSITION 2.5. *Let A be a Banach algebra with the property* $(\mathbb{B})$ *and let I be a closed two-sided ideal of A. Then the following assertions hold*:

(i) *A/I has the property* $(\mathbb{B})$.
(ii) *If $\overline{\mathrm{lin}(AI)} = \overline{\mathrm{lin}(IA)} = I$ (in particular, if either A or I has an approximate identity), then I has the property* $(\mathbb{B})$.

Proof. Let X be a Banach space. Let $\phi\colon A/I \times A/I \to X$ be a continuous linear map preserving zero products for some Banach space X. We then define $\psi\colon A \times A \to X$ by $\psi(a,b) = \phi(a+I, b+I)$ $(a, b \in A)$, which preserves zero products, and so the property $(\mathbb{B})$ gives $\psi(ab, c) = \psi(a, bc)$ for all $a, b, c \in A$. This clearly forces $\phi(xy, z) = \phi(x, yz)$ for all $x, y, z \in A/I$.

We now take a continuous bilinear map $\phi\colon I \times I \to X$ preserving zero products. Pick $u, v \in I$ and define $\psi_{u,v}\colon A \times A \to X$ by $\psi_{u,v}(a,b) = \phi(ua, bv)$ for all $a, b \in A$. We see that $\psi_{u,v}$ preserves zero products and so our assumption implies that $\psi_{u,v}(ab, c) = \psi_{u,v}(a, bc)$, and hence $\phi(uab, cv) = \phi(ua, bcv)$ for all $a, b, c \in A$. Using $\overline{\mathrm{lin}(AI)} = \overline{\mathrm{lin}(IA)} = I$ we arrive at the required identity for ϕ. ∎

PROPOSITION 2.6. *Let A be a Banach algebra with the property* $(\mathbb{B})$, *let B be a Banach algebra, and let $\Phi\colon A \to B$ be a continuous homomorphism with dense range. Then B has the property* $(\mathbb{B})$.

Proof. Let $\psi\colon B \times B \to X$ be a continuous bilinear map preserving zero products, for some Banach space X. We then define $\phi\colon A \times A \to X$ by $\phi(a,b) = \psi(\Phi(a), \Phi(b))$ for all $a, b \in A$. It is clear that ϕ is a continuous bilinear map preserving zero products and so $\phi(ab, c) = \phi(a, bc)$ for all $a, b, c \in A$. This can be rewritten as $\psi(\Phi(a)\Phi(b), \Phi(c)) = \psi(\Phi(a), \Phi(b)\Phi(c))$ for all $a, b, c \in A$. Since the range of Φ is dense, it follows that $\psi(uv, w) = \psi(u, vw)$ for all $u, v, w \in B$. ∎

PROPOSITION 2.7. *Let A and B be Banach algebras with the property* $(\mathbb{B})$. *Then the projective tensor product $A \otimes_\pi B$ has the property* $(\mathbb{B})$.

Proof. Let $\phi\colon (A \otimes_\pi B) \times (A \otimes_\pi B) \to X$ be a continuous bilinear map preserving zero products, for some Banach space X. For $u, v \in B$ we define $\phi_{u,v}\colon A \times A \to X$ by $\phi_{u,v}(a,b) = \phi(a \otimes u, b \otimes v)$ for all $a, b \in A$. Then $\phi_{u,v}$ preserves zero products and so $\phi_{u,v}(ab, c) = \phi_{u,v}(a, bc)$ for all $a, b, c \in A$. We thus get

$$\phi(ab \otimes u, c \otimes v) = \phi(a \otimes u, bc \otimes v) \quad (a, b, c \in A,\ u, v \in B). \tag{2.2}$$

Likewise we show that

$$\phi(a \otimes uv, b \otimes w) = \phi(a \otimes u, b \otimes vw) \quad (a, b \in A,\ u, v, w \in B). \tag{2.3}$$

If $a, b, c \in A$ and $u, v, w \in B$, then we deduce from (2.2) and (2.3) that

$$\begin{aligned} \phi((a \otimes u)(b \otimes v), c \otimes w) &= \phi(ab \otimes uv, c \otimes w) \\ &= \phi(a \otimes uv, bc \otimes w) = \phi(a \otimes u, bc \otimes vw) \\ &= \phi(a \otimes u, (b \otimes v)(c \otimes w)). \end{aligned} \tag{2.4}$$

Since the linear span of the elementary tensors $a \otimes u$ ($a \in A$, $u \in B$) is dense in $A \otimes_\pi B$ and ϕ is continuous, (2.4) implies that ϕ satisfies $\phi(xy, z) = \phi(x, yz)$ for all $x, y, z \in A \otimes_\pi B$. ∎

Let Ω be a locally compact Hausdorff space. We denote by $C_0(\Omega, A)$ the Banach algebra of all continuous functions from Ω into a Banach algebra A which vanish at infinity.

PROPOSITION 2.8. *Let Ω be a locally compact Hausdorff space and let A be a Banach algebra with the property* $(\mathbb{B})$. *Then $C_0(\Omega, A)$ has the property* $(\mathbb{B})$.

Proof. It is well known that the map $f \otimes a \mapsto f(\cdot)a$ induces an isometric algebra isomorphism from the injective tensor product $C_0(\Omega) \otimes_\epsilon A$ onto $C_0(\Omega, A)$. We then consider the canonical map $\kappa\colon C_0(\Omega) \otimes_\pi A \to C_0(\Omega) \otimes_\epsilon A$, which is a homomorphism with dense range. On the other hand, $C_0(\Omega)$ is a C^*-algebra and so it has the property $(\mathbb{A})$ (Example 1.3(iii)) and therefore the property $(\mathbb{B})$ (see Theorem 2.11 below). By Proposition 2.7, $C_0(\Omega) \otimes_\pi A$ has the property $(\mathbb{B})$. Proposition 2.6 now shows that $C_0(\Omega) \otimes_\epsilon A$ has the property $(\mathbb{B})$. ∎

Let $(A_i)_{i \in I}$ be a family of Banach algebras, and take $1 \le p \le \infty$. Then we write $\ell^p(I, A_i)$ and $c_0(I, A_i)$ for the Banach algebras defined in [14, Example 2.1.18(iii)].

PROPOSITION 2.9. *Let $(A_i)_{i \in I}$ be a family of Banach algebras with the property* $(\mathbb{B})$. *Then $\ell^p(I, A_i)$ with $1 \le p < \infty$ and $c_0(I, A_i)$ have the property* $(\mathbb{B})$.

Proof. Let A denote either $\ell^p(I, A_i)$ or $c_0(I, A_i)$ and let A_0 be the subalgebra of A consisting of the families $(a_i)_{i\in I}$ which are zero except at a finite number of elements of I. Observe that A_0 is dense in A. We also consider the natural embeddings $\epsilon_i\colon A_i \to A$ with $i \in I$.

Let $\phi\colon A\times A \to X$ be a continuous bilinear map preserving zero products. For each $i \in I$, consider the map $\phi_i\colon A_i \times A_i \to X$ given by $\phi_i(a_i, b_i) = \phi(\epsilon_i(a_i), \epsilon_i(b_i))$ for all $a_i, b_i \in A_i$. It is clear that ϕ_i preserves zero products and so $\phi(\epsilon_i(a_ib_i), \epsilon_i(c_i)) = \phi(\epsilon_i(a_i), \epsilon_i(b_ic_i))$ for all $a_i, b_i, c_i \in A_i$. This clearly entails that $\phi(ab, c) = \phi(a, bc)$ for all $a, b, c \in A_0$. Since A_0 is dense in A and ϕ is continuous we finally arrive at $\phi(ab, c) = \phi(a, bc)$ for all $a, b, c \in A$. ∎

2.3. *Banach algebras with the property* $(\mathbb{A})$ *have the property* $(\mathbb{B})$. The key tool for proving our fundamental result that the property $(\mathbb{A})$ implies the property $(\mathbb{B})$ is the following lemma concerning $\ell^1(\mathbb{Z})$. In our recent paper a similar result was obtained for the algebra $C(I)$ of continuous functions on a compact interval I [2, Lemma 2.1]. In the following, $\mathcal{F}$ denotes the Fourier transform and $\mathbb{T}$ stands for the circle group.

LEMMA 2.10. *Let $\phi\colon \ell^1(\mathbb{Z}) \times \ell^1(\mathbb{Z}) \to X$ be a continuous bilinear map with the property*

$$a, b \in \ell^1(\mathbb{Z}),\ \operatorname{supp}(\mathcal{F}(a)) \cap \operatorname{supp}(\mathcal{F}(b)) = \emptyset \ \Rightarrow\ \phi(a, b) = 0. \tag{2.5}$$

Then ϕ is symmetric, i.e. $\phi(a, b) = \phi(b, a)$ for all $a, b \in \ell^1(\mathbb{Z})$.

Proof. Throughout the proof we shall consider the usual elements δ_j in $\ell^1(\mathbb{Z})$ defined by

$$\delta_j(k) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \qquad (j, k \in \mathbb{Z}).$$

The map ϕ gives rise to a continuous linear operator Φ on the projective tensor product $\ell^1(\mathbb{Z}) \otimes_\pi \ell^1(\mathbb{Z})$ defined through

$$\Phi(a \otimes b) = \phi(a, b) \qquad (a, b \in \ell^1(\mathbb{Z})).$$

On the other hand, the Banach algebra $\ell^1(\mathbb{Z}) \otimes_\pi \ell^1(\mathbb{Z})$ may be identified with the Banach algebra $\ell^1(\mathbb{Z} \times \mathbb{Z})$ by defining

$$(a \otimes b)(j, k) = a(j)b(k)$$

for all $a, b \in \ell^1(\mathbb{Z})$ and $j, k \in \mathbb{Z}$. Hence Φ may be thought of as a continuous linear operator on $\ell^1(\mathbb{Z} \times \mathbb{Z})$.

Our first objective is to prove that $\Phi(x) = 0$ whenever $x \in \ell^1(\mathbb{Z} \times \mathbb{Z})$ is such that

$$\operatorname{supp}(\mathcal{F}(x)) \cap \{(z, z)\colon z \in \mathbb{T}\} = \emptyset. \tag{2.6}$$

Let $x \in \ell^1(\mathbb{Z}\times\mathbb{Z})$ satisfy (2.6), write $x = \sum_{n=1}^\infty u_n \otimes v_n$ with $u_n, v_n \in \ell^1(\mathbb{Z})$, let $\delta > 0$ be such that

$$\delta \le |z - u| + |w - u| \qquad ((z, w) \in \operatorname{supp}(\mathcal{F}(x)),\ u \in \mathbb{T}) \tag{2.7}$$

and let $m \in \mathbb{N}$ be such that $8\sin(\pi/m) < \delta$. We now consider the open subsets of $\mathbb{T}$ defined by

$$U_k = \{z \in \mathbb{T}\colon |z - e^{2\pi k/mi}| < 2\sin(\pi/m)\} \quad (k = 1, \dots, m).$$

Since $\bigcup_{k=1}^m U_k = \mathbb{T}$, it is well known that there exist smooth functions $\omega_1, \dots, \omega_m$ on $\mathbb{T}$ with $\omega_1 + \dots + \omega_m = \mathbf{1}$ and $\operatorname{supp}(\omega_k) \subset U_k$ for $k = 1, \dots, m$. By the smoothness, we have $\omega_k = \mathcal{F}(a_k)$ for some $a_k \in \ell^1(\mathbb{Z})$ for each $k = 1, \dots, m$. We thus get elements $a_1, \dots, a_m \in \ell^1(\mathbb{Z})$ such that

$$a_1 + \dots + a_m = \delta_0 \quad \text{and} \quad \operatorname{supp}(\mathcal{F}(a_k)) \subset U_k \quad (k = 1, \dots, m).$$

It is easily seen that

$$\operatorname{supp}(\mathcal{F}(x)) \cap (U_j \times U_k) = \emptyset$$

whenever U_j and U_k are such that $U_j \cap U_k \neq \emptyset$. This last property, together with the fact that $\sum_{k=1}^m a_k = \delta_0$, implies that

$$\begin{aligned} x &= \sum_{j=1}^m \sum_{k=1}^m x * (a_j \otimes a_k) = \sum_{U_j \cap U_k = \emptyset} x * (a_j \otimes a_k) \\ &= \sum_{U_j \cap U_k = \emptyset} \sum_{n=1}^\infty (u_n * a_j) \otimes (v_n * a_k). \end{aligned}$$

Finally, we observe that

$$\Phi(x) = \sum_{U_j \cap U_k = \emptyset} \sum_{n=1}^\infty \phi(u_n * a_j, v_n * a_k) = 0,$$

because of (2.5) and

$$\begin{aligned} &\operatorname{supp}(\mathcal{F}(u_n * a_j)) \cap \operatorname{supp}(\mathcal{F}(v_n * a_k)) \\ &\qquad = \operatorname{supp}(\mathcal{F}(u_n)\mathcal{F}(a_j)) \cap \operatorname{supp}(\mathcal{F}(v_n)\mathcal{F}(a_k)) \\ &\qquad \subset \operatorname{supp}(\mathcal{F}(a_j)) \cap \operatorname{supp}(\mathcal{F}(a_k)) \subset U_j \cap U_k. \end{aligned}$$

Our next goal is to show that $\Phi(x) = 0$ whenever $x \in \ell^1(\mathbb{Z} \times \mathbb{Z})$ is such that

$$\mathcal{F}(x)(z, z) = 0 \quad \forall z \in \mathbb{T}.$$

Let $x \in \ell^1(\mathbb{Z} \times \mathbb{Z})$ satisfy this condition. Since $\{(z, z)\colon z \in \mathbb{T}\}$ is a set of synthesis for $\ell^1(\mathbb{Z} \times \mathbb{Z})$ (see [31, 7.5.1 and Theorem 7.5.2]), there exists a sequence (x_n) in $\ell^1(\mathbb{Z} \times \mathbb{Z})$ such that x_n satisfies (2.6) for each $n \in \mathbb{N}$ and $\lim x_n = x$. From what has previously been proved, it follows that $\Phi(x) = \lim \Phi(x_n) = 0$.

We finally proceed to show that $\phi(a, b) = \phi(b, a)$ for all $a, b \in \ell^1(\mathbb{Z})$. Of course, we only need to consider the case where $a = \delta_j$ and $b = \delta_k$ for some $j, k \in \mathbb{Z}$. Set

$$x = \delta_j \otimes \delta_k - \delta_k \otimes \delta_j$$

and observe that

$$\mathcal{F}(x)(z,w) = z^{-j}w^{-k} - z^{-k}w^{-j} \quad (z,w \in \mathbb{T}),$$

in particular, $\mathcal{F}(x)(z,z) = 0$ for each $z \in \mathbb{T}$. This entails that $\Phi(x) = 0$ and therefore $\phi(\delta_j, \delta_k) = \phi(\delta_k, \delta_j)$, as required. ∎

THEOREM 2.11. *Let A be a faithful Banach algebra, let X be a Banach space, and let $\phi\colon A \times A \to X$ be a continuous bilinear map preserving zero products. Then*

$$\phi(a\mu, b) = \phi(a, \mu b)$$

for all $a, b \in A$ and $\mu \in \overline{\mathcal{D}(A)}^{\mathrm{so}}$. In particular, if A has the property $(\mathbb{A})$, *then A has the property* $(\mathbb{B})$.

Proof. We begin by proving that

$$\phi(a\mu, b) = \phi(a, \mu b) \tag{2.8}$$

for all $a, b \in A$ and $\mu \in \mathcal{D}(A)$. Clearly it suffices to prove (2.8) for all doubly power-bounded elements μ in $\mathcal{M}(A)$.

Pick $a, b \in A$ and a doubly power-bounded element $\mu \in \mathcal{M}(A)$. We define a continuous linear operator

$$\sigma_\mu\colon \ell^1(\mathbb{Z}) \to \mathcal{M}(A), \quad \sigma_\mu(\alpha) = \sum_{k=-\infty}^{\infty} \alpha(k)\mu^k,$$

for each $\alpha \in \ell^1(\mathbb{Z})$. We claim that

$$\sigma_\mu(\alpha * \beta) = \sigma_\mu(\alpha)\sigma_\mu(\beta) \tag{2.9}$$

for all $\alpha, \beta \in \ell^1(\mathbb{Z})$. Indeed,

$$\begin{aligned}
\sigma_\mu(\alpha * \beta) &= \sum_{k=-\infty}^{\infty} (\alpha * \beta)(k)\mu^k = \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{\infty} \alpha(j)\beta(k-j) \Big)\mu^k \\
&= \sum_{j=-\infty}^{\infty} \Big(\alpha(j)\mu^j \Big(\sum_{k=-\infty}^{\infty} \beta(k-j)\mu^{k-j} \Big) \Big) \\
&= \sum_{j=-\infty}^{\infty} \alpha(j)\mu^j \sigma_\mu(\beta) = \sigma_\mu(\alpha)\sigma_\mu(\beta).
\end{aligned}$$

We now consider the continuous bilinear map

$$\psi\colon \ell^1(\mathbb{Z}) \times \ell^1(\mathbb{Z}) \to X, \qquad \psi(\alpha, \beta) = \phi(a\sigma_\mu(\alpha), \sigma_\mu(\beta)b) \qquad (\alpha, \beta \in \ell^1(\mathbb{Z})).$$

If $\alpha, \beta \in \ell^1(\mathbb{Z})$ are such that $\operatorname{supp}(\mathcal{F}(\alpha)) \cap \operatorname{supp}(\mathcal{F}(\beta)) = \emptyset$, then $\alpha * \beta = 0$ and, according to (2.9), we have

$$(a\sigma_\mu(\alpha))(\sigma_\mu(\beta)b) = a\sigma_\mu(\alpha * \beta)b = 0$$

and so $\phi(a\sigma_\mu(\alpha), \sigma_\mu(\beta)b) = 0$. This shows that ψ satisfies (2.5). On account of Lemma 2.10, we have $\psi(\alpha, \beta) = \psi(\beta, \alpha)$ for all $\alpha, \beta \in \ell^1(\mathbb{Z})$. By taking $\alpha = e_0$ and $\beta = e_1$ we arrive at $\phi(a, \mu b) = \phi(a\mu, b)$, as required.

We now fix $a, b \in A$ and $\mu \in \overline{\mathcal{D}(A)}^{\mathrm{so}}$. Let $(\mu_i)_{i\in I}$ be a net in $\mathcal{D}(A)$ converging to μ with respect to the strong operator topology on $\mathcal{M}(A)$. Since $\lim a\mu_i = a\mu$ and $\lim \mu_i b = \mu b$, the identity (2.8) together with the continuity of ϕ yields

$$\phi(a\mu, b) = \lim_{i\in I} \phi(a\mu_i, b) = \lim_{i\in I} \phi(a, \mu_i b) = \phi(a, \mu b). \ \blacksquare$$

We now see that the class of Banach algebras with the property ($\mathbb{B}$) is really large, it contains algebras mentioned in Subsection 1.3, and has the extra advantage of being stable under standard constructions of Banach algebras.

3. Linear maps preserving zero products

Definition 3.1. Let A and B be Banach algebras and let $T\colon A \to B$ be a linear map. As mentioned in the introduction, T is said to *preserve zero products* if

$$a, b \in A,\ ab = 0 \ \Rightarrow\ T(a)T(b) = 0.$$

It is obvious that homomorphisms from A into B preserve zero products. The standard problem is to show that a map preserving zero products is "close" to a homomorphism. More precisely, one usually wants to describe T as being a weighted homomorphism.

3.1. *Weighted homomorphisms.* Let A be an algebra. A *centralizer* on A is a linear operator $W\colon A \to A$ satisfying

$$W(ab) = W(a)b = aW(b) \qquad (a, b \in A).$$

The set $\Gamma(A)$ of all centralizers on A is called the *centroid* of A. When endowed with the obvious operations, $\Gamma(A)$ becomes an algebra and the map $z \mapsto L_z$ $(= R_z)$ gives an embedding of the centre $Z(A)$ of A into $\Gamma(A)$. If A is faithful, then the map $W \mapsto (W, W)$ yields an isomorphism from $\Gamma(A)$ onto $Z(\mathcal{M}(A))$, the centre of $\mathcal{M}(A)$. On the other hand, if A is a faithful Banach algebra, then $\Gamma(A)$ is a closed subalgebra of $\mathcal{B}(A)$ and W is injective whenever $W \in \Gamma(A)$ is such that $\overline{W(A)} = A$, which follows from $\ker(W)W(A) = W(A)\ker(W) = \{0\}$.

Definition 3.2. Let A and B be Banach algebras. As usual, a *homomorphism* from A into B is a linear map $\Phi\colon A \to B$ such that $\Phi(ab) = \Phi(a)\Phi(b)$ $(a, b \in A)$. We call a map $T\colon A \to B$ a *weighted homomorphism* if there exist a homomorphism $\Phi\colon A \to B$ and an invertible centralizer $W \in \Gamma(B)$ such that $T = W\Phi$.

Note that any weighted homomorphism preserves zero products.

PROPOSITION 3.3. *Let A be a Banach algebra with a left approximate identity $(\rho_i)_{i\in I}$, let B be a faithful Banach algebra, and let $T\colon A \to B$ be a continuous weighted epimorphism. Then the decomposition $T = W\Phi$ with W invertible in $\Gamma(B)$ and $\Phi\colon A \to B$ an epimorphism is unique and the following assertions are equivalent*:

(i) *T is a homomorphism.*
(ii) *$(T(\rho_i))_{i\in I}$ is a left approximate identity in B.*

In particular, if A has an identity, then T is a homomorphism if and only if $T(\mathbf{1}) = \mathbf{1}$.

Proof. Suppose that $T = U\Phi = V\Psi$ for some invertible centralizers $U, V \in \Gamma(B)$ and epimorphisms $\Phi, \Psi\colon A \to B$. Since U and V are necessarily continuous, so are Φ and Ψ. Moreover, $\Phi = W\Psi$, where $W = U^{-1}V \in \Gamma(B)$. We now observe that

$$\begin{aligned} W^2(\Psi(ab)) &= W(\Psi(a))W(\Psi(b)) = \Phi(a)\Phi(b) \\ &= \Phi(ab) = W(\Psi(ab)) \qquad (a, b \in A). \end{aligned}$$

By replacing a by ρ_i and taking limits we arrive at $W(\Psi(b)) = W^2(\Psi(b))$ for each $b \in A$. Since $\Psi(A) = B$, it follows that $W = W^2$ and so W is the identity operator.

Assume that T is a homomorphism. If $u \in B$, then there is $a \in A$ with $T(a) = u$ and therefore $T(\rho_i)u = T(\rho_i a) \to T(a)$. Conversely, assume that $(T(\rho_i))_{i\in I}$ is a left approximate identity for B. Since Φ is an epimorphism, $(\Phi(\rho_i))_{i\in I}$ is a left approximate identity for B and then, for each $u \in A$, we have

$$u = \lim_{i\in I} T(\rho_i)u = \lim_{i\in I} W(\Phi(\rho_i))u = \lim_{i\in I} W(\Phi(\rho_i)u) = W(u).$$

We thus get $T = \Phi$ and so T is a homomorphism. ∎

3.2. *Characterizing homomorphisms through zero products*

LEMMA 3.4. *Let A and B be Banach algebras, and let $T\colon A \to B$ be a continuous linear map preserving zero products. If A has the property $(\mathbb{B})$, then*

$$T(ab)T(c) = T(a)T(bc) \qquad (a, b, c \in A). \tag{3.1}$$

Accordingly, if A and B are unital and $T(\mathbf{1}) = \mathbf{1}$, then T is a homomorphism.

Proof. Define a continuous bilinear map $\phi\colon A \times A \to B$ by

$$\phi(a, b) = T(a)T(b) \qquad (a, b \in A).$$

Then $\phi(a,b)=0$ whenever $a,b\in A$ are such that $ab=0$, and so the property $(\mathbb{B})$ gives the required identity. If A and B are unital and $T(\mathbf{1})=\mathbf{1}$, then setting $c=\mathbf{1}$ in (3.1) we conclude that T is a homomorphism. ∎

Lemma 3.4 shows that the problem we are considering is trivial if $T(\mathbf{1})=\mathbf{1}$. However, group algebras rarely have an identity and therefore this assumption is too strong for us. Without this assumption the problem is challenging from the technical point of view and we require some information about partially defined multipliers.

Let X and Y be Banach spaces. A linear operator $T\colon \mathrm{dom}(T)\subset X\to Y$ is *closed* if its graph $\{(x,T(x))\colon x\in\mathrm{dom}(T)\}$ is closed in $X\times Y$. Of course, this property is equivalent to the following condition:

$$x_n\in\mathrm{dom}(T)\ \forall n\in\mathbb{N},\ x_n\to x\in X,\ T(x_n)\to y \\ \Rightarrow\ x\in\mathrm{dom}(T)\text{ and }T(x)=y.$$

The operator T is *closable* if it has a closed extension, in which case the closure in $X\times Y$ of its graph is the graph of a linear operator $\overline{T}$, and that operator is the smallest closed extension of T, called the *closure* of T. It is readily seen that T is closable if and only if the following property holds:

$$x_n\in\mathrm{dom}(T)\ \forall n\in\mathbb{N},\ x_n\to 0,\ T(x_n)\to y\ \Rightarrow\ y=0.$$

LEMMA 3.5. *Let A be a left and right faithful Banach algebra, let A_U and A_V be dense subalgebras of A, and let $U\colon A_U\to A$, $V\colon A_V\to A$ be linear operators satisfying the conditions*

$$\begin{aligned} U(ab)&=U(a)b && (a,b\in A_U),\\ V(ab)&=aV(b) && (a,b\in A_V),\\ bU(a)&=V(b)a && (a\in A_U,\ b\in A_V). \end{aligned}$$

Then the operators U and V are closable, the domain $I_{\overline{U}}$ of $\overline{U}$ is a right ideal of A, the domain $I_{\overline{V}}$ of $\overline{V}$ is a left ideal of A, and the following identities hold:

$$\begin{aligned} \overline{U}(ab)&=\overline{U}(a)b && (a\in I_{\overline{U}},\ b\in A),\\ \overline{V}(ab)&=a\overline{V}(b) && (a\in A,\ b\in I_{\overline{V}}),\\ b\overline{U}(a)&=\overline{V}(b)a && (a\in I_{\overline{U}},\ b\in I_{\overline{V}}). \end{aligned}$$

Furthermore, if there exists a set $S\subset I_U\cap I_V$ which is dense in A and such that $U(a)=V(a)$ for each $a\in S$, then both $I_{\overline{U}}$ and $I_{\overline{V}}$ are two-sided ideals of A and

$$\begin{aligned} \overline{U}(ba)&=b\overline{U}(a) && (a\in I_{\overline{U}},\ b\in A),\\ \overline{V}(ba)&=\overline{V}(b)a && (a\in A,\ b\in I_{\overline{V}}). \end{aligned}$$

Proof. Let (a_n) be a sequence in A_U with $\lim a_n = 0$ and $\lim U(a_n) = a$ for some $a \in A$. For every $b \in A_V$, we have

$$ba = \lim bU(a_n) = \lim V(b)a_n = 0.$$

Since A_V is dense in A, it follows that $Aa = \{0\}$ and therefore $a = 0$. In the same way we check that V is also closable.

Let $\overline{U}\colon I_{\overline{U}} \to A$ and $\overline{V}\colon I_{\overline{V}} \to A$ be the corresponding closures. Let $a \in I_{\overline{U}}$, $b \in I_{\overline{V}}$, and $c \in A$. Then there exist sequences (a_n), (c_n) in I_U and (b_n), (d_n) in I_V such that $\lim a_n = a$, $\lim U(a_n) = \overline{U}(a)$, $\lim b_n = b$, $\lim V(b_n) = \overline{V}(b)$, and $\lim c_n = \lim d_n = c$. We now observe that $(a_n c_n)$ and $(d_n b_n)$ are sequences in I_U and I_V, respectively, with

$$U(a_n c_n) = U(a_n)c_n \to \overline{U}(a)c \quad \text{and} \quad V(d_n b_n) = d_n V(b_n) \to c\overline{V}(b).$$

This shows that $ac \in I_{\overline{U}}$ with $\overline{U}(ac) = \overline{U}(a)c$ and $cb \in I_{\overline{V}}$ with $\overline{V}(cb) = c\overline{V}(b)$. On the other hand,

$$b\overline{U}(a) = \lim b_n U(a_n) = \lim V(b_n)a_n = \overline{V}(b)a,$$

as required.

Let S satisfy the condition in the lemma. Let $a \in I_{\overline{U}}$ and $b \in A$. We take a sequence (b_n) in S with $\lim b_n = b$. The sequence $(b_n a)$ lies in $I_{\overline{U}}$, $\lim b_n a = ba$, and

$$\overline{U}(b_n a) = \overline{U}(b_n)a = U(b_n)a = V(b_n)a = \overline{V}(b_n)a = b_n\overline{U}(a) \to b\overline{U}(a).$$

Since $\overline{U}$ is closed, the preceding property implies that $ba \in I_{\overline{U}}$ and $\overline{U}(ba) = b\overline{U}(a)$. In the same way we check the corresponding property for $\overline{V}$. ∎

THEOREM 3.6. *Let A be a Banach algebra with the property* $(\mathbb{B})$ *and having a left approximate identity, and let $T\colon A \to B$ be a continuous linear map on some Banach algebra B preserving zero products. Then there exist a right ideal I_V of B containing $T(A^2)$ and a linear operator $V\colon I_V \to B$ such that*

$$V(uv) = V(u)v, \qquad V(zw) = zV(w)$$
$$(u \in I_V,\ v \in B,\ w \in \operatorname{alg} T(A^2),\ z \in \operatorname{alg} T(A))$$

and

$$V(T(ab)) = T(a)T(b) \qquad (a, b \in A).$$

Furthermore, the following assertions hold:

(i) *If B is faithful and $\overline{T(A)} = B$, then there exist a two-sided ideal I_W of B containing $T(A^2)$ and a closed linear operator $W\colon I_W \to B$ such that*

$$W(uv) = W(u)v \quad \textit{and} \quad W(vu) = vW(u) \qquad (u \in I_W,\ v \in B)$$

and

$$W(T(ab)) = T(a)T(b) \quad (a, b \in A).$$

Moreover, if B is essential, then $\overline{W(B)} = B$ and W is injective.

(ii) *If B is faithful and factors weakly, and $T(A) = B$, then the operator W given in* (i) *is invertible and there exists a continuous epimorphism $\Phi\colon A \to I_W$ such that $T = W\Phi$.*

Proof. Let $(\rho_i)_{i\in I}$ be a left approximate identity for A. Let $I_V = \{u \in B\colon (T(\rho_i)u)_{i\in I}$ converges$\}$ and define $V\colon I_V \to B$ by

$$V(u) = \lim_{i\in I} T(\rho_i)u \quad (u \in I_V).$$

It is clear that I_V is a right ideal of B and that

$$V(uv) = V(u)v \quad (u \in I_V,\ v \in B). \tag{3.2}$$

By (3.1) we have

$$T(a)T(b) = \lim_{i\in I} T(\rho_i a)T(b) = \lim_{i\in I} T(\rho_i)T(ab) \quad (a, b \in A). \tag{3.3}$$

This shows that $T(A^2) \subset I_V$ and

$$V(T(ab)) = T(a)T(b) \quad (a, b \in A). \tag{3.4}$$

On account of (3.1), we have $T(A)T(A^2) = T(A^2)T(A)$ so that $T(A)T(A^2) \subset I_V$ and moreover

$$\begin{aligned} V\big(T(a_1)\cdots T(a_n)T(bc)\big) &= \lim_{i\in I} T(\rho_i)T(a_1)\cdots T(a_n)T(bc) \\ &= \lim_{i\in I} T(\rho_i a_1)\cdots T(a_n)T(b)T(c) \\ &= T(a_1)\cdots T(a_n)T(b)T(c) \\ &= T(a_1)\cdots T(a_n)V(T(bc)) \end{aligned}$$

for all $a_1, \ldots, a_n, b, c \in A$. Consequently,

$$V(zw) = zV(w) \quad (z \in \operatorname{alg} T(A),\ w \in \operatorname{alg} T(A^2)). \tag{3.5}$$

We now assume that B is faithful and that $\overline{T(A)} = B$. Since A has a left approximate identity, it follows that A^2 is dense in A, and since $T(A)$ is dense in B, so is $\operatorname{alg} T(A^2)$. On account of (3.2), (3.5) and Lemma 3.5, the restriction of the operator V to $\operatorname{alg} T(A^2)$ is closable and the closure $W\colon I_W \to B$ of that operator has the properties given in (i). Moreover, from (3.4) we deduce that $T(A)^2 \subset W(T(A^2)) \subset W(B)$ and so $\overline{B^2} \subset \overline{W(B)}$. This shows that $\overline{W(B)} = B$ in the case where B is essential.

We now assume that B factors weakly and $T(A) = B$. Then (3.4) clearly entails that W is surjective. If, in addition, B is faithful, then W is injective. Indeed, for every $u \in \ker W$ and $v \in I_W$ we have

$$uW(v) = W(uv) = W(u)v = 0$$

and similarly $W(v)u = 0$. Since $W(I_W) = B$ and B is faithful, it follows that $u = 0$. It is straightforward to check that $W^{-1} \in \Gamma(B)$, which in particular entails that W^{-1} is continuous. We now define $\Phi = W^{-1}T$. Of course, Φ is surjective and continuous. We proceed to show that Φ is a homomorphism. Let $a, b \in A$. On account of $W(T(ab)) = T(a)T(b)$, we have

$$\begin{aligned}\Phi(ab) &= W^{-1}(T(ab)) = W^{-1}(W^{-1}(T(a)T(b)))\\ &= W^{-1}(T(a)W^{-1}(T(b))) = W^{-1}(T(a)\Phi(b))\\ &= W^{-1}(T(a))\Phi(b) = \Phi(a)\Phi(b). \ \blacksquare\end{aligned}$$

Let A be a *semiprime algebra*, i.e. $I^2 \neq \{0\}$ whenever I is a nonzero two-sided ideal of A; this is equivalent to the condition that $aAa = \{0\}$, where $a \in A$, implies $a = 0$, so in particular A is both left and right faithful. An *essentially defined centralizer* on A is a linear map $V\colon I_V \to A$ whose domain is a two-sided ideal I_V of A with $I_V \cap J \neq \{0\}$ for every nonzero two-sided ideal J of A, and satisfying the following identities:

$$V(ab) = V(a)b \quad \text{and} \quad V(ba) = bV(a) \qquad (a \in I,\, b \in A).$$

Of course, every centralizer is an essentially defined centralizer. The relation $\simeq$ defined on the set $\mathcal{C}(A)$ of all essentially defined centralizers on A by $U \simeq V$ if and only if there exists $W \in \mathcal{C}(A)$ such that both U and V extend W, is an equivalence relation. The set $C(A)$ of all equivalence classes under the obvious operations of partially defined operators is called the *extended centroid* of A, and was introduced by W. S. Martindale [27, 28] in the case where A is a *prime algebra*, i.e. $IJ \neq 0$ whenever I and J are nonzero two-sided ideals of A. In that case $C(A)$ is a field extension of the base field. It is interesting to note that $C(A)$ can be identified with the centre of the *symmetric algebra of quotients* of A: the definition is very similar to that of $\mathcal{M}(A)$, operating with (equivalence classes of) partially defined multipliers.

A prime algebra A is said to be *centrally closed* if the canonical embedding of the base field $\mathbb{F}$ into $C(A)$ is surjective. This means that for every partially defined centralizer $W\colon I \to A$ there is $\lambda \in \mathbb{F}$ such that $W(a) = \lambda a$ for each $a \in A$. Every primitive Banach algebra and every prime C^*-algebra is centrally closed. The first example follows from [27, 28] and the standard theory of Banach algebras, while the second one follows from [29].

Corollary 3.7. *Let A be a Banach algebra with a left approximate identity and the property* $(\mathbb{B})$, *let B be a centrally closed prime Banach algebra, and let $T\colon A \to B$ be a continuous linear map preserving zero products. If $\overline{T(A)} = B$, then there exist a nonzero complex number λ and a continuous homomorphism $\Phi\colon A \to B$ such that $T = \lambda\Phi$. Accordingly, $\overline{\Phi(A)} = B$, Φ is an epimorphism if $T(A) = B$, and Φ is an isomorphism if T is bijective.*

Proof. Let W be the partially defined centralizer given in assertion (i) of Theorem 3.6. Since B is centrally closed, there exists a complex number α such that $W(u) = \alpha u$ for each $u \in I_W$. Hence $\alpha T(ab) = T(a)T(b)$ for all $a, b \in A$. This clearly forces that $\alpha \neq 0$ and that $\Phi = \alpha^{-1}T$ is a homomorphism. ∎

It is worth pointing out that primitive Banach algebras and prime C^*-algebras are centrally closed (see Section 2), so Corollary 3.7 covers some interesting situations.

Roughly speaking, Theorem 3.6 shows that T is "almost" a weighted homomorphism. When the left approximate identity is bounded, the conclusion of Theorem 3.6 can be strengthened, and in particular it can be shown that under suitable additional assumptions T is actually a weighted homomorphism.

THEOREM 3.8. *Let A be a Banach algebra with the property* ($\mathbb{B}$) *and having a bounded left approximate identity, and let $T\colon A \to B$ be a continuous linear map on some Banach algebra B preserving zero products. Then there exist a closed right ideal I_V of B containing $T(A)$ and a continuous linear operator $V\colon I_V \to B$ such that*

$$\|V\| \leq \|T\| \sup_{i\in I} \|\rho_i\|,$$

$$V(uv) = V(u)v, \qquad V(zw) = zV(w) \qquad (u \in I_V,\ v \in B,\ w, z \in \overline{\operatorname{alg} T(A)}),$$

and

$$V(T(ab)) = T(a)T(b) \qquad (a, b \in A).$$

Furthermore, the following assertions hold:

(i) *If $\overline{T(A)} = B$, then there exists $W \in \Gamma(B)$ such that*

$$\|W\| \leq \|T\| \sup_{i\in I} \|\rho_i\| \quad \text{and} \quad W(T(ab)) = T(a)T(b) \quad (a, b \in A).$$

If B is essential, then $\overline{W(B)} = B$. Moreover, if B is essential and faithful, then W is injective.

(ii) *If B is faithful and factors weakly, and $T(A) = B$, then the operator W given in assertion* (i) *is invertible and there exists a continuous epimorphism $\Phi\colon A \to B$ such that $T = W\Phi$. Moreover, if T is bijective, then Φ is an isomorphism and*

$$\|\Phi^{-1}\| \leq \|T\|\,\|T^{-1}\| \sup_{i\in I} \|\rho_i\|.$$

Proof. We define I_V and V just as in the proof of Theorem 3.6. The boundedness of the net $(T(\rho_i))_{i\in I}$ clearly implies that I_V is closed. Since A has a bounded left approximate identity, it follows that $A^2 = A$ so that

$T(A) = T(A^2) \subset I_V$ and

$$V(zw) = zV(w) \qquad (w, z \in \overline{\operatorname{alg} T(A)}).$$

On the other hand,

$$\|T(\rho_i)u\| \le \sup_{i\in I} \|\rho_i\| \, \|T\| \, \|u\| \qquad (i \in I)$$

and so $\|V(a)\| \le \sup_{i\in I} \|\rho_i\| \, \|T\| \, \|a\|$, which shows that the operator V is continuous with $\|V\| \le \|T\| \sup_{i\in I} \|\rho_i\|$.

We now assume that $\overline{T(A)} = B$. Since $T(A) \subset I_V$ and I_V is closed, it follows that V is defined on B. Since V is continuous, it follows that the operator W given in Theorem 3.6 is nothing other than V. ∎

A rough summary of Theorem 3.8 can be given as follows.

COROLLARY 3.9. *Let A be a Banach algebra with the property* ($\mathbb{B}$) *and having a bounded left approximate identity, and let B be a Banach algebra which is faithful and factors weakly. If $T\colon A \to B$ is a continuous surjective linear map preserving zero products, then T is a weighted epimorphism.*

In order to apply the preceding results to group algebras, recall that $\Gamma(L^1(G)) = Z(M(G))$ for any locally compact group G. Consequently, we arrive at the following result.

COROLLARY 3.10. *Let G_1 and G_2 be locally compact groups. If $T\colon L^1(G_1) \to L^1(G_2)$ is a continuous surjective linear map preserving zero products, then there are an invertible measure μ in the centre of the measure algebra $M(G_2)$ and a continuous epimorphism $\Phi\colon L^1(G_1) \to L^1(G_2)$ with $T = \mu\Phi$.*

On the other hand, the information about $\|\Phi^{-1}\|$ given in assertion (ii) of Theorem 3.8 is very useful in the case of group algebras. G. V. Wood proved in [36] that if G_1 and G_2 are locally compact groups, and if $\Phi\colon L^1(G_1) \to L^1(G_2)$ is an isomorphism with $\|\Phi\| \le \frac{1}{2}(1+\sqrt{3})$, then G_1 and G_2 are isomorphic. In a joint paper with N. J. Kalton [24] they showed that when the groups are abelian or connected, the result still holds true when $\|\Phi\| < \sqrt{2}$. According to Theorem 3.8, if G_1 and G_2 are locally compact groups with the property that there exists a continuous bijective map $T\colon L^1(G_1) \to L^1(G_2)$ preserving zero products and $\|T\| \, \|T^{-1}\| \le \frac{1}{2}(1+\sqrt{3})$, then G_1 and G_2 are isomorphic.

J. J. Font and S. Hernández proved in [17] that if G_1 and G_2 are locally compact abelian groups, then every bijective linear map $T\colon L^1(G_1) \to L^1(G_2)$ preserving zero products is automatically continuous and can be described as a weighted homomorphism.

4. Characterizing derivations through zero products. Let A be a Banach algebra and let X be a Banach A-bimodule. As usual, a *derivation* from A into X is a linear operator $D\colon A \to X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A).$$

Throughout this section we shall be concerned with the following properties of a continuous linear map $D\colon A \to X$:

$$\text{(D1)} \qquad a, b, c \in A,\ ab = bc = 0 \ \Rightarrow\ a \cdot D(b) \cdot c = 0;$$

$$\text{(D2)} \qquad a, b \in A,\ ab = 0 \ \Rightarrow\ D(a) \cdot b + a \cdot D(b) = 0.$$

Our purpose is to investigate whether each of the above conditions characterizes derivations. In this regard we will arrive at a type of derivation-like operators, the so-called generalized derivations. They are closely related to the notion of a generalized derivation that first appeared in [7]. We point out that there are several papers which are concerned with the characterization of (generalized) derivations by their action on the zero product elements [2, 9, 18, 23, 33, 37].

4.1. *Introducing generalized derivations.* Let A be a Banach algebra and let X be a Banach A-bimodule. The bimodule X is *essential* if $X = \overline{\operatorname{lin}\{a \cdot x \cdot b\colon a, b \in A,\ x \in X\}}$. We write $Z_A(X)$ for the centre of X:

$$Z_A(X) = \{x \in X\colon a \cdot x = x \cdot a\ \forall a \in A\},$$

and we define the *annihilator* of A on X by

$$\operatorname{Ann}_X(S) = \{x \in X\colon a \cdot x \cdot b = 0\ \forall a, b \in S\}.$$

Note that X^{**} is a Banach A-bimodule, and we may regard X as a closed submodule of X^{**}. In fact, the natural embedding $\iota_X\colon X \to X^{**}$ is an A-bimodule homomorphism and so is the natural projection $\pi_X\colon X^{***} \to X^*$, which is nothing other than the dual operator ι_X^* of ι_X.

Definition 4.1. Let A be a Banach algebra and let X be a Banach A-bimodule. A linear operator $D\colon A \to X$ is said to be a *generalized derivation* if there exists $\xi \in X^{**}$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot \xi \cdot b \qquad (a, b \in A);$$

Generalized derivations from a Banach algebra into itself appear in [22, 23, 37].

It should be noted that if D is a generalized derivation, then the map

$$d\colon A \to X^{**}, \qquad d(a) = D(a) - a \cdot \xi \qquad (a \in A)$$

is a derivation. Accordingly, we can write $D = d + R_\xi$, where $R_\xi\colon A \to X^{**}$ is defined by $R_\xi(a) = a \cdot \xi$ for each $a \in A$. It is interesting to note that we can also write $D = d' + L_\xi$ for some derivation $d'\colon A \to X^{**}$, where $L_\xi\colon A \to X^{**}$ is now defined by $L_\xi(a) = \xi \cdot a$ $(a \in A)$. In fact, we have $R_\xi = \operatorname{ad}_\xi + L_\xi$,

where $\mathrm{ad}_\xi\colon A \to X^{**}$ is the derivation defined by $\mathrm{ad}_\xi(a) = a\cdot\xi - \xi\cdot a$ for each $a \in A$.

In some situations the element ξ defining a generalized derivation can be chosen in the given module instead of requiring its second dual, in which case the corresponding derivation maps into X.

PROPOSITION 4.2. *Let A be a Banach algebra with identity, and let $D\colon A \to X$ be a generalized derivation on some Banach A-bimodule X. Then there exists an element $\xi \in X$ such that*

$$D(ab) = D(a)b + aD(b) - a\cdot\xi\cdot b \qquad (a,b \in A)$$

and so there is a decomposition $D = d + R_\xi$ for some derivation $d\colon A \to X$. Furthermore, the following assertions are equivalent:

(a) *D is a derivation.*
(b) $\mathbf{1}\cdot D(\mathbf{1})\cdot\mathbf{1} = 0$.

Proof. According to the definition, there exists $\zeta \in X^{**}$ such that

$$D(ab) = D(a)\cdot b + a\cdot D(b) - a\cdot\zeta\cdot b \qquad (a,b \in A). \tag{4.1}$$

Note that this, in particular, entails that $a\cdot\zeta\cdot b \in X$ for all $a,b \in A$. Define $\xi = \mathbf{1}\cdot\zeta\cdot\mathbf{1} \in X$. It is clear that (4.1) holds with ζ replaced by ξ. On account of (4.1), we have

$$\xi = D(\mathbf{1})\cdot\mathbf{1} + \mathbf{1}\cdot D(\mathbf{1}) - D(\mathbf{1}). \tag{4.2}$$

Since obviously $\xi = \mathbf{1}\cdot\xi\cdot\mathbf{1}$, it follows from (4.2) that $\xi = \mathbf{1}\cdot D(\mathbf{1})\cdot\mathbf{1}$. Therefore (4.1) can be written as

$$D(ab) = D(a)\cdot b + a\cdot D(b) - a\cdot D(\mathbf{1})\cdot b \qquad (a,b \in A). \tag{4.3}$$

We now observe that if D is a derivation, then $D(\mathbf{1}) = D(\mathbf{1}\mathbf{1}) = D(\mathbf{1})\cdot\mathbf{1} + \mathbf{1}\cdot D(\mathbf{1})$ and so $\mathbf{1}\cdot D(\mathbf{1})\cdot\mathbf{1} = 0$. On the other hand, from (4.3) we deduce that D is a derivation when $\mathbf{1}\cdot D(\mathbf{1})\cdot\mathbf{1} = 0$. ∎

PROPOSITION 4.3. *Let A be a Banach algebra, and let $D\colon A \to X$ be a generalized derivation on some dual Banach A-bimodule X. Then there exists an element $\xi \in X$ such that*

$$D(ab) = D(a)b + aD(b) - a\cdot\xi\cdot b \qquad (a,b \in A)$$

and so there is a decomposition $D = d + R_\xi$ for some derivation $d\colon A \to X$. If A has a bounded approximate identity $(\rho_i)_{i\in I}$ and D is continuous, then the following assertions are equivalent:

(a) *D is a derivation.*
(b) $\lim_{i\in I} a\cdot D(\rho_i)\cdot b = 0$ *for all $a,b \in A$.*

If $\mathrm{Ann}_X(A) = \{0\}$, then (a) *and* (b) *are also equivalent to*

(c) $\sigma(X, X_*)\text{-}\lim_{i\in I} D(\rho_i) = 0$, *where X_* is a predual A-bimodule of X.*

Proof. Pick $\zeta \in X^{**}$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot \zeta \cdot b \quad (a, b \in A). \tag{4.4}$$

Let X_* be a predual A-bimodule of X and let $\pi_{X_*} \colon X^{**} \to X$ be the natural projection. Set $\xi = \pi_{X_*}(\zeta) \in X$. By applying π_{X_*} to (4.4), and taking into account that π_{X_*} is an A-bimodule homomorphism and $\pi_{X_*}(X) \subset X$, we arrive at

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot \xi \cdot b \quad (a, b \in A). \tag{4.5}$$

Throughout the rest of the proof we assume that A has a bounded approximate identity $(\rho_i)_{i \in I}$. Then we apply (4.5) to obtain

$$\begin{aligned} D(a) \cdot b &= \lim_{i \in I} D(a\rho_i) \cdot b = \lim_{i \in I} (D(a) \cdot \rho_i b + a \cdot D(\rho_i) \cdot b - a\xi \cdot \rho_i b) \\ &= D(a) \cdot b + \lim_{i \in I} a \cdot D(\rho_i) \cdot b - a \cdot \xi \cdot b \end{aligned}$$

and so

$$\lim_{i \in I} a \cdot D(\rho_i) \cdot b = a \cdot \xi \cdot b \quad (a, b \in A). \tag{4.6}$$

The equivalence of (a) and (b) is now clear from (4.5) and (4.6).

We finally assume that $\mathrm{Ann}_X(A) = \{0\}$. We claim that X_* is essential. On the contrary, suppose that $\overline{A \cdot X_* \cdot A} \neq X_*$. Then there exists $x \in X \setminus \{0\}$ such that $\langle a \cdot x_* \cdot b, x \rangle = 0$ for all $a, b \in A$ and $x_* \in X_*$. Therefore $\langle x_*, b \cdot x \cdot a \rangle = 0$ for all $a, b \in A$ and $x_* \in X_*$ and so $b \cdot x \cdot a = 0$ for all $a, b \in A$. Since $\mathrm{Ann}_X(A) = \{0\}$, we see that $x = 0$, a contradiction.

If condition (b) holds and $x_* \in X_*$, then there are $a, b \in A$ and $y_* \in X_*$ such that $x_* = a \cdot y_* \cdot b$ and therefore

$$\lim_{i \in I} \langle x_*, D(\rho_i) \rangle = \lim_{i \in I} \langle y_*, b \cdot D(\rho_i) \cdot a \rangle = 0,$$

which gives (c). Conversely, if (c) holds, then (4.6) yields

$$\langle x_*, a \cdot \xi \cdot b \rangle = \lim_{i \in I} \langle x_*, a \cdot D(\rho_i) \cdot b \rangle = \lim_{i \in I} \langle b \cdot x_* \cdot a, D(\rho_i) \rangle = 0$$

for all $a, b \in A$ and $x_* \in X_*$. This implies that $a \cdot \xi \cdot b = 0$ $(a, b \in A)$ and (4.5) now shows that D is a derivation. ∎

PROPOSITION 4.4. *Let A be a Banach algebra with a bounded approximate identity $(\rho_i)_{i \in I}$, and let $D \colon A \to X$ be a continuous generalized derivation on some Banach A-bimodule X. Then the following assertions are equivalent*:

(a) *D is a derivation.*
(b) $\lim_{i \in I} a \cdot D(\rho_i) \cdot b = 0$ *for all $a, b \in A$.*

Proof. The proof can be done in more or less the same way as for the corresponding assertions in Proposition 4.3. ∎

4.2. *Characterizing derivations through the condition* (D1). The condition (D1) was considered before in [2, 18, 33]. The motivation for its study arises from the connection with the concept of a local derivation.

THEOREM 4.5. *Let A be a Banach algebra with the property* ($\mathbb{B}$) *and having a bounded approximate identity, let X be a Banach A-bimodule such that* $\mathrm{Ann}_X(A) = \{0\}$, *and let $D\colon A \to X$ be a continuous linear operator satisfying*

$$a, b, c \in A,\ ab = bc = 0 \ \Rightarrow\ a \cdot D(b) \cdot c = 0.$$

Then D is a generalized derivation, i.e. there exists $\xi \in X^{**}$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot \xi \cdot b \qquad (a, b \in A).$$

Furthermore, ξ can be chosen in X in each of the following cases:

(i) *A has an identity.*
(ii) *X is a dual A-bimodule.*

Proof. We begin by proving that

$$D(abc) = D(ab) \cdot c + a \cdot D(bc) - a \cdot D(b) \cdot c \qquad (a, b, c \in A). \tag{4.7}$$

Pick $a_2, b_2 \in A$ with $a_2 b_2 = 0$. We consider the bilinear map $\phi\colon A \times A \to X$ defined by

$$\phi(a, b) = a \cdot D(ba_2) \cdot b_2 \qquad (a, b \in A).$$

On account of the given hypothesis on D, we see at once that ϕ preserves zero products. Consequently, the property ($\mathbb{B}$) yields

$$a_1 b_1 \cdot D(c_1 a_2) \cdot b_2 = a_1 \cdot D(b_1 c_1 a_2) \cdot b_2 \qquad (a_1, b_1, c_1 \in A). \tag{4.8}$$

We now fix $a_1, b_1, c_1 \in A$ and consider the bilinear map $\psi\colon A \times A \to X$ given by

$$\psi(a, b) = a_1 b_1 \cdot D(c_1 a) \cdot b - a_1 \cdot D(b_1 c_1 a) \cdot b \qquad (a, b \in A).$$

On account of (4.8), ψ preserves zero products and therefore the property ($\mathbb{B}$) gives

$$\begin{aligned} a_1 b_1 \cdot D(c_1 a_2 b_2) \cdot c_2 - a_1 \cdot D(b_1 c_1 a_2 b_2) \cdot c_2 \\ + a_1 \cdot D(b_1 c_1 a_2) \cdot b_2 c_2 - a_1 b_1 \cdot D(c_1 a_2) \cdot b_2 c_2 = 0 \end{aligned} \tag{4.9}$$

for all $a_1, b_1, c_1, a_2, b_2, c_2 \in A$. Note that (4.9) can be rewritten as

$$c \cdot (a \cdot D(eb) - D(aeb) + D(ae) \cdot b - a \cdot D(e) \cdot b) \cdot d = 0$$

for all $a, b, c, d \in A$ and $e \in A^2$, which gives (4.7).

Let $(\rho_i)_{i \in I}$ be a bounded approximate identity of A. Since the net $(D(\rho_i))_{i \in I}$ is bounded, we can assume that it converges to $\xi \in X^{**}$ with respect to the topology $\sigma(X^{**}, X^*)$.

On account of (4.7), for all $a, b \in A$ we have

$$D(a\rho_i b) = D(a\rho_i) \cdot b + a \cdot D(\rho_i b) - a \cdot D(\rho_i) \cdot b,$$

so that

$$a \cdot D(\rho_i) \cdot b = D(a\rho_i) \cdot b + a \cdot D(\rho_i b) - D(a\rho_i b)$$

and therefore the net $(a \cdot D(\rho_i) \cdot b)_{i \in I}$ converges to $D(a) \cdot b + a \cdot D(b) - D(ab)$ with respect to the norm topology. On the other hand, for each $x_* \in X_*$ we have

$$\begin{aligned}\langle x_*, \lim_{i \in I} a \cdot D(\rho_i) \cdot b \rangle &= \lim_{i \in I} \langle x_*, a \cdot D(\rho_i) \cdot b \rangle = \lim_{i \in I} \langle b \cdot x_* \cdot a, D(\rho_i) \rangle \\ &= \langle b \cdot x_* \cdot a, \xi \rangle = \langle x_*, a \cdot \xi \cdot b \rangle.\end{aligned}$$

We thus get

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot \xi \cdot b \quad (a, b \in A). \blacksquare$$

4.3. *Characterizing derivations through the condition* (D2). The condition (D2) has been considered in [9, 23, 37].

THEOREM 4.6. *Let A be a Banach algebra with the property* ($\mathbb{B}$) *and having a bounded approximate identity, let X be a Banach A-bimodule such that* $\mathrm{Ann}_X(A) = \{0\}$, *and let $D\colon A \to X$ be a continuous linear operator satisfying*

$$a, b \in A,\ ab = 0 \ \Rightarrow\ D(a) \cdot b + a \cdot D(b) = 0.$$

Then D is a generalized derivation and there exists $\xi \in Z_A(X^{**})$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot \xi \cdot b \quad (a, b \in A).$$

Accordingly, there exists a continuous derivation $d\colon A \to X^{**}$ such that

$$D(a) = d(a) + a \cdot \xi \quad (a \in A).$$

Furthermore, ξ can be chosen in $Z_A(X)$ in each of the following cases:

(i) *A has an identity.*
(ii) *X is a dual A-bimodule.*

Proof. Suppose that $a, b, c \in A$ are such that $ab = bc = 0$. According to the hypothesis, we have

$$0 = (D(a) \cdot b + a \cdot D(b))c = D(a) \cdot bc + a \cdot D(b) \cdot c = a \cdot D(b) \cdot c.$$

Let $\xi \in X^{**}$ be given by Theorem 4.5. If $a, b \in A$ are such that $ab = 0$, then

$$0 = D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot \xi \cdot b = -a \cdot \xi \cdot b$$

and therefore we can apply the property ($\mathbb{B}$) to the bilinear map $(a, b) \mapsto a \cdot \xi \cdot b$. We thus get

$$ab \cdot \xi \cdot c = a \cdot \xi \cdot bc \quad (a, b, c \in A). \tag{4.10}$$

In cases (i) and (ii) we can use the assumption $\mathrm{Ann}_X(A) = \{0\}$ to conclude from (4.10) that $b \cdot \xi = \xi \cdot b$ for each $b \in A$.

We now turn to the general case. Let $(\rho_i)_{i\in I}$ be a bounded approximate identity in A. Since the net $(\rho_i \cdot \xi \cdot \rho_i)_{i\in I}$ is bounded, we can assume that it converges to $\zeta \in X^{**}$ with respect to the topology $\sigma(X^{**}, X^*)$. We claim that

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot \zeta \cdot b \quad (a, b \in A). \tag{4.11}$$

Of course, it suffices to prove that $a \cdot \zeta \cdot b = a \cdot \xi \cdot b$ for all $a, b \in A$. Pick $a, b \in A$. Then

$$\begin{aligned}\langle x^*, a \cdot \zeta \cdot b\rangle &= \langle b \cdot x^* \cdot a, \zeta\rangle = \lim_{i\in I} \langle b \cdot x^* \cdot a, \rho_i \cdot \xi \cdot \rho_i\rangle \\ &= \lim_{i\in I} \langle \rho_i b \cdot x^* \cdot a\rho_i, \xi\rangle = \langle b \cdot x^* \cdot a, \xi\rangle = \langle x^*, a \cdot \xi \cdot b\rangle\end{aligned}$$

for each $x^* \in X^*$.

We finally show that $\zeta \in Z_A(X^{**})$. Set $b \in A$. On account of (4.10), we have

$$\begin{aligned}\rho_i^2 b \cdot (\rho_i \cdot \xi \cdot \rho_i) &= \rho_i(\rho_i b \rho_i) \cdot \xi \cdot \rho_i = \rho_i \cdot \xi \cdot (\rho_i b \rho_i)\rho_i \\ &= (\rho_i \cdot \xi \cdot \rho_i) \cdot b\rho_i^2\end{aligned} \tag{4.12}$$

for each $i \in I$. On the other hand, for every $x^* \in X^*$, we have

$$\begin{aligned}&\langle x^*, \rho_i^2 b \cdot (\rho_i \cdot \xi \cdot \rho_i)\rangle - \langle x^*, b \cdot \zeta\rangle \\ &\qquad = \langle x^*, (\rho_i^2 b - b) \cdot (\rho_i \cdot \xi \cdot \rho_i)\rangle + \langle x^* \cdot b, \rho_i \cdot \xi\rho_i\rangle - \langle x^* \cdot b, \zeta\rangle.\end{aligned}$$

Since

$$\begin{aligned}|\langle x^*, (\rho_i^2 b - b) \cdot (\rho_i \cdot \xi \cdot \rho_i)\rangle| &\le \|x^*\| \, \|\rho_i^2 b - b\| \, \|\rho_i \cdot \xi \cdot \rho_i\| \\ &\le \|x^*\|(\|\rho_i\| + 1)\|\rho_i b - b\| \, \|\rho_i\|^2 \sup_{\|u\|=\|v\|=1} \|u \cdot \xi \cdot v\| \to 0\end{aligned}$$

and

$$\lim_{i\in I}(\langle x^* \cdot b, \rho_i \cdot \xi\rho_i\rangle - \langle x^* \cdot b, \zeta\rangle) = 0,$$

it follows that

$$\lim_{i\in I} \langle x^*, \rho_i^2 b \cdot (\rho_i \cdot \xi \cdot \rho_i)\rangle = \langle x^*, b \cdot \zeta\rangle.$$

In the same way we check that

$$\lim_{i\in I} \langle x^*, (\rho_i \cdot \xi \cdot \rho_i) \cdot b\rho_i^2\rangle = \langle x^*, \zeta \cdot b\rangle$$

and therefore (4.12) gives $\langle x^*, b \cdot \zeta\rangle = \langle x^*, \zeta \cdot b\rangle$ for each $x^* \in X^*$ and so $b \cdot \zeta = \zeta \cdot b$, as claimed. ∎

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