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The tensor algebra of power series spaces

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Abstract. The linear isomorphism type of the tensor algebra $T(E)$ of Fréchet spaces and, in particular, of power series spaces is studied. While for nuclear power series spaces of infinite type it is always s, the situation for finite type power series spaces is more complicated. The linear isomorphism $T(s) \cong s$ can be used to define a multiplication on s which makes it a Fréchet m-algebra $s_\bullet$. This may be used to give an algebra analogue to the structure theory of s, that is, characterize Fréchet m-algebras with (Ω) as quotient algebras of $s_\bullet$ and Fréchet m-algebras with (DN) and (Ω) as quotient algebras of $s_\bullet$ with respect to a complemented ideal.

In [8] we calculated the linear isomorphism type of the space s of rapidly decreasing sequences and all of its complemented subspaces E. It was shown that $T(E) \cong s$ whenever $\dim E \geq 2$. This includes all of the so-called power series spaces of infinite type, including the space $H(\mathbb{C}^d)$ of entire functions for any dimension d. In the present work we study the tensor algebra of Fréchet spaces in a more general context.

We use these results to give another proof for the isomorphism theorem in the case of infinite type power series spaces. The isomorphism $T(s) \cong s$ defines a multiplication on s which turns it into a Fréchet m-algebra which we call $s_\bullet$. Its quotient algebras are all nuclear Fréchet m-algebras with property (Ω) or, equivalently, which are linearly isomorphic to a quotient of s, and its quotient algebras with respect to a complemented ideal are the nuclear Fréchet m-algebras with properties (DN) and (Ω). This gives an "algebra equivalent" to the structure theory of s as developed in [9].

In the last section we extend our study to the finite type power series spaces, including e.g. the spaces s_0 of very slowly increasing functions and the spaces $H(\mathbb{D}^d)$ of holomorphic functions on the d-dimensional polydisc. There the situation is much more complicated. But it is interesting to observe that for $E = s_0$ and $E = H(\mathbb{D}^d)$ the results are different, however in the second case the result does not depend on d. In fact we obtain coincidence

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of $T(E)$ on the same class of *power comparable* E (or E of *type* (I)) as in [2], where the symmetric tensor algebras of infinite type power series spaces have been investigated.

It should be finally mentioned that Fréchet m-algebras which are linearly isomorphic to s play a prominent role in [3] from where some of the notation below is taken. It might also be interesting to study the continuity of multiplicative forms on $s_\bullet$, which would solve Michael's conjecture (for a similar approach see [6]).

1. The tensor algebra of a Fréchet space. A *Fréchet space* is a complete metrizable locally convex space; its topology can be given by a fundamental system of seminorms $\|\cdot\|_1 \le \|\cdot\|_2 \le \|\cdot\|_3 \le \cdots$. A *Fréchet m-algebra* is an algebra over $\mathbb{C}$ which is a Fréchet space and admits a fundamental system of submultiplicative seminorms.

For a Fréchet space E we set

$$E^{\otimes n} := E \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} E,$$

the n-fold complete π-tensor product, and for any continuous seminorm p on E we denote by $p^{\otimes n}$ the n-fold π-tensor product of p.

With this notation we define

$$T(E) = \Big\{ x = (x_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} E^{\oplus n} : \\ \|x\|_k = \sum_{n=1}^{\infty} e^{kn} \|x_n\|_k^{\oplus n} < \infty \text{ for all } k \in \mathbb{N} \Big\}. \tag{1.1}$$

By defining

$$x_1 \otimes \cdots \otimes x_n \times y_1 \otimes \cdots \otimes y_m := x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m,$$

$T(E)$ becomes a Fréchet m-algebra. In a natural way $E \subset T(E)$ and every continuous linear map $E \to A$, where A is a Fréchet m-algebra, extends to a uniquely determined continuous algebra homomorphism $T(E) \to A$. If E and F are Fréchet spaces then every continuous linear map $\varphi : E \to F$ extends to a continuous algebra homomorphism $T(\varphi) : T(E) \to T(F)$.

Obviously, the definition of $T(E)$ is independent of the fundamental system of seminorms on E, and $T(E)$ is determined, up to bicontinuous algebra isomorphism, by its universal property. It is called the *tensor algebra* of E.

If $\|\cdot\|_1 \le \|\cdot\|_2 \le \cdots$ is a fundamental system of seminorms for E then we denote by E_k the respective local Banach spaces and by $j_m^n : E_m \to E_n$ the canonical linking maps for $m \ge n$ (see [5]). The local Banach spaces of

$T(E)$ have the following representation:

$$(1.2) \qquad T(E)_k = \Big\{x = (x_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} E_k^{\oplus n} : \|x\|_k = \sum_{n=1}^{\infty} e^{kn}\|x_n\|_k^{\oplus n} < \infty\Big\}.$$

The following facts are easy consequences of the definition of the tensor algebra and its universal property:

(1) If $E \cong F$ then $T(E) \cong T(F)$ with a bicontinuous algebra isomorphism.
(2) If $q : E \to F$ is a continuous surjective map, then $T(q) : T(E) \to T(F)$ is a continuous surjective algebra homomorphism.
(3) If F is a complemented subspace of E and P the projection, then $T(F)$ is a complemented subalgebra of $T(E)$ with the algebra homomorphism $T(P)$ as projection.

If A is a Fréchet m-algebra then the identity id_A extends to a surjective algebra homomorphism $q_A : T(A) \to A$ and we have an exact sequence

$$0 \to J_A \to T(A) \xrightarrow{q_A} A \to 0$$

where J_A is an ideal in $T(A)$. Obviously, q_A has $A \hookrightarrow T(A)$ as a continuous linear right inverse.

If therefore the Fréchet m-algebras A and B are linearly isomorphic, they can be considered as quotient algebras of the same algebra $T(A)$ with respect to different complemented ideals. The same holds for algebras B which are linearly isomorphic to a quotient of A. However, there the ideals need not be complemented.

We can improve the assertion (1) above.

Proposition 1.1. *Let E and F be Fréchet spaces, then the following are equivalent*:

(1) *$T(E) \cong T(F)$ with a bicontinuous algebra isomorphism.*
(2) *$E \cong F$.*

Proof. We only have to show (1)$\Rightarrow$(2). Let $\varphi : T(E) \to T(F)$ be a bicontinuous algebra isomorphism. We refer to formula (1.1) and set

$$T_2(E) = \{x \in T(E) : x_1 = 0\} = \overline{T(E)^2},$$

and likewise for F. Then it is obvious that φ maps $T_2(E)$ bijectively onto $T_2(F)$. Let π_F be the projection in $T(F)$ onto F with kernel $T_2(F)$. Then $\pi_F \circ \varphi|_E$ is an isomorphism from E onto F. ■

We close this section by two simple examples, which we take from [8]:

- $T(\mathbb{C}) = H_0(\mathbb{C})$, the space of all entire functions on $\mathbb{C}$ which vanish at 0.
- $T(\mathbb{C}^2) \cong s$, the space of all rapidly decreasing sequences (see below).

While the first example is the representation (1.1), the second is only a linear isomorphism. Because we will be using the second assertion, we repeat, for the convenience of the reader, its simple proof.

We think of $\mathbb{C}^2$ as equipped with the ℓ_1-norm. Then $(\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{2^n}$ again equipped with the ℓ_1-norm (see [4, Chap. I, §2, n° 2, Cor. 4, p. 61]). So we obtain, labelling the natural basis of $\mathbb{C}^{2^n}$ from 2^n to $2^{n+1}-1$:

$$T(\mathbb{C}^2) = \Big\{x = (x_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|x\|_k = \sum_{n=1}^{\infty} 2^{kn}\Big(\sum_{j=2^n}^{2^{n+1}-1} |x_j|\Big) \text{ for all } k\Big\}.$$

For $2^n \le j < 2^{n+1}$ we have $2^{kn} \le j^k < 2^k \cdot 2^{kn}$. Therefore $T(\mathbb{C}^2) \cong s$. ∎

2. Power series spaces. For any sequence α with $0 \le \alpha_1 \le \alpha_2 \le \cdots < \infty$ and for $r = 0$ or $r = \infty$ we set

$$\Lambda_r(\alpha) = \Big\{x = (x_j)_{j\in\mathbb{N}} : |x|_t = \sum_{j=0}^{\infty} |x_j| e^{t\alpha_j} < \infty \text{ for all } t < r\Big\}.$$

Then $\Lambda_r(\alpha)$ with the norms $|\cdot|_t$ is a Fréchet space. It is called a *power series space,* for $r = 0$ of *finite type,* for $r = \infty$ of *infinite type.*

Of particular importance is the case $\alpha_n = \log n$. In this case we set $\Lambda_\infty(\log n) =: s$ and $\Lambda_0(\log n) =: s_0$. We call s the space of *rapidly decreasing sequences*, and s_0 the space of *very slowly increasing sequences.*

Moreover, we note that for $\alpha_n = n$ we have $\Lambda_\infty(n) \cong H(\mathbb{C})$, the space of all entire functions in one complex variable, and $\Lambda_0(n) \cong H(\mathbb{D})$, the space of holomorphic functions on the open unit disc.

We set $r_k = k$ if $r = \infty$ and $r_k = -1/k$ if $r = 0$. For the tensor algebra $T(\Lambda_r(\alpha))$ we obtain

$$T(\Lambda_r(\alpha)) = \Big\{x = (x_{n,j})_{n\in\mathbb{N}, j\in\mathbb{N}^n} : \\ \|x\|_k = \sum_{n=1}^{\infty}\sum_{j\in\mathbb{N}^n} |x_{n,j}| e^{kn + r_k(\alpha_{j_1}+\cdots+\alpha_{j_n})} < \infty \text{ for all } k \in \mathbb{N}\Big\}.$$

We set $\nu_n(j) = \alpha_{j_1} + \cdots + \alpha_{j_n}$ for $j \in \mathbb{N}^n$ and denote by $\beta_n(\nu)$, $\nu = 1, 2, \ldots,$ an increasing enumeration of the numbers $\nu_n(j)$, $j \in \mathbb{N}^n$.

Then we obtain

$$T(\Lambda_r(\alpha)) \cong \Big\{x = (x_{n,\nu})_{n,\nu\in\mathbb{N}} : \\ \|x\|_k = \sum_{n,\nu} |x_{n,\nu}| e^{kn + r_k\beta_n(\nu)} < \infty \text{ for all } k \in \mathbb{N}\Big\}. \tag{2.1}$$

From this representation we immediately derive:

LEMMA 2.1. *The tensor algebra of an infinite type power series space is again an infinite type power series space.*

We remark that $\Lambda_r(\alpha) \cong \Lambda_r(\beta)$ if and only if $\Lambda_r(\alpha) = \Lambda_r(\beta)$ and this is equivalent to the existence of a constant $C > 0$ such that $C^{-1}\alpha_j \le \beta_j \le C\alpha_j$ for large j (see [5, Proposition 29.1]).

At some point we will assume that $\log n = O(\alpha_n)$, that is,

$$\limsup_n \frac{\log n}{\alpha_n} < \infty. \tag{2.2}$$

For $r = \infty$ this is equivalent to the nuclearity of $\Lambda_\infty(\alpha)$. All infinite type power series spaces which are relevant in analysis belong to this class. It is not difficult to show that under this condition there is a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\Lambda_r(\alpha) = \Lambda_r(\log n_k)$.

3. Linear topological properties of tensor algebras of Fréchet spaces. Throughout this section $\cong$ will always denote a linear topological isomorphism. First we see that nuclearity of a Fréchet space is inherited by its tensor algebra.

THEOREM 3.1. *A Fréchet space E is nuclear if and only if $T(E)$ is nuclear.*

Proof. If $T(E)$ is nuclear then so is E as an (even complemented) subspace. To prove the converse we assume that E is a nuclear Fréchet space. This means that for every k we find p such that $j^k_{k+p} : E_{k+p} \to E_k$ is nuclear. We have to show that for some q also the linking map $J^k_{k+q} : T(E)_{k+q} \to T(E)_k$ is nuclear.

Now with j^k_{k+p} also $(j^k_{k+p})^{\otimes n} : E^{\otimes n}_{k+p} \to E^{\otimes n}_k$ is nuclear and $\nu((j^k_{k+p})^{\otimes n}) \le \nu(j^k_{k+p})^n$, where $\nu(\cdot)$ denotes the nuclear norm of an operator. We choose $q \ge p$ such that $\nu(j^k_{k+p}) < e^q$ and remark that $\nu(j^k_{k+q}) \le \nu(j^k_{k+p})$. Then a straightforward calculation shows that J^k_{k+q} is nuclear. ∎

For our general discussion we need the following lemma.

LEMMA 3.2. *For any two Fréchet spaces E and F the space $T(E) \mathbin{\widehat{\otimes}} T(F)$ is isomorphic to a complemented subspace of $T(E \oplus F)$.*

Proof. We have

$$T(E) \mathbin{\widehat{\otimes}} T(F) = \Big\{ u = (u_{l,k}) \in \prod_{l,k\in\mathbb{N}} E^{\otimes l} \otimes F^{\otimes k} : \\ \|u\|_m = \sum_n e^{mn} \sum_{l+k=n} \|u_{l,k}\|_m < \infty \Big\}.$$

This space arises in a natural way as a complemented subspace of $T(E \oplus F)$ by decomposing each summand $(E \oplus F)^{\otimes n}$ into 2^n direct summands following from the direct decomposition $E \oplus F$. ∎

We call a Fréchet space E *shift-stable* if $E \cong \mathbb{C} \oplus E$. A power series space is shift-stable if and only if $\limsup_{n\to\infty} \alpha_{n+1}/\alpha_n < \infty$. In this case we also call α shift-stable. Notice that all concrete spaces we will be considering are shift-stable.

LEMMA 3.3. *If E is shift-stable then $T(E) \cong s \mathbin{\widehat{\otimes}} T(E)$.*

Proof. By assumption we have $E \cong \mathbb{C}^2 \oplus E$. Since, by Theorem 4.2, $T(\mathbb{C}^2) \cong s$, we deduce from Lemma 3.2 that $s \mathbin{\widehat{\otimes}} T(E) \cong T(\mathbb{C}^2) \mathbin{\widehat{\otimes}} T(E)$ is a complemented subspace of $T(\mathbb{C}^2 \oplus E) \cong T(E)$.

On the other hand, $T(E)$ is obviously isomorphic to a complemented subspace of $s \mathbin{\widehat{\otimes}} T(E)$. Now [7, Proposition 1.2] yields the result. ∎

By using [7, Proposition 1.2] again, Lemma 3.3 implies:

PROPOSITION 3.4. *If E and F are Fréchet spaces, E shift-stable, F isomorphic to a complemented subspace of $T(E)$ and $T(E)$ isomorphic to a complemented subspace of F, then $F \cong T(E)$.*

We will use the following simple remark:

LEMMA 3.5. *For any Fréchet space E we have*

$$T(E) \cong E \oplus T(E \mathbin{\widehat{\otimes}} E) \oplus E \mathbin{\widehat{\otimes}} T(E \mathbin{\widehat{\otimes}} E).$$

This shows, in particular, that $T(E \mathbin{\widehat{\otimes}} E)$ is a complemented subspace of $T(E)$. We will use Proposition 3.4 to show:

THEOREM 3.6. *For any shift-stable Fréchet space E we have $T(E) \cong T(E \mathbin{\widehat{\otimes}} E)$.*

Proof. By Lemma 3.5, $T(E \mathbin{\widehat{\otimes}} E)$ is isomorphic to a complemented subspace of $T(E)$. Since E is clearly isomorphic to a complemented subspace of $E \mathbin{\widehat{\otimes}} E$, the space $T(E)$ is isomorphic to a complemented subspace of $T(E \mathbin{\widehat{\otimes}} E)$. As E is shift-stable, Proposition 3.4 yields the result. ∎

Moreover, from Lemma 3.5 and Theorem 3.6 we get the following coincidences:

COROLLARY 3.7. *For any shift-stable Fréchet space E we have*

$$T(E) \cong E \oplus T(E) \cong E \mathbin{\widehat{\otimes}} T(E) \cong T(E) \oplus T(E).$$

Proof. By Theorem 3.6 we can write the identity in Lemma 3.5 as

$$T(E) \cong E \oplus T(E) \oplus E \mathbin{\widehat{\otimes}} T(E). \tag{3.1}$$

So $T(E)$ contains $E \oplus T(E)$ and $E \widehat{\otimes} T(E)$ as complemented subspaces. As both contain $T(E)$ as complemented subspace, the first two isomorphisms follow from Proposition 3.4. The last one then follows from the first two and equation (3.1). ∎

Finally, we call a Fréchet space E *stable* if $E \oplus E \cong E$. A power series space is stable if and only if $\limsup_{n\to\infty} \alpha_{2n}/\alpha_n < \infty$. In this case we also call α stable. Clearly, every stable power series space is shift-stable. Notice that all concrete spaces we will be considering are stable.

Theorem 3.8. *For any stable Fréchet space E we have*

$$T(E) \widehat{\otimes} T(E) \cong T(E).$$

Proof. By Lemma 3.2, $T(E) \widehat{\otimes} T(E)$ is isomorphic to a complemented subspace of $T(E \oplus E) \cong T(E)$. As $T(E)$ is clearly isomorphic to a complemented subspace of $T(E) \widehat{\otimes} T(E)$, Proposition 3.4 gives the result. ∎

4. The tensor algebra of an infinite type power series space. The following theorem is contained in [8]. However, its proof here will be different.

Theorem 4.1. *If $\log n = O(\alpha_n)$, that is, $\Lambda_\infty(\alpha)$ is nuclear, then $T(\Lambda_\infty(\alpha)) \cong s$.*

Proof. By Lemma 2.1 and Theorem 3.1 we know that there is β such that $T(\Lambda_\infty(\alpha)) \cong \Lambda_\infty(\beta)$ and $\log n = O(\beta_n)$.

On the other hand, let π be the canonical projection in $\Lambda_\infty(\alpha)$ onto the span F of the first two natural basis vectors $\{e_1, e_2\}$ in $\Lambda_\infty(\alpha)$; then $T(\pi)$ is a projection in $T(\Lambda_\infty(\alpha))$ onto $T(F) \cong T(\mathbb{C}^2) \cong s$. We remark that the basis vectors in $T(\mathbb{C}^2)$ which yield the isomorphism to s are products of e_1 and e_2, therefore a subset of the basis vectors in $T(\Lambda_\infty(\alpha))$ which yield the isomorphism to $\Lambda_\infty(\beta)$. This implies that there is a subsequence β_{n_k} which is equivalent to $\log k$. In particular, we have $\beta_k \le \beta_{n_k} = O(\log k)$. Therefore $\Lambda_\infty(\beta) = s$. ∎

As a consequence, we have:

Corollary 4.2. *$T(s) \cong s$.*

We obtain a complete characterization of the Fréchet spaces E with $T(E) \cong s$.

Theorem 4.3. *For any Fréchet space E the following are equivalent*:

(1) *$T(E) \cong s$.*
(2) *E is isomorphic to a complemented subspace of s and $\dim E \ge 2$.*

Proof. (1) implies (2) because E is a complemented subspace of $T(E)$. If, on the other hand, (2) is satisfied, then we may assume that E is a complemented subspace of s. Let P be the projection. Then $T(P)$ is a projection in $T(s) \cong s$ onto $T(E)$. So $T(E)$ is a complemented subspace of s. We choose a 2-dimensional subspace $F \subset E$. Let π be a projection in E onto F. Then $T(\pi)$ is a projection in $T(E)$ onto $T(F) \cong s$. Therefore s is isomorphic to a complemented subspace of $T(E)$. By [7, Proposition 1.2] we conclude that $T(E) \cong s$. ∎

Clearly, Theorem 4.3 contains Theorem 4.1. However, the latter gives in its special case a more precise description of an isomorphism which we will now use.

The isomorphism in Theorem 4.1, hence also in Corollary 4.2, is of a special type: the basis vectors in $T(s)$ which yield the isomorphism to s are pure (tensor) products of canonical basis vectors in s. They are ordered in such a way that the norms are increasing. We now fix such an ordering, that is, a special such isomorphism. Then this isomorphism equips the space s with a multiplication which turns it into a Fréchet m-algebra, which we call $s_\bullet$. The multiplication in $s_\bullet$ is of special form: products of basis vectors are basis vectors.

In what follows, isomorphisms with s and its quotients or complemented subspaces are understood to be linear isomorphisms, and isomorphisms with $s_\bullet$ and its quotients to be Fréchet algebra isomorphisms.

The multiplicative equivalent of Theorem 4.3 is:

THEOREM 4.4. *For a Fréchet space E the following are equivalent*:

(1) $T(E) \cong s_\bullet$.
(2) $E \cong s$.

Proof. This is a consequence of Proposition 1.1 and Corollary 4.2. ∎

The considerations at the end of Section 1 and the characterization of the quotient spaces and complemented subspaces of s in [9] (see also [5, §31]) now lead immediately to the following results, which give a Fréchet algebra equivalent to these characterizations.

THEOREM 4.5. *For a Fréchet m-algebra A the following are equivalent*:

(1) *A is isomorphic to a quotient of s.*
(2) *A is nuclear and has property* (Ω).
(3) *A is isomorphic to a quotient of $s_\bullet$ with respect to a closed ideal.*

THEOREM 4.6. *For a Fréchet m-algebra A the following are equivalent*:

(1) *A is isomorphic to a complemented subspace of s.*

(2) *A is nuclear and has properties* (DN) *and* (Ω).
(3) *A is isomorphic to a quotient of* $s_\bullet$ *with respect to a complemented ideal.*

For the definition and basic properties of the invariants (DN) and (Ω) see e.g. [5, §29].

5. The tensor algebra of finite type power series spaces. Now we will determine the linear isomorphism type of the tensor algebra for certain power series spaces of finite type.

THEOREM 5.1. $T(s_0) \cong H(\mathbb{C}) \mathbin{\widehat{\otimes}} s_0$.

Proof. We refer to formula (2.1) and want to estimate the numbers $\beta_n(\nu)$. For that we put, for $r \geq 0$,

$$m_n(r) = \#\{j \in \mathbb{N}^n : \nu_n(j) \leq r\} = \#\Big\{j \in \mathbb{N}^n : \sum_{\nu=1}^{n} \log j_\nu \leq r\Big\}.$$

Counting only the j with $j_2 = \cdots = j_n = 1$ we see that

$$m_n(r) \geq [e^r] \geq e^r - 1.$$

For a reverse estimate we refer to the proof of [8, Theorem 5] and obtain

$$m_n(r) \leq e^r \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{r^k}{k!} \leq 2^{n-1} e^r \sum_{k=0}^{n-1} \frac{r^k}{k!} \leq e^{n+2r}.$$

This yields

$$\nu \leq m_n(\beta_n(\nu)) \leq e^{n+2\beta_n(\nu)},$$

hence

$$\beta_n(\nu) \geq \frac{1}{2}(\log \nu - n).$$

On the other hand, for all $r < \beta_n(\nu)$ we have

$$\nu \geq m_n(r) \geq e^r - 1.$$

Hence $\nu \geq e^{\beta_n(\nu)} - 1$ and therefore

$$\beta_n(\nu) \leq \log(\nu + 1) \leq \log 2 + \log \nu.$$

Consequently,

$$kn - \frac{1}{k}\beta_n(\nu) \leq \Big(k + \frac{1}{2}\Big)n - \frac{1}{2k}\log \nu \leq 2kn - \frac{1}{2k}\log \nu$$

and

$$kn - \frac{1}{k}\beta_n(\nu) \geq -\frac{1}{k}\log 2 + kn - \frac{1}{k}\log \nu.$$

If for $x = (x_{n,\nu})_{n,\nu\in\mathbb{N}}$ we set

$$|x|_k = \sum_{n,\nu} |x_{n,\nu}| e^{kn - k^{-1}\log\nu},$$

i.e. the standard norms in $H(\mathbb{C}) \mathbin{\widehat{\otimes}_\pi} s_0$, then we have

$$\frac{1}{2}\,|x|_k \le \|x\|_k \le |x|_{2k}.$$

This completes the proof. ■

Due to Lemma 3.3 we can put Theorem 5.1 in a more symmetric form.

Corollary 5.2. $T(s_0) \cong s \mathbin{\widehat{\otimes}} s_0$.

It is impossible for $T(s_0)$ to be the common tensor algebra for all complemented subspaces of s_0. This has nuclearity reasons and is due to Theorem 3.1.

Recall that $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We will now study the tensor algebra of $H(\mathbb{D}) \cong \Lambda_0(n)$. Proceeding as in the proof of Theorem 5.1 we have to estimate

$$m_n(r) = \#\Big\{j \in \mathbb{N}^n : \sum_{\nu=1}^{n} j_\nu \le r\Big\}.$$

Comparing with the volume of the standard simplex in $\mathbb{R}^n$ we obtain

$$m_n(r) \le \frac{r^n}{n!} \le m_n(r + \sqrt{n})$$

and therefore

$$\nu \le m_n(\beta_n(\nu)) \le \frac{\beta_n(\nu)^n}{n!},$$

which implies

$$\frac{1}{e}\, n\nu^{1/n} \le (n!\nu)^{1/n} \le \beta_n(\nu)$$

as a lower estimate. We get an upper estimate from

$$\nu \ge m_n(r) \ge \frac{(r - \sqrt{n})^n}{n!}$$

for all $r < \beta_n(\nu)$, which implies, upon replacing r with $\beta_n(\nu)$,

$$\beta_n(\nu) \le (n!\nu)^{1/n} + \sqrt{n} \le 2n\nu^{1/n}.$$

So we have

$$\frac{1}{3}\, n\nu^{1/n} \le \beta_n(\nu) \le 2n\nu^{1/n}.$$

DEFINITION. We set

$$T\Lambda_0 := \Big\{x = (x_{n,\nu})_{n,\nu\in\mathbb{N}} : \|x\|_k = \sum_{n,\nu} |x_{n,\nu}| e^{n(k-k^{-1}\nu^{1/n})} < \infty \text{ for all } k \in \mathbb{N}\Big\}.$$

Hence we have proved

LEMMA 5.3. $T(H(\mathbb{D}))) \cong T\Lambda_0$.

Since for the d-dimensional polydisc $\mathbb{D}^d$ we have $H(\mathbb{D}^d) \cong H(\mathbb{D})^{\otimes d}$ we immediately deduce from Theorem 3.6 and Lemma 5.3:

THEOREM 5.4. *$T(H(\mathbb{D}^d)) \cong T\Lambda_0$ for all dimensions d.*

We will generalize this result to a larger class of finite type power series spaces.

DEFINITION. We call an exponent sequence α *power comparable* if there are $c > 0$ and $0 < a \le b$ so that

$$\frac{1}{c}\, n^a \le \alpha_n \le cn^b$$

for all $n \in \mathbb{N}$.

To exploit this condition we need the following well known lemma (cf. [1]), for which we give a simple proof for the convenience of the reader.

LEMMA 5.5. *If α is stable and $\alpha_n \le C\,\beta_n$ for all n and some $C > 0$ then there is a subsequence k_n of $\mathbb{N}$ and $D > 0$ such that*

$$\frac{1}{D}\,\alpha_{k_n} \le \beta_n \le D\alpha_{k_n}$$

for all $n \in \mathbb{N}$. In particular, $\Lambda_r(\beta)$ is a complemented subspace of $\Lambda_r(\alpha)$ for $r = 0, \infty$.

Proof. We set

$$m_n := \sup\{m : \alpha_m \le C\beta_n\};$$

then $m_n \ge n$. We put $k_n = m_n + n$ and obtain

$$\alpha_{k_n} \le \alpha_{2m_n} \le \lambda\alpha_{m_n} \le \lambda C\beta_n \le \lambda\alpha_{m_n+1} \le \lambda\alpha_{k_n},$$

which completes the proof. ∎

THEOREM 5.6. *If α is stable and power comparable then $T(\Lambda_0(\alpha)) \cong T\Lambda_0$.*

Proof. First we note that for any $\nu \in \mathbb{Z}$ we have

$$\Lambda_0(n^{2^\nu}) \mathbin{\widehat{\otimes}} \Lambda_0(n^{2^\nu}) \cong \Lambda_0(n^{2^{\nu-1}})$$

and $H(\mathbb{D}) \cong \Lambda_0(n^{2^\nu})$ with $\nu = 0$.

From Theorem 3.6 we deduce that $T(\Lambda_0(n^{2^\nu})) \cong T(\Lambda_0(n)) \cong T\Lambda_0$ for all $\nu \in \mathbb{Z}$. Since α is power comparable we may assume that for some $\nu \in \mathbb{N}$ and $c > 0$ we have

$$\frac{1}{c} n^{2^{-\nu}} \le \alpha_n \le c n^{2^\nu}$$

for all $n \in \mathbb{N}$.

From Lemma 5.5 applied to the stable sequences $n^{2^{-\nu}}$ and α_n we deduce that $T(\Lambda_0(\alpha))$ is isomorphic to a complemented subspace of $T(\Lambda_0(n^{2^{-\nu}})) \cong T\Lambda_0$, and $T\Lambda_0 \cong T(\Lambda_0(n^{2^\nu}))$ is isomorphic to a complemented subspace of $T(\Lambda_0(\alpha))$. From Proposition 3.4 we obtain the result. ■

We also consider the following condition:

Definition. We say that α is of *type* (I) if there is $p \in \mathbb{N}$ so that

$$1 < \inf_n \frac{\alpha_{pn}}{\alpha_n} \le \sup_n \frac{\alpha_{pn}}{\alpha_n} < \infty.$$

This condition was introduced in [2, Proposition 4.7] and shown to be equivalent to α being stable and $(n^{-\beta}\alpha_n)_n$ increasing for some $\beta > 0$.

To consider exponent sequences of type (I) is interesting in our context as for those sequences the linear isomorphism type of the symmetric tensor algebra $S(\Lambda_\infty(\alpha))$ has been calculated in [2, Theorem 5.5] as $S(\Lambda_\infty(\alpha)) \cong \Lambda_\infty(\beta)$ where $\beta_n = \alpha_{[\log(n+1)]} \log(n+1)$.

Lemma 5.7. *If α is of type* (I) *then it is stable and power comparable.*

Proof. We may assume that $\alpha_1 = 1$. If

$$1 < q = \inf_n \frac{\alpha_{pn}}{\alpha_n} \le \sup_n \frac{\alpha_{pn}}{\alpha_n} = Q$$

then we have

$$q^\nu \le \alpha_{p^\nu} \le Q^\nu$$

for all $\nu \in \mathbb{N}_0$. For $p^\nu \le n \le p^{\nu+1}$ this gives, with $a = \frac{\log q}{\log p}$ and $b = \frac{\log Q}{\log p}$,

$$\frac{1}{q} n^a \le q^\nu \le \alpha_{p^\nu} \le \alpha_n \le \alpha_{p^{\nu+1}} \le Q^{\nu+1} \le Q n^b,$$

which completes the proof. ■

Hence we obtain

Corollary 5.8. *If α is of type* (I) *then $T(\Lambda_0(\alpha)) \cong T\Lambda_0$.*

6. The space $T\Lambda_0$.

The space $T\Lambda_0$ is a nuclear (see Theorem 3.1) double indexed sequence space of the class of power spaces of mixed type. This class was introduced by Zahariuta in [10] (see also Zahariuta [11, §2]). They are Köthe spaces given by matrices of the form $b_{j,k} = e^{(\lambda_j k - 1/k)a_j}$ with $\limsup \lambda_j > 0$, $\liminf \lambda_j = 0$. In our case, with $j = (n,\nu)$, we have $a_{n,\nu} = n\nu^{1/n}$ and $\lambda_{n,\nu} = \nu^{-1/n}$.

To understand the asymptotics of the $a_{n,\nu}$ and to write the space in a single-indexed form we estimate the increasing arrangement γ_μ, $\mu \in \mathbb{N}$, of the set $\{n\nu^{1/n} : n, \nu \in \mathbb{N}\}$.

For this we need to estimate

$$M(t) = \#\{(n,\nu) : n\nu^{1/n} \le t\} = \#\{(n,\nu) : \nu \le (t/n)^n\}.$$

For every $n \le t$ we have $[(t/n)^n]$ pairs (n,ν) satisfying the estimates. Therefore

$$M(t) = \sum_{n=1}^{[t]} [(t/n)^n].$$

This immediately gives $M(t) \le e^t$. For a lower estimate we use the summand with $n = [t/e]$. We obtain

$$M(t) \ge [e^n] \ge e^{t/e-1} - 1.$$

Proceeding as in the calculations in the previous section we obtain

$$\log\mu \le \gamma_\mu \le \log(\mu+1) + (e+1).$$

Therefore we may replace γ_μ by $\log\mu$ and write $T\Lambda_0$ in the following form:

$$T\Lambda_0 = \Big\{x = (x_\mu)_{\mu\in\mathbb{N}} : \|x\|_k = \sum_{\mu=1}^{\infty} |x_\mu| e^{kn_\mu - k^{-1}\log\mu} < \infty\Big\}.$$

The sequence $(n_\mu)_{\mu\in\mathbb{N}}$ consists of integers and it takes on every integer value infinitely often. Moreover, without the kn_μ-term we would have s_0, which is not nuclear. The kn_μ-term provides nuclearity.

Finally, let us remark that all the finite type power series spaces in question are complemented subspaces of s_0, hence $T\Lambda_0$ is a complemented subspace of $T(s_0) \cong H(\mathbb{C}) \widehat{\otimes} s_0$. If we visualize the elements of this space as matrices, then $T\Lambda_0$ in the above representation consists of matrices which have in every column exactly one entry and in every row infinitely many sparsely distributed entries.

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