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L_{h}^2-domains of holomorphy and the Bergman kernel

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Abstract. We give a characterization of L_{h}^2-domains of holomorphy with the help of the boundary behavior of the Bergman kernel and geometric properties of the boundary, respectively.

For $\lambda_0 \in \mathbb{C}$, $r > 0$ we define $\triangle(\lambda_0, r) := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\}$. We also put $E := \triangle(0,1)$. Moreover, the set of all plurisubharmonic (respectively, subharmonic) functions on an open set $D \subset \mathbb{C}^n$ is denoted by $\mathrm{PSH}(D)$ (respectively, $\mathrm{SH}(D)$). We allow the (pluri)subharmonic functions to be identically $-\infty$ on connected components of D.

Following [Kli] for a domain $D \subset \mathbb{C}^n$ define

$$g_D(p,z) := \sup\{u(z)\}, \qquad p, z \in D,$$

where the supremum is taken over all negative $u \in \mathrm{PSH}(D)$ such that $u(\cdot) - \log \|\cdot - p\|$ is bounded from above near p. We call the function $g_D(p,\cdot)$ the *pluricomplex Green function* (with the logarithmic pole at p). We also define

$$A_D(p;X) := \limsup_{\lambda \to 0} \frac{\exp(g_D(p, p+\lambda X))}{|\lambda|}, \qquad p \in D,\ X \in \mathbb{C}^n.$$

Following [Jar-Pfl] the function A_D is called *the Azukawa pseudometric*.

For a boundary point w of a bounded domain $D \subset \mathbb{C}$ we introduce the notion of regularity. Namely, we say that D is *regular* at w if there exist a neighborhood U of w and a subharmonic function u on $U \cap D$ with $u < 0$ on $U \cap D$ and $\lim_{U\cap D \ni \lambda \to w} u(\lambda) = 0$.

A set $P \subset \mathbb{C}^n$ is called *pluripolar* if for any point $z \in P$ there exist a connected neighborhood $U = U(z)$ and a function $u \in \mathrm{PSH}(U)$, $u \not\equiv -\infty$, such that $P \cap U \subset \{z \in U : u(z) = -\infty\}$. In case $n = 1$ we call such a

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set P *polar*. It is well known (cf. [Kli], Josefson theorem) that a set $P \subset \mathbb{C}^n$ is pluripolar if and only if there is a function $u \in \mathrm{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$, such that $P \subset \{z \in \mathbb{C}^n : u(z) = -\infty\}$.

A bounded domain $D \subset \mathbb{C}^n$ is said to be *hyperconvex* if there exists a negative and continuous plurisubharmonic exhaustion function of D.

Denote the class of square integrable holomorphic functions on an open set D by $L^2_{\mathrm{h}}(D)$. It is a Hilbert space with the standard scalar product induced from $L^2(D)$. Let us recall the definition of the *Bergman kernel*:

$$K_D(z) := \sup\left\{\frac{|f(z)|^2}{\|f\|^2_{L^2_{\mathrm{h}}(D)}} : f \not\equiv 0,\ f \in L^2_{\mathrm{h}}(D)\right\}.$$

If D is a bounded domain then $\log K_D$ is smooth and strictly plurisubharmonic. Therefore, for a bounded domain D one may define the *Bergman metric* β_D:

$$\beta_D(z;X) := \sqrt{\sum_{j,k=1}^{n} \frac{\partial^2 \log K_D(z)}{\partial z_j \bar{\partial} z_k} X_j \overline{X}_k}, \qquad z \in D,\ X \in \mathbb{C}^n,$$

and set

$$b_D(w,z) := \inf\{L_{\beta_D}(\alpha)\}, \qquad w, z \in D,$$

where $L_{\beta_D}(\alpha) = \int_0^1 \beta_D(\alpha(t);\alpha'(t))\,dt$ and the infimum is taken over all piecewise C^1-curves $\alpha : [0,1] \to D$ such that $\alpha(0) = w$, $\alpha(1) = z$. We call b_D the *Bergman distance*. If (D, b_D) is a complete metric space we say that D is *Bergman complete*.

A domain $D \subset \mathbb{C}^n$ is called a *domain* (resp. an L^2_{h}-*domain*) *of holomorphy* if there are no domains $D_0, D_1 \subset \mathbb{C}^n$ with $\emptyset \neq D_0 \subset D_1 \cap D$, $D_1 \not\subset D$ such that for any $f \in \mathcal{O}(D)$ (resp. $f \in L^2_{\mathrm{h}}(D)$) there exists an $\widetilde{f} \in \mathcal{O}(D_1)$ with $\widetilde{f} = f$ on D_0.

Let us recall several results concerning the above-mentioned notions, which show a close relationship between the theory of square integrable holomorphic functions and pluripotential theory.

For a bounded pseudoconvex domain D consider the following properties:

(1) D is hyperconvex,
(2) for any $w \in \partial D$, $\lim_{D \ni z \to w} K_D(z) = \infty$,
(3) D is Bergman complete,
(4) D is an L^2_{h}-domain of holomorphy.

All the relations between the properties (1)–(4) are known. Namely, (1)⇒(2) (see [Ohs 1]), (1)⇒(3) (see [Bło-Pfl], [Her]), and (3)⇒(4). The implication (2)⇒(1) does not hold in general (take the Hartogs triangle in $\mathbb{C}^2$ or consider some one-dimensional Zalcmann-type domains—see [Ohs 1]). The one-dimensional counterexample to the implication (3)⇒(1) is given in [Chen 1].

Recall that any bounded pseudoconvex fat domain is an L^2_{h}-domain of holomorphy (see [Pfl]). Thus the Hartogs triangle is an L^2_{h}-domain of holomorphy in $\mathbb{C}^2$ which is not Bergman complete. Moreover, there also exists a fat domain in the complex plane that is not Bergman complete (see [Jar-Pfl-Zwo]). Thus, the implication (4)⇒(3) does not hold even for fat pseudoconvex domains. In dimension one the implication (2)⇒(3) does hold (see [Chen 2]) but in higher dimensions this is no longer the case (take the Hartogs triangle once more). As far as (3)⇒(2) is concerned one may find a counterexample already in dimension one (see [Zwo 2]).

Let us have a closer look at the last example. The counterexamples belong to the following class of domains:

$$D := E \setminus \Big(\bigcup_{j=1}^{\infty} \overline{\triangle}(z_j, r_j) \cup \{0\} \Big),$$

where $z_j \to 0$, $r_j > 0$, $\overline{\triangle}(z_j, r_j) \subset E \setminus \{0\}$, $\overline{\triangle}(z_j, r_j) \cap \overline{\triangle}(z_k, r_k) = \emptyset$, $j \neq k$. It is easy to see that for any $w \in \partial D$, $w \neq 0$, we have $\lim_{D \ni z \to w} K_D(z) = \infty$. The point is that the sequences can be chosen so that $\liminf_{D \ni z \to 0} K_D(z) < \infty$ and the domain is still Bergman complete. On the other hand one may easily see that $\limsup_{z \to 0} K_D(z) = \infty$. So the natural problem arises whether one may construct an example of a Bergman complete domain such that for some $w \in \partial D$ we have $\limsup_{z \to w} K_D(z) < \infty$. Below we show that this is impossible. Let us write down explicitly the condition we are interested in (as some kind of complement to properties (1)–(4)):

(5) for any $w \in \partial D$ we have $\limsup_{D \ni z \to w} K_D(z) = \infty$.

The main aim of this paper is to present the following characterizations of L^2_{h}-domains of holomorphy.

Theorem 1. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then* (4) *is equivalent to* (5), *i.e. D is an L^2_{h}-domain of holomorphy if and only if for any $w \in \partial D$ we have* $\limsup_{D \ni z \to w} K_D(z) = \infty$.

Making use of Theorem 1 and a result of A. Sadullaev we also get the following characterization of bounded L^2_{h}-domains of holomorphy.

Theorem 2. *Let D be a bounded pseudoconvex domain. Then D is an L^2_{h}-domain of holomorphy if and only if for any $w \in \partial D$ and for any neighborhood U of w the set $U \setminus D$ is not pluripolar.*

Before proving Theorem 1 let us recall some properties of the notions just defined that we need in what follows.

We shall start by considering L^2_{h}-domains of holomorphy in $\mathbb{C}$ ($n = 1$). First we list a number of properties of polar sets in $\mathbb{C}$ that we shall use (see [Ran], [Con]).

Let D be an open set in $\mathbb{C}$ and let $K \subset D$ be a polar set relatively closed in D. Then:

- if D is additionally connected then so is $D \setminus K$,
- for any $\lambda \in D$ and for any $0 < s$ with $\triangle(\lambda, s) \subset\subset D$ there is an $s < r$ with $\triangle(\lambda, r) \subset\subset D$ and $\partial\triangle(\lambda, r) \cap K = \emptyset$,
- for any $f \in L^2_{\mathrm{h}}(D \setminus K)$ there is an $\widetilde{f} \in \mathcal{O}(D)$ such that $\widetilde{f}|_{D\setminus K} = f$.

There is also a precise description of L^2_{h}-domains of holomorphy in $\mathbb{C}$.

Theorem 3 (see [Con], Theorem 9.9, p. 351). *Let D be a bounded domain in $\mathbb{C}$ and let $z \in \partial D$. Then there is an open neighborhood U of z such that any $f \in L^2_{\mathrm{h}}(D)$ extends holomorphically to $D \cup U$ if and only if there is a neighborhood V of z such that the set $V \setminus D$ is polar.*

One may easily get from Theorem 3 the following description of L^2_{h}-domains of holomorphy in $\mathbb{C}$.

Theorem 4. *Let D be a bounded domain in $\mathbb{C}$. Then D is an L^2_{h}-domain of holomorphy iff for any $w \in \partial D$ and for any neighborhood U of w the set $U \setminus D$ is not polar.*

Note that Theorem 2 is the exact higher-dimensional counterpart of Theorem 4.

Let us now recall some basic properties of regular points and the Green function. For a domain $D \subset \mathbb{C}^n$ we have $g_D(p, \cdot) \in \mathrm{PSH}(D)$, $g_D(p, \cdot) < 0$. A bounded domain D is hyperconvex iff $g_D(p, \cdot)$ is a continuous exhaustive function of D.

In the case of bounded planar domains it is well known that the Green function is symmetric (as a function of two variables) and $g_D(p, \cdot)$ is harmonic on $D \setminus \{p\}$. Moreover, a point $w \in \partial D$ is regular iff for some (any) $p \in D$, $g_D(p, \lambda) \to 0$ as $D \ni \lambda \to w$. Consequently, a bounded domain $D \subset \mathbb{C}$ is hyperconvex iff any point of its boundary is regular. The set of irregular points of any bounded domain in $\mathbb{C}$ is polar.

Below we shall need some estimate for the Bergman kernel in the one-dimensional case that will enable us to prove Theorem 1 in dimension one.

Theorem 5 (see [Ohs 2]). *Let D be a domain in $\mathbb{C}$. Then there is a positive constant C such that*

$$\sqrt{K_D(z)} \geq C A_D(z; 1), \qquad z \in D.$$

Our first aim is to obtain the following exhaustion property of the Bergman kernel at regular points.

Proposition 6. *Let D be a bounded domain in $\mathbb{C}$. Assume that $w \in \partial D$ is a regular point. Then $K_D(z) \to \infty$ as $D \ni z \to w$.*

Proof. In view of Theorem 5 it is sufficient to show that

$$r(p) \to 0 \quad \text{as } p \to w, \tag{6}$$

where $r := r(p) := \operatorname{diam} D(p)$, $D(p) := \{z \in D : g_D(p,z) < -1\}$. In fact, assuming the last property we get (see [Zwo 1])

$$A_D(p;1) = eA_{D(p)}(p;1) \ge eA_{\triangle(p,r)}(p;1) = \frac{e}{r} \to \infty \quad \text{as } p \to w.$$

Suppose that (6) does not hold. Then one easily finds an $\varepsilon > 0$, sequences $D \ni p_\nu \to w$ and $D \ni z_\nu \to z \in \overline{D}$ such that $|p_\nu - z_\nu| \ge \varepsilon$ and $g_D(p_\nu, z_\nu) < -1$. Taking $\widetilde{D} := D \cup V$, where V is some small disc around z such that $w \notin \overline{V}$, we get $g_D(p_\nu, z_\nu) \ge g_{\widetilde{D}}(p_\nu, z_\nu)$ and $z \in \widetilde{D}$. In other words, it is sufficient to show that $g_{\widetilde{D}}(p_\nu, z_\nu) \to 0$. But because of the pointwise convergence of $g_{\widetilde{D}}(p_\nu, \cdot) = g_{\widetilde{D}}(\cdot, p_\nu)$ to 0 (as $\nu \to \infty$), the harmonicity of $g_{\widetilde{D}}(p_\nu, \cdot)$ near z and the Vitali theorem, we conclude that $g_{\widetilde{D}}(p_\nu, \cdot)$ tends uniformly to 0 on some neighborhood of z. ∎

REMARK 7. In view of property (6) it follows from the estimates in [Die-Her] that for any bounded domain in $\mathbb{C}$ the convergence $\beta_D(z;1) \to \infty$ as $z \to w \in \partial D$ holds for any regular point $w \in \partial D$.

LEMMA 8. *Let D be a bounded domain in $\mathbb{C}$, $w \in \partial D$. Then the following conditions are equivalent*:

(7) $\limsup_{D \ni z \to w} K_D(z) < \infty$,
(8) *there is an open neighborhood U of w such that the set $U \setminus D$ is polar.*

Proof. Let us first make a general remark: $U \setminus D$ being polar is equivalent to $U \cap \partial D$ being polar.

(8)⇒(7). If U satisfies (8) then without loss of generality one may assume that $K := U \cap \partial D \subset\subset U$. So there is a domain $\widetilde{D}$ with $D = \widetilde{D} \setminus K$, $w \in \widetilde{D}$, where K is a compact polar set. Then

$$L^2_{\mathrm{h}}(D) = L^2_{\mathrm{h}}(\widetilde{D})|_D$$

and, consequently, $K_D = K_{\widetilde{D}}|_D$, which implies (7).

(7)⇒(8). Suppose that for any neighborhood U of w the set $U \cap \partial D$ is not polar. Then there is a sequence $w_\nu \to w$, $w_\nu \in \partial D$, such that D is regular at w_ν. In view of Proposition 6 we have $K_D(z) \to \infty$ as $D \ni z \to w_\nu$, which easily finishes the proof. ∎

We are now able to study the situation in $\mathbb{C}^n$ $(n > 1)$.

LEMMA 9. *Let D be a domain in $\mathbb{C}^n$, $n \ge 2$. Fix $0 < r < t$. For any $z' \in \mathbb{C}^{n-1}$ define $A(z') := \{z_n \in tE : (z', z_n) \in D\} = tE \setminus K(z')$. Assume that $K(0')$ is polar and there is a neighborhood $0' \in V$ such that for almost any $z' \in V$ (with respect to the $(2n-2)$-dimensional Lebesgue measure) the* *set $K(z')$ is polar. Then there is a neighborhood $0' \in V' \subset V$ such that for any $f \in L^2_{\mathrm{h}}(D)$ there exists a function $F \in \mathcal{O}(V' \times rE)$ with $F = f$ on $(V' \times rE) \cap D$.*

Proof. Because $K(0')$ is polar there is an s with $0 < r < s < t$ such that $K(0') \cap \partial(sE) = \emptyset$. Then there is a neighborhood $0' \in V' \subset V$ such that for any $\zeta' \in V'$ we have $K(\zeta') \cap \partial(sE) = \emptyset$.

Define

$$F(\zeta', z_n) := \frac{1}{2\pi i} \int\limits_{\partial(sE)} \frac{f(\zeta', \lambda)}{\lambda - z_n}\, d\lambda, \qquad (\zeta', z_n) \in V' \times sE.$$

Then F is a holomorphic function on $V' \times sE$.

On the other hand by the square integrability of f, the Fubini theorem and the assumptions of the lemma, for almost all $\zeta' \in V'$ (with respect to the $(2n-2)$-dimensional Lebesgue measure) the function $f(\zeta', \cdot)$ is in $L^2_{\mathrm{h}}(tE \setminus K(\zeta'))$ and $K(\zeta')$ is polar. Since closed polar sets are removable for L^2_{h}-functions, for almost all $\zeta' \in V'$ the function $f(\zeta', \cdot)$ extends to a holomorphic function on tE. So the Cauchy formula applies and we obtain the equality $f(\zeta', z_n) = F(\zeta', z_n)$, $(\zeta', z_n) \in (V' \times sE) \cap D$, for almost all $\zeta' \in V'$. Since the equality holds on a dense subset of $(V' \times sE) \cap D$, it holds on the whole set. ■

Before we start the proof of Theorem 1 let us formulate, in the form that we need, the most powerful tool we shall use, namely the Ohsawa–Takegoshi extension theorem.

THEOREM 10 (see [Ohs-Tak]). *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let L be a complex line. Then there is a constant $C > 0$ such that for any $f \in L^2_{\mathrm{h}}(D \cap L)$ there is an $F \in L^2_{\mathrm{h}}(D)$ with $\|F\|_{L^2_{\mathrm{h}}(D)} \le C\|f\|_{L^2_{\mathrm{h}}(D\cap L)}$ and $F|_{D\cap L} = f$.*

Note that Theorem 10 directly leads to the following inequality for the Bergman kernel:

$$K_{D\cap L}(z) \le C^2 K_D(z), \qquad z \in D \cap L.$$

This inequality will often be used below. Note only that the set $D \cap L$ on the left-hand side is open (as a subset of $\mathbb{C}$) but not necessarily connected.

We now prove our main result.

Proof of Theorem 1. First note that the result for $n = 1$ follows from Theorem 4 and Lemma 8, so assume that $n \ge 2$.

(5)⇒(4). Suppose that D is not an L^2_{h}-domain of holomorphy. Then there are a polydisc $P \subset D$ with $\partial P \cap \partial D \ne \emptyset$ and a polydisc $\widetilde{P} \supset\supset P$,

$\widetilde{P} \not\subset D$, such that for every function $f \in L^2_{\rm h}(D)$ there is a function $\widehat{f} \in H^\infty(\widetilde{P})$ with $f = \widehat{f}$ on P.

We claim that for any $z \in P$ and for any complex line L passing through z we have

$$L \cap D \cap \widetilde{P} = (L \cap \widetilde{P}) \setminus K(z), \quad \text{where } K(z) \text{ is a polar set.}$$

Suppose that $L \cap D \cap \widetilde{P} = (L \cap \widetilde{P}) \setminus K(z)$, where $K(z)$ is not a polar set. Choose a compact non-polar set $K' \subset K(z) \subset (L \cap \widetilde{P}) \setminus D$ such that $V_0 = L \setminus \widehat{K}'$ (where $\widehat{K}'$ denotes the polynomial hull of K') contains $L \cap P$. Then there is a function $f \in L^2_{\rm h}(V_0)$ which does not extend holomorphically through $\widehat{K}'$ (cf. Theorem 3). Let $\{V_j\}_{j=1}^N$, where $0 \le N \le \infty$, be the family of bounded components of $L \setminus K'$. Additionally, we let f be identically 0 on $\bigcup_{j=1}^N V_j$.

In view of the Ohsawa–Takegoshi extension theorem there exists an $F \in L^2_{\rm h}(D)$ such that $F|_{L\cap D} = f|_{L\cap D}$. But then there is an $\widehat{F} \in H^\infty(\widetilde{P})$ such that $\widehat{F}|_P = F|_P$. Consequently, $\widehat{F}|_{L\cap\widetilde{P}}$ is a holomorphic extension of $f|_{L\setminus\widehat{K}'}$ through $\widehat{K}'$, a contradiction.

It follows from the above claim that $\widetilde{P} \cap D$ is connected. Consequently, for any function $f \in L^2_{\rm h}(D)$ its (unique) extension $\widehat{f} \in H^\infty(\widetilde{P})$ satisfies the equality $f = \widehat{f}$ on $D \cap \widetilde{P}$.

Consider the space

$$A := \{(f, \widehat{f}) : f \in L^2_{\rm h}(D)\} \subset L^2_{\rm h}(D) \times H^\infty(\widetilde{P})$$

with the norm $\|(f,\widehat{f})\| := \|f\|_{L^2_{\rm h}(D)} + \|\widehat{f}\|_{H^\infty(\widetilde{P})}$. It is easily seen that A is a Banach space. Consider the mapping $\pi : A \ni (f,\widehat{f}) \mapsto f \in L^2_{\rm h}(D)$. Then π is a one-to-one surjective continuous linear mapping. Hence, in view of the Banach open mapping theorem, π^{-1} is a continuous linear mapping. In other words, there is a constant $C > 1$ such that

$$\|(f,\widehat{f})\| \le C\|f\|_{L^2_{\rm h}(D)}, \quad f \in L^2_{\rm h}(D);$$

in particular, $\|\widehat{f}\|_{H^\infty(\widetilde{P})} \le C\|f\|_{L^2_{\rm h}(D)}$. Consequently,

$$\sup_{z\in\widetilde{P}\cap D} K_D(z) = \sup\left\{\frac{|f(z)|^2}{\|f\|^2_{L^2_{\rm h}(D)}} : z \in \widetilde{P}\cap D,\ f \not\equiv 0,\ f \in L^2_{\rm h}(D)\right\} \le C^2,$$

which contradicts (5) for any $w \in \partial P \cap \partial D \neq \emptyset$.

(4)$\Rightarrow$(5). Fix $w \in \partial D$. First consider the case $w \notin \operatorname{int}(\overline{D})$. Then there is a sequence $z_\nu \to w$ with $z_\nu \notin \overline{D}$. Let B_ν be the largest open ball centered at z_ν disjoint from $\overline{D}$. Choose $w_\nu \in \partial B_\nu \cap \partial D$. Obviously, $w_\nu \to w$.

Note that for any ν, D satisfies at w_ν the "outer cone condition" (see [Pfl]). Therefore, for any ν we have $\lim_{D\ni z\to w_\nu} K_D(z)=\infty$ (see [Pfl]), which easily implies (5).

Assume now that $w\in\operatorname{int}(\overline{D})$. Suppose that (5) does not hold at w. Then there is a polydisc P with center at w such that $\sup\{K_D(z): z\in D\cap P\}<\infty$. Without loss of generality we may assume that $P\subset\subset\operatorname{int}(\overline{D})$. Consider any complex line L intersecting P. We claim that $L\cap P\cap D$ is equal to $(L\cap P)\setminus K$, where K is a polar set or $K=L\cap P$. In fact if this were not the case then $\sup_{z\in L\cap P\cap D} K_{L\cap D}(z)=\infty$ (the Bergman kernel is here understood as that of a one-dimensional set) (use Lemma 8) and, consequently, in view of the Ohsawa–Takegoshi extension theorem we would get $\sup_{z\in L\cap P\cap D} K_D(z)=\infty$, a contradiction.

Note that there is a complex line L passing through w such that $L\cap P\cap D$ is not empty. Assume that $w=0$. Making a linear change of coordinates and shrinking P if necessary we may assume that $P=E^n$ and that $\{\lambda\in E: (0,\dots,0,\lambda)\in D\}$ is not empty.

Therefore, the assumptions of Lemma 9 are satisfied (with some neighborhood $V\subset E^{n-1}$ of $0'\in\mathbb{C}^{n-1}$) and there is a neighborhood $0'\in V'\subset E^{n-1}$ such that for any $f\in L^2_{\mathrm{h}}(D)$ there is a function $F\in\mathcal{O}\big(V'\times\frac12 E\big)$ with $F=f$ on $\big(V'\times\frac12 E\big)\cap D$, a contradiction. ∎

Proof of Theorem 2. Because of Theorem 4 we may assume that $n\ge 2$.

($\Rightarrow$) Suppose that for some $w\in\partial D$ there is a polydisc P such that $P\setminus D$ is pluripolar. Let $u\in\operatorname{PSH}(P)$ be such that $u\not\equiv-\infty$ and $P\setminus D\subset\{u=-\infty\}$. Take a non-empty open set $U\subset D\cap P$ and consider all complex lines connecting w to some point from U. It is easy to see that there is a complex line L such that $u\not\equiv-\infty$ on $L\cap P$. Assume that $w=0$. Making a linear change of coordinates and shrinking P if necessary, we may assume that $P=E^n$ and $\{z_n\in E:(0',z_n)\notin D\}$ is polar. Because of the local integrability of u, for almost any $z'\in E^{n-1}$ (with respect to the $(2n-2)$-dimensional Lebesgue measure) the function $u(z',\cdot)$ is not identically $-\infty$ on E. Consequently, for almost every $z'\in E^{n-1}$ the set $\{z_n\in E:(z',z_n)\notin D\}$ is polar. Applying Lemma 9 we obtain the existence of an open set $0\in Q$ such that for any $f\in L^2_{\mathrm{h}}(D)$ there exists an $F\in\mathcal{O}(Q)$ with $f=F$ on $D\cap Q$, a contradiction.

($\Leftarrow$) Suppose that the implication does not hold, so in view of Theorem 1 there is a $w\in\partial D$ such that $\limsup_{D\ni z\to w} K_D(z)<\infty$. In other words there is a polydisc P with center at w such that $\sup_{z\in D\cap P} K_D(z)<\infty$.

First note that for any complex line L with $L\cap P\ne\emptyset$ we have $L\cap P\cap D=\emptyset$ or $L\cap P\cap D=(L\cap P)\setminus K$, where K is a polar set. Actually, if there were L such that $L\cap P\cap D=(L\cap P)\setminus K$, with $K\ne L\cap P$ and K not polar, then for some $U\subset\subset L\cap P$, $\sup_{z\in U\cap D} K_{D\cap L}(z)=\infty$ (use Lemma 8). Therefore, in view of the Ohsawa–Takegoshi theorem, $\sup_{z\in U\cap D} K_D(z)=\infty$, a contradiction.

Consequently, one may apply a result of A. Sadullaev (see [Sad 2] and also [Sad 1]) to deduce that the set $P\setminus D$ is pluripolar, a contradiction. ∎

It follows from the reasoning in the proofs of Theorems 1 and 2 that the following higher-dimensional counterpart of Lemma 8 holds.

LEMMA 11. *Let D be a bounded pseudoconvex domain and let $w\in\partial D$. Then $\limsup_{D\ni z\to w} K_D(z)<\infty$ if and only if for any neighborhood U of w the set $U\setminus D$ is pluripolar.*

The known examples of L^2_{h}-domains of holomorphy include bounded pseudoconvex fat domains and bounded pseudoconvex balanced domains. The characterization of L^2_{h}-domains of holomorphy given by us yields many examples of such domains. Below we give an example of a new class of domains having this property.

For a bounded pseudoconvex domain $D\subset\mathbb{C}^n$ we define the following Hartogs domain with m-dimensional balanced fibers:

$$G_D:=\{(w,z)\in\mathbb{C}^{n+m}: H(z,w)<1\},$$

where $\log H$ is plurisubharmonic on $D\times\mathbb{C}^m$, $H(z,\lambda w)=|\lambda|H(z,w)$, $(z,w)\in D\times\mathbb{C}^m$, $\lambda\in\mathbb{C}$, and G_D is bounded (i.e. $H(z,w)\ge C\|w\|$ for some $C>0$, $(z,w)\in D\times\mathbb{C}^m$). Then G_D is a bounded pseudoconvex domain.

PROPOSITION 12. *Let D be a bounded L^2_{h}-domain of holomorphy. Then G_D is an L^2_{h}-domain of holomorphy.*

Proof. Take $(z^0,w^0)\in\partial G_D$. If $z^0\in D$ then

$$\lim_{G_D\ni(z,w)\to(z^0,w^0)} K_{G_D}(z,w)=\infty$$

(use Theorem 3.1(i) from [Jar-Pfl-Zwo]).

Assume now that $z^0\in\partial D$. Let V be any neighborhood of (z^0,w^0). In view of Lemma 11 and Theorem 1 it is sufficient to show that $V\setminus G_D$ is not pluripolar. We may assume that $V=V_1\times V_2\subset\mathbb{C}^{n+m}$. Because D is an L^2_{h}-domain of holomorphy Theorem 2 applies and $V_1\setminus D$ is not pluripolar. Since $V\setminus G_D\supset(V_1\setminus D)\times V_2$ and the latter set is not pluripolar, the proof is finished. ∎

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After the paper had been finished the authors learnt about the existence of a paper of J. Siciak (see [Sic]) in which a similar result to that of Lemma 9 was proven (but with other methods).

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