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The space $S_{\alpha,\beta}$ and σ-core

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Abstract. We give some new properties of the space $S_{\alpha,\beta}$ and we apply them to the σ-core theory. These results generalize those by Choudhary and Yardimci.

1. Notations and preliminary results. For a given infinite matrix $A = (a_{nm})_{n,m\geq 1}$ we define the operators A_n for any integer $n \geq 1$ by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm} x_m \tag{1}$$

where $X = (x_n)_{n\geq 1}$ and the series is assumed to be convergent. So we are led to the study of the *infinite linear system*

$$A_n(X) = y_n, \quad n = 1, 2, \ldots, \tag{2}$$

where $Y = (y_n)_{n\geq 1}$ is a one-column matrix and X the unknown (see [4, 6–10, 12]). The equations (2) can be written in the form

$$AX = Y, \quad \text{where} \quad AX = (A_n(X))_{n\geq 1}.$$

In this paper we shall also consider A as an operator from a sequence space into another sequence space.

We will write s and l_∞ for the sets of all sequences and of all bounded sequences, respectively. We shall use the set

$$U^{+*} = \{(u_n)_{n\geq 1} \in s : u_n > 0 \text{ for all } n\}.$$

Using Wilansky's notation [16], for any sequence $\alpha = (\alpha_n)_{n\geq 1} \in U^{+*}$ we define the set

$$s_\alpha = (1/\alpha)^{-1} * l_\infty = \{(x_n)_{n\geq 1} \in s : (x_n/\alpha_n)_n \in l_\infty\}.$$

The set s_α is a Banach space normed by

$$\|X\|_{s_\alpha} = \sup_{n\geq 1} |x_n|/\alpha_n. \tag{3}$$

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Now let $\alpha = (\alpha_n)_{n\geq 1}$ and $\beta = (\beta_n)_{n\geq 1} \in U^{+*}$. Then $S_{\alpha,\beta}$ is the set of infinite matrices $A = (a_{nm})_{n,m\geq 1}$ such that $\sup_n \beta_n^{-1} \sum_{m=1}^{\infty} |a_{nm}|\alpha_m < \infty$. The set $S_{\alpha,\beta}$ is a linear space normed by

$$\|A\|_{S_{\alpha,\beta}} = \sup_{n\geq 1} \frac{1}{\beta_n} \sum_{m=1}^{\infty} |a_{nm}|\alpha_m.$$

Let E and F be any subsets of s. When A maps E into F we shall write $A \in (E, F)$ (see [5]). It was shown in [3] that $A \in (s_\alpha, s_\beta)$ if and only if $A \in S_{\alpha,\beta}$. So we can write that $(s_\alpha, s_\beta) = S_{\alpha,\beta}$.

When $s_\alpha = s_\beta$ we obtain the Banach algebra with identity $S_{\alpha,\beta} = S_\alpha$ (see [4, 9, 10, 12]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$.

We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$. If $\|I - A\|_{S_\alpha} < 1$ (where $I = (\delta_{nm})_{n,m\geq 1}$, with $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ otherwise), we shall say that $A \in \Gamma_\alpha$. The set S_α being a Banach algebra with identity, we have the useful result: if $A \in \Gamma_\alpha$, then A is bijective from s_α into itself.

If $\alpha = (r^n)_{n\geq 1}$, then Γ_α, S_α and s_α are denoted by Γ_r, S_r and s_r respectively (see [4, 6–9, 11]). When $r = 1$, we obtain $s_1 = l_\infty$ and putting $e = (1, 1, \dots)$ we have $S_1 = S_e$. It is well known that if c_0 and c are the sets of all sequences that are convergent to zero and convergent respectively, then

$$(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1 \tag{4}$$

(see [5]). We will write $e_n = (0, \dots, 1, \dots)$, where 1 is in the nth position.

For any subset E of s, we put

$$AE = \{Y \in s : \text{there is } X \in E \text{ with } Y = AX\}. \tag{5}$$

If F is a subset of s, we write

$$F(A) = F_A = \{X \in s : AX \in F\}. \tag{6}$$

2. Other properties of the space $S_{\alpha,\beta}$. For the study of σ-core, we need some properties of the set $S_{\alpha,\beta}$. First, we define the following sets of sequences. Let $M \in (s_\gamma, s_\alpha)$ and $N \in (s_\beta, s_\gamma)$ for $\alpha, \beta, \gamma \in U^{+*}$ and consider the linear spaces

$$S_{\alpha,\beta}.M = \{AM : A \in S_{\alpha,\beta}\}, \qquad N.S_{\alpha,\beta} = \{NA : A \in S_{\alpha,\beta}\}.$$

For any sequence $\xi = (\xi_n)_n$ such that $\xi_n \neq 0$ for all n, we put

$$D_\xi = (\xi_n \delta_{nm})_{n,m\geq 1}.$$

It can be easily shown that

$$D_\xi s_\alpha = \left(\frac{1}{\xi}\right)^{-1} * \left(\left(\frac{1}{\alpha}\right)^{-1} * s_1\right) = s_{|\xi|\alpha}$$

(see [12]). Now we can assert the following:

THEOREM 1.

(i) *Let $\alpha, \beta, \alpha', \beta' \in U^{+*}$. Then*

(a) $\alpha_n = O(\beta_n)$ $(n \to \infty)$ *if and only if* $s_\alpha \subset s_\beta$;

(b) $\alpha_n = O(\beta_n)$ *and* $\beta_n = O(\alpha_n)$ $(n \to \infty)$ *if and only if* $s_\alpha = s_\beta$;

(c) $s_\alpha = s_\beta$ *if and only if there exist* $K_1, K_2 > 0$ *such that*

$$K_1\alpha_n \le \beta_n \le K_2\alpha_n \quad \text{for all } n; \tag{7}$$

(d) *the identity* $S_{\alpha,\beta} = S_{\alpha',\beta'}$ *is equivalent to* $s_\alpha = s_{\alpha'}$ *and* $s_\beta = s_{\beta'}$.

(ii) *Let $\alpha, \beta, \gamma, \mu \in U^{+*}$. Then*

(a) $S_{\alpha,\beta}$ *is a Banach space with respect to the norm* $\| \, \|_{S_{\alpha,\beta}}$;

(b) $A(BC) = (AB)C$ *for every* $A \in S_{\gamma,\mu}$, $B \in S_{\beta,\gamma}$ *and* $C \in S_{\alpha,\beta}$;

(c) $\|AB\|_{S_{\gamma,\beta}} \le \|B\|_{S_{\gamma,\alpha}} \|A\|_{S_{\alpha,\beta}}$ *for* $A \in S_{\alpha,\beta}$ *and* $B \in S_{\gamma,\alpha}$;

(d) *the set*

$$S_{\alpha,\beta}.S_{\gamma,\alpha} = \bigcup_{M \in S_{\gamma,\alpha}} S_{\alpha,\beta}.M$$

is a Banach space with the norm $\| \, \|_{S_{\gamma,\beta}}$ *and*

$$S_{\alpha,\beta}.S_{\gamma,\alpha} = S_{\gamma,\beta};$$

(e) *if* $M \in (s_\gamma, s_\alpha)$ *is bijective, then*

$$S_{\alpha,\beta}.M = S_{\gamma,\beta},$$

and if $N \in (s_\beta, s_\gamma)$ *is bijective, then*

$$N.S_{\alpha,\beta} = S_{\alpha,\gamma}.$$

Proof. (i)(a) Assume that $\alpha_n = O(\beta_n)$ $(n \to \infty)$. If $X = (x_n)_n \in s_\alpha$, then

$$\frac{x_n}{\beta_n} = \frac{x_n}{\alpha_n}\frac{\alpha_n}{\beta_n} = O(1) \quad (n \to \infty)$$

and so $X \in s_\beta$. So $s_\alpha \subset s_\beta$. Conversely, $\alpha \in s_\alpha \subset s_\beta$ implies $\alpha_n/\beta_n = O(1)$ $(n \to \infty)$ and so $\alpha_n = O(\beta_n)$ $(n \to \infty)$.

(i)(b) is obvious.

(i)(c) Condition (7) is equivalent to $\alpha_n = O(\beta_n)$ and $\beta_n = O(\alpha_n)$ $(n \to \infty)$.

(i)(d) The sufficiency being obvious, we prove the necessity. Suppose that $S_{\alpha,\beta} = S_{\alpha',\beta'}$. First, we shall prove that $S_{\alpha,\beta} = S_{\alpha',\beta}$. For this, denote by $\widetilde{c}_1 = (c_{nm})_{n,m\ge 1}$ the infinite matrix defined by $c_{n1} = \beta_n/\alpha_1$ for all $n \ge 1$ and $c_{nm} = 0$ otherwise. We see immediately that $\widetilde{c}_1 \in S_{\alpha,\beta}$ and since $S_{\alpha,\beta} = S_{\alpha',\beta'}$, we get $\widetilde{c}_1 \in S_{\alpha',\beta'}$. So $\widetilde{c}_1\alpha' = (\beta_n\alpha_1'/\alpha_1)_{n\ge 1} \in s_{\beta'}$, i.e.

$$\beta_n = \beta_n' O(1) \quad (n \to \infty)$$

and from (i)(a) we conclude $s_\beta \subset s_{\beta'}$. By a similar argument, taking $\widetilde{c}_1' = (c_{nm}')_{n,m\ge 1}$, with $c_{n1}' = \beta_n'/\alpha_1'$ for all $n \ge 1$ and $c_{nm}' = 0$ otherwise, we get $\widetilde{c}_1'\alpha = (\beta_n'\alpha_1/\alpha_1')_{n\ge 1} \in s_\beta$ and $s_{\beta'} \subset s_\beta$. Thus we have shown $s_\beta = s_{\beta'}$, so $S_{\alpha,\beta} = S_{\alpha',\beta'}$ implies $S_{\alpha,\beta} = S_{\alpha',\beta}$. It remains to show that the latter equality implies $s_\alpha = s_{\alpha'}$. For this, consider the matrix $D_{\beta/\alpha} \in S_{\alpha,\beta}$. Since $S_{\alpha,\beta} = S_{\alpha',\beta}$, we deduce that

$$D_{\beta/\alpha} s_{\alpha'} = s_{\beta\alpha'/\alpha} \subset s_\beta \tag{8}$$

and $\alpha'_n/\alpha_n = O(1)$ $(n \to \infty)$. So, from (i)(a), $s_\alpha \subset s_{\alpha'}$. Similarly, since $D_{\beta/\alpha'} \in S_{\alpha',\beta} = S_{\alpha,\beta}$, we get

$$D_{\beta/\alpha'} s_\alpha = s_{\beta\alpha/\alpha'} \subset s_\beta. \tag{9}$$

So $\alpha_n = O(\alpha'_n)$ $(n \to \infty)$ and $s_{\alpha'} \subset s_\alpha$. We conclude that $s_\alpha = s_{\alpha'}$ and (i)(d) is proved.

(ii)(b) Letting $A = D_\mu A_1 D_{1/\gamma}$, $B = D_\gamma B_1 D_{1/\beta}$ and $C = D_\beta C_1 D_{1/\alpha}$ it can be easily seen that $A_1, B_1, C_1 \in S_1$. So

$$\begin{aligned} A(BC) &= (D_\mu A_1 D_{1/\gamma})(D_\gamma B_1 D_{1/\beta} D_\beta C_1 D_{1/\alpha}) \\ &= (D_\mu A_1 D_{1/\gamma})(D_\gamma B_1 C_1 D_{1/\alpha}) \end{aligned}$$

and since $D_{1/\gamma}$ and D_γ are diagonal matrices and S_1 is a Banach algebra, we deduce that

$$\begin{aligned} A(BC) &= D_\mu (A_1 B_1) C_1 D_{1/\alpha} = (D_\mu A_1 D_{1/\gamma} D_\gamma B_1 D_{1/\beta})(D_\beta C_1 D_{1/\alpha}) \\ &= (AB)C. \end{aligned}$$

(ii)(c) Since S_1 is a Banach algebra, we see that

$$\|AB\|_{S_{\gamma,\beta}} = \|D_{1/\beta} A D_\alpha D_{1/\alpha} B D_\gamma\|_{S_1} \le \|D_{1/\beta} A D_\alpha\|_{S_1} \|D_{1/\alpha} B D_\gamma\|_{S_1},$$

that is, $\|AB\|_{S_{\gamma,\beta}} \le \|B\|_{S_{\gamma,\alpha}} \|A\|_{S_{\alpha,\beta}}$.

(ii)(a) The set $S_{\alpha,\beta}$ being a vector space, it is enough to show that $S_{\alpha,\beta}$ is complete. Let $(A_i)_i$ be a Cauchy sequence in $S_{\alpha,\beta}$. For any given real $\varepsilon > 0$, there is an integer n_0 such that

$$\|A_i - A_j\|_{S_{\alpha,\beta}} = \|D_{\alpha/\beta} A_i - D_{\alpha/\beta} A_j\|_{S_\alpha} \le \varepsilon \quad \text{for } i, j \ge n_0.$$

The set S_α being a Banach space, there is an infinite matrix $M \in S_\alpha$ such that $D_{\alpha/\beta} A_i \to M$ $(i \to \infty)$. Then from (ii)(c) we get

$$\|A_i - D_{\beta/\alpha} M\|_{S_{\alpha,\beta}} \le \|D_{\beta/\alpha}\|_{S_{\alpha,\beta}} \|D_{\alpha/\beta} A_i - M\|_{S_\alpha},$$

where $\|D_{\beta/\alpha}\|_{S_{\alpha,\beta}} = 1$ and $\|D_{\alpha/\beta} A_i - M\|_{S_\alpha} = o(1)$ $(i \to \infty)$, so we conclude that $A_i \to D_{\beta/\alpha} M$ $(i \to \infty)$ in $S_{\alpha,\beta}$ and $S_{\alpha,\beta}$ is a Banach space.

(ii)(d) It is enough to show $S_{\alpha,\beta}.S_{\gamma,\alpha} = S_{\gamma,\beta}$. Take any $A = BC \in S_{\alpha,\beta}.S_{\gamma,\alpha}$. Since C maps s_γ into s_α and B maps s_α into s_β, we conclude easily that A maps s_γ into s_β, i.e. $A \in S_{\gamma,\beta}$. So $S_{\alpha,\beta}.S_{\gamma,\alpha} \subset S_{\gamma,\beta}$. Furthermore for every $A \in S_{\gamma,\beta}$,

$$A = (AD_{\gamma/\alpha}) D_{\alpha/\gamma} \quad \text{with } D_{\alpha/\gamma} \in S_{\gamma,\alpha} \text{ and } AD_{\gamma/\alpha} \in S_{\alpha,\beta}.$$

We conclude that $S_{\gamma,\beta} \subset S_{\alpha,\beta}.S_{\gamma,\alpha}$ and $S_{\alpha,\beta}.S_{\gamma,\alpha} = S_{\gamma,\beta}$.

(ii)(e) The inclusion $S_{\alpha,\beta}.M \subset S_{\gamma,\beta}$ comes from (ii)(d). Let $A \in S_{\gamma,\beta}$ be any infinite matrix. Since M is invertible and $M^{-1} \in (s_\alpha, s_\gamma)$, from (ii)(b) we get

$$A = (AM^{-1})M$$

where $AM^{-1} \in (s_\alpha, s_\beta)$. So $A \in S_{\alpha,\beta}.M$ and $S_{\gamma,\beta} \subset S_{\alpha,\beta}.M$. We conclude that $S_{\alpha,\beta}.M = S_{\gamma,\beta}$. Let us prove $N.S_{\alpha,\beta} = S_{\alpha,\gamma}$. From (ii)(d), we have $N.S_{\alpha,\beta} \subset S_{\alpha,\gamma}$. Take now $A \in S_{\alpha,\gamma}$. Reasoning as above we see that there exists $B \in S_{\alpha,\beta}$ such that $A = NB$, where $B = N^{-1}A \in S_{\alpha,\beta}$. This gives the conclusion. ∎

Remark 1. Note that the identity $(E, s_\beta) = S_{e,\beta} = (s_1, s_\beta)$, where E is any given set of sequences, does not imply $E = s_1$. Indeed, from (4) it can be deduced that $(c_0, s_\beta) = S_{e,\beta}$ and $c_0 \neq s_1$.

We deduce from (ii)(e) of Theorem 1 the following.

Corollary 2. *Let $\alpha, \beta, \tau \in U^{+*}$. Then*

(i)(a) $M \in \Gamma_\alpha$ *implies* $S_{\alpha,\beta}.M = S_{\alpha,\beta}$;
(b) $N \in \Gamma_\beta$ *implies* $N.S_{\alpha,\beta} = S_{\alpha,\beta}$.
(ii)(a) $S_{\alpha,\beta}.D_\tau = S_{\alpha/\tau,\beta}$;
(b) $D_\tau.S_{\alpha,\beta} = S_{\alpha,\beta\tau}$.

Proof. $M \in \Gamma_\alpha$ implies that M is bijective from s_α into itself, so applying Theorem 1(ii)(e) we obtain $S_{\alpha,\beta}.M = S_{\alpha,\beta}$. Similarly, since $N \in \Gamma_\beta$, N is bijective from s_β into itself and we get (i)(b) by applying (ii)(e).

Next, D_τ is bijective from $s_{\alpha/\tau}$ into s_α, so $S_{\alpha,\beta}.D_\tau = S_{\alpha/\tau,\beta}$ from (ii)(e), and we obtain (ii)(b) by a similar argument. ∎

3. σ-core.

In this section, we apply the results of Sections 1 and 2 to the σ-core. Among other things, we will give some properties of the product of two infinite matrices AB^{-1}.

3.1. *Some known results on the σ-core.* First, denote by σ a one-to-one mapping of $\mathbb{N}$ and define for a given sequence $X = (x_n)_{n\geq 1}$ the sequence

$$t_{np}(X) = \frac{x_n + x_{\sigma(n)} + \cdots + x_{\sigma^p(n)}}{p+1} \quad \text{for } p \geq 0 \text{ and } n \geq 1.$$

We shall assume throughout this paper that $\sigma^j(n) \neq n$ for all $j \geq 1$ and $n \geq 1$. As in [15] we define

$$V_\sigma = \{X \in s_1 : \lim_{p\to\infty} \sup_{n\geq 1} |t_{np}(X) - l| = 0 \text{ for some } l \in \mathbb{C}\}$$

and write $l = \sigma\text{-}\lim X$. The matrix $A = (a_{nm})_{n,m\geq 1}$ is said to be *σ-regular* if

$$AX \in V_\sigma \quad \text{and} \quad \lim X = \sigma\text{-}\lim AX \quad \text{for all } X \in c$$

(see [14]). Furthermore, $A=(a_{nm})_{n,m\geq 1}$ is said to be *strongly σ-regular* if

$$AX \in V_\sigma \quad \text{and} \quad \sigma\text{-}\lim X = \sigma\text{-}\lim AX \qquad \text{for all } X \in V_\sigma.$$

$A=(a_{nm})_{n,m\geq 1}$ is said to be *σ-uniformly positive* if

$$\lim_{p\to\infty}\sup_{n\geq 1}\Big|\sum_{m=1}^{\infty} a^{-}(n,p,m)\Big| = 0,$$

where

$$a^{-}(n,p,m) = \frac{1}{p+1}\sum_{j=0}^{p} a^{-}_{\sigma^j(n),m},$$

with the notation $\lambda^{-}=\max(-\lambda,0)$.

Let V be the map from s_1 into $\mathbb{R}$ defined by

$$V(X) = \sup_{n\geq 1}(\overline{\lim_{p\to\infty}}\, t_{np}(X)).$$

For any given X, set

$$\sigma\text{-core}\{X\} = [-V(-X), V(X)].$$

As a direct consequence of a theorem due to Mishra, Rath and Satapathy [13], in which we use the equivalence $D_{1/\beta}AD_\alpha \in (s_1,s_1)$ if and only if $A\in(s_\alpha,s_\beta)$, we obtain

Lemma 3. *Let $A\in(s_\alpha,s_\beta)$ and $X=(x_n)_{n\geq 1}\in s_1$. Then*

$$\sigma\text{-core}\{(D_{1/\beta}AD_\alpha)X\} \subset \sigma\text{-core}\{X\}$$

if and only if $D_{1/\beta}AD_\alpha$ is strongly σ-regular and σ-uniformly positive.

It is well known that for a given matrix M, σ-core$\{MX\}\subset\sigma$-core$\{X\}$ if and only if $V(MX)\leq V(X)$ for all $X\in s_1$.

Now from a theorem due to Choudhary [1] with B replaced by $D_{1/\alpha}B$, we get the following result. So the condition $BX\in s_\alpha$ is equivalent to $D_{1/\alpha}BX\in s_1$. Throughout this section we shall suppose that B is invertible and we shall write $B^{-1}=(b'_{nm})_{n,m\geq 1}$.

Lemma 4. *Let n_0 be a given integer. Then the following conditions are equivalent*:

(i) *The condition $X\in s_\alpha(B)$ implies*

$$A_{n_0}(X) = \sum_{m=1}^{\infty} a_{n_0 m}x_m \text{ is convergent for all } X\in s.$$

(ii)(a) $\displaystyle\sum_{m=1}^{\infty}\Big|\sum_{k=m}^{\infty} a_{nk}b'_{km}\Big|\alpha_m < \infty$ *for all* $n\geq 1$;

(b) $\lim_{j\to\infty}\sum_{m=1}^{j}\Big|\sum_{k=j+1}^{\infty} a_{n_0k}b'_{km}\Big|\alpha_m = 0\ (j\to\infty)$.

Recall now a result which can be obtained from a theorem due to Yardimci [17] by repolacing A and B by $D_{1/\beta}A$ and $D_{1/\alpha}B$ and which is a consequence of the previous lemma. We will write $L(X) = \overline{\lim}_{n\to\infty} x_n$.

LEMMA 5. *Let B be a triangle and A any matrix. Consider the condition*

(a)(α) $s_\alpha(B) \subset s_\beta(A)$;
(β) $V((D_{1/\beta}A)X) \le L((D_{1/\alpha}B)X)$ *for all* $X \in s$.

Condition (a) *is equivalent to*

(i) *the product* $C = (D_{1/\beta}A)(B^{-1}D_\alpha)$ *exists*;
(ii) C *is σ-regular*;
(iii) C *is σ-uniformly positive*;
(iv) $\lim_{j\to\infty}\sum_{m=1}^{j}\Big|\sum_{k=j+1}^{\infty} a_{nk}b'_{km}\Big|\alpha_m = 0$ *for all* n.

3.2. *The main results.* In this subsection we shall see that under some conditions on A and B, conditions (a)(α) and (iv) of Lemma 5 are satisfied. Then we obtain necessary and sufficient conditions for $D_{1/\beta}AB^{-1}D_\alpha$ to be σ-regular and σ-uniformly positive.

In the following we shall suppose that $B = (b_{nm})_{n,m\ge1}$ is a *triangle*, that is, $b_{nm} = 0$ for $m > n$ and $b_{nn} \ne 0$ for all n (see [2]).

To simplify, we shall write $b = (b_{nn})_{n\ge1}$, $D_{1/b} = (\delta_{nm}/b_{nn})_{n,m\ge1}$ and suppose $1/b \in s_1$ throughout. Consider the following additional conditions on A and B:

$$A \in S_{\alpha,\beta}; \tag{10}$$

$$\sup_{n\ge2}\sum_{m=1}^{n-1}\left|\frac{b_{nm}}{b_{nn}}\right|\frac{\alpha_m}{\alpha_n} < 1. \tag{11}$$

Now we can state the following

THEOREM 6. *Let A and B satisfy conditions* (10) *and* (11). *Then*

(i) $s_\alpha(B) \subset s_\beta(A)$;

(ii) $\lim_{j\to\infty}\sum_{m=1}^{j}\Big|\sum_{k=j+1}^{\infty} a_{nk}b'_{km}\Big|\alpha_m = 0$ *for all* n;

(iii) $(D_{1/\beta}A)(B^{-1}D_\alpha) \in S_{|b|,e}$.

Proof. Condition (11) means that $D_{1/b}B \in \Gamma_\alpha$. So $(D_{1/b}B)^{-1} = (b'_{nm}b_{mm})_{n,m\geq 1} \in S_\alpha$, that is,

$$\sup_{n\geq 2}\sum_{m=1}^{n} |b'_{nm}|\,|b_{mm}|\alpha_m/\alpha_n < \infty. \tag{12}$$

From (12) we see that $B^{-1} \in S_{\alpha|b|,\alpha}$, so using Corollary 2(ii)(a), we obtain

$$B^{-1}D_\alpha \in S_{\alpha|b|,\alpha}.D_\alpha = S_{|b|,\alpha}.$$

Since $A \in S_{\alpha,\beta}$ and $D_{1/\beta} \in S_{\beta,e}$ we deduce from Corollary 2(ii)(b) that $D_{1/\beta}A \in S_{\alpha,e}$; thus using Theorem 1(ii)(d) we get

$$(D_{1/\beta}A)(B^{-1}D_\alpha) \in S_{\alpha,e}.S_{|b|,\alpha} = S_{|b|,e}.$$

Then (iii) and condition (i) of Lemma 5 hold.

Let us show (i) holds. Take any X such that $Y = D_{1/\alpha}BX \in s_1$. Since $B^{-1}D_\alpha \in S_{|b|,\alpha}$ and the condition $1/b \in s_1$ implies $S_{|b|,\alpha} \subset S_{e,\alpha}$, we get

$$(B^{-1}D_\alpha)Y \in s_\alpha.$$

Furthermore, since $D_{1/\beta}A \in S_{\alpha,e} = (s_\alpha, s_1)$ we obtain

$$(D_{1/\beta}A)X = (D_{1/\beta}A)[(B^{-1}D_\alpha)Y] \in s_1$$

and (i) holds.

It remains to show (ii) holds. First, (12) and the condition $1/b \in s_1$ imply that there are two reals $K_1, K_2 > 0$ such that

$$K_1\sum_{m=1}^{n} |b'_{nm}|\alpha_m \leq \sum_{m=1}^{n} |b'_{nm}|\,|b_{mm}|\alpha_m \leq K_2\alpha_n \quad \text{for all } n\geq 1. \tag{13}$$

Then from (10) and (13) we deduce that there exists $K_3 > 0$ such that for every $Y = (y_n)_{n\geq 1} \in s_1$,

$$\begin{aligned}\frac{1}{\beta_n}\Big(\sum_{k=1}^{\infty} |a_{nk}|\Big(\sum_{m=1}^{k} |b'_{km}|\alpha_m|y_m|\Big)\Big) &\leq \frac{K_3}{\beta_n}\Big(\sum_{k=1}^{\infty} |a_{nk}|\alpha_k\Big) \\ &= O(1) \quad (n\to\infty).\end{aligned} \tag{14}$$

Letting

$$\tau_k = \sum_{m=1}^{k} |b'_{km}|\alpha_m, \tag{15}$$

we deduce from (14), in which $Y = e$, that for any fixed n,

$$\sum_{k=j+1}^{\infty} |a_{nk}|\tau_k = o(1) \quad (j\to\infty), \tag{16}$$

and

$$\sum_{k=j+1}^{\infty} |a_{nk}| \Big(\sum_{m=1}^{j} |b'_{km}| \alpha_m \Big) \le \sum_{k=j+1}^{\infty} |a_{nk}| \tau_k \quad \text{for all } j \ge 1. \tag{17}$$

From (16), (17) and the inequality

$$\sum_{m=1}^{j} \Big| \sum_{k=j+1}^{\infty} a_{nk} b'_{km} \Big| \alpha_m \le \sum_{k=j+1}^{\infty} |a_{nk}| \Big(\sum_{m=1}^{j} |b'_{km}| \alpha_m \Big) \quad \text{for all } n, j \ge 1$$

we conclude that (ii) holds. ∎

REMARK 2. Since $A \in S_{\alpha,\beta}$, we have seen that $D_{1/\beta} \in S_{\beta,e}$ and by assumption $B^{-1} D_\alpha \in S_{|b|,\alpha}$. By Theorem 1(ii)(a) we then have

$$(D_{1/\beta} A)(B^{-1} D_\alpha) = D_{1/\beta} (AB^{-1}) D_\alpha \in S_{|b|,e} \subset S_1.$$

So

$$(D_{1/\beta} A)[(B^{-1} D_\alpha) Y] = [D_{1/\beta} (AB^{-1}) D_\alpha] Y \in s_1 \quad \text{for all } Y \in s_1. \tag{18}$$

PROPOSITION 7. *Assume that A and B satisfy* (10) *and* (11). *The condition*

(a) $BX \in s_\alpha$ *implies*

$$V((D_{1/\beta} A) X) \le L((D_{1/\alpha} B) X) \quad \textit{for all } X$$

is equivalent to

(i) $C = D_{1/\beta} AB^{-1} D_\alpha$ *is σ-regular*;
(ii) C *is σ-uniformly positive.*

Proof. From Theorem 6 we see that $BX \in s_\alpha$ implies $AX \in s_\beta$. So by Lemma 5, conditions (i) and (ii) then hold. Conversely, from Theorem 6, conditions (10) and (11) imply (i) and (iv) of Lemma 5. Finally, again from Lemma 5, (i) and (ii) imply condition (a). ∎

PROPOSITION 8. *Assume that A and B satisfy* (10) *and* (11). *The condition*

(a) $D_{1/\alpha} BX \in s_1$ *implies*

$$V((D_{1/\beta} A) X) \le V((D_{1/\alpha} B) X) \quad \textit{for all } X \tag{19}$$

is equivalent to

(i) $C = D_{1/\beta} AB^{-1} D_\alpha$ *is strongly σ-regular*;
(ii) C *is σ-uniformly positive.*

Proof. Necessity. Take $Y \in s_1$. Since $D_{1/\alpha} B$ is a triangle,

$$X = (D_{1/\alpha} B)^{-1} Y = B^{-1} D_\alpha Y$$

satisfies the equation $Y = (D_{1/\alpha}B)X$; and from Remark 2, (10) and (11) together imply (18), that is,

$$(D_{1/\beta}A)X = CY. \tag{20}$$

Then from (19), $V(CY) \le V(Y)$. Using Lemma 3, we conclude that (i) and (ii) hold.

Sufficiency. First, note that as above (20) holds. So we obtain (19) from Lemma 3. ∎

As a direct consequence of Propositions 7 and 8, we obtain

COROLLARY 9. *Assume that $A \in S_{\alpha,\beta}$ and $B \in S_{\beta,\alpha}$ are triangles satisfying*

$$\sup_{n\ge 2}\sum_{m=1}^{n-1}\left|\frac{b_{nm}}{b_{nn}}\right|\frac{\alpha_m}{\alpha_n} < 1 \quad \textit{and} \quad \sup_{n\ge 2}\sum_{m=1}^{n-1}\left|\frac{a_{nm}}{a_{nn}}\right|\frac{\beta_m}{\beta_n} < 1. \tag{21}$$

Then the condition $V((D_{1/\beta}A)X) = L((D_{1/\alpha}B)X)$ for all $X \in s_\beta(A) \cap s_\alpha(B)$ is equivalent to

(a) *$D_{1/\beta}AB^{-1}D_\alpha$ and $D_{1/\alpha}BA^{-1}D_\beta$ are σ-regular*;
(b) *$D_{1/\beta}AB^{-1}D_\alpha$ and $D_{1/\alpha}BA^{-1}D_\beta$ are σ-uniformly positive.*

COROLLARY 10. *Assume that the matrices $A \in S_{\alpha,\beta}$ and $B \in S_{\beta,\alpha}$ satisfy the conditions given in* (21). *Then*

$$V((D_{1/\beta}A)X) = V((D_{1/\alpha}B)X) \quad \textit{for all } X \in s_\beta(A) \cap s_\alpha(B)$$

if and only if condition (b) *of Corollary* 9 *holds and $D_{1/\beta}AB^{-1}D_\alpha$ and $D_{1/\alpha}BA^{-1}D_\beta$ are strongly σ-regular.*

REMARK 3. Assume that there exist $K_1, K_2, K_1', K_2' > 0$ such that

$$K_1 \le |b_{nn}| \le K_2 \quad \text{and} \quad K_1' \le |a_{nn}| \le K_2' \quad \text{for all } n.$$

If $\xi_n = \inf(\alpha_n, \beta_n)$, then $s_\beta(A) \cap s_\alpha(B) = s_\xi$ in Corollaries 9 and 10. Indeed, from (21) we deduce that $D_{1/b}B$ is bijective from s_α into itself. So

$$D_{1/b}B.s_\alpha = s_\alpha$$

and as we have seen in Section 2,

$$B.s_\alpha = D_b.s_\alpha = s_{\alpha|b|}.$$

Since

$$K_1\alpha_n \le |b_{nn}|\alpha_n \le K_2\alpha_n \quad \text{for all } n,$$

by Theorem 1(i)(c) we deduce that $s_{\alpha|b|} = s_\alpha$ and

$$s_\alpha(B) = B^{-1}s_{\alpha|b|} = B^{-1}s_\alpha = s_\alpha.$$

By a similar argument A is bijective from s_β into $s_{|a|\beta}$ (with $a = (a_{nn})_{n\ge1}$), so $s_{|a|\beta} = s_\beta$ and $s_\beta(A) = s_\beta$. We conclude that

$$s_\beta(A) \cap s_\alpha(B) = s_\beta \cap s_\alpha = s_{\inf(\alpha,\beta)}.$$

3.3. *An application.* In order to give an application of the previous results, recall [4, 7, 11] that we can associate to any power series $f(z) = \sum_{k=0}^\infty a_k z^k$ defined in the open disk $|z| < R$ the upper triangular infinite matrix $A = \varphi(f) \in \bigcup_{0<r<R} S_r$ defined by

$$\varphi(f) = \begin{pmatrix} a_0 & a_1 & a_2 & . \\ & a_0 & a_1 & . \\ 0 & & a_0 & . \\ & & & . \end{pmatrix}.$$

We shall write $\varphi[f(z)]$ instead of $\varphi(f)$. We have

Lemma 11.

(i) *The map $\varphi : f \to A$ is an isomorphism from the algebra of all power series defined in $|z| < R$ into the algebra of the corresponding matrices $\bar{A}$.*

(ii) *Let $f(z) = \sum_{k=0}^\infty a_k z^k$, with $a_0 \neq 0$, and assume that $1/f(z) = \sum_{k=0}^\infty a'_k z^k$ has radius of convergence $R' > 0$. Then*

$$\varphi\left(\frac{1}{f}\right) = [\varphi(f)]^{-1} \in \bigcup_{0<r<R'} S_r.$$

We can give an application using the well known operator of first difference $\Delta = (\varphi(1-z))^t$. For any real r we will write $D_r = (r^n \delta_{nm})_{n,m\ge1}$ for short.

Example 1. Let χ be a complex number satisfying $0 < |\chi| \le 1$, let $R \ge 1$ and consider $\Lambda = (\Lambda_{nm})_{n,m\ge1}$ and $\Lambda' = (\Lambda'_{nm})_{n,m\ge1}$ defined by

$$\Lambda_{nm} = \begin{cases} \dfrac{|\chi|^{n-m} - |\chi|^{n-m-1}}{R^{m-n}} & \text{for } m < n, \\ 1 & \text{for } m = n, \\ 0 & \text{otherwise;} \end{cases} \qquad \Lambda'_{nm} = \begin{cases} \dfrac{1-|\chi|}{R^{m-n}} & \text{for } m < n, \\ 1 & \text{for } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the condition

$$V((D_{1/R}\Delta)X) = L((D_{|\chi|/R}\Delta D_{1/|\chi|})X) \quad \text{for all } X \in s_R$$

is equivalent to

(i) Λ and Λ' are σ-regular;

(ii) Λ is σ-uniformly positive.

Proof. It is enough to apply Corollary 9 to the matrices $A = \Delta$ and $B = D_{|\chi|}\Delta D_{1/|\chi|}$, with $\alpha = \beta = (R^n)_n$. First, (21) holds since $\|I - A\|_{S_R} = 1/R < 1$ and

$$\|I - B\|_{S_R} = \|D_{|\chi|}(I - \Delta)D_{1/|\chi|}\|_{S_R} = \|I - \Delta\|_{S_{R/|\chi|}} = \frac{|\chi|}{R} < 1.$$

So A and B are bijective from s_R to itself and $s_\beta(A) \cap s_\alpha(B) = s_R$. Furthermore, we have

$$(\Delta^{-1})^t = \varphi(1/(1-z)) = \varphi\Big(\sum_{n=0}^{\infty} z^n\Big) \quad \text{for } |z| < 1.$$

So

$$(B^{-1})^t = (D_{|\chi|}\Delta^{-1}D_{1/|\chi|})^t = \varphi\Big(\sum_{n=0}^{\infty}(|\chi|z)^n\Big)$$

with $|z| < 1/|\chi|$ and

$$(AB^{-1})^t = \varphi\Big(\sum_{n=0}^{\infty}(|\chi|z)^n\Big)\varphi(1-z) = \varphi\Big[(1-z)\Big(\sum_{n=0}^{\infty}(|\chi|z)^n\Big)\Big].$$

Since

$$(1-z)\Big(\sum_{n=0}^{\infty}(|\chi|z)^n\Big) = 1 + \sum_{n=1}^{\infty}(|\chi|^n - |\chi|^{n-1})z^n,$$

we get $\Lambda = AB^{-1}$. Similarly, we obtain $\Lambda' = BA^{-1}$, using the identity

$$(BA^{-1})^t = \varphi\Big[(1-|\chi|z)\Big(\sum_{n=0}^{\infty} z^n\Big)\Big] = \varphi\Big[1 + (1-|\chi|)\Big(\sum_{n=1}^{\infty} z^n\Big)\Big].$$

Note that Λ' is σ-uniformly positive since all its entries are positive. This concludes the proof. ∎

Remark 4. Let χ and R be reals with $0 < |\chi| \leq 1$ and $R > 1$. It can be easily seen that one of the conditions (i) or (ii) in the previous proposition is false if and only if there is $X_0 \in s_R$ such that

$$V((D_{1/R}\Delta)X_0) \neq L(D_{|\chi|/R}\Delta D_{1/|\chi|})X_0.$$

References

[1] B. Choudhary, *An extension of Knopp's core theorem*, J. Math. Anal. Appl. 132 (1988), 226–233.

[2] R. G. Cooke, *Infinite Matrices and Sequence Spaces*, McMillan, 1950.

[3] E. Jürimäe, *Matrix mappings between rate-spaces and spaces with speed*, Tartu Ül. Toimetised 970 (1994), 29–52.

[4] R. Labbas and B. de Malafosse, *On some Banach algebra of infinite matrices and applications*, Demonstratio Math. 31 (1998), 153–168.

[5] I. J. Maddox, *Infinite Matrices of Operators*, Springer, Berlin, 1980.
[6] B. de Malafosse, *Some properties of the Cesàro operator in the space s_r*, Comm Fac. Sci. Univ. Ankara Ser. A1 Math. Statist. 48 (1999), 53–71.
[7] —, *Bases in sequence spaces and expansion of a function in a series of power series*, Mat. Vesnik 52 (2000), 99–112.
[8] —, *Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ*, Hokkaido Math. J. 31 (2002), 283–299.
[9] —, *Recent results in the infinite matrix theory*, *and application to Hill equation*, Demonstratio Math. 35 (2002), 11–26.
[10] —, *Variation of an element in the matrix of the first difference operator and matrix transformations*, Novi Sad J. Math. 32 (2002), 141–158.
[11] —, *Application of the sum of operators in the commutative case to the infinite matrix theory*, Soochow J. Math. 27 (2001), 405–421.
[12] B. de Malafosse and E. Malkowsky, *Sequence spaces and inverse of an infinite matrix*, Rend. Circ. Mat. Palermo Ser. (2) 51 (2002), 277–294.
[13] S. L. Mishra, N. Rath and B. Satapathy, *Invariant means and σ-core*, J. Indian Math. Soc. 60 (1994), 151–158.
[14] Mursaleen, *On some new invariant matrix methods of summability*, Quart. J. Math. Oxford (2) 34 (1983), 77–86.
[15] P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc. 36 (1972), 104–110.
[16] A. Wilansky, *Summability through Functional Analysis*, North-Holland Math. Stud. 85, North-Holland, 1984.
[17] Ş. Yardimci, *σ-core for real bounded sequences*, Demonstratio Math. 35 (2002), 111–116.

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