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# Boundedness and growth orders of means of discrete and continuous semigroups of operators

by

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**Abstract.** We discuss implication relations for boundedness and growth orders of Cesàro means and Abel means of discrete semigroups and continuous semigroups of linear operators. Counterexamples are constructed to show that implication relations between two Cesàro means of different orders or between Cesàro means and Abel means are in general strict, except when the space has dimension one or two.

**1. Introduction.** Let $X$ be a real or complex Banach space, and let $T$ be a bounded linear operator on $X$. One of the important issues of the ergodic theory of $T$ is concerned with convergence of various means of the discrete semigroup $\{T^n; n \geq 0\}$. The Cesàro means of order $\gamma$ (or $\gamma$-*Cesàro means*) with $\gamma > -1$ are defined by

$$C_n^\gamma(T) = \frac{1}{\sigma_n^\gamma} \sum_{k=0}^{n} \sigma_{n-k}^{\gamma-1} T^k, \qquad n = 0, 1, \ldots,$$

where $\sigma_n^\gamma = \binom{\gamma+n}{n} = (\gamma+n)(\gamma+n-1)\cdots(\gamma+1)/n!$, $n \geq 1$, and $\sigma_0^\gamma = 1$ for $\gamma \in \mathbb{R} \setminus (-\mathbb{N})$ (see [25, Chapter 3]). These include two particular means: $C_n^0(T) = T^n$ and $C_n^1(T) = C_{n+1}(T) := \frac{1}{n+1} S_{n+1}(T)$ for $n \geq 0$, where $S_{n+1}(T) = \sum_{k=0}^{n} T^k$.

The *Abel means* of $T$ for $x \in X$ are defined as

$$A_r(T)x := (1-r) \lim_{N\to\infty} \sum_{n=0}^{N} r^n T^n x$$

for those $r \in [0, 1)$ such that the series converges. The series $(1-r)\sum_{n=0}^{\infty} r^n T^n$ converges absolutely to $A_r(T)$ in operator norm (we will simply say

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that $A_r(T)$ converges absolutely) for all $0 \le |r| < 1/r(T)$, where $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ denotes the spectral radius of $T$. Clearly, $r(T) \le 1$ if and only if $A_r(T)$ converges absolutely for all $0 \le |r| < 1$, if and only if it converges absolutely for all $0 \le r < 1$. Moreover, in this case, $A_r(T) = (1-r)(I-rT)^{-1}$ for each $0 \le |r| < 1$.

It is known (cf. [25, Chapter 3]) that if $0 < \gamma < \beta < \infty$ then

$$\sup_{n\ge 0} \|T^n\| \ge \sup_{n\ge 0} \|C_n^\gamma(T)\| \ge \sup_{n\ge 0} \|C_n^\beta(T)\| \ge \sup_{0<r<1} \|A_r(T)\|. \tag{1.1}$$

In some particular cases, some of the above inequalities become equalities. But, in general, the inequalities are strict. It is interesting to see when an inequality becomes an equality and for what concrete examples an inequality is strict. Results in this direction can be found e.g. in [6], [7], [8], [14], [17], [18], [21], and [23].

More generally, we are interested in implication relations between growth orders of means, that is, implication relations between the properties: $\|C_n^\gamma(T)\| = O(n^\alpha)$, $-1 < \gamma < \infty$, and $\|A_r(T)\| = O((1-r)^\alpha)$ $(r \uparrow 1)$, for $\alpha \ge 0$.

Similar questions can be asked about a continuous semigroup $T(\cdot) \equiv (T(t))_{t>0}$ of operators on $X$. By definition, $T(s+t) = T(s)T(t)$ for all $s, t > 0$ and $t \mapsto T(t)x$ is strongly continuous on $(0,\infty)$ for every $x \in X$. If, in addition, $T(t)$ converges strongly to $T(0) := I$, the identity operator, as $t \to 0$, then $(T(t))_{t\ge 0}$ is called a $C_0$-*semigroup* (cf. [10, 12, 19]). In this case, the infinitesimal generator $A$ of $T(\cdot)$, defined by $Ax := \lim_{t\downarrow 0} t^{-1}(T(t)x - x)$, is a densely defined closed linear operator.

The *Cesàro means of order* $\gamma$ (or $\gamma$-*Cesàro means*) $C_t^\gamma$, $\gamma \ge 0$, of $T(\cdot)$ are the operators defined by $C_t^0 = T(t)$ and $C_t^\gamma := \gamma t^{-\gamma} \int_0^t (t-s)^{\gamma-1} T(s)\,ds$ for $\gamma > 0$. In particular, $C_t^1 = C_t := S(t)/t$ where $S(t) = \int_0^t T(s)\,ds,\ t > 0$.

Suppose $\int_0^1 \|T(s)x\|\,ds < \infty$ for all $x \in X$. For given $x \in X$ and $\lambda \in \mathbb{C}$, we define

$$A_\lambda x = \lambda \int_0^\infty e^{-\lambda s} T(s)x\,ds := \lim_{t\to\infty} \lambda \int_0^t e^{-\lambda s} T(s)x\,ds$$

if the limit exists. If $\lambda \in \mathbb{C}$ is such that $A_\lambda x$ exists for all $x \in X$, then, by the uniform boundedness principle, $A_\lambda$ is a bounded linear operator, and is called an *Abel mean* of $T(\cdot)$. Let us denote by $\sigma$ the *abscissa of convergence* of the Laplace integral of $T(\cdot)$:

$$\begin{aligned}\sigma &:= \inf\{u \in \mathbb{R};\ A_\lambda \text{ exists for all } \lambda \text{ with } \operatorname{Re}\lambda > u\}\\ &= \inf\{\operatorname{Re}\lambda;\ A_\lambda \text{ exists}\}.\end{aligned}$$

Then we have $\sigma \le w_0$, where $w_0 := \lim_{t\to\infty} (\ln \|T(t)\|)/t$ is the *type* (or *exponential growth bound*) of $T(\cdot)$. If $T(\cdot)$ is a $C_0$-semigroup with generator $A$, then $\lambda \in \varrho(A)$ and $A_\lambda = \lambda(\lambda - A)^{-1}$ for all $\lambda$ with $\operatorname{Re}\lambda > \sigma$. The *spectral bound* $s(A)$ of $A$ is defined by

$$s(A) := \sup\{\operatorname{Re}\lambda;\ \lambda \in \sigma(A)\}.$$

Thus $s(A) \le \sigma \le w_0$ [10]. It is possible that $\sigma < w_0$. In fact, there exists a $C_0$-semigroup of positive operators on a Banach lattice which is uniformly Cesàro ergodic so that $\|C_t^1\| = O(1)$ $(t \to \infty)$ and hence $\|A_\lambda\| = O(1)$ $(\lambda \downarrow 0)$, but satisfies $-\infty = s(A) = \sigma < 0 < w_0$ (cf. [11], [22, p. 62]). On the other hand, $s(A) = \sigma = w_0$ if $T(\cdot)$ is an eventually norm-continuous semigroup (see [1, Theorem 5.1.12], [10, Corollary 4.3.11]). Thus, $s(A) = \sigma = w_0$ holds in particular when $A$ is bounded, and hence holds on finite-dimensional spaces.

Results on relations between boundedness of a $C_0$-semigroup and of its Cesàro and Abel means can be found e.g. in [9], [15], [16], [20], and [24]. It is interesting to consider implication relations between growth orders of means, i.e., between the properties: $\|C_t^\gamma\| = O(t^\alpha)$ $(t \to \infty)$, $-1 < \gamma < \infty$, and $\|A_\lambda\| = O(\lambda^{-\alpha})$ $(\lambda \downarrow 0)$, for $\alpha \ge 0$.

In this paper, we obtain some results on the two subjects: 1) implication relations between growth orders of 0-Cesàro, 1-Cesàro, and Abel means, for both discrete and continuous semigroups; 2) relations between boundedness of $\gamma$-Cesàro means $(-1 < \gamma < \infty)$ and of Abel means for discrete semigroups. Some results on 1) for $\gamma$-Cesàro means $(0 \le \gamma < \infty)$ of continuous semigroups, and on 2) for continuous semigroups will appear in [5].

In Section 2, we will investigate general implication relations between growth orders of 0-Cesàro means, 1-Cesàro means, and Abel means for discrete and continuous semigroups. In general, for each $\alpha \ge 0$ we have:

$$\begin{aligned}
[\|T^n\| = O(n^\alpha)] &\Rightarrow [\|C_n^1(T)\| = O(n^\alpha)] \\
&\Rightarrow [r(T) \le 1 \text{ and } \|A_r(T)\| = O((1-r)^{-\alpha})\ (r \uparrow 1)] \\
&\Rightarrow [r(T) \le 1]; \\
[\|T(t)\| = O(t^\alpha)\ (t \to \infty)] &\Rightarrow [\|C_t^1\| = O(t^\alpha)\ (t \to \infty)] \\
&\Rightarrow [\sigma \le 0 \text{ and } \|A_\lambda\| = O(\lambda^{-\alpha})\ (\lambda \downarrow 0)] \\
&\Rightarrow [\sigma \le 0].
\end{aligned}$$

If $\dim X = 1$, then all the properties are equivalent to $\|T\| \le 1$ (resp. $T(t) = e^{at}$ with $\operatorname{Re} a \le 0$) (Corollary 2.2). If $\dim X = 2$, then the second implication is reversible for all $0 \le \alpha \le 1$; in particular, in this case, Abel-mean-boundedness is equivalent to $\gamma$-Cesàro-mean-boundedness for any $\gamma \ge 1$ (Proposition 2.5). But the second implication can be strict when $\dim X \ge 3$ (Proposition 2.8). All other implications can be strict once $\dim X \ge 2$ (Proposition 2.3). We also prove in Proposition 2.10 that if $m = \dim X < \infty$ and $r(T) \le 1$ (resp. $w_0 \le 0$), then we must have $\|T^n\| = O(n^{m-1})$ (resp. $\|T(t)\| = O(t^{m-1})$ $(t \to \infty)$).

In Section 3, we consider the situation for positive operators on Banach lattices. In this case, the equivalence of $\|C_n^1(T)\| = O(n^\alpha)$ (resp. $\|C_t^1\| = O(t^\alpha)$ $(t \to \infty)$) and $\|A_r(T)\| = O((1-r)^{-\alpha})$ $(r \uparrow 1)$ (resp. $\|A_\lambda\| = O(\lambda^{-\alpha})$ $(\lambda \downarrow 0)$) under the assumption $r(T) \le 1$ (resp. $\sigma \le 0$) is shown to hold for all $\alpha > -1$ (Corollary 3.2).

Section 4 will be mainly concerned with implication relations between boundedness of $\gamma$-Cesàro means $\{C_n^\gamma(T); n \ge 0\}$ for $\gamma \in (-1, \infty)$. It is possible to find an invertible positive linear isometry on an $L_1$-space such that $\sup_{n\ge 0} \|C_n^\gamma(T)\| = \infty$ for all $-1 < \gamma < 0$ (Proposition 4.1). For any $0 < \gamma < 1$, one can also find a positive linear operator $T$ on an $L_1$-space such that $\sup_{n\ge 0} \|C_n^\gamma(T)\| = \infty$ but $\sup_{n\ge 0} \|C_n^\beta(T)\| < \infty$ for all $\beta > \gamma$ (Proposition 4.3). Finally, for any integer $k \ge 0$, there exists an operator $T$ on a Banach space such that $\sup_{n\ge 0} \|C_n^{k+1}(T)\| < \infty$ but $\sup_{n\ge 0} \|C_n^\gamma(T)\| = \infty$ for every $\gamma$ with $0 \le \gamma < k+1$ (Proposition 4.4(i)), and there exists an operator $T$ on a Banach space with $r(T) = 1$ such that $\sup_{0<r<1} \|A_r(T)\| < \infty$ but $\sup_{n\ge 0} \|C_n^\gamma(T)\| = \infty$ for every $0 \le \gamma < \infty$ (Proposition 4.4(ii)). The authors do not know whether the above integer $k \ge 0$ can be replaced with any nonnegative real number.

Note that, for $C_0$-semigroups and cosine operator functions, the continuous analog of Proposition 4.3 holds for both cases $0 < \gamma < 1$ and $\gamma = 0$ (see [5, Theorems 4.2 and 4.4]), and the continuous analog of Proposition 4.4 is also true (see [5, Theorems 3.6 and 3.8]). It can be seen that the validity of Proposition 4.3 for $\gamma = 0$ follows immediately from the proof of Theorem 4.4 in [5]. We also refer the reader to [5] for the behavior of growth orders of $\gamma$-Cesàro means, $0 \le \gamma < \infty$, of $C_0$-semigroups.

As will be shown in (2.2), for $\alpha \ge 0$, $\|n^{-\alpha} C_n^1(T)x\| = O(1)$ $(n \to \infty)$ implies $\|(1-r)^\alpha A_r(T)x\| = O(1)$ $(r \uparrow 1)$. One would naturally ask about relations between the existence of $\lim_{n\to\infty} n^{-\alpha} C_n^1(T)x$ and the existence of $\lim_{r\uparrow 1} (1-r)^\alpha A_r(T)x$. The answer to this question is as follows:

*For $\alpha > -2$, if the limit $y := \lim_{n\to\infty} n^{-\alpha} C_n^1(T)x$ (resp. $\lim_{t\to\infty} t^{-\alpha} C_t^1 x$) exists, then*

$$\lim_{r\uparrow 1} \frac{(1-r)^\alpha}{\Gamma(\alpha+2)} A_r(T)x = y \quad \left(\text{resp. } \lim_{\lambda\downarrow 0} \frac{\lambda^\alpha}{\Gamma(\alpha+2)} A_\lambda x = y.\right)$$

*In general, the converse is not true. But, if $\|T^n\| = O(n^\alpha)$ (resp. $T(\cdot)$ is locally integrable and $\|T(t)\| = O(t^\alpha)(t \to \infty)$), or if $T$ is a positive operator (resp. $T(\cdot)$ is a locally integrable semigroup of positive operators) on a Banach lattice, then the converse also holds for $\alpha > -1$.*

These will appear as Propositions 5.1, 5.2, and 6.1 in [15], wherein more general convergence theorems and Tauberian theorems for functions and sequences are to be discussed.

**2. Growth orders of Abel means and $\gamma$-Cesàro means for $\gamma = 0, 1$.** We start with general implication relations.

PROPOSITION 2.1. (i) *Let $d(0) = 0$ and $d(\alpha) := \max\{2\alpha^\alpha, 1\}$ for $\alpha > 0$. The following implications hold for all $\alpha \geq 0$ and $x \in X$:*

$$\tag{2.1} \|T^n x\| \leq M_x n^\alpha \text{ for some } M_x \geq \|x\| \text{ and all } n \geq 1$$
$$\Rightarrow \begin{cases} \|C_n(T)x\| \leq M_x n^\alpha \quad \text{for all } n \geq 1, \\ A_r(T)x \text{ converges absolutely and} \\ \|A_r(T)x\| \leq M_x[d(\alpha) + \Gamma(\alpha+1)](1-r)^{-\alpha} \text{ for all } r \in [0,1); \end{cases}$$

$$\tag{2.2} \|C_n(T)x\| \leq M_x n^\alpha \text{ for some } M_x \geq \|x\| \text{ and all } n \geq 1$$
$$\Rightarrow A_r(T)x \text{ converges absolutely and}$$
$$\|A_r(T)x\| \leq M_x(\alpha+1)2^\alpha[d(\alpha) + \Gamma(\alpha+1)](1-r)^{-\alpha} \text{ for } r \in [0,1);$$

$$\tag{2.3} \|T^n\| \leq Mn^\alpha \text{ for some } M \geq 1 \text{ and all } n \geq 1$$
$$\Rightarrow \begin{cases} \|C_n(T)\| \leq Mn^\alpha \quad \text{for all } n \geq 1; \\ A_r(T) \text{ converges absolutely for all } r \in [0,1) \ (\text{i.e., } r(T) \leq 1); \\ \|A_r(T)\| \leq M[d(\alpha) + \Gamma(\alpha+1)](1-r)^{-\alpha} \text{ for all } r \in [0,1); \end{cases}$$

$$\tag{2.4} \|C_n(T)\| \leq Mn^\alpha \text{ for some } M \geq 1 \text{ and all } n \geq 1$$
$$\Rightarrow r(T) \leq 1 \text{ and } \|A_r(T)\| \leq M(\alpha+1)2^\alpha[d(\alpha) + \Gamma(\alpha+1)](1-r)^{-\alpha}$$
$$\text{for all } r \in [0,1).$$

(ii) *The following implications hold for all $\alpha \geq 0$ and $x \in X$:*

$$\tag{2.5} \|T(t)x\| \leq M_x(1+t^\alpha) \text{ for all } t > 0$$
$$\Rightarrow \|C_t x\| \leq M_x\left(1 + \frac{t^\alpha}{\alpha+1}\right) \text{ for all } t > 0;$$

$$\tag{2.6} \|C_t x\| \leq M_x\left(1 + \frac{t^\alpha}{\alpha+1}\right) \text{ for all } t > 0$$
$$\Rightarrow A_\lambda x \text{ exists and } \|A_\lambda x\| \leq M_x(1 + \Gamma(\alpha+1)\lambda^{-\alpha}) \text{ for all } \lambda > 0;$$

$$\tag{2.7} \|T(t)\| \leq M(1+t^\alpha) \text{ for all } t > 0$$
$$\Rightarrow \|C_t\| \leq M\left(1 + \frac{t^\alpha}{\alpha+1}\right) \text{ for all } t > 0;$$

$$\tag{2.8} \|C_t\| \leq M\left(1 + \frac{t^\alpha}{\alpha+1}\right) \text{ for all } t > 0$$
$$\Rightarrow \sigma \leq 0 \text{ and } \|A_\lambda\| \leq M(1 + \Gamma(\alpha+1)\lambda^{-\alpha}) \text{ for all } \lambda > 0.$$

*Proof.* (i) The first estimate in (2.1) is obvious, so we prove the second. Under the assumption $\|T^n x\| \leq M_x n^\alpha$ with $M_x \geq \|x\|$, we have

$$\|(1-r)^\alpha A_r(T)x\| \leq M_x(1-r)^{\alpha+1}\Big(1 + \sum_{n=1}^{\infty} r^n n^\alpha\Big), \qquad 0 \leq r < 1,$$

the series being convergent, by the ratio test. If $\alpha = 0$ or $r = 0$, the right-hand side becomes $M_x$.

For $\alpha > 0$ and $0 < r < 1$, let $t_1 := \alpha/(-\ln r) > 0$. It is easily seen that $r^{t_1} t_1^\alpha = \max_{t \geq 0} r^t t^\alpha$.

If $t_1 \geq 1$, then

$$1 + \sum_{n=1}^{\infty} r^n n^\alpha = 1 + \sum_{n=1}^{[t_1]} r^n n^\alpha + r^{[t_1]+1}([t_1]+1)^\alpha + \sum_{[t_1]+2}^{\infty} r^n n^\alpha,$$

so that

$$\begin{aligned}
&(1-r)^{\alpha+1}\Big(1 + \sum_{n=1}^{\infty} r^n n^\alpha\Big) \\
&\qquad < (1-r)^{\alpha+1}\left[\frac{1-r^{[t_1]+1}}{1-r}\, t_1^\alpha + r^{t_1}\, t_1^\alpha + \int\limits_{[t_1]+1}^{\infty} e^{-t(-\ln r)} t^\alpha \, dt\right] \\
&\qquad \leq \left(\frac{1-r}{-\ln r}\right)^{\alpha} (1 + (1-r) r^{t_1}) \alpha^\alpha + \left(\frac{1-r}{-\ln r}\right)^{\alpha+1} \Gamma(\alpha+1) \\
&\qquad \leq 2\alpha^\alpha + \Gamma(\alpha+1).
\end{aligned}$$

Here we have used the fact that $\frac{1-r}{-\ln r} \leq 1$ for $0 < r < 1$. If $0 < t_1 < 1$, then

$$1 + \sum_{n=1}^{\infty} r^n n^\alpha = 1 + r + \sum_{n=2}^{\infty} r^n n^\alpha \leq 1 + r + \int\limits_{1}^{\infty} r^t t^\alpha \, dt,$$

and a similar estimation to the above gives

$$(1-r)^{\alpha+1}\Big(1 + \sum_{n=1}^{\infty} r^n n^\alpha\Big) \leq 1 + \Gamma(\alpha+1).$$

Combining all these possible estimates, we have

$$(1-r)^{\alpha+1}\Big(1 + \sum_{n=1}^{\infty} r^n n^\alpha\Big) \leq d(\alpha) + \Gamma(\alpha+1) \tag{2.9}$$

for all $\alpha \geq 0$. This shows (2.1).

To show (2.2) under the assumption $\|C_n(T)x\| \leq M_x n^\alpha$ with $M_x \geq \|x\|$, we first note that the series $\sum r^n S_n(T)x$ is absolutely convergent for $0 \leq |r| < 1$. So $A_r(T)x$ also converges absolutely for $0 \leq |r| < 1$ and we can write

$$\|(1-r)^\alpha A_r(T)x\| = (1-r)^{\alpha+1}\Big\| x + \sum_{n=1}^{\infty} r^n (S_{n+1}(T) - S_n(T))x \Big\|$$

$$
\begin{aligned}
&= (1-r)^{\alpha+1}\Big\|\sum_{n=1}^{\infty}(r^{n-1}-r^n)S_n(T)x\Big\| \\
&\le M_x(1-r)^{\alpha+1}\sum_{n=1}^{\infty}(r^{n-1}-r^n)n^{\alpha+1} \\
&= M_x(1-r)^{\alpha+1}\sum_{n=0}^{\infty}((n+1)^{\alpha+1}-n^{\alpha+1})r^n \\
&\le M_x(1-r)^{\alpha+1}\sum_{n=0}^{\infty}(\alpha+1)(n+1)^{\alpha}r^n \\
&\qquad\qquad\qquad\qquad \text{(by the mean value theorem)} \\
&= M_x(\alpha+1)(1-r)^{\alpha+1}\Big[1+\sum_{n=1}^{\infty}(n+1)^{\alpha}r^n\Big] \\
&\le M_x(\alpha+1)2^{\alpha}(1-r)^{\alpha+1}\Big[1+\sum_{n=1}^{\infty}n^{\alpha}r^n\Big] \\
&\le M_x(\alpha+1)2^{\alpha}[d(\alpha)+\Gamma(\alpha+1)] \quad \text{(by (2.9))}
\end{aligned}
$$

for $r \in [0,1)$. The proofs of (2.3) and (2.4) are similar to the above proofs of (2.1) and (2.2), respectively.

(ii) The implication (2.5) is obvious from the definition of $C_t$. To show (2.6), we first note that the assumption $\|C_t x\| \le M_x(1+t^{\alpha}/(\alpha+1))$ implies that $S(t)x$ is polynomially bounded so that $\int_0^{\infty} e^{-\lambda s}S(s)x\,ds$ exists for all $\lambda > 0$. Then, using integration by parts and the polynomial boundedness of $S(t)x$, we see that, for all $\lambda > 0$,

$$
\begin{aligned}
A_\lambda x &= \lambda \lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\,ds = \lambda\lim_{t\to\infty}\lambda\int_0^t e^{-\lambda s}S(s)x\,ds \\
&= \lambda^2\int_0^{\infty} e^{-\lambda s}S(s)x\,ds.
\end{aligned}
$$

Now (2.6) is obtained using the following estimates:

$$
\begin{aligned}
\|A_\lambda x\| &= \Big\|\lambda^2\int_0^{\infty}e^{-\lambda t}S(t)x\,dt\Big\| \\
&\le M_x\lambda^2\int_0^{\infty}e^{-\lambda t}\Big(t+\frac{t^{\alpha+1}}{\alpha+1}\Big)\,dt = M_x\Big[1+\frac{1}{\alpha+1}\,\lambda^{-\alpha}\Gamma(2+\alpha)\Big] \\
&= M_x(1+\Gamma(\alpha+1)\lambda^{-\alpha}) \qquad \text{for all } \lambda > 0.
\end{aligned}
$$

The implications (2.7) and (2.8) follow from (2.5) and (2.6), respectively. ∎

REMARK. For the case $\alpha = 0$, (2.7) and (2.8) reduce to the following inequalities:

$$\sup_{t>0} \|T(t)\| \geq \sup_{t>0} \|C_t^1\| \geq \sup_{\lambda>0} \|A_\lambda\|.$$

In fact, it can be shown (cf. [5, Theorem 2.3]) that

$$(2.10)\quad \begin{aligned} &\sup_{t>0} \|T(t)\| \geq \sup_{t>0} \|C_t^\gamma\| \geq \sup_{t>0} \|C_t^\beta\| \quad \text{for } 0 < \gamma < \beta < \infty; \\ &\sup_{t>0} \|C_t^\beta\| \geq \sup_{\lambda>0} \|A_\lambda\| \quad \text{in case } T(\cdot) \text{ is subexponential (i.e. } w_0 \leq 0). \end{aligned}$$

COROLLARY 2.2. *If* $\dim X = 1$, *or* $T$ *is a normal operator* (*resp.* $T(\cdot)$ *is an eventually norm-continuous* $C_0$*-semigroup of normal operators*) *on a Hilbert space*, *or* $T$ *is a hermitian operator* (*resp.* $T(\cdot)$ *is an eventually norm-continuous* $C_0$*-semigroup of hermitian operators*) *on a Banach space*, *then*, *for* $\alpha \geq 0$, *each of the conditions*: $\|T^n\| = O(n^\alpha)$, $\|C_n(T)\| = O(n^\alpha)$, *and* $\|A_r(T)\| = O((1-r)^{-\alpha})$ $(r \uparrow 1)$ *with* $r(T) \leq 1$ (*resp.* $\|T(t)\| = O(t^\alpha)$ $(t \to \infty)$, $\|C_t\| = O(t^\alpha)$ $(t \to \infty)$, *and* $\|A_\lambda\| = O(\lambda^{-\alpha})$ $(\lambda \downarrow 0)$ *with* $\sigma = w_0 \leq 0$) *is equivalent to* $T$ *being a contraction* (*resp.* $T(\cdot)$ *being a contraction semigroup*).

*Proof.* Recall that a hermitian operator $T$ on a Banach space $X$ is a linear operator which has its algebra numerical range $V(T) := \{F(T) \mid F \in B(X)^*, F(I) = \|F\| = 1\}$ contained in $\mathbb{R}$ (cf. [3, 4]). It is known that $r(T) = \|T\|$ when $T$ is hermitian (see [4, Theorem 26.2]). Proposition 2.1 asserts that each of the above conditions implies that $r(T) \leq 1$ (resp. $w_0 = \sigma \leq 0$). Hence $\|T\| \leq 1$ (resp. $\|T(t)\| \leq 1$ for all $t \geq 0$) follows from the fact that, in each of the above cases, $\|T\| = r(T)$ and $\|T(t)\| = r(T(t)) = e^{w_0 t}$ for all $t \geq 0$ (cf. [10, p. 251]). The converse is obvious. ∎

The next proposition shows that when $\dim X \geq m \geq 2$, the converse of each of the implications (2.3) and (2.7) in general does not hold, and $r(T) \leq 1$ (resp. $\sigma \leq 0$) does not guarantee that $\|A_r(T)\| = O((1-r)^{-\alpha})$ $(r \uparrow 1)$ (resp. $\|A_\lambda\| = O(\lambda^{-\alpha})$ $(\lambda \downarrow 0)$) for some $\alpha \geq 0$.

PROPOSITION 2.3. *If* $\dim X \geq m \geq 2$, *then the following hold*:

(i) *There exists an operator* $T$ (*resp. a uniformly continuous* $C_0$*-semigroup* $T(\cdot)$) *on* $X$ *such that* $\|C_n(T)\| \leq Mn^{m-2}$ *for all* $n \geq 1$ (*resp.* $\|C_t\| \leq Mt^{m-2}$ *for all* $t > 0$) *but* $T^n/n^{m-1}$ (*resp.* $T(t)/t^{m-1}$) *does not converge to* 0 *strongly as* $n \to \infty$ (*resp.* $t \to \infty$), *and hence* $\|T^n\| \neq O(n^\alpha)$ (*resp.* $\|T(t)\| \neq O(t^\alpha)$ $(t \to \infty)$) *for any* $\alpha \in [0, m-1)$.

(ii) *There exists an operator* $T$ (*resp. a uniformly continuous* $C_0$*-semigroup* $T(\cdot)$) *on* $X$ *such that* $r(T) \leq 1$ (*resp.* $\sigma = w_0 \leq 0$) *but* $(1-r)^{m-1} A_r(T)$ (*resp.* $\lambda^{m-1} A_\lambda$) *does not converge to* 0 *strongly as* $r \uparrow 1$ (*resp.* $\lambda \downarrow 0$), *and* *hence* $\|A_r(T)\| \neq O((1-r)^{-\alpha})$ $(r \uparrow 1)$ *(resp.* $\|A_\lambda\| \neq O(\lambda^{-\alpha})$ $(\lambda \downarrow 0))$ *for any* $\alpha \in [0, m-1)$.

*Proof.* Since $\dim X \geq m$, there exists a nilpotent operator $N$ on $X$ such that $N^m = 0$ but $N^{m-1} \neq 0$.

*The discrete case.* The operator $T := \mu I + N$ has spectrum $\sigma(T) = \{\mu\}$. We see from the binomial theorem that $T^n = \sum_{k=0}^{m-1} \binom{n}{k} \mu^{n-k} N^k$ and

$$\begin{aligned} n(I-T)C_n(T) &= n(I-T)\,\frac{1}{n}\sum_{k=0}^{n-1} T^k = I - T^n \\ &= (1-\mu^n)I - \sum_{k=1}^{m-1} \binom{n}{k} \mu^{n-k} N^k \qquad \text{for every } n \geq 1. \end{aligned}$$

(i) If $|\mu| \leq 1$, $\mu \neq 1$, then $I - T$ is invertible and

$$\begin{aligned} \|C_n(T)\|/n^{m-2} &\leq \|(I-T)^{-1}\|\,\|(I-T)C_n(T)\|/n^{m-2} \\ &\leq \|(I-T)^{-1}\|\left(\frac{2}{n^{m-1}} + \frac{1}{n^{m-1}}\sum_{k=1}^{m-1}\binom{n}{k}\|N^k\|\right) \\ &\leq \|(I-T)^{-1}\|\left(2 + \sum_{k=1}^{m-1}\frac{1}{k!}\,\|N^k\|\right) \end{aligned}$$

for all $n \geq 1$. But, when $|\mu| = 1$, we have

$$\begin{aligned} \|T^n\|/n^{m-1} &\geq \left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-2}{n}\right)\frac{\|N^{m-1}\|}{(m-1)!} - \frac{1}{n^{m-1}}\sum_{k=0}^{m-2}\binom{n}{k}\|N^k\| \\ &\to \|N^{m-1}\|/(m-1)! > 0 \qquad \text{as } n \to \infty. \end{aligned}$$

(ii) If $\mu = 1$, then $\sigma(T) = \{1\}$ and

$$\begin{aligned} A_r(T) &= (1-r)\sum_{n=0}^{\infty} r^n \sum_{k=0}^{m-1}\binom{n}{k} N^k = (1-r)\sum_{k=0}^{m-1}\left[\sum_{n=k}^{\infty}\binom{n}{k} r^n\right] N^k \\ &= (1-r)\sum_{k=0}^{m-1}\frac{r^k}{k!}\left(\frac{d}{dr}\right)^k\left(\frac{1}{1-r}\right) N^k \\ &= (1-r)\sum_{k=0}^{m-1} r^k (1-r)^{-k-1} N^k, \end{aligned}$$

so that $r(T) = 1$ and

$$(1-r)^{m-1}A_r(T) = r^{m-1}N^{m-1} + (1-r)\sum_{k=0}^{m-2} r^k (1-r)^{m-2-k} N^k \to N^{m-1}$$

as $r \to 1^-$.

*The continuous case.* Let $T(t) = e^{\mu t}e^{tN} = e^{\mu t}(\sum_{k=0}^{m-1}(t^k/k!N^k))$, $t \geq 0$. Then $T(\cdot)$ is a $C_0$-semigroup with generator $A := \mu I + N$.

(i) If $\operatorname{Re}\mu \leq 0$ and $\mu \neq 0$, then the operator $A$ is invertible and

$$\begin{aligned}\|C_t\| \leq \|A^{-1}\|\,\|AC_t\| &= \|A^{-1}\|\,\frac{1}{t}\,\|T(t) - I\| \\ &\leq \|A^{-1}\|\,\frac{1}{t}\Big(\sum_{k=1}^{m-1}\frac{t^k}{k!}\,\|N^k\| + |e^{\mu t} - 1|\Big) \\ &\leq \|A^{-1}\|\,\frac{1}{t}\Big(\sum_{k=1}^{m-1}\frac{t^k}{k!}\,\|N^k\| + 2\Big)\end{aligned}$$

so that $\|C_t\| = O(t^{m-2})$ $(t \to \infty)$. But, when $\operatorname{Re}\mu = 0$, we have

$$\begin{aligned}\|T(t)\|/t^{m-1} &\geq \frac{1}{(m-1)!}\,\|N^{m-1}\| - t^{-m+1}\sum_{k=0}^{m-2}\frac{t^k}{k!}\,\|N^k\| \\ &\to \frac{1}{(m-1)!}\,\|N^{m-1}\| > 0 \quad \text{as } t \to \infty.\end{aligned}$$

(ii) If $\mu = 0$, then

$$T(t) = \sum_{k=0}^{m-1}\frac{t^k}{k!}\,N^k \quad \text{and} \quad w_0 \leq \lim_{t\to\infty}\frac{1}{t}\ln\Big(\sum_{k=0}^{m-1}\frac{t^k}{k!}\,\|N^k\|\Big) = 0.$$

For $\lambda > 0$,

$$\begin{aligned}\lambda^{m-1}\|A_\lambda\| &= \lambda^m\Big\|\int_0^\infty e^{-\lambda t}\Big(\sum_{k=0}^{m-1}\frac{t^k}{k!}\,N^k\Big)dt\Big\| \\ &= \Big\|\sum_{k=0}^{m-1}\lambda^{m-1-k}N^k\Big\| \to \|N^{m-1}\| \neq 0 \quad \text{as } \lambda \downarrow 0. \ \blacksquare\end{aligned}$$

The formulation for the special case that $m = 2$ and $\alpha = 0$ is as follows.

COROLLARY 2.4. *Let $X$ be a Banach space of dimension more than one.*

(i) *A Cesàro-mean-bounded operator $T$ (resp. a uniformly continuous $C_0$-semigroup $T(\cdot)$) on $X$ need not have the property that $T^n/n \to 0$ strongly (resp. $T(t)/t \to 0$ strongly as $t \to \infty$), and hence is not necessarily power-bounded (resp. uniformly bounded).*

(ii) *$r(T) \leq 1$ (resp. $\sigma \leq 0$) is not sufficient for $T$ (resp. $T(\cdot)$) to be Abel-mean-bounded.*

REMARKS. (1) Since there are examples of Cesàro-mean-ergodic operators which are not power-bounded (cf. [7, p. 255], [6, p. 451]), a Cesàro-mean-bounded operator is not necessarily power-bounded. On the other hand, every Cesàro-mean-bounded positive operator on a finite-dimensional space is power-bounded (cf. [6, p. 449], [21, Chap. 1, Sec. 3]).

(2) Since the norm convergence of $T(t)$ to $I$ as $t \to 0$ implies that $\int_0^\delta T(s)\,ds$ is invertible for small $\delta > 0$, from the identity $\frac{t+\delta}{t} C_{t+\delta} - C_t = t^{-1}T(t)\int_0^\delta T(s)\,ds$ it follows that if a uniformly continuous $C_0$-semigroup $T(\cdot)$ is Cesàro-mean-ergodic, then $T(t)/t \to 0$ strongly as $t \to \infty$. Similarly, if $T$ is Cesàro-mean-ergodic, then $T^n/n \to 0$ strongly as $n \to \infty$. Thus, by Corollary 2.4(i), a Cesàro-mean-bounded operator $T$ (resp. uniformly continuous $C_0$-semigroup $T(\cdot)$) is not necessarily Cesàro-mean-ergodic.

(3) Any upper triangular $n \times n$ matrix which has all its diagonal entries equal to $\mu = e^{i\theta}$ for some $\theta$ is of the form $\mu I + N$ with $N$ a nilpotent matrix and $|\mu| = 1$. Hence, as shown in the proof of Proposition 2.3(i), all such matrices $T = \mu I + N$ with $\mu \neq 1$ (resp. semigroups $T(t) = e^{t(\mu I+N)}$ with $\operatorname{Re}\mu = 0$ and $\mu \neq 0$) satisfy assertion (i) of Proposition 2.3 (in particular, (i) of Corollary 2.4, in case $N^2 = 0$). In particular, the matrix $\left[\begin{smallmatrix} -1 & 2 \\ 0 & -1 \end{smallmatrix}\right]$ has been given in [9, p. 10]. Thus the semigroup

$$T(t) := \begin{bmatrix} e^{\mu t} & te^{\mu t}a \\ 0 & e^{\mu t} \end{bmatrix}$$

with $a \neq 0$, $\operatorname{Re}\mu = 0$ and $\mu \neq 0$ satisfies $\sup_{t>0}\|C_t\| < \infty$ and $\sup_{t\geq 0}\|T(t)\| = \infty$. On the other hand, it is known that if a $C_0$-semigroup $T(\cdot)$ satisfies $M := \sup_{t>0}\|C_t\| \leq 1$, then $\sup_{t\geq 0}\|T(t)\| \leq 1$ (see [9, Theorem 1.10]).

(4) If $T$ is a positive operator on a reflexive Banach lattice, then Abel-mean-boundedness of $T$ implies that $T$ is Cesàro-mean-ergodic and hence $T^n/n \to 0$ strongly (cf. [9, 15]).

PROPOSITION 2.5. *Suppose* $\dim X = 2$. *Then for an operator* $T$ (*resp. a* $C_0$*-semigroup* $T(\cdot)$) *on* $X$ *such that* $r(T) \leq 1$ (*resp.* $\sigma = w_0 \leq 0$), *the following hold*:

(i) $\|T^n\| = O(n)$ (*resp.* $\|T(t)\| = O(t)$ $(t \to \infty)$); *and therefore both* $\{(1-r)A_r(T);\ 0 < r < 1\}$ *and* $\{C_n(T)/n;\ n \geq 1\}$ (*resp.* $\{\lambda A_\lambda;\ 0 < \lambda \leq 1\}$ *and* $\{C_t/t;\ t \geq 1\}$) *are bounded.*

(ii) *For* $0 \leq \alpha < 1$, $\{(1-r)^\alpha A_r(T);\ 0 < r < 1\}$ (*resp.* $\{\lambda^\alpha A_\lambda;\ 0 < \lambda \leq 1\}$) *is bounded if and only if* $\{C_n(T)/n^\alpha;\ n \geq 1\}$ (*resp.* $\{C_t/t^\alpha;\ t \geq 1\}$) *is bounded. In particular*, $T$ (*resp.* $T(\cdot)$) *is Abel-mean-bounded if and only if it is* $\gamma$*-Cesàro-mean-bounded for any* $\gamma \geq 1$.

*Proof.* First, we consider the case where $X$ is a two-dimensional complex Banach space. One can easily see that the same kind of means of two similar matrices have the same growth order. Hence it suffices to assume that $T$ is of the Jordan canonical form $T = \left[\begin{smallmatrix} \mu_1 & a \\ 0 & \mu_2 \end{smallmatrix}\right]$, where $a$ is assumed to be zero when $\mu_1 \neq \mu_2$ (cf. [13, p. 126]). Then

$$T^n = \begin{bmatrix} \mu_1^n & \sum_{k=0}^{n-1} \mu_1^k \mu_2^{n-1-k} a \\ 0 & \mu_2^n \end{bmatrix},$$

$$C_n(T) = \frac{1}{n} \begin{bmatrix} p_{n-1}(\mu_1) & \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \mu_1^j \mu_2^{k-1-j} a \\ 0 & p_{n-1}(\mu_2) \end{bmatrix},$$

where $p_n(t) = 1 + t + \cdots + t^n$. For every $0 < r < 1$,

$$(I - rT)^{-1} = \begin{bmatrix} \frac{1}{1-r\mu_1} & \frac{ra}{(1-r\mu_1)(1-r\mu_2)} \\ 0 & \frac{1}{1-r\mu_2} \end{bmatrix}.$$

Since $\sigma(T) = \{\mu_1, \mu_2\}$, the assumption $r(T) \leq 1$ implies that $0 \leq |\mu_1|, |\mu_2| \leq 1$, and $|p_{n-1}(\mu_i)| \leq n,\ i = 1, 2$.

Case 1: $a = 0$. Then $\|T^n\| \leq 1$ for all $n \geq 1$, and hence both $C_n(T)$ and $A_r(T)$ are uniformly bounded.

Case 2: $a \neq 0$. Then $\mu_1 = \mu_2 = \mu$, so that

$$T^n = \begin{bmatrix} \mu^n & n\mu^{n-1} a \\ 0 & \mu^n \end{bmatrix}, \qquad C_n(T) = \frac{1}{n} \begin{bmatrix} p_{n-1}(\mu) & \sum_{k=1}^{n-1} k\mu^{k-1} a \\ 0 & p_{n-1}(\mu) \end{bmatrix},$$

and

$$A_r(T) = (1-r) \begin{bmatrix} \frac{1}{1-r\mu} & \frac{ra}{(1-r\mu)^2} \\ 0 & \frac{1}{1-r\mu} \end{bmatrix}.$$

Hence $\|T^n\| = O(n)$. This with Case 1 shows the first part of assertion (i); and the second part follows from Proposition 2.1(i).

Case 2.1: $\mu \neq 1$, $|\mu| \leq 1$. Then $p_{n-1}(\mu)/n \to 0$ and

$$\begin{aligned} \frac{1}{n}\Big|\sum_{k=1}^{n-1} k\mu^{k-1}\Big| &= \frac{1}{n}\Big|\frac{d}{d\mu}\Big(\sum_{k=0}^{n-1} \mu^k\Big)\Big| = \frac{1}{n}\Big|\frac{d}{d\mu}\Big(\frac{1-\mu^n}{1-\mu}\Big)\Big| \\ &= \Big|\frac{p_{n-1}(\mu)/n - \mu^{n-1}}{1-\mu}\Big| \to \begin{cases} 1/|1-\mu| & \text{for } |\mu| = 1, \\ 0 & \text{for } |\mu| < 1, \end{cases} \end{aligned}$$

so that

$$\limsup_{n\to\infty} \|C_n(T)\|/n^\alpha \leq \begin{cases} 1/|1-\mu| & \text{for } \alpha = 0, \\ 0 & \text{for } \alpha > 0. \end{cases}$$

We also have $\lim_{r\to 1^-} (1-r)^\alpha A_r(T) = 0$ for all $\alpha \geq 0$.

Case 2.2: $\mu = 1$. Then

$$C_n(T) = \begin{bmatrix} 1 & 2^{-1}(n-1)a \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad (1-r)A_r(T) = \begin{bmatrix} 1-r & ra \\ 0 & 1-r \end{bmatrix},$$

so that

$$\|C_n(T)\|/n^\alpha \to \begin{cases} \frac{1}{2}\|T-I\| & \text{for } \alpha = 1 \\ \infty & \text{for } \alpha \in [0,1) \end{cases} \qquad \text{as } n \to \infty$$

and

$$(1-r)^\alpha \|A_r(T)\| \to \begin{cases} \|T-I\| & \text{for } \alpha = 1 \\ \infty & \text{for } \alpha \in [0,1) \end{cases} \qquad \text{as } r \uparrow 1.$$

The above, together with (1.1), proves assertion (ii) for the discrete case.

Next, we consider the continuous case. A $C_0$-semigroup $T(\cdot)$ can be written in the Jordan form:

$$T(t) := e^{tA} = \begin{bmatrix} e^{\mu_1 t} & \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n-1} \mu_1^k \mu_2^{n-1-k} a \\ 0 & e^{\mu_2 t} \end{bmatrix}$$

with generator

$$A := \begin{bmatrix} \mu_1 & a \\ 0 & \mu_2 \end{bmatrix}, \qquad \text{where } a = 0 \text{ when } \mu_1 \neq \mu_2.$$

If $w_0 \leq 0$, then $\operatorname{Re} \mu_i \leq 0$, $j = 1, 2$. For $\lambda > 0$ we have

$$A_\lambda = \lambda(\lambda - A)^{-1} = \begin{bmatrix} \frac{\lambda}{\lambda - \mu_1} & \frac{\lambda a}{(\lambda-\mu_1)(\lambda-\mu_2)} \\ 0 & \frac{\lambda}{\lambda-\mu_2} \end{bmatrix}.$$

CASE 1: $a = 0$. Then $\|T(t)\| \leq 1$, $\|C_t\| \leq 1$, and $\|A_\lambda\| \leq 1$.

CASE 2: $a \neq 0$. Then $\mu_1 = \mu_2 = \mu$.

CASE 2.1: $\mu \neq 0$, $\operatorname{Re} \mu \leq 0$. Then $\lim_{\lambda \to 0^+} A_\lambda = 0$ and $\lim_{\lambda \to \infty} A_\lambda = I$, so that $A_\lambda$ is uniformly bounded on $(0, \infty)$. In this case we have

$$T(t) = \begin{bmatrix} e^{\mu t} & \sum_{n=1}^{\infty} \frac{t^n}{n!} \mu^{n-1} n a \\ 0 & e^{\mu t} \end{bmatrix} = \begin{bmatrix} e^{\mu t} & t e^{\mu t} a \\ 0 & e^{\mu t} \end{bmatrix}.$$

Hence $\|T(t)\| = O(t)$ $(t \to \infty)$, and

$$C_t = \begin{bmatrix} \frac{1}{t}\int_0^t e^{\mu u}\, du & f(t) a \\ 0 & \frac{1}{t}\int_0^t e^{\mu u}\, du \end{bmatrix},$$

with

$$f(t) := \frac{1}{t}\int_0^t s e^{\mu s}\, ds = \frac{1}{\mu^2}\left(\mu e^{\mu t} + \frac{1 - e^{\mu t}}{t}\right) \to \frac{1}{\mu^2}(\mu - \mu) = 0$$

as $t \downarrow 0$. Since $\operatorname{Re} \mu \leq 0$ implies $|e^{\mu t}| \leq 1$, we also have

$$|f(t)| \leq \frac{1}{|\mu|^2}\left(\frac{2}{t} + |\mu|\right) \to \frac{1}{|\mu|} \qquad \text{as } t \to \infty.$$

Hence $C_t$ is uniformly bounded on $(0, \infty)$.

CASE 2.2: $\mu = 0$. Then $T(t) = I + tA$, $C_t = I + \frac{1}{2}tA$, and $A_\lambda = \left[\begin{smallmatrix} 1 & \lambda^{-1}a \\ 0 & 1 \end{smallmatrix}\right]$, so that

$$t^{-\alpha}\|C_t\| \to \begin{cases} \frac{1}{2}\|A\| & \text{for } \alpha = 1 \\ \infty & \text{for } \alpha \in [0,1) \end{cases} \qquad \text{as } t \to \infty,$$

$$\lambda^{\alpha}\|A_\lambda\| \to \begin{cases} \|A\| & \text{for } \alpha = 1 \\ \infty & \text{for } \alpha \in [0,1) \end{cases} \qquad \text{as } \lambda \downarrow 0.$$

Hence $t^{-1}C_t$ is uniformly bounded on $[1,\infty)$ and $\lambda A_\lambda$ is uniformly bounded on $(0,1]$. This, together with (2.10), proves assertion (ii) for the continuous case. ■

That the assertion also holds true on a two-dimensional real space $X$ follows from the fact that all norms on $X$ are equivalent and the next well-known lemma (cf. [2, pp. 68–71] for a similar version).

LEMMA 2.6. *Let $T$ be a linear operator on the real Hilbert space $\mathbb{R}^2$, and denote by the same symbol $T$ its canonical extension to the complex Hilbert space $\mathbb{C}^2$. Let $\|T\|_{\mathbb{R}^2}$ and $\|T\|_{\mathbb{C}^2}$ denote the respective operator norms. Then $\|T\|_{\mathbb{R}^2} = \|T\|_{\mathbb{C}^2}$.*

*Proof.* Clearly it suffices to show that $\|T\|_{\mathbb{R}^2} \geq \|T\|_{\mathbb{C}^2}$. We may assume that $\|T\|_{\mathbb{C}^2} = 1$. Then there exists $(a_1 + ia_2, b_1 + ib_2) \in \mathbb{C}^2$ with

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1 \quad \text{and} \quad \|T(a_1 + ia_2, b_1 + ib_2)\| = 1.$$

Write $T(a_1 + ia_2, b_1 + ib_2) = (\alpha_1 + i\alpha_2, \beta_1 + i\beta_2)$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Thus

$$T(a_1, b_1) = (\alpha_1, \beta_1),\ T(a_2, b_2) = (\alpha_2, \beta_2) \quad \text{and} \quad \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1.$$

When $(a_1, b_1) = (0,0)$, or $(a_2, b_2) = (0,0)$, the conclusion is obvious.

Suppose $(a_1, b_1) \neq (0,0) \neq (a_2, b_2)$. Since $\|T\|_{\mathbb{R}^2} \leq \|T\|_{\mathbb{C}^2}$, it is impossible that $\|T(a_1, b_1)\| > \|(a_1, b_1)\|$. Similarly, $\|T(a_1, b_1)\| < \|(a_1, b_1)\|$ is impossible, because together with

$$1 = \|T\|_{\mathbb{C}^2} = |\alpha_1 + i\alpha_2|^2 + |\beta_1 + i\beta_2|^2 = \|T(a_1, b_1)\|_{\mathbb{R}^2}^2 + \|T(a_2, b_2)\|_{\mathbb{R}^2}^2$$

it would imply $\|T(a_2, b_2)\| > \|(a_2, b_2)\|$, a contradiction. Consequently, $\|T(a_1, b_1)\| = \|(a_1, b_1)\|$ and $\|T\|_{\mathbb{R}^2} \geq 1 = \|T\|_{\mathbb{C}^2}$. ■

COROLLARY 2.7. *An Abel-mean-bounded operator $T$ with $r(T) \leq 1$ (resp. a $C_0$-semigroup $T(\cdot)$ with $w_0 \leq 0$) on a two-dimensional space is Cesàro-mean-ergodic if and only if $T^n/n \to 0$ (resp. $T(t)/t \to 0$) as $n \to \infty$ (resp. $t \to \infty$).*

The following proposition shows in particular that Abel-mean-boundedness does not imply Cesàro-mean-boundedness on spaces of dimension more than two.

PROPOSITION 2.8. *If* $\dim X \geq m \geq 3$, *then there exists an operator* $T$ (*resp. a* $C_0$*-semigroup* $T(\cdot)$) *on* $X$ *such that* $r(T) \leq 1$ *and* $\|A_r(T)\| \leq 1 - r$ *for all* $r \in (0,1)$ (*resp.* $\sigma = w_0 \leq 0$ *and* $\|A_\lambda\| \leq m \min\{1, \lambda\}$ *for all* $\lambda > 0$) *but* $C_n(T)/n^{m-2}$ (*resp.* $C_t/t^{m-2}$) *does not converge strongly to* $0$ *as* $n \to \infty$ (*resp.* $t \to \infty$), *so that* $\|C_n(T)\| \neq O(n^\alpha)$ (*resp.* $\|C_t\| \neq O(t^\alpha)$ $(t \to \infty)$) *for all* $\alpha \in [0, m-2)$.

*Proof.* We give counterexamples for the case $m = 3$. The case $m > 3$ can be shown similarly. Let $N$ be a nilpotent operator on $X$ of order 3, i.e., $N^3 = 0$ and $N^2 \neq 0$.

*The discrete case.* Let $T := -I + N$. We may choose $N$ in such a way that $\|N\| < 1$. Then $T$ is dissipative, so that $(I - rT)^{-1}$ is a contraction for every $0 \leq r < 1$ and hence $\|A_r(T)\| \leq 1 - r \leq 1$ for all $0 \leq r < 1$ (cf. e.g. [19, pp. 13–14]). One can also see this directly from the following estimates:

$$
\begin{aligned}
\|A_r(T)\| &= (1-r)\Big\|\sum_{n=0}^{\infty} r^n T^n\Big\| \\
&= (1-r)\Big\|\sum_{n=0}^{\infty} (-r)^n \Big(I - nN + \frac{n(n-1)}{2}\, N^2\Big)\Big\| \\
&= (1-r)\Big\|\frac{1}{1+r}\, I - \frac{r}{(1+r)^2}\, N + \frac{r^2}{(1+r)^3}\, N^2\Big\| \\
&\leq (1-r)\Big[\frac{1}{1+r} + \frac{r}{(1+r)^2} + \frac{r^2}{(1+r)^3}\Big] \\
&\leq (1-r)\, \frac{1 + 3r + 3r^2}{(1+r)^3} \leq 1 - r.
\end{aligned}
$$

On the other hand, using the identities

$$
\begin{aligned}
n(I-T)C_n(T) &= I - T^n = I - (-1)^n (I - N)^n \\
&= (1 - (-1)^n) I + (-1)^n nN - (-1)^n \frac{n(n-1)}{2}\, N^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\|C_n(T)\|/n &\geq \|I - T\|^{-1} \Big[\frac{n-1}{2n}\, \|N^2\| - \frac{2}{n^2} - \|N\|/n\Big] \\
&\to \frac{1}{2}\, \|I - T\|^{-1} \|N^2\| > 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

*The continuous case.* Let $A := iI + N$ and $T(t) := e^{tA} = e^{it} e^{tN} = e^{it}\big(I + tN + \frac{t^2}{2} N^2\big)$. Then

$$\begin{aligned}\|A_\lambda\| &= \Big\|\lambda\int_0^\infty e^{-\lambda t}e^{it}\Big(I+tN+\frac{t^2}{2}N^2\Big)\,dt\Big\| \\ &= \lambda\Big\|\frac{1}{\lambda-i}\,I+\frac{1}{(\lambda-i)^2}\,N+\frac{1}{(\lambda-i)^3}\,N^2\Big\| \\ &\le 3\min\{1,\lambda\} \quad \text{for all } \lambda>0\end{aligned}$$

and meanwhile

$$\begin{aligned}\|C_t\|/t &\ge (1+\|N\|)^{-1}\|AC_t\|/t = \frac{1}{t^2}(1+\|N\|)^{-1}\|T(t)-I\| \\ &\ge \frac{1}{t^2}\,(1+\|N\|)^{-1}\Big[\frac{t^2}{2}\,\|N^2\|-t\|N\|-2\Big] \\ &\to (1+\|N\|)^{-1}\,\frac{1}{2}\,\|N^2\|>0 \quad \text{as } t\to\infty.\end{aligned}$$

Hence the assertion is true. ∎

Propositions 2.3 and 2.8 imply the next corollary.

COROLLARY 2.9. *Suppose* $\dim X=\infty$. *The following hold.*

(i) *For any* $\alpha\ge 0$, *there exists an operator* $T$ (*resp. a* $C_0$*-semigroup* $T(\cdot)$) *on* $X$ *such that* $\|C_n(T)\|=O(n^\alpha)$ (*resp.* $\|C_t\|=O(t^\alpha)$ $(t\to\infty)$) *but* $\|T^n\|\ne O(n^\alpha)$ (*resp.* $\|T(t)\|\ne O(t^\alpha)$ $(t\to\infty)$), *and there exists an operator* $T$ (*resp. a* $C_0$*-semigroup* $T(\cdot)$) *on* $X$ *such that* $r(T)\le 1$ (*resp.* $w_0\le 0$) *but* $\|A_r\|\ne O((1-r)^{-\alpha})$ $(r\uparrow 1)$ (*resp.* $\|A_\lambda\|\ne O(\lambda^{-\alpha})$ $(\lambda\downarrow 0)$).

(ii) *For any* $m\ge 3$ *there exists an operator* $T$ (*resp. a* $C_0$*-semigroup* $T(\cdot)$) *on* $X$ *such that* $r(T)\le 1$ *and* $\|A_r(T)\|\le 1-r$ *for all* $r\in(0,1)$ (*resp.* $\sigma=w_0\le 0$ *and* $\|A_\lambda\|\le m\min\{1,\lambda\}$ *for all* $\lambda>0$) *but* $\|C_n(T)\|\ne O(n^\alpha)$ $(n\to\infty)$ (*resp.* $\|C_t\|\ne O(t^\alpha)$ $(t\to\infty)$) *for all* $\alpha\in[0,m-2)$. *In particular, Abel-mean-boundedness does not imply Cesàro-mean-boundedness.*

It follows from Corollary 2.2 and Propositions 2.5(ii) and 2.8 that the converse statements of (2.4) and (2.8) hold if and only if $\dim X\le 2$.

It follows from Propositions 2.1 and 2.3(i) that if $\dim X=m\ge 2$, there exists an operator $T$ (resp. $C_0$-semigroup $T(\cdot)$) on $X$ such that $r(T)\le 1$ (resp. $w_0\le 0$) and $\|T^n\|\ne O(n^\alpha)$ (resp. $\|T(t)\|\ne O(t^\alpha)$ $(t\to\infty)$) for all $\alpha\in[0,m-1)$. However, as the next proposition shows, they are of order $O(n^{m-1})$ and $O(t^{m-1})$, respectively.

PROPOSITION 2.10. *Suppose* $\dim X=m$ *with* $1\le m<\infty$.

(i) *If* $T\in B(X)$ *with* $r(T)\le 1$, *then* $\|T^n\|=O(n^{m-1})$.
(ii) *If* $T(\cdot)$ *is a* $C_0$*-semigroup on* $X$ *with* $w_0\le 0$, *then there is an* $M\ge 1$ *such that* $\|T(t)\|\le M(1+t^{m-1})$ *for all* $t\ge 0$.

*Proof.* This proposition follows from the proof of Theorem 1.3.2 and the estimate (1.3.11) of [2]. For completeness, we give a proof. The validity of the assertion for $\dim X = 1$ is obvious. The case $\dim X = 2$ has been verified in Proposition 2.5(i). Assume that the assertion holds for $m-1$. By an extension of Lemma 2.6 to dimension $m$, we may assume that $T$ is an $m\times m$ complex matrix in Jordan form, so that all entries below the diagonal are zeroes. Thus $T = \left[\begin{smallmatrix} A & B \\ 0 & c\end{smallmatrix}\right]$, where $A \in \mathbb{C}^{(m-1)\times(m-1)}$ is in Jordan form, $B \in \mathbb{C}^{(m-1)\times 1}$, $0 \in \mathbb{C}^{1\times(m-1)}$, and $c \in \mathbb{C}$. Then $r(A) \le 1$ and $|c| \le 1$. By the induction hypothesis on $m-1$, there is a constant $M > 0$ such that $\|A^n\| \le Mn^{m-2}$ for all $n \ge 1$. Thus for every $n = 1, 2, \ldots,$

$$T^n = \begin{bmatrix} A^n & \sum_{j=0}^{n-1} c^j A^{n-1-j}B \\ 0 & c^n \end{bmatrix}$$

and

$$\begin{aligned}\Big\|\sum_{j=0}^{n-1} c^j A^{n-1-j}B\Big\| &\le \sum_{j=0}^{n-1} \|A^{n-1-j}\|\cdot\|B\| \\ &\le \sum_{j=0}^{n-1} M(n-1-j)^{m-2}\|B\| \le Mn^{m-1}\|B\|.\end{aligned}$$

Therefore $\|T^n\| = O(n^{m-1})$.

(ii) Note that $r(T(t)) = e^{w_0 t}$ for all $t \ge 0$ (cf. [10, p. 251]), so that $w_0 \le 0$ is equivalent to $r(T(t)) \le 1$ for some (and all) $t > 0$. In particular, $r(T(1)) \le 1$. By (i), we have $\|T(n)\| = O(n^{m-1})$. Therefore there is a constant $M_1 \ge 1$ such that

$$\|T(n)\| \le M_1 n^{m-1} \quad \text{for all } n = 1, 2, \ldots.$$

Now, for any $t > 1$, set $n := [t]$. Then $n \le t < n+1$ and

$$\begin{aligned}\|T(t)\| \le \|T(t-n)\|\cdot\|T(n)\| &\le M_1 \sup_{0\le s\le 1}\|T(s)\|\cdot n^{m-1} \\ &\le M_1 \sup_{0\le s\le 1}\|T(s)\|\cdot t^{m-1}.\end{aligned}$$

Let $M := M_1 \sup_{0\le s\le 1}\|T(s)\|$ ($\ge M_1 \ge 1$). Then $\|T(t)\| \le M(1+t^{m-1})$ for all $t \ge 0$. ∎

Remark. The number $m-1$ in Proposition 2.10 is sharp. This is explained by Proposition 2.3(ii) (together with Proposition 2.1).

**3. Growth orders of means for positive operators in Banach lattices.** In this section, we show that the conclusion of Proposition 2.5(ii) also holds for positive operators on Banach lattices and for all $\alpha > -1$. We begin with the following equivalence theorem for positive vector-valued sequences and functions.

PROPOSITION 3.1. *Assume that $X$ is a Banach lattice. Then the following hold*:

(i) *Let $\{x_n\}$ be a sequence of positive elements of $X$ such that $\sum_{k=0}^{\infty} r^k x_k$ exists for all $r \in (0,1)$. Then, for every $\gamma > 0$, $\{(1-r)^\gamma \sum_{k=0}^{\infty} r^k x_k;\ 0 < r < 1\}$ is bounded if and only if $\{n^{-\gamma} \sum_{k=0}^{n-1} x_k;\ n \ge 1\}$ is bounded.*

(ii) *Let $x : (0,\infty) \to X$ be a strongly measurable positive function such that the integral $\int_0^\infty e^{-\lambda t} x(t)\, dt$ exists for all $\lambda > 0$. Then, for every $\gamma > 0$, $\{\lambda^\gamma \int_0^\infty e^{-\lambda t} x(t)\, dt;\ \lambda > 0\}$ is bounded if and only if $\{t^{-\gamma} \int_0^t x(s)\, ds;\ t > 0\}$ is bounded.*

*Proof.* (i) Suppose $n^{-\gamma} \| \sum_{k=0}^{n-1} x_k \| \le M$ for all $n \ge 1$. Then, letting $t = -\ln r$ $(0 < r < 1)$, we have

$$\begin{aligned}
\Big\|(1-r)^\gamma \sum_{n=0}^{\infty} r^n x_n\Big\| &= (1-r)^\gamma \Big\|(1-r) \sum_{n=0}^{\infty} r^n \Big(\sum_{k=0}^{n} x_k\Big)\Big\| \\
&\le (1-r)^{\gamma+1} \sum_{n=0}^{\infty} r^n M (n+1)^\gamma \\
&= (1-r)^{\gamma+1} (M/r) \sum_{n=1}^{\infty} r^n n^\gamma \\
&= (1-r)^{\gamma+1} (M/r) \sum_{n=1}^{\infty} e^{-tn} (tn)^\gamma t \, \frac{1}{t^{\gamma+1}} \\
&= \Big(\frac{1-r}{-\ln r}\Big)^{\gamma+1} (M/r) \sum_{n=1}^{\infty} e^{-tn} (tn)^\gamma t \\
&\to M \int_0^\infty e^{-x} x^\gamma \, dx = M\Gamma(\gamma+1) \quad \text{as } r \uparrow 1.
\end{aligned}$$

It follows that $\{(1-r)^\gamma \sum_{k=0}^{\infty} r^k x_k;\ 0 < r < 1\}$ is bounded.

To show the necessity, suppose $\|(1-r)^\gamma \sum_{k=0}^{\infty} r^k x_k\| \le M$ for all $0 < r < 1$. Then

$$M \ge (1-r)^\gamma \sum_{k=0}^{n-1} r^k x_k \ge (1-r)^\gamma r^{n-1} \sum_{k=0}^{n-1} x_k$$

for all $r \in (0,1)$ and $n \ge 1$. If we take $r = 1 - 1/n$, then we obtain

$$n^{-\gamma} \sum_{k=0}^{n-1} x_k \le M \Big(1 - \frac{1}{n}\Big)^{-(n-1)} = M \Big(1 + \frac{1}{n-1}\Big)^{n-1} \le Me$$

for all $n \ge 2$.

(ii) Suppose $t^{-\gamma} \| \int_0^t x(s)\, ds \| \le M$ for all $t > 0$. Then, using integration by parts, for all $\lambda > 0$ we have

$$
\begin{aligned}
\Big\|\lambda^{\gamma}\int_0^{\infty} e^{-\lambda t}x(t)\,dt\Big\| &= \lambda^{\gamma+1}\Big\|\int_0^{\infty} e^{-\lambda t}\int_0^{t} x(s)\,ds\,dt\Big\| \\
&= \lambda^{\gamma+1}\Big\|\int_0^{\infty} e^{-\lambda t}t^{\gamma}\Big(\frac{1}{t^{\gamma}}\int_0^{t} x(s)\,ds\Big)dt\Big\| \\
&\le \lambda^{\gamma+1}M\int_0^{\infty} e^{-\lambda t}t^{\gamma}\,dt = M\Gamma(\gamma+1).
\end{aligned}
$$

It follows that $\{\lambda^{\gamma}\int_0^{\infty} e^{-\lambda t}x(t)\,dt;\ \lambda>0\}$ is bounded.

For the converse implication, we have

$$
M \ge \lambda^{\gamma}\int_0^{\infty} e^{-\lambda t}x(t)\,dt \ge \lambda^{\gamma}\int_0^{t} e^{-\lambda s}x(s)\,ds \ge \lambda^{\gamma}e^{-\lambda t}\int_0^{t} x(s)\,ds
$$

for all $\lambda, t>0$. If we take $\lambda = 1/t$, then we have $t^{-\gamma}\int_0^t x(s)\,ds \le Me$ for all $t>0$. ■

The following corollary is seen immediately from Proposition 3.1 with (1.1) and (2.10).

COROLLARY 3.2. *Let $T$ be a positive operator* (*resp. a positive $C_0$-semigroup $T(\cdot)$*) *on a Banach lattice $X$ such that $r(T)\le 1$* (*resp. $w_0\le 0$*). *For any $\alpha>-1$, $\{(1-r)^{\alpha}A_r(T);\ 0<r<1\}$* (*resp. $\{\lambda^{\alpha}A_{\lambda};\ \lambda>0\}$*) *is bounded if and only if $\{C_n(T)/n^{\alpha};\ n\ge 1\}$* (*resp. $\{C_t/t^{\alpha};\ t>0\}$*) *is bounded. In particular, $T$* (*resp. $T(\cdot)$*) *is Abel-mean-bounded if and only if it is $\gamma$-Cesàro-mean-bounded for any $\gamma\ge 1$.*

REMARKS. (1) However, Cesàro-mean-boundedness does not imply power-boundedness (see the examples constructed in [18, Chapter 3.3]); for examples of non-power-bounded positive Cesàro-mean-bounded operators, see [7, p. 255] and [9, p. 14]); the growth of powers of positive Cesàro-mean-bounded operators on $L^1$ was studied in [14]. On the other hand, (unbounded) positive Cesàro-mean-bounded $C_0$-semigroups were treated in [20]; see also [24, p. 254] for an example of a group $T(\cdot)$ of positive operators on $L^2(\mathbb{R})$ which is unbounded but satisfies

$$
\sup\Big\{(2t)^{-1}\int_{-t}^{t}\|T(s)f\|_2^2\,ds;\ t>0\Big\}<\infty
$$

for all $f\in L^2(\mathbb{R})$, and hence is obviously Cesàro-mean-bounded; (unbounded) Cesàro-mean-bounded $C_0$-groups can also be found in [16].

(2) When the Banach lattice $X$ is reflexive, an Abel-mean-bounded positive operator on $X$ is even Cesàro-mean-ergodic (cf. [9, Theorem 4.2], [15, Proposition 6.2(iii)]). From this or Corollary 3.2, together with Remark (1)

after Corollary 2.4, one can infer that every Abel-mean-bounded positive $n \times n$ matrix has to be power-bounded.

(3) $r(T) \le 1$ does not imply Abel-mean-boundedness and Cesàro-mean-boundedness either. A simple example is the matrix $\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$ with $a > 0$ (see the proof of Proposition 2.3(ii)). The following is an infinite-dimensional example. Let $\mathbb{Z}$ be the integers and define a measure $\mu$ on $\mathbb{Z}$ by $\mu(\{k\}) = 1$ for $k \le -1$, and $\mu(\{k\}) = k+1$ for $k \ge 0$. Let $T : L_1(\mathbb{Z}, \mu) \to L_1(\mathbb{Z}, \mu)$ be such that $Tf(k) = f(k-1)$. Then $\|T^n\| = n+1$ and thus $r(T) = 1$, but

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k \right\| = \frac{n+1}{2} \qquad (n \ge 1).$$

(4) If the positivity of $T$ is not assumed then for every $\gamma \ge 0$ there exists an example of $T$ such that $T$ is Abel-mean-bounded but not Cesàro-mean-bounded of order $\gamma$, i.e., $\sup_{n\ge 0} \|C_n^\gamma(T)\| = \infty$. This will be seen in Proposition 4.4(ii).

## 4. Boundedness of $\gamma$-Cesàro means for $\gamma > -1$

PROPOSITION 4.1. *There is an invertible positive linear isometry $T$ on an $L_1$-space such that*

$$\sup_{n\ge 0} \|C_n^\gamma(T)\| = \infty \qquad \textit{for all } -1 < \gamma < 0.$$

*Proof.* Let $-1 < \gamma < 0$. In this case the Cesàro means

$$C_n^\gamma(T) = \frac{1}{\sigma_n^\gamma} \sum_{k=0}^{n} \sigma_{n-k}^{\gamma-1} T^k$$

have the properties

$$\sigma_n^{\gamma-1} = \frac{\gamma(\gamma+1)\cdots(\gamma-1+n)}{n!} < 0 \qquad (n \ge 1),$$

and

$$\sigma_n^\gamma = \sum_{k=0}^{n} \sigma_k^{\gamma-1} = 1 + \sum_{k=1}^{n} \sigma_k^{\gamma-1} \qquad (n \ge 1),$$

$$\sigma_n^\gamma = \frac{(\gamma+1)\cdots(\gamma+n)}{n!} \downarrow 0 \qquad (n \to \infty),$$

where the last property comes from the fact that $\lim_{n\to\infty} n^\gamma/\sigma_n^\gamma = \Gamma(\gamma+1)$ (cf. [25, Chapter 3]). It follows that

$$\lim_{n\to\infty} \frac{1}{\sigma_n^\gamma} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \sigma_k^{\gamma-1}}{\sigma_n^\gamma} = -\infty.$$

Let $\mu$ be the counting measure on $\mathbb{Z}$, and consider the invertible positive linear isometry $T$ on $L_1(\mathbb{Z}, \mu)$ defined by $Tf(m) := f(m-1)$ $(m \in \mathbb{Z})$. If

$f = \chi_{\{0\}}$, then, since $T^k f = \chi_{\{k\}}$ for $k \geq 0$, we have

$$\|C_n^\gamma(T)f\|_1 = \frac{1}{\sigma_n^\gamma}\Big\|\chi_{\{n\}} + \sum_{k=0}^{n-1}\sigma_{n-k}^{\gamma-1}\chi_{\{k\}}\Big\|_1,$$

so that

$$\|C_n^\gamma(T)f\|_1 = \frac{1}{\sigma_n^\gamma}\Big(1 - \sum_{k=0}^{n-1}\sigma_{n-k}^{\gamma-1}\Big) \uparrow \infty \quad (n \to \infty). \ \blacksquare$$

The following elementary lemma will be used in the proof of Proposition 4.3. (Since it is proved in Lemma 4.3(a) of [5], we only quote it here.)

LEMMA 4.2. *Let $\xi_i, \eta_i > 0$, $\sum_{i=1}^n \xi_i = 1 = \sum_{i=1}^n \eta_i$, and $\xi_i/\xi_{i+1} \geq \eta_i/\eta_{i+1}$ $(1 \leq i \leq n-1)$. Then $\lambda_1 \geq \cdots \geq \lambda_n$ implies*

$$\sum_{i=1}^n \lambda_i \xi_i \geq \sum_{i=1}^n \lambda_i \eta_i.$$

PROPOSITION 4.3. *For any $\gamma$ with $0 < \gamma < 1$, there exists a positive linear operator $T$ on an $L_1$-space such that*

$$\sup_{n\geq 0}\|C_n^\gamma(T)\| = \infty, \quad \textit{but} \quad \sup_{n\geq 0}\|C_n^\beta(T)\| < \infty \quad \textit{for all } \beta > \gamma.$$

*Proof.* Let $a := a(\gamma) = 2^{1/\gamma}$. It follows that $a > 2$. Next, for $j \geq 1$, let

$$(4.1)\quad X_j := [0, a^j) \quad \text{and} \quad w_j(s) := \begin{cases} 2^j, & 0 \leq s < a^0 = 1, \\ 2^{j-k}, & a^{k-1} \leq s < a^k,\ 1 \leq k \leq j. \end{cases}$$

Let $\vartheta_j : X_j \to X_j$ be the point transformation defined by $\vartheta_j(s) := s + 1$ (mod $a^j$), and put $T_j f(s) := f(\vartheta_j(s))$ for $f$ on $X_j$. Define a measure $\mu_j$ on $X_j$ by $\mu_j := w_j(s)ds$.

Let

$$(4.2)\qquad X := \bigcup_{j=1}^\infty X_j \quad \text{(regarded as disjoint union)},$$

and let $\mu$ be the measure on $X$ defined by $\mu|_{X_j} := \mu_j$ for each $j \geq 1$. Define an operator $T : L_1(X,\mu) \to L_1(X,\mu)$ by $(Tf)|_{X_j} := T_j(f|_{X_j})$ for $j \geq 1$. Thus $T$ is a positive linear operator on $L_1(X,\mu)$, with $\|T\| = 2$.

We will prove that $\sup_{n\geq 0}\|C_n^\gamma(T)\| = \infty$, and that $\sup_{n\geq 0}\|C_n^\beta(T)\| < \infty$ for all $\beta > \gamma$. To do this, let $0 < \alpha < 1$. We consider the Cesàro means $C_n^\alpha(T)$ of order $\alpha$, and define

$$(4.3)\qquad \alpha(X_j) := \sup\{\|C_n^\alpha(T)f\|_1/\|f\|_1;\ 0 \neq f \in L_1(X_j,\mu_j),\ n \geq 0\}.$$

It suffices to show that $\sup_{j\geq 1}\alpha(X_j) < \infty$ for $\alpha = \beta > \gamma$, and $\lim_{j\to\infty}\alpha(X_j) = \infty$ for $0 < \alpha \leq \gamma$. To estimate $\alpha(X_j)$, we need to estimate $\|C_K^\alpha(T)f\|_1$ for $f \in L_1(X_j,\mu_j)$ and $K \geq 0$. Let $f_d := \delta^{-1}2^{-(j-k)}\chi_{[d-\delta,\,d)}$ with $1 \leq k \leq j$,

$d \in (a^{k-1}, a^k]$ and $0 < \delta < 1$ such that $1 \le a^{k-1} \le d - \delta < d \le a^k$. It follows that $\|f_d\|_1 = 1$. Since $C_n^\alpha(T)$, $n \ge 0$, are positive operators, and the set of all convex combinations of these $f_\delta$-functions is dense in the set $\{f \in L_1(X_j);\ f \ge 0, \|f\|_1 = 1\}$, it suffices to estimate $\|C_K^\alpha(T) f_d\|_1$ for all these $f_d$. Letting

$$N(d) := \begin{cases} d-1 & \text{if } d \text{ is an integer,} \\ [d] & \text{if } d \text{ is not an integer,} \end{cases} \tag{4.4}$$

where $[d]$ denotes the largest integer not exceeding $d$, we easily see that:

$$T^{N(d)} f_d = \delta^{-1} 2^{-(j-k)} \chi_{[d-\delta-N(d), d-N(d))} \pmod{a^j}, \tag{4.5}$$

$$[d - \delta - N(d), d - N(d)) \cap [0,1) \ne \emptyset, \tag{4.6}$$

$$T^{N(d)+1} f_d = \delta^{-1} 2^{-(j-k)} \chi_{[a^j + d - \delta - N(d) - 1, a^j + d - N(d) - 1)}, \tag{4.7}$$

$$[a^j + d - \delta - N(d) - 1, a^j + d - N(d) - 1) \subset [a^j - 2, a^j). \tag{4.8}$$

Hence if $0 \le i \le N(d)$, then $2^{r+1} \ge \|T^i f_d\|_1 > 2^r$ for some integer $0 \le r \le k-1$; and furthermore $2^{r+1} \ge \|T^i f_d\|_1 > 2^r$ is equivalent to $a^{k-r-2} \le d - i - \delta < a^{k-r-1}$, where we let $a^{-1} := 0$ for convenience.

(i) First we estimate $\|C_{N(d)}^\alpha(T) f_d\|_1$. To do this, we introduce another function $g_d$ corresponding to $f_d$ by

$$g_d := 2^{-(j-k)} \chi_{[d-1,d)}. \tag{4.9}$$

It is clear that $\|f_d\|_1 = 1 \le \|g_d\|_1 \le 2$. Furthermore, since

$$0 \le d - l - (1-s) < d - l - \delta(1-s) < d - l \le a^j$$

for $0 \le s < 1$, and the function $w_j$ is nonincreasing on $[0, a^j)$, clearly

$$\begin{aligned} \delta^{-1} \int_{d-l-\delta}^{d-l} w_j(s)\, ds &= \int_0^1 w_j(d - l - \delta(1-s))\, ds \le \int_0^1 w_j(d - l - (1-s))\, ds \\ &= \int_{d-l-1}^{d-l} w_j(s)\, ds. \end{aligned}$$

Hence, with $w_j = 2^j$ on $[0,1]$, $\mu = w_j(s) ds$, it follows that

$$\|T^l f_d\|_1 = 2^{-(j-k)} \delta^{-1} \int_{d-l-\delta}^{d-l} w_j(s)\, ds \le 2^{-(j-k)} \int_{d-l-1}^{d-l} w_j(s)\, ds = \|T^l g_d\|_1$$

for $0 \le l \le N(d) - 1$, and that $\|T^{N(d)} f_d\|_1 \le 2^k$ and $\|T^{N(d)} g_d\|_1 \le 2^k$.

Therefore,

$$\begin{aligned}(4.10)\qquad \|C^\alpha_{N(d)}(T)f_d\|_1 &\le \frac{1}{\sigma^\alpha_{N(d)}}\sum_{l=0}^{N(d)}\sigma^{\alpha-1}_{N(d)-l}\|T^l f_d\|_1\\ &\le \frac{1}{\sigma^\alpha_{N(d)}}\Big(\sum_{l=0}^{N(d)-1}\sigma^{\alpha-1}_{N(d)-l}\|T^l g_d\|_1 + 2^k\Big)\\ &= \frac{1}{\sigma^\alpha_{N(d)}}\Big(\sum_{l=0}^{N(d)-1}\sigma^{\alpha-1}_{N(d)-l}\mu_j([d-l-1,d-l))\,\frac{1}{2^{j-k}}\Big) + \frac{1}{\sigma^\alpha_{N(d)}}\,2^k\\ &=: I(N(d)) + II(N(d)).\end{aligned}$$

Since $\lim_{n\to\infty} n^\tau/\sigma^\tau_n = \Gamma(\tau+1)$ for $\tau > -1$, it follows that

$$(4.11)\qquad \sigma^\alpha_n \sim n^\alpha \quad (n \ge 1),$$

where $a(n) \sim b(n)$ $(n \ge 1)$ means that $\sup_{n\ge1}|a(n)/b(n)| < \infty$ and $\sup_{n\ge1}|b(n)/a(n)| < \infty$. Using this and the facts that $a^{k-1} < d \le a^k$ and $N(d) < d \le 2N(d)$, we find

$$\begin{aligned}(4.12)\qquad II(N(d)) = \frac{1}{\sigma^\alpha_{N(d)}}\,2^k \sim \frac{2^k}{N(d)^\alpha} &\le 2^\alpha\cdot\frac{2^k}{d^\alpha}\\ &\le 2^{\alpha+1}\Big(\frac{2}{a^\alpha}\Big)^{k-1} \quad (1 \le k \le j).\end{aligned}$$

Next, we estimate $I(N(d))$. To do this we prepare some elementary inequalities. First, using $0 < d - N(d) \le 1$, we easily see that

$$(4.13)\qquad 1 < \frac{d-l}{N(d)-l} \le 2 \quad (0 \le l \le N(d)-1),$$

and that

$$(4.14)\qquad 1 \le \frac{d-l}{s} \le \frac{d-l}{d-l-1} < 2 \quad \text{for } s \in [d-l-1, d-l] \quad (0 \le l \le N(d)-2).$$

If $l = N(d)-1$, then, using $2^j \ge w_j(s) \ge 2^{j-1}$ on $[d-N(d), d-N(d)+1] \subset (0,2]$, we see that

$$2^{j-1} \le \frac{\int_{d-N(d)}^{d-N(d)+1}(d-N(d)+1)^{\alpha-1}w_j(s)\,ds}{(d-N(d)+1)^{\alpha-1}} \le 2^j,$$

and that

$$\frac{2^{j-1}}{\alpha} \le \frac{\int_{d-N(d)}^{d-N(d)+1}s^{\alpha-1}w_j(s)\,ds}{(d-N(d)+1)^\alpha-(d-N(d))^\alpha} \le \frac{2^j}{\alpha}.$$

Hence

$$\text{(4.15)}\qquad \frac{\alpha}{2}\cdot\frac{(d-N(d)+1)^{\alpha-1}}{(d-N(d)+1)^{\alpha}-(d-N(d))^{\alpha}} \le \frac{\int_{d-N(d)}^{d-N(d)+1}(d-N(d)+1)^{\alpha-1}w_j(s)\,ds}{\int_{d-N(d)}^{d-N(d)+1}s^{\alpha-1}w_j(s)\,ds} \le 2\alpha\cdot\frac{(d-N(d)+1)^{\alpha-1}}{(d-N(d)+1)^{\alpha}-(d-N(d))^{\alpha}}.$$

Furthermore, we notice that, since $0<\alpha<1$ and $1<d-N(d)+1\le 2$, it follows that

$$2^{\alpha-1}\le (d-N(d)+1)^{\alpha-1}\le 1.$$

Also, the function $f(t):=t^{\alpha}-(t-1)^{\alpha}$ is decreasing on $[1,\infty)$, so that

$$2^{\alpha}-1=f(2)\le (d-N(d)+1)^{\alpha}-(d-N(d))^{\alpha}<f(1)=1.$$

Thus,

$$2^{\alpha-1}<\frac{(d-N(d)+1)^{\alpha-1}}{(d-N(d)+1)^{\alpha}-(d-N(d))^{\alpha}}\le\frac{1}{2^{\alpha}-1}.$$

From this and (4.15) we obtain

$$\text{(4.16)}\qquad \frac{\alpha}{2}\cdot 2^{\alpha-1}\le\frac{\int_{d-N(d)}^{d-N(d)+1}(d-N(d)+1)^{\alpha-1}w_j(s)\,ds}{\int_{d-N(d)}^{d-N(d)+1}s^{\alpha-1}w_j(s)\,ds}\le\frac{2\alpha}{2^{\alpha}-1}.$$

Therefore,

$$\begin{aligned}
\text{(4.17)}\qquad I(N(d)) &= \frac{1}{\sigma_{N(d)}^{\alpha}}\sum_{l=0}^{N(d)-1}\sigma_{N(d)-l}^{\alpha-1}\mu_j([d-l-1,d-l))\,\frac{1}{2^{j-k}}\\
&= \frac{2^{-(j-k)}}{\sigma_{N(d)}^{\alpha}}\sum_{l=0}^{N(d)-1}\int_{d-l-1}^{d-l}\sigma_{N(d)-l}^{\alpha-1}w_j(s)\,ds\\
&\sim \frac{2^{-(j-k)}}{\sigma_{N(d)}^{\alpha}}\sum_{l=0}^{N(d)-1}\int_{d-l-1}^{d-l}(N(d)-l)^{\alpha-1}w_j(s)\,ds \qquad \text{(by (4.11))}\\
&\sim \frac{2^{-(j-k)}}{\sigma_{N(d)}^{\alpha}}\sum_{l=0}^{N(d)-1}\int_{d-l-1}^{d-l}(d-l)^{\alpha-1}w_j(s)\,ds \qquad \text{(by (4.13))}\\
&\sim \frac{2^{-(j-k)}}{\sigma_{N(d)}^{\alpha}}\sum_{l=0}^{N(d)-1}\int_{d-l-1}^{d-l}s^{\alpha-1}w_j(s)\,ds \qquad \text{(by (4.14), (4.16))}
\end{aligned}$$

$$= \frac{2^{-(j-k)}}{\sigma^{\alpha}_{N(d)}} \int\limits_{d-N(d)}^{d} s^{\alpha-1} w_j(s)\,ds$$

$$= \frac{2^{-(j-k)}}{\sigma^{\alpha}_{N(d)}} \Big( \int\limits_{a^{k-1}}^{d} + \int\limits_{1}^{a^{k-1}} + \int\limits_{d-N(d)}^{1} \Big) s^{\alpha-1} w_j(s)\,ds$$

$$=: I(N(d);1) + I(N(d);2) + I(N(d);3).$$

Since $w_j(s) = 2^{j-k}$ on the interval $[a^{k-1}, d)$ ($\subset [a^{k-1}, a^k)$) and $2N(d) \geq d > 1$, it follows that

$$I(N(d);1) = \frac{1}{\sigma^{\alpha}_{N(d)}} \int\limits_{a^{k-1}}^{d} s^{\alpha-1}\,ds \sim \frac{1}{N(d)^{\alpha}} \int\limits_{a^{k-1}}^{d} s^{\alpha-1}\,ds \tag{4.18}$$

$$= \frac{1}{N(d)^{\alpha}} \cdot \frac{d^{\alpha} - a^{(k-1)\alpha}}{\alpha} \leq \frac{1}{N(d)^{\alpha}} \cdot \frac{a^{k\alpha} - a^{(k-1)\alpha}}{\alpha}$$

$$\leq \frac{2^{\alpha}}{d^{\alpha}} \cdot \frac{(a^{\alpha}-1)a^{(k-1)\alpha}}{\alpha} \leq \frac{2^{\alpha}}{a^{(k-1)\alpha}} \cdot \frac{(a^{\alpha}-1)a^{(k-1)\alpha}}{\alpha} = 2^{\alpha} \cdot \frac{a^{\alpha}-1}{\alpha}.$$

Similarly,

$$I(N(d);3) = \frac{2^{-(j-k)}}{\sigma^{\alpha}_{N(d)}} \int\limits_{d-N(d)}^{1} s^{\alpha-1} w_j(s)\,ds \tag{4.19}$$

$$\sim \frac{2^k}{N(d)^{\alpha}} \int\limits_{d-N(d)}^{1} s^{\alpha-1}\,ds \leq \frac{2^k}{N(d)^{\alpha}} \int\limits_{0}^{1} s^{\alpha-1}\,ds$$

$$= \frac{2^k}{N(d)^{\alpha}} \cdot \frac{1}{\alpha} \leq \frac{2^{\alpha}}{a^{(k-1)\alpha}} \cdot \frac{2^k}{\alpha} = \frac{2^{\alpha+1}}{\alpha} \Big(\frac{2}{a^{\alpha}}\Big)^{k-1}$$

$$= O\Big(\Big(\frac{2}{a^{\alpha}}\Big)^{k-1}\Big) = O\Big(\Big(\frac{2}{a^{\alpha}}\Big)^{j-1}\Big).$$

Lastly, we find

$$I(N(d);2) = \frac{2^{-(j-k)}}{\sigma^{\alpha}_{N(d)}} \int\limits_{1}^{a^{k-1}} s^{\alpha-1} w_j(s)\,ds \sim \frac{2^{-(j-k)}}{N(d)^{\alpha}} \int\limits_{1}^{a^{k-1}} s^{\alpha-1} w_j(s)\,ds \tag{4.20}$$

$$= \frac{2^{-(j-k)}}{N(d)^{\alpha}} \cdot \frac{1}{\alpha} \{ (a^{(k-1)\alpha} - a^{(k-2)\alpha}) \cdot 2^{j-k+1}$$

$$+ (a^{(k-2)\alpha} - a^{(k-3)\alpha}) \cdot 2^{j-k+2} + \cdots + (a^{\alpha}-1) 2^{j-1} \}$$

$$= \frac{1}{N(d)^{\alpha}} \cdot \frac{a^{\alpha}-1}{\alpha} \{ a^{(k-2)\alpha} \cdot 2 + a^{(k-3)\alpha} \cdot 2^2 + \cdots + a^0 \cdot 2^{k-1} \}$$

$$=: I(N(d);2)^*.$$

Since $2N(d) \geq d \geq a^{k-1}$, we have

$$(4.21)\qquad I(N(d);2)^* \\ \leq \frac{2^\alpha}{a^{(k-1)\alpha}} \cdot \frac{a^\alpha - 1}{\alpha} \{a^{(k-2)\alpha} \cdot 2 + a^{(k-3)\alpha} \cdot 2^2 + \cdots + a^0 \cdot 2^{k-1}\} \\ = \frac{a^\alpha - 1}{\alpha} \cdot 2^\alpha \left\{ \left(\frac{2}{a^\alpha}\right) + \left(\frac{2}{a^\alpha}\right)^2 + \cdots + \left(\frac{2}{a^\alpha}\right)^{k-1} \right\}.$$

We also notice that, since $N(d) < d \leq a^k$, the reverse inequality holds:

$$(4.22)\qquad I(N(d);2)^* \\ \geq \frac{1}{a^{k\alpha}} \cdot \frac{a^\alpha - 1}{\alpha} \{a^{(k-2)\alpha} \cdot 2 + 2^{(k-3)\alpha} \cdot 2^2 + \cdots + a^0 \cdot 2^{k-1}\} \\ = \frac{a^\alpha - 1}{\alpha} \cdot \frac{1}{a^\alpha} \left\{ \left(\frac{2}{a^\alpha}\right) + \left(\frac{2}{a^\alpha}\right)^2 + \cdots + \left(\frac{2}{a^\alpha}\right)^{k-1} \right\}.$$

It follows from (4.20)–(4.22) that

$$(4.23)\qquad I(N(d);2) = O\left(\sum_{i=1}^{k-1} \left(\frac{2}{a^\alpha}\right)^i\right),$$

and it follows from (4.12), (4.17)–(4.19), and (4.22) that

$$(4.24)\qquad \|C^\alpha_{N(d)}(T)f_d\|_1 \\ = O\left(\left(\frac{2}{a^\alpha}\right)^{k-1}\right) + O(1) + O\left(\left(\frac{2}{a^\alpha}\right)^{k-1}\right) + O\left(\sum_{i=1}^{k-1} \left(\frac{2}{a^\alpha}\right)^i\right) \\ = O\left(\sum_{i=1}^{j-1} \left(\frac{2}{a^\alpha}\right)^i\right).$$

(ii) We next estimate $\|C^\alpha_K(T)f_d\|_1$ for $0 \leq K \leq N(d)$. Since $\|f_d\|_1 = 1$ and $\|Tf_d\|_1 \leq 2$, it is clear that $\|C^\alpha_0(T)f_d\|_1 = \|f_d\|_1 = 1$ and $\|C^\alpha_1(T)f_d\| \leq 2$. We then consider the case $2 \leq K \leq N(d)$. Let $l$ be an integer, with $1 \leq l \leq k$, such that $a^{l-1} < K \leq a^l$, and let $0 < \delta' < 1$ be such that $a^{l-1} \leq K - \delta' < K \leq a^l$. Define the function $f'_K$ by

$$(4.25)\qquad f'_K := \delta'^{-1} 2^{-(j-l)} \chi_{[K-\delta', K)}.$$

Then $\|f'_K\|_1 = 1$ and $T^i f'_K = \delta'^{-1} 2^{-(j-l)} \chi_{[K-\delta'-i,\, K-i)}$ for $0 \leq i \leq K-1 = N(K)$.

If $0 \leq i \leq K-1 = N(K)$, then there is an integer $0 \leq r \leq k-1$ such that $2^{r+1} \geq \|T^i f_d\|_1 > 2^r$, which is equivalent to $a^{k-r-2} \leq d - i - \delta < a^{k-r-1}$, where, as before, we let $a^{-1} := 0$. We will only discuss the case $r = 1$; the arguments for $r \geq 2$ are similar. So, assume that $4 \geq \|T^i f_d\|_1 > 2$. Then we must have $a^{k-3} \leq d - \delta - i < a^{k-2}$, and thus $a^{k-1} - i \leq d - \delta - i < a^{k-2}$;

consequently,

$$a^{k-1} - a^{k-2} < i.$$

Since $1 \le l \le k$, either $1 \le l \le k-1$ or $l = k$. If $1 \le l \le k-1$ (so that $a^{l-1} < K \le a^l < d \le a^k$, by the previous assumptions on $l$ and $k$), then

$$K - i \le a^l - i < a^l - (a^{k-1} - a^{k-2}) \le a^l - (a^l - a^{l-1}) = a^{l-1},$$

from which $\|T^i f'_K\|_1 \ge 2$ follows at once. Thus $\|T^i f_d\|_1 \le 4 \le 2\|T^i f'_K\|_1$. On the other hand, if $l = k$, then $f'_K = \delta'^{-1} 2^{-(j-k)} \chi_{[K-\delta', K)}$ and $a^{k-1} = a^{l-1} < K \le N(d) < d \le a^k$. From these and the fact that $w_j$ is nonincreasing, it is clear that $\|T^i f'_K\|_1 \ge \|T^i f_d\|_1$ for all $0 \le i \le N(K)$. Hence we always have

$$\|T^i f_d\|_1 \le 2\|T^i f'_K\|_1 \quad (0 \le i \le N(K)). \tag{4.26}$$

Using (4.26), we can apply the estimation done in (4.10) for $f_d$ to the function $f'_K$ to obtain

$$\|C^\alpha_K(T) f_d\|_1 \le \frac{2}{\sigma^\alpha_{N(K)}} \sum_{i=0}^{N(K)} \sigma^{\alpha-1}_{N(K)-i} \|T^i f'_K\|_1 = O\bigg(\sum_{i=1}^{j-1} \Big(\frac{2}{a^\alpha}\Big)^i\bigg). \tag{4.27}$$

(iii) Lastly, we estimate $\|C^\alpha_K(T) f_d\|_1$ for $K > N(d)$. To do this we divide the interval $\{0, 1, \dots, K\}$ of integers into several intervals. We can write

$$\{0, 1, \dots, K\} = \bigcup_{n=0}^{j(d,K)} I_n, \tag{4.28}$$

where $I_n = \{a_n, a_n+1, \dots, a_n+N_n\}$, $0 \le n \le j(d,K)$, are disjoint intervals of integers, with $a_0 = 0$ and $a_{n+1} = a_n + N_n + 1 \ge a_n + 1$ for $0 \le n \le j(d,K) - 1$, such that

$$a^j - 1 < d - a_n + na^j < a^j \quad \text{for } 1 \le n \le j(d,K), \tag{4.29}$$

$$\begin{cases} 0 < d - (a_n + N_n) + na^j \le 1 & \text{for } 0 \le n \le j(d,K) - 1, \\ 0 < d - (a_{j(d,K)} + N_{j(d,K)}) + j(d,K) a^j < a^j & \text{for } n = j(d,K). \end{cases} \tag{4.30}$$

We note that these mean the following:

$$\begin{cases} N_n = N(d - a_n + na^j) & \text{for } 0 \le n \le j(d,K) - 1, \\ 0 \le N_{j(d,K)} \le N(d - a_{j(d,K)} + j(d,K) a^j) & \text{for } n = j(d,K), \end{cases} \tag{4.31}$$

where $N(\cdot)$ is defined in (4.4).

Then we have

$$\begin{aligned} \|C^\alpha_K(T) f_d\|_1 &= \frac{1}{\sigma^\alpha_K} \sum_{i=0}^{K} \sigma^{\alpha-1}_{K-i} \|T^i f_d\|_1 \\ &= \frac{1}{\sigma^\alpha_K} \sum_{n=0}^{j(d,K)} \left( \sum_{i=a_n}^{a_n+N_n} \sigma^{\alpha-1}_{K-i} \cdot \frac{\sum_{i=a_n}^{a_n+N_n} \sigma^{\alpha-1}_{K-i} \|T^i f_d\|_1}{\sum_{i=a_n}^{a_n+N_n} \sigma^{\alpha-1}_{K-i}} \right); \end{aligned} \tag{4.32}$$

and using Lemma 4.2 we prove below the inequality

$$\frac{\sum_{i=a_n}^{a_n+N_n} \sigma_{K-i}^{\alpha-1}\|T^i f_d\|_1}{\sum_{i=a_n}^{a_n+N_n} \sigma_{K-i}^{\alpha-1}} \le \frac{\sum_{i=0}^{N_n-1} \sigma_{N_n-i}^{\alpha-1}\|T^i(T^{a_n} f_d)\|_1 + 2^k}{\sigma_{N_n}^{\alpha}} \tag{4.33}$$

for $0 \le n \le j(d,K) - 1$.

To do this, for $0 \le n \le j(d,K) - 1$ let

$$\xi_i := \frac{\sigma_i^{\alpha-1}}{\sigma_{N_n}^{\alpha}}, \qquad \eta_i := \frac{\sigma_{K-a_n-N_n+i}^{\alpha-1}}{\sum_{p=0}^{N_n} \sigma_{K-a_n-N_n+p}^{\alpha-1}} \qquad (0 \le i \le N_n). \tag{4.34}$$

It is clear that $\xi_i, \eta_i > 0$, and $\sum_{i=0}^{N_n} \xi_i = 1 = \sum_{i=0}^{N_n} \eta_i$. Furthermore, since

$$\frac{\xi_i}{\xi_{i+1}} = \frac{\sigma_i^{\alpha-1}}{\sigma_{i+1}^{\alpha-1}} = \frac{i+1}{i+\alpha}, \qquad \frac{\eta_i}{\eta_{i+1}} = \frac{\sigma_{K-a_n-N_n+i}^{\alpha-1}}{\sigma_{K-a_n-N_n+i+1}^{\alpha-1}} = \frac{K-a_n-N_n+i+1}{K-a_n-N_n+i+\alpha},$$

it follows from the inequalities $0 < \alpha < 1$ and $K - a_n - N_n \ge 1$ that $\xi_i/\xi_{i+1} \ge \eta_i/\eta_{i+1}$ for $0 \le i \le N_n - 1$. In addition, by (4.29) and (4.30), we see that $2^k \ge \|T^{a_n+N_n} f_d\|_1$ and

$$2^k \ge \|T^{a_n+N_n-1} f_d\|_1 \ge \|T^{a_n+N_n-2} f_d\|_1 \ge \cdots \ge \|T^{a_n} f_d\|_1. \tag{4.35}$$

Thus we can apply Lemma 4.2 to infer that

$$\sum_{i=0}^{N_n} \eta_i \|T^{a_n+N_n-i} f_d\|_1 \le \sum_{i=1}^{N_n} \xi_i \|T^{a_n+N_n-i} f_d\|_1 + \xi_0 2^k. \tag{4.36}$$

This establishes (4.33).

If $n = j(d,K)$ and $N_n = N(d - a_n + na^j)$, then, since $K = a_n + N_n$, we have

$$\sigma_{K-i}^{\alpha-1}\|T^i f_d\|_1 = \sigma_{N_n-(i-a_n)}^{\alpha-1}\|T^{i-a_n}(T^{a_n} f_d)\|_1 \quad \text{for } a_n \le i \le a_n + N_n - 1,$$

and

$$\sigma_{K-(a_n+N_n)}^{\alpha-1}\|T^{(a_n+N_n)} f_d\|_1 = \sigma_0^{\alpha-1}\|T^{N_n}(T^{a_n} f_d)\|_1 = \|T^{N_n}(T^{a_n} f_d)\|_1 \le 2^k.$$

Hence, inequality (4.33) is also true in this case. On the other hand, if $n = j(d,K)$ and $N_n < N(d - a_n - na^j)$, then it is clear that

$$\frac{\sum_{i=a_n}^{a_n+N_n} \sigma_{K-i}^{\alpha-1}\|T^i f_d\|_1}{\sum_{i=a_n}^{a_n+N_n} \sigma_{K-i}^{\alpha-1}} = \frac{\sum_{i=0}^{N_n} \sigma_{N_n-i}^{\alpha-1}\|T^i(T^{a_n} f_d)\|_1}{\sigma_{N_n}^{\alpha}}. \tag{4.37}$$

Since $\|T^{a_0} f_d\|_1 = \|f_d\|_1 = 1$, and $\|T^{a_n} f_d\|_1 = 2^{-(j-k)} \le 1$ for $1 \le n \le j(d,K)$ by (4.29), we can apply the results in (i) and (ii), together with (4.31), (4.33) and (4.37), to infer that

$$\frac{\sum_{i=a_n}^{a_n+N_n} \sigma_{K-i}^{\alpha-1}\|T^i f_d\|_1}{\sum_{i=a_n}^{a_n+N_n} \sigma_{K-i}^{\alpha-1}} = O\Big(\sum_{i=1}^{j-1}\Big(\frac{2}{a^{\alpha}}\Big)^i\Big) \qquad (0 \le n \le j(d,K)). \tag{4.38}$$

Next, using $\sigma_K^\alpha = \sum_{i=0}^K \sigma_{K-i}^{\alpha-1} = \sum_{n=0}^{j(d,K)} (\sum_{i=a_n}^{a_n+N_n} \sigma_{K-i}^{\alpha-1})$ and (4.32), we see that if $K > N(d)$, then

$$\|C_K^\alpha(T)f_d\|_1 = \frac{1}{\sigma_K^\alpha} \sum_{n=0}^{j(d,K)} \Big( \sum_{i=a_n}^{a_n+N_n} \sigma_{K-i}^{\alpha-1} \cdot O\Big( \sum_{i=1}^{j-1} \Big(\frac{2}{a^\alpha}\Big)^i \Big)\Big) = O\Big( \sum_{i=1}^{j-1} \Big(\frac{2}{a^\alpha}\Big)^i \Big). \tag{4.39}$$

This and the estimation in (ii), together with the inequalities $1 \le \|g_d\|_1 \le 2$, show that

$$\sup_{a^{j-1}<d\le a^j} \|C_{N(d)}^\alpha(T)g_d\|_1 \le 2\alpha(X_j) = O\Big( \sum_{i=1}^{j-1} \Big(\frac{2}{a^\alpha}\Big)^i \Big) \quad (j \ge 1). \tag{4.40}$$

If $\gamma < \alpha < 1$, then $a^\alpha = 2^{\alpha/r} > 2$, and so $\sup_{j\ge1} \alpha(X_j) < \infty$.

If $0 < \alpha \le \gamma$, then $a^\alpha \le 2$. From (4.10), (4.17), (4.20) and (4.22) we see that for $a^{k-1} < d < a^k$ with $1 \le k \le j$,

$$\begin{aligned}\|C_{N(d)}^\alpha(T)g_d\|_1 &\ge I(N(d)) \sim I(N(d);1) + I(N(d);2) + I(N(d);3) \\ &\ge I(N(d);2) \sim I(N(d);2)^* \ge \frac{a^\alpha - 1}{\alpha} \cdot \frac{1}{a^\alpha} \sum_{i=1}^{k-1} \Big(\frac{2}{a^\alpha}\Big)^i.\end{aligned}$$

Hence

$$2\alpha(X_j) \ge \sup_{a^{j-1}<d\le a^j} \|C_{N(d)}^\alpha(T)g_d\|_1 \ge G \sum_{i=1}^{j-1} \Big(\frac{2}{a^\alpha}\Big)^i,$$

where $G > 0$ is an absolute constant depending only on $0 < \alpha \le \gamma$, and so $\lim_{j\to\infty} \alpha(X_j) = \infty$. ∎

Proposition 4.4. *Let* $\dim X = \infty$. *Then the following hold*:

(i) *For any integer* $k \ge 0$, *there exists a bounded linear operator* $T$ *on* $X$ *such that*

$$\sup_{n\ge0} \|C_n^{k+1}(T)\| < \infty \quad \textit{and} \quad \sup_{n\ge0} \|C_n^\gamma(T)\| = \infty \quad (0 \le \gamma < k+1).$$

(ii) *There exists a bounded linear operator* $T$ *on* $X$, *with* $r(T) = 1$, *such that*

$$\sup_{0<r<1} \|A_r(T)\| < \infty \quad \textit{and} \quad \sup_{n\ge0} \|C_n^\gamma(T)\| = \infty \quad (0 \le \gamma < \infty).$$

*Proof.* (i) Let $N : X \to X$ be such that $N^{k+1} \ne 0$, $N^{k+2} = 0$ and $\|N\| < 1$. Define $T := -(I+N)$. We recall the fundamental relation

$$(T-I)C_n^\gamma(T) = \frac{\gamma}{n+1}\,[C_{n+1}^{\gamma-1}(T) - I] \quad \text{for every } \gamma > 0 \text{ and } n \ge 0.$$

(This can be proved by an elementary calculation.) Using this, we can write

$$(T-I)C_n^\gamma(T) = \frac{\gamma}{n+1}\,[C_{n+1}^{\gamma-1}(T) - I] = \frac{\gamma}{n+1}\,C_{n+1}^{\gamma-1}(T) + D_n^1(T),$$

where $\|D_n^1(T)\| = o(1)$; next,

$$(T-I)^2C_n^\gamma(T) = \frac{\gamma}{n+1}\cdot\frac{\gamma-1}{n+2}\,C_{n+2}^{\gamma-2}(T) + D_n^2(T) \quad \text{with } \|D_n^2(T)\| = o(1);$$

and lastly, for $k \le [\gamma] \le \gamma$,

$$(T-I)^kC_n^\gamma(T) = \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{(n+1)(n+2)\cdots(n+k)}\,C_{n+k}^{\gamma-k}(T) + D_n^k(T),$$

with $\|D_n^k(T)\| = o(1)$.

To show $\sup_{n\ge0}\|C_n^\gamma(T)\| = \infty$ for all $\gamma \in [0, k+1)$, it suffices to show this for all $\gamma \in [k, k+1)$.

First, consider the case $\gamma = k$. Since

$$T^n = (-1)^n(I+N)^n = (-1)^n\sum_{l=0}^{k+1}\binom{n}{l}N^l \qquad (n \ge 0),$$

it follows that

$$C_{n+k}^0(T) = T^{n+k} = (-1)^{n+k}\biggl(\sum_{l=0}^{k}\binom{n+k}{l}N^l + \binom{n+k}{k+1}N^{k+1}\biggr).$$

Then

$$\begin{aligned}
&\left\|\frac{k!}{(n+1)(n+2)\cdots(n+k)}\,C_{n+k}^0(T)\right\| \\
&\quad\ge \frac{n}{k+1}\,\|N^{k+1}\| - \frac{k!}{(n+1)(n+2)\cdots(n+k)} \\
&\qquad\qquad \cdot\sum_{l=0}^{k}\frac{(n+k)(n+k-1)\cdots(n+k-l+1)}{l!}\,\|N^l\| \\
&\quad\to\infty \qquad \text{as } n\to\infty.
\end{aligned}$$

Consequently, $\lim_{n\to\infty}\|(T-I)^kC_n^k(T)\| = \infty$ and so $\sup_{n\ge0}\|C_n^k(T)\| = \infty$.

Next, let $k < \gamma < k+1$ and put $\beta = \gamma - k$ and

$$C_n^\beta(T) = \frac{1}{\sigma_n^\beta}\sum_{k=0}^{n}\sigma_{n-k}^{\beta-1}T^k.$$

Since $0 < \beta < 1$, we have:

$$\sigma_n^{\beta-1}\downarrow 0 \qquad (n\to\infty); \quad \sigma_n^{\beta-1} \sim \frac{n^{\beta-1}}{\Gamma(\beta)} \qquad (n \ge 1);$$

$$0 < \sigma_n^\beta \uparrow \infty \quad (n\to\infty); \quad \sigma_n^\beta \sim \frac{n^\beta}{\Gamma(\beta+1)} \quad (n\ge1);$$

$$\sigma_n^\beta = \sum_{k=0}^n \sigma_{n-k}^{\beta-1}; \qquad 0 < \sigma_n^{\beta-1} \downarrow 0.$$

Using these, we find that

$$\begin{aligned} C_{n+k}^{\gamma-k}(T) = C_{n+k}^\beta(T) &= \frac{1}{\sigma_{n+k}^\beta} \sum_{s=0}^{n+k} \sigma_{n+k-s}^{\beta-1} T^s \\ &= \frac{1}{\sigma_{n+k}^\beta} \sum_{s=0}^{n+k} \sigma_{n+k-s}^{\beta-1} \Big[(-1)^s \sum_{l=0}^k \binom{s}{l} N^l + (-1)^s \binom{s}{k+1} N^{k+1}\Big], \end{aligned}$$

where

$$\begin{aligned} &\Big\| \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{(n+1)\cdots(n+k)} \cdot \frac{1}{\sigma_{n+k}^\beta} \sum_{s=0}^{n+k} \sigma_{n+k-s}^{\beta-1} \Big[(-1)^s \sum_{l=0}^k \binom{s}{l} N^l\Big] \Big\| \\ &\qquad\qquad \leq \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{(n+1)\cdots(n+k)} \sum_{l=0}^k \binom{n+k}{l} = O(1). \end{aligned}$$

Now, consider the series

$$\sum_{s=0}^{n+k} \sigma_{n+k-s}^{\beta-1} (-1)^s \cdot s(s-1)\cdots(s-k)$$

$$\Big(= \sum_{s=k+1}^{n+k} \sigma_{n+k-s}^{\beta-1} (-1)^s \cdot s(s-1)\cdots(s-k)\Big).$$

Since

$$\begin{aligned} 0 &< \sigma_{n+k-s}^{\beta-1} \cdot s(s-1)\cdots(s-k) \\ &< \sigma_{n+k-(s+1)}^{\beta-1} \cdot (s+1)(s+1-1)\cdots(s+1-k) \end{aligned}$$

for $k+1 \leq s < n+k$, it follows that

$$\begin{aligned} &\Big| \sum_{s=0}^{n+k-1} \sigma_{n+k-s}^{\beta-1} (-1)^s \cdot s(s-1)\cdots(s-k) \Big| \\ &\qquad\qquad < \sigma_1^{\beta-1} (n+k-1)(n+k-2)\cdots(n+k-1-k), \end{aligned}$$

so that

$$\begin{aligned} &\Big| \sum_{s=0}^{n+k} \sigma_{n+k-s}^{\beta-1} (-1)^s \cdot s(s-1)\cdots(s-k) \Big| \\ &\qquad > \sigma_0^{\beta-1} (n+k)(n+k-1)\cdots n - \sigma_1^{\beta-1} (n+k-1)\cdots(n-1) \\ &\qquad = (n+k)(n+k-1)\cdots n - \beta(n+k-1)\cdots(n-1) \\ &\qquad > (1-\beta)(n+k)(n+k-1)\cdots n. \end{aligned}$$

Hence

$$\begin{aligned}
&\left\|\frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{(n+1)\cdots(n+k)}\cdot\frac{1}{\sigma_{n+k}^{\beta}}\left(\sum_{s=0}^{n+k}\sigma_{n+k-s}^{\beta-1}(-1)^s\binom{s}{k+1}\right)N^{k+1}\right\|\\
&\qquad\geq\|N^{k+1}\|\cdot\frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{(n+1)\cdots(n+k)}\cdot\frac{1}{\sigma_{n+k}^{\beta}}\\
&\qquad\quad\cdot(1-\beta)\cdot\frac{(n+k)(n+k-1)\cdots n}{(k+1)!}\\
&\qquad=\|N^{k+1}\|\cdot\frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{(k+1)!}\,(1-\beta)\,\frac{n}{\sigma_{n+k}^{\beta}}\to\infty\qquad\text{as } n\to\infty,
\end{aligned}$$

because $0<\beta<1$ implies that

$$\lim_{n\to\infty}\frac{n}{\sigma_{n+k}^{\beta}}=\lim_{n\to\infty}n\cdot\frac{\Gamma(\beta+1)}{(n+k)^{\beta}}=\infty.$$

Thus, for every $\gamma$ with $k<\gamma<k+1$, we have

$$\|(T-I)^kC_n^{\gamma}(T)\|\geq\left\|\frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{(n+1)(n+2)\cdots(n+k)}\,C_{n+k}^{\gamma-k}(T)\right\|-\|D_n^k(T)\|\to\infty$$

as $n\to\infty$, which implies

$$\sup_{n\geq0}\|C_n^{\gamma}(T)\|=\infty.$$

We next prove that $\sup_{n\geq0}\|C_n^{k+1}(T)\|<\infty$. In fact, we can write

$$(T-I)^{k+1}C_n^{k+1}(T)=\frac{(k+1)!}{(n+1)\cdots(n+k+1)}\,C_{n+k+1}^{0}(T)+D_n^{k+1}(T),$$

where $\|D_n^{k+1}(T)\|=o(1)\ \ (n\to\infty)$. Then, as above,

$$\begin{aligned}
C_{n+k+1}^{0}(T)=T^{n+k+1}&=(-1)^{n+k+1}(I+N)^{n+k+1}\\
&=(-1)^{n+k+1}\left(\sum_{l=0}^{k+1}\binom{n+k+1}{l}N^l\right),
\end{aligned}$$

whence

$$\begin{aligned}
&\left\|\frac{(k+1)!}{(n+1)(n+2)\cdots(n+k+1)}\,C_{n+k+1}^{0}(T)\right\|\\
&\qquad\leq\frac{(k+1)!}{(n+1)(n+2)\cdots(n+k+1)}\\
&\qquad\quad\cdot\left[1+\sum_{l=1}^{k+1}\frac{(n+k+1)(n+k)\cdots(n+k-l+2)}{l!}\,\|N^l\|\right]\\
&\qquad=O(1).
\end{aligned}$$

This proves that $\|(T-I)^{k+1}C_n^{k+1}(T)\| = O(1)$ $(n\to\infty)$, and thus

$$\|C_n^{k+1}(T)\| \le \|(T-I)^{-(k+1)}\|\cdot\|(T-I)^{k+1}C_n^{k+1}(T)\| = O(1) \quad (n\to\infty).$$

This completes the proof of (i).

(ii) For each $k\ge 1$, there exists an operator $T_k : \mathbb{C}^{k+2}\to\mathbb{C}^{k+2}$ of the form $T_k = -(I_k+N_k)$, where $I_k$ is the identity operator on $\mathbb{C}^{k+2}$ and $N_k$ satisfies $\|N_k\| < k^{-1}$, $N_k^{k+1}\ne 0$, $N_k^{k+2}=0$ on $\mathbb{C}^{k+2}$. As is already shown (cf. the proof of Proposition 2.8), $T_k$ is dissipative, and thus $\|(I-rT_k)x\|\ge\|x\|$ for every $x\in X_k$ and $0<r<1$. Furthermore, $\sigma(T_k) = -1-\sigma(N_k) = -1-\{0\} = \{-1\}$. It follows that $\|(I-rT_k)^{-1}\|\le 1$ for every $0<r<1$, and $r(T_k)\le 1$. Hence

$$\|A_r(T_k)\| = \Big\|(1-r)\sum_{n=0}^{\infty} r^n T_k^n\Big\| = \|(1-r)(I-rT_k)^{-1}\| \le 1-r \quad (0<r<1).$$

From (i) we also know that $\sup_{n\ge 0}\|C_n^k(T_k)\|=\infty$ and $\sup_{n\ge 0}\|C_n^{k+1}\|<\infty$.

Let $X := \{x=(x_1,x_2,\ldots);\ x_k\in\mathbb{C}^{k+2},\ \sum_{k=1}^{\infty}\|x_k\|<\infty\}$ and $\|x\| := \sum_{k=1}^{\infty}\|x_k\|$ for $x\in X$. Then $X$ becomes a Banach space. Define an operator $T : X\to X$ by $Tx := (T_1x_1, T_2x_2,\ldots)$ for $x=(x_1,x_2,\ldots)\in X$. Then we have

$$\|T\| = \sup_{k\ge 1}\|T_k\| \le \sup_{k\ge 1}(1+1/k) = 2,$$
$$\|A_r(T)\| = \sup_{k\ge 1}\|A_r(T_k)\| \le 1-r \quad (0<r<1)$$

and

$$\sup_{n\ge 0}\|C_n^k(T)\| \ge \sup_{n\ge 0}\|C_n^k(T_k)\| = \infty$$

for every $k\ge 1$. This completes the proof of (ii). ∎

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