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Weakly mixing rank-one transformations conjugate to their squares

by

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Abstract. Utilizing the cut-and-stack techniques we construct explicitly a weakly mixing rigid rank-one transformation T which is conjugate to T^2. Moreover, it is proved that for each odd q, there is such a T commuting with a transformation of order q. For any n, we show the existence of a weakly mixing T conjugate to T^2 and whose rank is finite and greater than n.

0. Introduction. Recently there has been progress in studying ergodic transformations isomorphic to their composition squares [Ag2], [Go2], [Go3] (see also [Go1]). In [Ag2], Ageev answered a well known question: he proved the existence of a weakly mixing rank-one transformation T conjugate to T^2. However, his proof given in the Baire category framework is not constructive. Thus, no concrete example of T is known so far. The main purpose of the present paper is to construct such a T via the cutting-and-stacking algorithm with explicitly described spacers. For this, we apply a *group action approach* suggested first by A. del Junco in [dJ3] to produce a counterexample in the theory of simple actions. The idea is to select an auxiliary countable group H and an element $h \in H$ and to construct via (C, F)-techniques a special funny rank-one action V of H in such a way that the transformation V_h has required dynamical properties. In our case, H is the group of 2-adic rationals and $h = 1$. The action V is constructed in §2. For other—sometimes unexpected—applications of the group action approach we refer to [Ma], [Ag1], [Da5], [Da6], [DdJ]. For the basics of the cutting-and-stacking (C, F)-techniques we refer to §2 below (see also [Da3], [Da4], [DaS] and a survey [Da7]).

A new short category proof of the existence theorem from [Ag2] is given below, in Section 1 (Theorem 1.3). Section 2 contains the main result of the present paper (Theorem 2.2). In Section 3 we discuss "elements" of the general theory of ergodic transformations T conjugate to T^2: generic aspects of

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the inclusion of T into actions of some related "larger" groups, application of the co-inducing procedure from [D–S] and some spectral properties of T. In Section 4, given T conjugate to T^2, we investigate which ergodic compact extensions of T are conjugate to their square and what is the structure of the conjugating maps. As an application, we show that for any n, there exists a weakly mixing transformation T conjugate to T^2 and such that the rank of T is finite and greater than n (Theorem 4.10). In [Go2] Goodson isolated the class of those T (conjugate to T^2) that have King's *weak closure property* [Ki1]. Such T do not commute with periodic transformations of even order. He raised the question whether they can commute with periodic transformations of odd order. We answer this affirmatively using Baire category arguments: for any odd $q > 0$, there exists a weakly mixing rank-one transformation T conjugate to T^2 and whose centralizer contains an element of order q (Theorem 4.9). Concrete examples can also be constructed via the (C, F)-techniques as in Theorem 2.2. We also note that some results from [Go2] and [Go3] are refined and generalized in Sections 3 and 4. In the final Section 5 we state several open problems.

1. Notation and short proof of Ageev's theorem. Let $(X, \mathfrak{B}, \mu)$ be a standard non-atomic probability space. Denote by $\mathrm{Aut}_0(X, \mu)$ the group of μ-preserving transformations of X. It is well known that $\mathrm{Aut}_0(X, \mu)$ endowed with the weak (operator) topology is a Polish group [Ha].

We recall that a transformation $T \in \mathrm{Aut}_0(X, \mu)$ has *rank* n (we will denote that by $\mathrm{rk}(T) = n$) if n is the smallest $r \in \mathbb{N}$ such that there exist measurable subsets $B_j^{(m)}$ and positive integers $h_j^{(m)}$ such that the subsets $T^i B_j^{(m)}$, $j = 1, \dots, r$, $i = 1, \dots, h_j^{(m)}$, are pairwise disjoint and *approximate the entire σ-algebra* $\mathfrak{B}$ as $m \to \infty$. The latter means that given $B \in \mathfrak{B}$, there are subsets $A^{(m)}$ such that $\nu(B \triangle A^{(m)}) \to 0$ and every $A^{(m)}$ is the union of several subsets $T^i B_j^{(m)}$ with $0 \le j \le r$ and $1 \le i \le h_j^{(m)}$.

It follows from the classical results of Katok and Stepin [KaS] that the subset $\mathcal{R}^1$ of rank-one transformations is residual in $\mathrm{Aut}_0(X, \mu)$. We need the following a bit subtler fact.

Proposition 1.1. *$\mathcal{R}^1$ is a G_δ in* $\mathrm{Aut}_0(X, \mu)$.

Proof. Take a countable subfamily $\mathfrak{B}_0 \subset \mathfrak{B}$ which is dense in $(\mathfrak{B}, \mu)$. Let $B_1, B_2, \dots$ be a sequence of measurable subsets in X such that each element of $\mathfrak{B}_0$ occurs infinitely often in this sequence. Now given $A \in \mathfrak{B}$, $n, m \in \mathbb{N}$ and $\varepsilon, \delta > 0$, we let

$$\mathcal{O}_{A,n,\varepsilon} := \{T \in \mathrm{Aut}_0(X, \mu) \mid \mu(T^j A \cap T^k A) < \varepsilon\mu(A)/n \text{ for all } 0 \le j, k < n\},$$
$$\mathcal{O}^{m,\delta}_{A,n,\varepsilon} := \Big\{T \in \mathcal{O}_{A,n,\varepsilon} \;\Big|\; \max_{0 \le i < m} \min_{J \subset \{0,\dots,n\}} \mu\Big(B_i \triangle \bigcup_{j \in J} T^j A\Big) < \delta\Big\}.$$

Clearly, the two subsets are open in $\mathrm{Aut}_0(X,\mu)$. It remains to notice that

$$\mathcal{R}^1 = \bigcap_{k=1}^{\infty} \bigcup_{A\in\mathfrak{B}_0} \bigcup_{n=1}^{\infty} \mathcal{O}^{k,1/k}_{A,n,1/k}. \ \blacksquare$$

From now on let H stand for the group of 2-adic rationals. We denote by G the semidirect product $H \rtimes \mathbb{Z}$ with the multiplication

$$(h,n)(h',n') := (h + 2^n h', n+n'), \qquad h,h' \in H,\ n,n' \in \mathbb{Z}.$$

Let $\mathcal{A}_G$ stand for the set of μ-preserving G-actions $V = (V_g)_{g\in G}$ on X. Of course, there is a one-to-one correspondence between $\mathcal{A}_G$ and the pairs of transformations $T,S \in \mathrm{Aut}_0(X,\mu)$ such that $STS^{-1} = T^2$. The correspondence is given by the formulae $V_{(1,0)} := T$ and $V_{(0,1)} := S$.

LEMMA 1.2. *If T is ergodic then V is free.*

Proof. Suppose first that there is a subset $A \subset X$ with $\mu(A) > 0$ such that $V_h x = x$ for all $x \in X$ and some $h \in H$, $h \neq 0$. Since V_h and T commute and T is ergodic, $V_h = \mathrm{Id}$. This implies in turn that a power of T is the identity, a contradiction. Thus the H-subaction $(V_h)_{h\in H}$ is free. Suppose now that there are $h \in H$ and $n \in \mathbb{N}\setminus\{0\}$ with $V_h x = S^n x$ on a set of positive measure. Since the transformation S^n normalizes the ergodic transformation group $\{V_h \mid h \in H\}$, there exists a measurable map $m : X \to H$ such that $S^n x = V_{m(x)} x$ for a.a. $x \in X$. Hence the relation $S^n T = T^{2^n} S^n$ plus the freeness of $(V_h)_{h\in H}$ yield now $m(Tx)+1 = m(x)+2^n$ for a.a. $x \in X$. Then for each $\lambda \in \mathbb{R}$, the function $X \ni x \mapsto \exp(2\pi i\lambda m(x)) \in \mathbb{T}$ is an eigenfunction of T corresponding to the eigenvalue $\exp(2\pi i\lambda(2^n-1))$. Since the discrete spectrum of T is at most countable, we obtain a contradiction. $\blacksquare$

Now we are ready to give a short proof of the main result from [Ag2].

THEOREM 1.3. *There exists a weakly mixing rank-one transformation conjugate to its square.*

Proof. Note that $\mathcal{A}_G$ is a closed subset of the infinite product space $\mathrm{Aut}_0(X,\mu)^G$ furnished with the product topology. Moreover, $\mathrm{Aut}_0(X,\mu)$ acts continuously on $\mathcal{A}_G$ by conjugation as follows: $(T\cdot V)_g := TV_gT^{-1}$, $g \in G$. We let

$$\mathcal{W}_G := \{V \in \mathcal{A}_G \mid V_{(1,0)} \text{ is weakly mixing}\},$$
$$\mathcal{R}^1_G := \{V \in \mathcal{A}_G \mid V_{(1,0)} \in \mathcal{R}^1\}.$$

Since the set of weakly mixing transformations and the set of rank-one transformations are both G_δ in $\mathrm{Aut}_0(X,\mu)$ (see [Gl] and Proposition 1.1) and the map $\mathcal{A}_G \ni V \mapsto V_{(1,0)} \in \mathrm{Aut}_0(X,\mu)$ is continuous, it follows that $\mathcal{W}_G$ and $\mathcal{R}^1_G$ are both G_δ in $\mathcal{A}_G$. We notice that $\mathcal{W}_G$ contains any Bernoullian shiftwise G-action. Any such action is free. Let us show that $\mathcal{R}^1_G$ also contains a free G-action. Since H is the inductive limit of the sequence $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots$, we can represent the compact dual group $\widehat{H}$ as the inverse limit of the sequence $\mathbb{T} \leftarrow \mathbb{T} \leftarrow \cdots$, where all the arrows denote the power two homomorphism. Take any $z_0 \in \mathbb{T}$ of infinite order and consider the infinite sequence $k' := (z_0, z_1, \dots) \in \widehat{H}$ with $z_n = z_{n+1}^2 \in \mathbb{T}$ for all n. Now we define two measure preserving transformations of $(\widehat{H}, \lambda_{\widehat{H}})$ by setting $Tk := kk'$ and $Sk = k^2$. Then $STS^{-1} = T^2$. Moreover, T is ergodic by the standard criterion of ergodicity for rotations on compact groups [CFS]. By Lemma 1.2, the action V of G corresponding to (T, S) is free. Since T has pure point spectrum, $T \in \mathcal{R}^1$ by [dJ1], i.e. $V \in \mathcal{R}^1_G$.

By [FoW], the $\mathrm{Aut}_0(X, \mu)$-orbit of any free G-action is dense in $\mathcal{A}_G$. Since $\mathcal{W}_G$ and $\mathcal{R}^1_G$ are $\mathrm{Aut}_0(X, \mu)$-invariant, it follows that they are both residual in $\mathcal{A}_G$. Hence their intersection is non-empty. ∎

Notice that the above example of an affine G-action $V \in \mathcal{R}^1_G$ appeared first (in a slightly different form) in [Go2]. It was reproduced later in [Ag2].

2. Explicit examples. We start this section by recalling the (C, F)-construction (see also [dJ3], [Da3]–[Da7], [DaS], [DdJ]). Let $(C_m)_{m=1}^\infty$ and $(F_m)_{m=0}^\infty$ be two sequences of finite subsets in H such that for each $m \geq 0$ the following properties are satisfied:

$$\text{(2-1)} \qquad \begin{aligned} &F_m + C_{m+1} \subset F_{m+1}, \qquad \#C_{m+1} > 1, \\ &\text{the sets } F_m + c,\ c \in C_{m+1}, \text{ are pairwise disjoint.} \end{aligned}$$

We put $X_m := F_m \times C_{m+1} \times C_{m+2} \times \cdots$, endow X_m with the (compact) product topology and define a continuous embedding $X_m \to X_{m+1}$ by setting

$$(f_m, c_{m+1}, c_{m+2}, \dots) \mapsto (f_m + c_{m+1}, c_{m+2}, \dots).$$

Then we have $X_1 \subset X_2 \subset \cdots$. Let $X := \bigcup_m X_m$ stand for the topological inductive limit of the sequence X_m. Clearly, X is a locally compact totally disconnected metrizable space without isolated points and X_m is clopen in X. Hence the corresponding Borel σ-algebra $\mathfrak{B}$ is standard. Assume in addition that

$$\text{(2-2)} \qquad \prod_{m=1}^{\infty} \frac{\#F_{m+1}}{\#F_m \#C_{m+1}} < \infty.$$

Then it is easy to see that there exists a unique probability measure μ on $(X, \mathfrak{B})$ such that the restriction of μ to each X_m is the infinite product measure

$$\tau_m \times \lambda_{m+1} \times \lambda_{m+2} \times \cdots,$$

where λ_j is the equidistribution on C_j and τ_m is a finite measure on F_m with $\tau_m(f) = \tau_m(f')$ for all $f, f' \in F_m$. Thus $(X, \mathfrak{B}, \mu)$ is a standard probability space. Given $h \in H$ and $m > 0$, we set

$$D_h^{(m)} := (F_m \cap F_m - h) \times C_{m+1} \times C_{m+2} \times \cdots \quad \text{and} \quad R_h^{(m)} := D_{-h}^{(m)}.$$

Clearly, $D_h^{(m)}$ and $R_h^{(m)}$ are clopen subsets of X_n. Moreover, $D_h^{(m)} \subset D_h^{(m+1)}$ and $R_h^{(m)} \subset R_h^{(m+1)}$. Define a map $T_h^{(m)} \colon D_h^{(m)} \to R_h^{(m)}$ by setting

$$T_h^{(m)}(f_m, c_{m+1}, \dots) := (h + f_m, c_{m+1}, \dots).$$

Clearly, it is a homeomorphism. Put

$$D_h := \bigcup_{m=1}^{\infty} D_h^{(m)} \quad \text{and} \quad R_h := \bigcup_{m=1}^{\infty} R_h^{(m)} = D_{h^{-1}}.$$

Then D_h and R_h are open subsets of X. Moreover, a homeomorphism $T_h \colon D_h \to R_h$ is well defined by $T_h \restriction D_h^{(m)} = T_h^{(m)}$ for all m. Suppose now that

$$(F_m)_{m \geq 0} \text{ is a Følner sequence in } G. \tag{2-3}$$

This implies $\mu(D_h^{(m)}) \to 1$ as $m \to \infty$. Hence $\mu(D_h) = \mu(R_h) = 1$. Since $\mu(O) > 0$ for each open subset $O \subset X$, it follows that the subset $D := \bigcap_{h \in H} D_h = \bigcap_{h \in H} R_h$ is a dense G_δ of full μ-measure. It is easy to see that $T_{h_2 h_1} = T_{h_2} T_{h_1}$ on D for all $h_1, h_2 \in H$. Thus $T := (T_h)_{h \in H}$ is a continuous H-action on the Polish (in the induced topology) space D. This action is minimal. To see this, just notice that for any m,

(i) the T-orbit equivalence relation restricted to $X_m \cap D$ coincides with the restriction of the tail equivalence relation $\mathcal{T}_m$ on the infinite product space X_m to D,

(ii) $D \cap X_m$ is invariant under $\mathcal{T}_m$.

Moreover, T preserves μ and T is free and ergodic.

Definition 2.1. We call T the *(C, F)-action of H associated to* $(C_{m+1}, F_m)_{m=0}^{\infty}$.

In the following we will not distinguish between sets, maps, and transformations which agree a.e. For each subset $A \subset F_m$, we let

$$[A]_m := \{x = (f_m, c_{m+1}, \dots) \in X_m \mid f_m \in A\}$$

and call it an *m-cylinder*. Then

$$\begin{aligned}
&[A \cap B]_m = [A]_m \cap [B]_m \quad \text{and} \quad [A \cup B]_m = [A]_m \cup [B]_m, \\
&[A]_m = [A + C_{m+1}]_{m+1} = \bigsqcup_{c \in C_{m+1}} [A + c]_{m+1}, \\
&T_h[A]_m = [h + A]_m \quad \text{if } h + A \subset F_m, \\
&\mu([A]_m) = \#C_{m+1} \cdot \mu([A + c]_{m+1}) \quad \text{for every } c \in C_{m+1}, \\
&\mu([A]_m) = \mu(X_m) \, \frac{\#A}{\#F_m},
\end{aligned}$$

where $\sqcup$ denotes the union of mutually disjoint sets. Moreover, given $Y \in \mathfrak{B}$,

$$\min_{A\subset F_m} \mu(Y \triangle [A]_m) \to 0 \quad \text{as } m \to \infty. \tag{2-4}$$

This means that T has *funny rank one* [Da3].

To construct a special (C, F)-action of H we need several definitions. Let $F = \{2^{-k}, 2\cdot 2^{-k}, \dots, L - 2^{-k}\} \subset H$ for some $k, L \in \mathbb{N}$. Then F is an arithmetic progression of length $2^k L - 1$. It is easy to see that for each $0 \le i < \#F$, there is a unique element $f_i \in F$ such that

$$\frac{1}{2^k} + i = f_i + m_i\left(L - \frac{1}{2^k}\right), \tag{2-5}$$

where m_i is a non-negative integer with $m_i \le 2^k$. Moreover, the map $i \mapsto f_i$ is one-to-one and hence $F = \{f_i \mid i = 0, \dots, \#F - 1\}$. Now for a given $p > 0$, we define the *p-good tiling set* C for F by setting

$$C := \{0, -2^{-k-1}\} + \{0, L - 2^{-k}, \dots, (2^k p - 1)(L - 2^{-k})\}$$

Then $(F + c) \cap (F + c') = \emptyset$ whenever $c \ne c' \in C$. The sum $F + C$ equals the arithmetic progression $\{2^{-k-1}, 2\cdot 2^{-k-1}, \dots, 2^k p(L - 2^{-k})\}$. We now set

$$C' := \{0, -2^{-k-1}\} + \{0, L - 2^{-k}, \dots, (2^k(p-1) - 1)(L - 2^{-k})\}$$

and call it the *reduced p-good tiling set* for F. It follows from (2-5) and the inequality $m_i \le 2^k$ that

$$\frac{\#C'}{\#C} = 1 - \frac{1}{p} \quad \text{and} \quad i + 2^{-k} + C' \subset f_i + C \tag{2-6}$$

for all $0 \le i < \#F$.

Consider now another case. Suppose that $F = \{2^{-k}, 2\cdot 2^{-k}, \dots, L\}$ for some integers $k, L > 0$. Then we define the *Chacon tiling set* C for F by setting

$$C := \{0, L, 2L + 1\}.$$

Of course, $(F + c) \cap (F + c') = \emptyset$ whenever $c \ne c' \in C$.

Now we are going to construct a special (C, F)-action $T = (T_h)_{h\in H}$ of H. Fix a sequence of positive integers p_n such that $\sum_{n=1}^{\infty} p_n^{-1} < \infty$. Then choose $(C_n)_{n=1}^{\infty}$ and $(F_n)_{n=0}^{\infty}$ recurrently as follows:

- $F_0 = \{0.5\}$,
- C_{2n+1} is a p_n-good tiling for F_{2n},
- F_{2n+1} is the shortest arithmetic progression which contains the arithmetic progression $F_{2n} + C_{2n+1}$ and satisfies the following two conditions: the largest element of F_{2n+1} is an integer and the cardinality of the subset $C^+_{2n+1} := \{c \in C_{2n+1} \mid 2\cdot F_{2n} + 2c \subset F_{2n+1}\}$ is $0.5\#C_{2n+1}$,
- C_{2n+2} is the Chacon tiling set for F_{2n+1},

- F_{2n+2} is the shortest arithmetic progression of positive 2-adic rationals such that $F_{2n+1}+C_{2n+2}\subset F_{2n+2}$ and the sum $\max F_{2n+2}+\min F_{2n+2}$ is an integer.

Now it is easy to see that the conditions (2-1)–(2-4) are all satisfied. Hence the associated rank-one (C,F)-action $T=(T_h)_{h\in H}$ is well defined.

THEOREM 2.2. *The transformation T_1 is weakly mixing and of rank one. Moreover, T_1 is conjugate to T_2.*

Proof. We first prove that T_1 is weakly mixing. Let $e:X\to\mathbb{T}$ be an eigenfunction of T_1 corresponding to a non-trivial eigenvalue $\lambda\in\mathbb{T}$, i.e. $e\circ T_1=\lambda e$, $|e|\equiv 1$ and $\lambda\neq 1$. Given $\varepsilon>0$, there is a subset $A\subset X$ of positive measure such that

$$|e(x)-e(y)|<\varepsilon \quad \text{for all } x,y\in A. \tag{2-7}$$

Now we can find $n>0$ and a cylinder $[f]_{2n+1}$ such that

$$\mu(A\cap[f]_{2n+1})>0.99\mu([f]_{2n+1}). \tag{2-8}$$

We recall that $F_{2n+1}=\{2^{-k_n},2\cdot 2^{-k_n},\dots,L_n\}$ and $C_{2n+2}=\{0,L_n,2L_n+1\}$ for some positive integers k_n and L_n. Since $[f]_{2n+1}$ is the union of the three disjoint subsets $[f]_{2n+2}$, $[f+L_n]_{2n+2}$ and $[f+2L_n+1]_{2n+2}$ of equal measure, and $[f+L_n]_{2n+2}=T_{L_n}[f]_{2n+2}$ and $[f+2L_n+1]_{2n+2}=T_{2L_n+1}[f]_{2n+2}$, we deduce from (2-8) that there exists a subset $B\subset[f]_{2n+2}$ of positive measure such that $B\sqcup T_{L_n}B\sqcup T_{2L_n+1}B\subset A$. Hence by (2-7),

$$|1-\lambda^{L_n}|=|e(b)-e(T_{L_n}b)|<\varepsilon \quad\text{and}\quad |1-\lambda^{2L_n+1}|=|e(b)-e(T_{2L_n+1}b)|<\varepsilon$$

for all $b\in B$. This yields $|\lambda-1|<3\varepsilon$. Since ε is arbitrary, we deduce that $\lambda=1$, a contradiction.

Now we will show that T_1 has rank one. Recall that for each $n>0$, there exist positive integers d_n and R_n such that $F_{2n}=\{2^{-d_n},2\cdot 2^{-d_n},\dots$ $\dots,R_n-2^{-d_n}\}$. We let $D_n:=[2^{-d_n}+C'_{2n+1}]_{2n+1}$, where $C'_{2n+1}\subset C_{2n+1}$ is the reduced p_n-good tiling set for F_{2n}. It follows from (2-6) that $T_iD_n\subset[f_i]_{2n}$ for all $i=0,\dots,\#F_{2n}-1$, where $\{f_i\}_{i=0}^{\#F_{2n}-1}$ is an enumeration of the elements of F_{2n}. Moreover, (2-6) yields

$$\sum_{i=0}^{\#F_{2n}-1}\mu([f_i]_{2n}\setminus T_iB_n)=\sum_{i=0}^{\#F_{2n}-1}\mu([C_{2n+1}\setminus C'_{2n+1}]_{2n+1})<p_n^{-1}.$$

Since the sequence of towers $[f]_{2n}$, $f\in F_{2n}$, approximates the entire σ-algebra $\mathfrak{B}$ by (2-4), it now follows that the sequence of towers T_iB_n, $i=0,\dots,\#F_{2n}-1$, also approximates $\mathfrak{B}$ as $n\to\infty$. Hence T_1 is of rank one.

It remains to establish that T_1 is conjugate to its square. To define a transformation S which conjugates T_1 and T_2 it suffices to show how it acts on the cylinders. For each $n>0$, we put $C^-_{2n+1}:=C_{2n+1}\setminus C^+_{2n+1}$. Now given $f \in F_{2n}$, we partition the $2n$-cylinder $[f]_{2n}$ into two $(2n+1)$-cylinders $[f + C^+_{2n+1}]_{2n+1}$ and $[f + C^-_{2n+1}]_{2n+1}$ of equal measure and define the image of the first one under S by setting

$$S[f + C^+_{2n+1}]_{2n+1} := [2f + 2 \cdot C^+_{2n+1}]_{2n+1}.$$

It is well defined due to our choice of F_{2n+1}. Next, we partition the second $(2n+1)$-cylinder into two $(2n+3)$-cylinders

$$[f + C^-_{2n+1} + C_{2n+2} + C^+_{2n+3}]_{2n+3} \quad \text{and} \quad [f + C^-_{2n+1} + C_{2n+2} + C^-_{2n+3}]_{2n+3}$$

of equal measure and define the image of the first one under S by setting

$$S[f+C^-_{2n+1}+C_{2n+2}+C^+_{2n+3}]_{2n+3} := [2f+2\cdot C^-_{2n+1}+2\cdot C_{2n+2}+2\cdot C^+_{2n+3}]_{2n+3}.$$

Continuing this procedure infinitely many times, we define the S-image of the entire $[f]_{2n}$, since

$$[f]_{2n} = \bigsqcup_{n\le q<\infty} \Big[f + \sum_{j=n}^{q-1}(C^-_{2j+1} + C_{2j+2}) + C^+_{2q+1}\Big]_{2q+1}.$$

It is straightforward that $\mu(S[f]_{2n}) = \mu([f]_{2n})$ and $S[f]_{2n} \cap S[f']_{2n} = \emptyset$ whenever $f \neq f' \in F_{2n}$. Hence $S \in \mathrm{Aut}_0(X,\mu)$. Moreover, it is easy to see that $ST_1 = T_2S$, as desired. ∎

Remark 2.3. It is easy to deduce from the proof of Theorem 2.1 that $T_{\#F_{2n}}D_n \subset [f_0]_{2n}$. This implies $T_1^{\#F_{2n}} \to \mathrm{Id}$ as $n \to \infty$. Hence T_1 is rigid.

Remark 2.4. Slightly modifying the construction of T one can produce explicitly a rigid weakly mixing rank-one transformation R conjugate to R^j for any $j > 1$. Moreover, the corresponding conjugating maps will pairwise commute. The existence of such a transformation can also be proved via Baire category arguments as in Theorem 1.3. The only difference in the proof is to consider the group $\mathbb{Q} \rtimes_\alpha \bigoplus_{j=1}^\infty \mathbb{Z}$ instead of G, where α denotes the following action of $\bigoplus_{j=1}^\infty \mathbb{Z}$ on $\mathbb{Q}$:

$$\alpha(n_1,\dots,n_k,0,0,\dots)q := 2^{n_1}\cdot 3^{n_2}\cdots(k+1)^{n_k}\cdot q.$$

3. Extending transformations to G-actions. Let T be an ergodic transformation of $(X,\mathfrak{B},\mu)$. We consider the problem of whether T is conjugate to its square, i.e. $STS^{-1} = T^2$. This can be "split" into two separate subproblems:

- ($\circ$) Does T embed into a μ-preserving action H-action $V = (V_h)_{h\in H}$ on the same measure space (we mean that $V_1 = T$)?
- ($\widehat{\circ}$) Given an H-action V, does it extend to a μ-preserving G-action on the same measure space?

Notice that the affirmative answer to ($\circ$) is stronger than the fact that T has roots of order 2^n for each $n > 0$. The corresponding counterexample was produced via the (C, F)-techniques in [Ma].

Some necessary and sufficient conditions for extending an action of a subgroup to an action of a "larger" group are given in [Da2, Proposition 2.5]. However, being of rather abstract orbital nature they are of little practical importance.

We now answer ($\circ$) and ($\widehat{\circ}$) generically, i.e. up to a meager subset in the Baire category sense. Let $\mathcal{A}_G$ and $\mathcal{A}_H$ denote the Polish spaces of μ-preserving G-actions and H-actions respectively. They are Polish $\mathrm{Aut}_0(X,\mu)$-spaces, and the natural projections

$$\pi_G : \mathcal{A}_G \ni W \mapsto (W_h)_{h\in H} \in \mathcal{A}_H, \qquad \pi_H : \mathcal{A}_H \ni V \mapsto V_1 \in \mathrm{Aut}_0(X,\mu)$$

are both continuous and $\mathrm{Aut}_0(X,\mu)$-equivariant (see §1). It follows from [dRdS] that

($\bullet$) a generic transformation from $\mathrm{Aut}_0(X,\mu)$ embeds into an H-action.

On the other hand,

($\widehat{\bullet}$) the subset $\pi_G(\mathcal{A}_G)$ of H-actions extending to G-actions is meager in $\mathcal{A}_H$.

Indeed, if ($\widehat{\bullet}$) does not hold then $\pi_G(\mathcal{A}_G)$ is residual in $\mathcal{A}_H$ by the 0-1 law from [GlK] (see also [FoW]). It follows from the argument presented in [dRdS, §3.2] that the *locally dense points* for π_H are dense in $\mathcal{A}_H$. Hence the set $\pi_H(\pi_G(\mathcal{A}_G))$ is residual in $\mathrm{Aut}_0(X,\mu)$ [Ki2]. However, this contradicts [dJ2], where it was shown that a generic transformation is disjoint from (hence, not conjugate to) its square.

We think that it is relevant to mention here that an important class of transformations conjugate to their squares—considered in [Go2] and [Ag2]—can be described in a canonical way via *co-inducing*. We recall the definition following [D–S].

DEFINITION 3.1. Let Γ be a countable group and Λ a subgroup of Γ. Let $\widehat{T} = (\widehat{T}_h)_{h\in\Lambda}$ be a measure preserving action of Λ on a standard probability space $(Y, \mathfrak{C}, \nu)$. Select a cross-section $\sigma : \Lambda\backslash\Gamma \to \Gamma$ of the quotient map $\Gamma \to \Lambda\backslash\Gamma$ with $\sigma(\Lambda) = 1_\Gamma$. Define an action $T = (T_g)_{g\in\Gamma}$ of Γ on the product space $(X, \mathfrak{B}, \mu) := (Y, \mathfrak{C}, \nu)^{\Lambda\backslash\Gamma}$ by setting

$$(T_g x)(\Lambda g') := \widehat{T}_{\sigma(\Lambda g')g\sigma(\Lambda g'g)^{-1}} x(\Lambda g'g)$$

for all maps $x : \Lambda\backslash\Gamma \to Y$ and $g \in \Gamma$. Then T is said to be *co-induced from* $\widehat{T}$.

It is easy to see that T does not depend (up to conjugacy) on the choice of σ. Moreover, if $\widehat{T}$ is free or ergodic then so is T. Notice that T is defined on a "larger" space than Y: the original action $\widehat{T}$ is only a factor of $(T_h)_{h\in\Lambda}$.

Suppose now that we are given an action $\widehat{T}$ of H on $(Y,\mathfrak{C},\nu)$. Then the co-induced action T of G is defined on $(Y,\nu)^{\mathbb{Z}}$. The generators $(1,0)$ and $(0,1)$ of G act as follows:

$$(T_{(1,0)}y)_n = \widehat{T}_{2^n}y_n \quad \text{and} \quad (T_{(0,1)}y)_n = y_{n+1}, \quad n\in\mathbb{Z}.$$

Thus we obtain the well-known examples from [Go2] and [Ag2].

It is interesting to note that (•) combined with [Ag1] implies that a generic transformation $T\in \mathrm{Aut}_0(X,\mu)$ embeds into an H-action $\widehat{V}$ such that the $(1,0)$-transformation $V_{(1,0)}$ of the corresponding G-action V co-induced from $\widehat{H}$ is weakly mixing and has a simple spectrum. However—as noticed by V. Ryzhikov [Ry]—it is never of rank one (cf. the example in Theorem 2.2). Indeed, if $V_{(1,0)}$ were of rank one then its factor $T\times T^2$ would also be of rank one. Hence by King's weak closure theorem [Ki1] the transformation $\mathrm{Id}\times T$ commuting with $T\times T^2$ is the weak limit of a sequence of powers of $T\times T^2$. Thus $\mathrm{Id}=\lim_{i\to\infty}T^{n_i}$ and $T=\lim_{i\to\infty}T^{2n_i}$, a contradiction.

Now we are going to discuss some aspects of spectral theory for the transformations embedding into G-actions (or, equivalently, transformations conjugate to their squares). Let $STS^{-1}=T^2$ and $V=(V_g)_{g\in G}$ stand for the corresponding G-action. Denote by $U: G\ni g\mapsto U_g\in\mathcal{U}(L^2(X,\mu))$ the Koopman unitary representation of G associated to G. Then by the spectral theorem for $U\restriction H$, there exist a probability measure σ on $\widehat{H}$ and a Borel map $m:\widehat{H}\ni w\mapsto m(w)\in\mathbb{N}\cup\{\infty\}$ such that the following decomposition holds (up to unitary equivalence):

$$L^2(X,\mu)=\int_{\widehat{H}}^{\oplus}\mathcal{H}_w\,d\sigma(w) \quad\text{and}\quad U_h=\int_{\widehat{H}} w(h)I_w\,d\sigma(w), \tag{3-1}$$

for each $h\in H$, where $w\mapsto\mathcal{H}_w$ is a Borel field of Hilbert spaces, $\dim\mathcal{H}_w = m(w)$ and I_w is the identity operator on $\mathcal{H}_w$. Let Q denote the power two automorphism $w\mapsto w^2$ of $\widehat{H}$. Since the unitary representation $H\ni h\mapsto U_{2h}\in\mathcal{U}(L^2(X,\mu))$ of H is unitarily equivalent to $U\restriction H$ via $U_{(0,1)}$, it follows that σ is quasi-invariant under Q and $m\circ Q=m$.

The canonical embedding $\mathbb{Z}\subset H$ induces a projection $\pi:\widehat{H}\to\mathbb{T}$. Let $\sigma=\int_{\mathbb{T}}\sigma_z\,d\widehat{\sigma}(z)$ denote the desintegration of σ relative to this projection. Then we derive from (3-1) the spectral decomposition for the operator $U_{(1,0)}$:

$$L^2(X,\mu)=\int_{\mathbb{T}}^{\oplus}\mathcal{H}'_z\,d\widehat{\sigma}(z) \quad\text{and}\quad U_{(1,0)}=\int_{\mathbb{T}} zI_z\,d\widehat{\sigma}(z),$$

where $\mathcal{H}'_z := \int_{\widehat{H}}^{\oplus} \mathcal{H}_w \, d\sigma_z(w)$. Let $l(z) := \dim \mathcal{H}'_z$, $z \in \mathbb{T}$. Then

$$l(z) = \begin{cases} \infty & \text{if } \sigma_z \text{ is not purely atomic,} \\ \sum_{\sigma_z(w)>0} m(w) & \text{otherwise.} \end{cases} \tag{3-2}$$

Since π intertwines Q with the power two endomorphism $\widehat{Q}$ of $\mathbb{T}$, we obtain (i) and (ii) of the following assertion.

Proposition 3.2. *If T is conjugate to T^2 then*

(i) *the measure $\widehat{\sigma}$ of the maximal spectral type of T is quasi-invariant under $\widehat{Q}$;*

(ii) *the spectral multiplicity function l of T satisfies the equation*

$$l(z) = \sum_{\widehat{\sigma}_z(v)>0} l(v) \tag{3-3}$$

for $\widehat{\sigma}$-a.a. z, where $\widehat{\sigma}_z$ is the conditional measure of $\widehat{\sigma}$ on the 2-point fiber $\widehat{Q}^{-1}z$ over Z;

(iii) *if l is bounded then $\widehat{\sigma}$ is singular to Lebesgue measure $\lambda_{\mathbb{T}}$;*

(iv) *if T has simple spectrum then π is one-to-one (on a σ-conull subset) and hence $\widehat{Q}$ is invertible (on a $\widehat{\sigma}$-conull subset).*

Proof. (iii) If there exists a subset $A \subset \mathbb{T}$ with $\lambda_{\mathbb{T}}(A) > 0$ such that $\widehat{\sigma}\restriction A$ is equivalent to $\lambda_{\mathbb{T}}\restriction A$ then—since $\widehat{\sigma}$ is quasi-invariant under $\widehat{Q}$—we can assume without loss of generality that $\lambda_{\mathbb{T}}(A) = 1$. It follows that $\widehat{\sigma}_z(v) > 0$ for the two points $v \in \widehat{Q}^{-1}z$ at $\lambda_{\mathbb{T}}$-a.a. $z \in \mathbb{T}$. Applying (3-3) several times we come to a contradiction with the boundedness of l.

(iv) Since $l(z) = 1$ a.e., it follows from (3-2) that σ_z is supported on a singleton for a.a. $z \in \mathbb{T}$. ∎

Remark 3.3. If a rank-one transformation T is conjugate to both T^2 and T^3 then by Proposition 3.2(i), each measure $\widehat{\sigma}$ of the maximal spectral type for T is quasi-invariant under $\widehat{Q}$ and the power 3 endomorphism of $\mathbb{T}$. Since T has a simple spectrum, we deduce from Proposition 3.2(iii) that $\widehat{\sigma}$ is singular to Lebesgue measure. This provides an answer to a “non-singular” counterpart of Furstenberg’s $(\times 2, \times 3)$-problem [Fu].

4. Compact extensions of transformations conjugate to their squares. Let $(T_h)_{h\in H}$ be a free measure preserving action of H on $(X, \mathfrak{B}, \mu)$. Suppose that the transformation T_1 is ergodic. Given a compact second countable group K, denote by $\mathcal{M}(X, K)$ the group of measurable maps from X to K. It is Polish when endowed with the topology of convergence in μ. Let $\alpha : H \ni h \mapsto \alpha_h \in \mathcal{M}(X, K)$ be a T-*cocycle*, i.e. $\alpha_{h_1 h_2} = \alpha_{h_1} \circ T_{h_2} \cdot \alpha_{h_2}$ for all $h_1, h_2 \in H$. Then a new H-action $T^\alpha = (T_h^\alpha)_{h\in H}$ is well defined on the product space $(X \times K, \mu \times \lambda_K)$:

$$T_h^\alpha(x,k) := (T_h x, \alpha_h(x)k).$$

It is called the *α-skew product extension* (or *an H-extension*) of T. Two cocycles α and β of T are *cohomologous* if there is a map $\phi \in \mathcal{M}(X,K)$ such that $\alpha_h = \phi \circ T_h \cdot \beta_h \cdot \phi^{-1}$ for all $h \in H$. If T^α is ergodic then α is called *ergodic*.

We first investigate a relation between the ergodicity of α and the ergodicity of a single transformation T_1^α.

PROPOSITION 4.1. *α is ergodic if and only if there exist a T-cocycle β cohomologous to α, a closed normal subgroup $K_0 \subset K$ and a group homomorphism*

$$p : H/\mathbb{Z} \ni h + \mathbb{Z} \mapsto p(h+\mathbb{Z}) \in K/K_0$$

such that the following conditions are satisfied:

(i) $\beta_1(x) \in K_0$ *for a.a.* $x \in X$,
(ii) *the transformation* $(x,k) \mapsto (T_1 x, \beta_1(x)k)$ *of the space* $(X \times K_0, \mu \times \lambda_{K_0})$ *is ergodic, where* λ_{K_0} *stands for the normalized Haar measure on* K_0,
(iii) *the range* $p(H/\mathbb{Z})$ *of* p *is dense in the quotient group* K/K_0; *and hence* K/K_0 *is Abelian*,
(iv) $\beta_h(x)H_0 = p(h+\mathbb{Z})H_0 \in K/K_0$ *for a.a.* $x \in X$ *and all* $h \in H$.

Proof. We need only verify the "only if" part. The "if" part is trivial. Thus let α be ergodic. By [Zi, Corollary 3.8], there exist a closed subgroup $K_0 \subset K$ and a cocycle β cohomologous to α such that (i) and (ii) are satisfied. Now it follows from [Da1, Lemma 2.1] that for each $h \in H$, there is $p(h) \in N_K(K_0)$ such that $\beta_h(x)K_0 = p(h)K_0$ for a.a. x. Recall that $N_K(K_0)$ denotes the normalizer of K_0 in K. The cocycle identity for β implies that p is a group homomorphism. Hence $\beta_h(X) \subset N_K(K_0)$ for all $h \in H$. Since α is ergodic, so is β. Therefore $N_K(K_0) = K$, i.e. K_0 is normal in K. The kernel of p includes $\mathbb{Z}$ by (i). Also, the range of the homomorphism $H \ni h \mapsto p(h)K_0 \in K/K_0$ is dense because β is ergodic. ∎

COROLLARY 4.2. *Suppose that the quotient of K by its commutant is totally disconnected* (*an interesting particular case is when K is finite*). *Then α is ergodic if and only if the transformation T_1^α is ergodic.*

Proof. Let α be ergodic and $K_0 \neq K$. Denote by K' the commutant of K. It follows from Proposition 4.1(iii) that $K_0 \supset K'$. Let $\pi : K/K' \to K/K_0$ stand for the canonical projection $kK' \mapsto kK_0$. Since K/K' is totally disconnected, there exists a subgroup O of finite index in K/K' such that $\pi(O) \neq K/K_0$. Notice that $\pi(O)$ is a subgroup of finite index in K/K_0. Then it follows from Proposition 4.1(iii) that $p^{-1}(\pi(O))$ is a subgroup of the very same finite index in $H/\mathbb{Z}$. We get a contradiction because $H/\mathbb{Z}$ has no proper subgroups of finite index. ∎

We also note that if $K=\mathbb{T}$ then it is easy to construct T and α such that α is ergodic but T_1^α is not.

Suppose now that $ST_hS^{-1}=T_{2h}$ for all $h\in H$ and a transformation $S\in \mathrm{Aut}_0(X,\mu)$. We say that S *can be lifted to a conjugacy of* T_1^α *and* T_2^α if there exists a transformation $\widetilde{S}\in \mathrm{Aut}_0(X\times K,\mu\times\lambda_k)$ with $\widetilde{S}T_1^\alpha\widetilde{S}^{-1}=T_2^\alpha$ and $\widetilde{S}(x,k)=(Sx,\cdot)$ for a.a. $(x,k)\in X\times K$.

PROPOSITION 4.3. *Let α be ergodic. Then S can be lifted to a conjugacy $\widetilde{S}$ of T_1^α and T_2^α if and only if there exist a map $a\in\mathcal{M}(X,K)$ and a continuous group automorphism $l:K\to K$ such that*

$$\alpha_1\circ(T_1S)\cdot\alpha_1\circ S=a\circ T_1\cdot l\circ\alpha_1\cdot a^{-1}. \tag{4-1}$$

Moreover, the corresponding lift of S is then of the form

$$\widetilde{S}(x,k)=(Sx,l(k)a(x))\quad \textit{for a.a. } (x,k)\in X\times K.$$

Proof. We first notice that S normalizes the T-orbit equivalence relation, i.e. $S\{T_hx\mid h\in H\}=\{T_hSx\mid h\in H\}$ for a.a. x. Hence we may apply some results on “lifting” from [Da1] which are of orbital nature. (Observe also that S does not normalize the T_1-orbit equivalence relation.) It follows from [Da1, Theorem 5.3] that S can be lifted to a conjugacy $\widetilde{S}$ of T_1^α and T_2^α if and only if there exists a map $a\in\mathcal{M}(X,K)$ and a continuous group automorphism $l:K\to K$ such that

$$\alpha_{2h}\circ S=a\circ T_h\cdot l\circ\alpha_h\cdot a^{-1}\quad \text{for all } h\in H \tag{4-2}$$

and $\widetilde{S}(x,k)=(Sx,l(k)a(x))$. Notice that the system of equations (4-2) is equivalent to a single equation (4-1), which, in turn, is equivalent to $\widetilde{S}T_1^\alpha=T_2^\alpha\widetilde{S}$. ∎

It is easy to see that the T-cocycles form a closed subset $\mathcal{C}$ of $\mathcal{M}(X,K)^H$ endowed with the (Polish) infinite product topology. Applying [Da1, Theorems 5.9 and Theorem 5.8] we deduce the following from Proposition 4.3.

COROLLARY 4.4.

(i) *There is a residual subset of ergodic cocycles $\alpha\in\mathcal{C}$ such that S cannot be lifted to a conjugacy of T_1^α and T_2^α.*
(ii) *For each automorphism $l\in\mathrm{Aut}\,K$, there exists an ergodic cocycle α of T such that* (4-1) *is satisfied, i.e. S can be lifted to a conjugacy of T_1^α and T_2^α.*

Recall that given a dynamical system $(X,\mathfrak{B},\mu,R)$, every R-invariant sub-σ-algebra is called a *factor* of R. For instance, $\mathfrak{B}\otimes\{\emptyset,K\}$ is a factor of T_1^α. Proposition 4.3 describes a “structure” of the conjugations between

T_1^α and T_2^α that preserve $\mathfrak{B}\otimes\{\emptyset, K\}$. Now we want to specify a condition on a factor of R under which *every conjugation* of R and R^2 preserves this factor. For this, we first recall a couple of definitions from the joining theory of dynamical systems [Gl].

Given two transformations $R_1, R_2 \in \mathrm{Aut}_0(X,\mu)$, a measure ν on $X\times X$ is called a *joining* of R_1 and R_2 if ν is $R_1\times R_2$-invariant and the coordinate marginals of ν are both equal to μ. For instance, the product $\mu_1\times\mu_2$ is a joining of T_1 and T_2. Another example is a *graph-joining* μ_S generated by a transformation conjugating T_1 with T_2. It is defined by $\mu_S(A\times B) := \mu(A\cap S^{-1}B)$, $A,B\in\mathfrak{B}$. Of course, μ_S (i.e. the dynamical system $(X\times X,\mu_S,R_1\times R_2)$) is ergodic. A transformation R is called 2-*fold simple* if every ergodic joining of R with itself is either the product measure or a graph-joining. For instance, each transformation with pure point spectrum is 2-fold simple.

Proposition 4.5. *Let T be an ergodic transformation and let $\mathfrak{F}$ be a proper factor of T. If $T\restriction\mathfrak{F}$ is 2-fold simple and conjugate to its square then every conjugation of T with T^2 preserves $\mathfrak{F}$ (and hence conjugates $T\restriction\mathfrak{F}$ and $T^2\restriction\mathfrak{F}$).*

Proof. Let $STS^{-1}=T^2$ and let μ_S denote the graph-joining of T and T^2 generated by S. Of course, the sub-σ-algebra $\mathfrak{F}\otimes\mathfrak{F}$ is a factor of $T\times T^2$. Hence $\mu_S\restriction(\mathfrak{F}\otimes\mathfrak{F})$ is an ergodic joining of $T\restriction\mathfrak{F}$ and $T^2\restriction\mathfrak{F}$. It follows from the assumption on $\mathfrak{F}$ that $\mu_S\restriction(\mathfrak{F}\otimes\mathfrak{F})$ is either the product measure or a graph-joining. It is rather easy to see that the former is impossible. Hence $\mu_S\restriction(\mathfrak{F}\otimes\mathfrak{F})$ is a graph-joining. This implies that S preserves $\mathfrak{F}$. ∎

The next proposition—partly following from Proposition 4.5—extends and refines [Go2, Theorems 3, 4, Corollary 1] and [Go3, Theorems 1, 5].

Proposition 4.6. *Let $STS^{-1}=T^2$ with T ergodic. Let $\mathfrak{F}$ denote the Kronecker factor of T, i.e. the maximal factor with pure discrete spectrum. Then*

(i) *the discrete spectrum $\Lambda(T)\subset\mathbb{T}$ of T is invariant under the power two map*;
(ii) *$\mathfrak{F}$ is invariant under S and hence $T\restriction\mathfrak{F}$ is conjugate to its square*;
(iii) *$S\restriction\mathfrak{F}$ is ergodic if and only if T is totally ergodic, i.e. $\Lambda(T)$ does not contain non-trivial roots of* 1;
(iv) *if T is totally ergodic but not weakly mixing then $S\restriction\mathfrak{F}$ is Bernoullian.*

Proof. (i) Since T and T^2 are isomorphic, we have $\Lambda(T)=\Lambda(T^2)=\{\lambda^2\mid\lambda\in\Lambda(T)\}$.

(ii) follows from (i) and Proposition 4.5.

(iii) Since the restriction of S to $\mathfrak{F}$ is isomophic to the "power two" automorphism of the compact dual group $\widehat{\Lambda(T)}$, we deduce (iii) from the standard ergodicity criterion for such maps [CFS, Chapter 3, §3, Theorem 1].

(iv) follows from (iii) and the fact that each ergodic automorphism of a compact group is Bernoullian. ∎

Recall that a transformation R has the *weak closure property* (WCP) if $C(T) = \mathrm{Cl}_w(\{R^n \mid n \in \mathbb{Z}\})$, where $\mathrm{Cl}_w(.)$ denotes the closure in the weak topology. It follows that $C(T)$ is a monothetic group. By [Ki1], each rank-one transformation has WCP.

We notice that if T is ergodic then $C(T)$ contains an element of order q if and only if T is a $\mathbb{Z}/q\mathbb{Z}$-extension of another transformation.

PROPOSITION 4.7. *Let T have WCT and $STS^{-1} = T^2$. Then the following are satisfied*:

(i) (cf. [Go2, Theorem 2]) *The map*

$$C(T) \ni R \mapsto SRS^{-1} = R^2 \in C(T)$$

is a group isomorphism. Therefore $C(T) = C(T^2)$.

(ii) (cf. [Go2, Proposition 7]) $C(T)$ *does not contain non-trivial elements of finite even order.*

Proof. (i) Let V be an action of H with $V_1 = T$. Now it suffices to notice that $C(T) = \mathrm{Cl}_w(\{V_h \mid h \in H\})$, $SV_hS^{-1} = V_{2h}$ for all $h \in H$ and the homomorphism $H \ni h \mapsto 2h \in H$ is a bijection of H.

(ii) Let $R \in C(T) \setminus \{\mathrm{Id}\}$ be of finite even order. Since $SRS^{-1} = R^2$ by (i), the orders of R and R^2 are the same, a contradiction. ∎

In view of the above, Goodson asks naturally in [Go2] and [Go3]:

($\circ$) *for T weakly mixing and satisfying the conditions of Proposition* 4.7, *can $C(T)$ contain elements of odd order?*

We first provide a preliminary "answer" in the class of transformations with pure point spectrum.

EXAMPLE 4.8. Let T be the ergodic transformation with pure point spectrum $\{\exp(i\theta n/2^m) \mid n, m \in \mathbb{Z}\}$ for an irrational θ. Then T is conjugate to T^2. Fix an odd q. Let R be a translation by 1 on the cyclic group $\mathbb{Z}/q\mathbb{Z}$. Since the power two automorphism of $\mathbb{Z}/q\mathbb{Z}$ conjugates R with R^2, the direct product $T \times R$ is conjugate to $T^2 \times S^2$. Moreover, $T \times R$ is ergodic with pure point spectrum. Hence $T \times R$ has WCP. It remains to notice that $\mathrm{Id} \times R$ commutes with $T \times R$.

Since each rank-one transformation has WCP, the following theorem answers ($\circ$) affirmatively.

Theorem 4.9. *Given an odd q, there exists a weakly mixing rank-one transformation T conjugate to T^2 and such that $C(T)$ contains an element of order q.*

Proof. Let E be the semidirect product $E := (H \times \mathbb{Z}/q\mathbb{Z}) \rtimes \mathbb{Z}$ with the multiplication

$$(h, i, n)(h', i', n') := (h + 2^n h', i + 2^n i', n + n').$$

Denote by $\mathcal{A}_E \subset \mathrm{Aut}_0(X, \mu)^E$ the closed subset of μ-preserving actions of E on (X, μ). As in the proof of Theorem 1.3, it is enough to show that the following two subsets contain free E-actions:

$$\mathcal{W}_E := \{V \in \mathcal{A}_E \mid V_{(1,0,0)} \text{ is weakly mixing}\},$$
$$\mathcal{R}_E^1 := \{V \in \mathcal{A}_E \mid V_{(1,0,0)} \text{ has rank one}\}.$$

Any Bernoullian E-action is free and belongs to $\mathcal{W}_E$. Since each ergodic transformation with pure point spectrum has rank one [dJ1], it follows from Example 4.7 that $\mathcal{R}_E^1$ contains a free E-action V given by $V_{(1,0,0)} := T$, $V_{(0,1,0)} := R$ and $V_{(0,0,1)} := S$, where S is a transformation conjugating $T \times R$ with $T^2 \times R^2$. ∎

While the above proof is not constructive, we also note that slightly modifying the argument given in the proof of Theorem 2.2, one can construct explicit cut-and-stack examples of T with the properties stated in Theorem 4.9.

We conclude this section with a "higher rank" analogue of Theorem 1.3.

Theorem 4.10. *For each $n > 0$, there exists a weakly mixing transformation R conjugate to its square and such that $n < \mathrm{rk}\, R < \infty$.*

Proof. Let $K_1, K_2, \dots$ be a sequence of simple finite groups such that the supremum of the dimensions of the irreducible representations of K_n is more than n. Let R be a rank-one weakly mixing transformation conjugate to its square. Then there exists an action V of H such that $V_1 = R$. By Corollary 4.4(ii), for each $n > 0$, there exists an ergodic cocycle α of V with values in K_n such that (4-1) is satisfied for $l = \mathrm{Id}$. Hence the transformation V_1^α is conjugate to its square. By Corollary 4.2, the transformation V_1^α is ergodic. Since R is weakly mixing and K_n has no non-trivial homomorphisms to $\mathbb{T}$, it follows that V_1^α is weakly mixing. Since R is of rank one, $\mathrm{rk}(V_1^\alpha) \le \# K_n$. Let M_n denote the essential supremum of the spectral multiplicity function of V_1^α with respect to the maximal spectral type. Since M_n is no less than the maximum of the dimensions of the irreducible representations of K_n by [Ro], we obtain $M_n \ge n$. It remains to use the fact that $\mathrm{rk}(V_1^\alpha) \ge M_n$ [Ch]. ∎

Note that V_1^α does not have WCP because $C(V_1^\alpha)$ contains the non-Abelian subgroup K_n.

5. Concluding remarks. The following classes of ergodic transformations (plus their direct products) conjugate to their squares are known so far:

(A) the transformations with pure point spectrum whose point spectrum is invariant under the power two homomorphism;
(B) the Bernoullian shifts of infinite entropy;
(C) the time-$(0, 1)$ transformations of the G-actions induced from H-actions;
(D) the time-1 transformations of the *horocycle flows* on compact connected orientable Riemannian surfaces of constant negative curvature [CFS] or, more generally, on surfaces whose sectional curvatures are negative [Mar] (such transformations are mixing and of infinite rank, some of them are simple);
(E) a class of *Gaussian transformations* whose spectral measure (not to be confused with the maximal spectral type) is non-atomic and quasi-invariant under the power two homomorphism [Go3] (they are weakly mixing and can have a simple spectrum);
(F) weakly mixing rank-one or, more generally, finite rank transformations constructed in this paper.

Notice that the class (F) is essentially new. It does not intersect any other class: $(E) \cap (F) = \emptyset$ since any Gaussian transformation is of infinite rank [dR]; the fact that $(C) \cap (F) = \emptyset$ was explained in Section 3; the rest is obvious.

It is interesting to find new examples. For instance:

($\circ$) Do there exist interval exchange transformations conjugate to their squares?
($\widehat{\circ}$) Are there mildly mixing non-mixing smooth models for ergodic T conjugate to T^2?

We also conjecture that

($\widetilde{\circ}$) the transformation T_1 constructed in Theorem 2.2 is simple and prime.

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