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n-supercyclic and strongly n-supercyclic operators in finite dimensions

by

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Abstract. We prove that on $\mathbb{R}^N$, there is no n-supercyclic operator with $1 \le n < \lfloor (N+1)/2 \rfloor$, i.e. if $\mathbb{R}^N$ has an n-dimensional subspace whose orbit under $T \in \mathcal{L}(\mathbb{R}^N)$ is dense in $\mathbb{R}^N$, then n is greater than $\lfloor (N+1)/2 \rfloor$. Moreover, this value is optimal. We then consider the case of strongly n-supercyclic operators. An operator $T \in \mathcal{L}(\mathbb{R}^N)$ is strongly n-supercyclic if $\mathbb{R}^N$ has an n-dimensional subspace whose orbit under T is dense in $\mathbb{P}_n(\mathbb{R}^N)$, the nth Grassmannian. We prove that strong n-supercyclicity does not occur non-trivially in finite dimensions.

Let T be a continuous linear operator on a Banach space X. The orbit of a set $E \in X$ under T is defined by

$$\mathcal{O}(E,T) := \bigcup_{n \in \mathbb{Z}_+} T^n(E).$$

Many authors have studied density properties of such orbits for different sets E. If E is a singleton and $\mathcal{O}(E,T)$ is dense in X, then T is said to be *hypercyclic*. Hypercyclicity was first studied by Birkhoff in 1929 and has been a subject of great interest during the last twenty years (see [2] and [6] for a survey). In 1974, Hilden and Wallen [9] considered $E = \mathbb{K}x$, a one-dimensional subspace of X; if $\mathcal{O}(E,T)$ is dense in X, then T is said to be *supercyclic*. Several generalisations of supercyclicity have been proposed since, like the one introduced by Feldman [5] in 2002. Rather than considering orbits of lines, Feldman defines an *n-supercyclic operator* as one for which there exists an n-dimensional subspace E such that $\mathcal{O}(E,T)$ is dense in X. This notion has been mainly studied in [1], [3] and [5]. In 2004, Bourdon, Feldman and Shapiro proved in the complex case that non-trivial n-supercyclicity is purely infinite-dimensional:

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Theorem (Bourdon, Feldman and Shapiro). *Let $N \geq 2$. Then there is no $(N-1)$-supercyclic operator on $\mathbb{C}^N$. In particular, there is no n-supercyclic operator on $\mathbb{C}^N$ for any $1 \leq n \leq N-1$.*

This theorem extends a result proved by Hilden and Wallen for supercyclic operators in the complex setting. On the other hand, Herzog [8] proved that there is no supercyclic operator on $\mathbb{R}^n$ for $n \geq 3$. Therefore, it is natural to ask about the existence of n-supercyclic operators in the real setting.

In 2008, Shkarin [13] introduced another generalisation of supercyclicity. Roughly speaking, an operator $T \in \mathcal{L}(X)$, where $\dim(X) \geq n$, is strongly n-supercyclic if there exists a subspace of dimension n whose orbit is dense in the set of n-dimensional subspaces of X. To be more precise, we need to define the topology of this set, which is called the nth *Grassmannian* of X. To do this, let us consider the open subset X_n of X^n, consisting of all linearly independent n-tuples of vectors in X with the topology induced from X^n, and let $\pi_n : X_n \to \mathbb{P}_n(X)$ be defined by $\pi_n(x) = \operatorname{span}\{x_1, \dots, x_n\}$. The topology on $\mathbb{P}_n(X)$ is the coarsest topology for which the map π_n is open and continuous. Let us now turn to the definition of strong n-supercyclicity: $M \in \mathbb{P}_n(X)$ is a *strongly n-supercyclic subspace* for T if every $T^k(M)$ is n-dimensional and if $\{T^k(M) : k \in \mathbb{Z}_+\}$ is dense in $\mathbb{P}_n(X)$. If such a subspace exists, then T is said to be *strongly n-supercyclic*. We denote by $\mathcal{ES}_n(T)$ the set of strongly n-supercyclic subspaces for an operator T.

An open question regarding n-supercyclic operators is whether they have the Ansari property: is it true that T^p is n-supercyclic for any $p \geq 2$ provided T itself is n-supercyclic? Shkarin [13] has shown that strongly n-supercyclic operators do have the Ansari property and he asks if n-supercyclicity and strong n-supercyclicity are equivalent. Indeed, this would solve the Ansari problem for n-supercyclic operators. Unfortunately, Shkarin did not go further in the study of strongly n-supercyclic operators. A study of general properties of strongly n-supercyclic operators can be found in [4].

In this paper, we study in detail n-supercyclicity and strong n-supercyclicity in finite-dimensional spaces. Of course, by the results of Bourdon, Feldman and Shapiro, we need only concentrate on the real Banach spaces. In particular, in Section 2 we prove the following theorem:

Theorem 1. *Let $N \geq 2$. There is no $(\lfloor (N+1)/2 \rfloor - 1)$-supercyclic operator on $\mathbb{R}^N$. Moreover there exist $(\lfloor (N+1)/2 \rfloor)$-supercyclic operators on $\mathbb{R}^N$.*

This theorem generalises Hilden and Wallen's and Herzog's results and is optimal for these operators. Actually, it is not difficult to prove that for every $\lfloor (N+1)/2 \rfloor \leq k \leq N$ there exists an operator on $\mathbb{R}^N$ which is k-supercyclic but not $(k-1)$-supercyclic. The proof of Theorem 1 is not easy and one needs to get familiar with specific notation to fully understand it. The proof is in several steps, from simplest matrices, called primary, to general ones.

Then, in Section 3, we completely solve the question of the existence of non-trivial strongly n-supercyclic operators in finite-dimensional vector spaces. In fact, we prove:

THEOREM 2. *For $N \geq 3$ and $1 \leq n < N$, there is no strongly n-supercyclic operator on $\mathbb{R}^N$.*

This result puts an end to the study of strong n-supercyclicity in finite-dimensions. It implies, in particular, that there exist n-supercyclic operators that are not strongly n-supercyclic and answers the question concerning the equivalence between n-supercyclicity and strong n-supercyclicity raised in [13]. The interested reader can refer to [4] for other properties of strongly n-supercyclic operators in the infinite-dimensional setting.

1. Preliminaries. It has been known for years that in the real setting, supercyclic operators are completely characterised, and they only appear on $\mathbb{R}$ or $\mathbb{R}^2$. Moreover, on $\mathbb{R}^2$, if π and θ are linearly independent over $\mathbb{Q}$, then R_θ, rotation with angle θ, is supercyclic. Building on this, one may easily see that any rotation on $\mathbb{R}^3$ around any one-dimensional subspace and with angle linearly independent from π over $\mathbb{Q}$ is 2-supercyclic. This simple example proves that the real setting is completely different from the complex one and raises hopes to find similar examples in higher dimensions. It seems clear that rotations distinguish the real case from the complex one. The next part is devoted to the real Jordan decomposition and highlights the role played by rotations in the real setting.

Jordan decomposition. In the complex setting, it is common to use the Jordan decomposition to obtain a matrix similar to T but with a better "shape". Bourdon, Feldman and Shapiro took advantage of this decomposition to prove that there is no $(N-1)$-supercyclic operator on $\mathbb{C}^N$. Recall that a Jordan block with eigenvalue μ and of size k is usually the $k \times k$ matrix with μ along the main diagonal, ones on the first super-diagonal and zeros everywhere else. For convenience, all along this paper we follow another convention which slightly improves notation but does not change the efficiency of this decomposition. Thus, in our convention, a *classical Jordan block* with eigenvalue μ and of size k will be the $k \times k$ matrix with μ along the main diagonal and along the first super-diagonal and zeros elsewhere.

The well-known Jordan decomposition for complex matrices cannot be applied without changes to the case of real matrices because of the existence of complex eigenvalues. However, there also exists an improved real version of the Jordan decomposition. In the real case, every matrix is similar to a direct sum of classical Jordan blocks and real Jordan blocks, where a *real Jordan block* of modulus μ and of size k is usually a $2k \times 2k$ matrix with μR_θ along the main diagonal, identity matrices along the first super-diagonal and zeros elsewhere. For the same reasons, our convention is different: for us, the terms along the first super-diagonal are the same as those on the main diagonal, i.e. μR_θ.

Let $\mathcal{B}$ be a classical (respectively real) Jordan block with eigenvalue (respectively modulus) μ and of size k, and let $\mathcal{A} = \mu$ (respectively $\mathcal{A} = \mu R_\theta$). Then the powers of $\mathcal{B}$ are easy to compute. Indeed, for all $n \in \mathbb{N}$,

$$\mathcal{B}^n = \begin{pmatrix} \mathcal{A}^n & \binom{n}{1}\mathcal{A}^n & \binom{n}{2}\mathcal{A}^n & \cdots & \binom{n}{k-1}\mathcal{A}^n \\ 0 & \mathcal{A}^n & \binom{n}{1}\mathcal{A}^n & \binom{n}{2}\mathcal{A}^n \cdots & \binom{n}{k-2}\mathcal{A}^n \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \binom{n}{1}\mathcal{A}^n \\ 0 & \cdots & 0 & 0 & \mathcal{A}^n \end{pmatrix}.$$

As we are going to repeatedly apply the Jordan decomposition, in both real and complex cases, we use the term modulus instead of eigenvalue. For more information on the Jordan decomposition see [10] or [12].

All along this paper, we will be interested in dynamical properties of such matrices. Consequently, when we consider a Jordan block, its modulus is supposed to be non-zero. To summarise, every operator on $\mathbb{R}^N$ is similar to one of the shape

$$\begin{pmatrix} \boxed{J_1} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \boxed{J_q} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \boxed{\mathcal{J}_1} & 0 & 0 \\ 0 & \cdots & \cdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{\mathcal{J}_r} \end{pmatrix}$$

where J_i are classical Jordan blocks:

$$J_i = \begin{pmatrix} \mu_i & \mu_i & 0 & 0 \\ 0 & \ddots & \ddots & \cdots \\ \vdots & 0 & \ddots & \mu_i \\ 0 & \cdots & 0 & \mu_i \end{pmatrix},$$

and $\mathcal{J}_i$ are real Jordan blocks:

$$\mathcal{J}_i = \begin{pmatrix} \lambda_i R_{\theta_i} & \lambda_i R_{\theta_i} & 0 & 0 \\ 0 & \ddots & \ddots & \cdots \\ \vdots & 0 & \ddots & \lambda_i R_{\theta_i} \\ 0 & \cdots & 0 & \lambda_i R_{\theta_i} \end{pmatrix}.$$

2. n-supercyclic operators on $\mathbb{R}^N$

2.1. Introduction. Bourdon, Feldman and Shapiro showed that there are n-supercyclic operators on $\mathbb{C}^N$ if and only if $n = N$. This completely characterises n-supercyclic operators in the complex finite-dimensional setting. In this section, we are going to apply the real Jordan decomposition to determine for which $n \in \mathbb{N}$ there are n-supercyclic operators on $\mathbb{R}^N$.

Actually, the following examples reveal how to provide $(\lfloor (N+1)/2 \rfloor)$-supercyclic operators on $\mathbb{R}^N$.

EXAMPLE 2.1. For all $N \geq 1$:

- On $\mathbb{R}^{2N}$, endomorphisms represented by matrices of the form

$$\begin{pmatrix} R_{\theta_1} & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ \vdots & \cdots & 0 & R_{\theta_N} \end{pmatrix}$$

are N-supercyclic if (and only if) $\{\pi, \theta_1, \ldots, \theta_N\}$ is a linearly independent family over $\mathbb{Q}$.

- On $\mathbb{R}^{2N+1}$, endomorphisms represented by matrices of the form

$$\begin{pmatrix} R_{\theta_1} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ \vdots & \cdots & 0 & R_{\theta_N} & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

are $(N+1)$-supercyclic if (and only if) $\{\pi, \theta_1, \ldots, \theta_N\}$ is a linearly independent family over $\mathbb{Q}$.

The proof of these statements relies on the fact that every rotation submatrix is supercyclic and the Kronecker density theorem [7] permits one to consider each submatrix separately.

These simple examples prove that our Theorem 1 is optimal. In the following, we are going to study n-supercyclic operators on $\mathbb{R}^N$ in order to prove Theorem 1. We proceed step by step. We begin by proving two special cases: the case of a real Jordan block matrix of size 2 is considered first because it is the simplest matrix that Bourdon, Feldman and Shapiro have not checked in [3]; then the case of a direct sum of rotation matrices is considered because it permits one to notice that something more is needed if one wants to go further. The proofs of these two results introduce some techniques involved in more general proofs. Then, we will establish a basis reduction which is of constant use all along the paper.

From that point on, our aim will be to find the best supercyclic constant for different types of matrices. We will begin by primary matrices which are direct sums of unimodular real and complex Jordan blocks of size one, and we will continue with the case of a single real Jordan block of arbitrary size. After that, we discuss the best supercyclic constant for matrices being direct sums of Jordan blocks with pairwise different moduli and then with the same modulus. Finally, we gather the results to give a general result having Theorem 1 as a corollary.

Let us begin with a real Jordan block of size 2.

PROPOSITION 2.2. *The operator*

$$T = \begin{pmatrix} R_\theta & R_\theta \\ 0 & R_\theta \end{pmatrix}$$

is not 2*-supercyclic on* $\mathbb{R}^4$.

Proof. Suppose that T is 2-supercyclic. Let $M = \operatorname{span}\{x, y\}$ be a 2-supercyclic subspace for T. One can suppose that either $x = (x_1, x_2, 0, 1)$ and $y = (y_1, y_2, 1, 0)$, or $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, 0, 0)$ where $(x_3, x_4) \neq (0, 0)$.

• If $x = (x_1, x_2, 0, 1)$ and $y = (y_1, y_2, 1, 0)$, then for any non-empty open sets U and V in $\mathbb{R}^2$, there exist a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ and two real sequences $(\lambda_{n_i})_{i\in\mathbb{N}}, (\mu_{n_i})_{i\in\mathbb{N}}$ such that

$$\begin{cases} R_\theta^{n_i}\left(\lambda_{n_i}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu_{n_i}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) + n_i R_\theta^{n_i}\begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in U, \\ R_\theta^{n_i}\begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in V, \end{cases}$$

which is equivalent to

$$\begin{cases} \lambda_{n_i}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu_{n_i}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + n_i\begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in R_\theta^{-n_i}(U), & (2.1) \\ \begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in R_\theta^{-n_i}(V). & (2.2) \end{cases}$$

Let $V = B\big(\binom{1}{0}, \varepsilon\big)$ be the open ball of radius ε centred at $\binom{1}{0}$ with $0 < \varepsilon < 1$, and U be any non-empty bounded open set. Then (2.2) implies that $0 \le |\lambda_{n_i}|, |\mu_{n_i}| < 1 + \varepsilon$ for all $i \in \mathbb{N}$. One may divide (2.1) by n_i to get

$$\frac{\lambda_{n_i}}{n_i}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{\mu_{n_i}}{n_i}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in \frac{R_\theta^{-n_i}(U)}{n_i}.$$

However, since the sequences $(\lambda_{n_i})_{i\in\mathbb{N}}$ and $(\mu_{n_i})_{i\in\mathbb{N}}$ are bounded, we have $\lambda_{n_i}/n_i \to 0$ and $\mu_{n_i}/n_i \to 0$ as $i \to \infty$, and since U is a bounded set, $(\lambda_{n_i})_{i\in\mathbb{N}}$ and $(\mu_{n_i})_{i\in\mathbb{N}}$ have to go to zero. This contradicts $\binom{\mu_{n_i}}{\lambda_{n_i}} \in R_\theta^{-n_i}(V)$.

• If $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, 0, 0)$, then one may suppose $\|(x_3, x_4)\| = 1$. By 2-supercyclicity of T, for any non-empty open sets U, V in $\mathbb{R}^2$, there exist a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ and two real sequences $(\lambda_{n_i})_{i\in\mathbb{N}}, (\mu_{n_i})_{i\in\mathbb{N}}$ such that

$$\begin{cases} R_\theta^{n_i}\left(\lambda_{n_i}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu_{n_i}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) + n_i\lambda_{n_i}R_\theta^{n_i}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in U, \\ \lambda_{n_i}R_\theta^{n_i}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in V, \end{cases}$$

which can be rewritten as

$$\begin{cases} \lambda_{n_i}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu_{n_i}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + n_i\lambda_{n_i}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(U), & (2.3) \\ \lambda_{n_i}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(V). & (2.4) \end{cases}$$

Let $V = B\big(\binom{r}{0}, \varepsilon\big)$ with $0 < \varepsilon < 1$ and $r > 1$. According to (2.4), one may observe that for every $i \in \mathbb{N}$, we have $r - \varepsilon < |\lambda_{n_i}| < r + \varepsilon$. Divide then (2.3) by $n_i\lambda_{n_i}$:

$$\frac{\mu_{n_i}}{n_i\lambda_{n_i}}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \xrightarrow[i\to\infty]{} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From this we deduce that the sequence $\big(\frac{\mu_{n_i}}{n_i\lambda_{n_i}}\big)_{i\in\mathbb{N}}$ is convergent to some $t \in \mathbb{R}$ because $(y_1, y_2) \neq (0, 0)$, so we have

$$t\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As $\binom{x_3}{x_4}$ is non-zero, this implies that $\binom{x_3}{x_4}, \binom{y_1}{y_2}$ are linearly dependent. Thus choosing an appropriate linear combination of x and y, one may assume

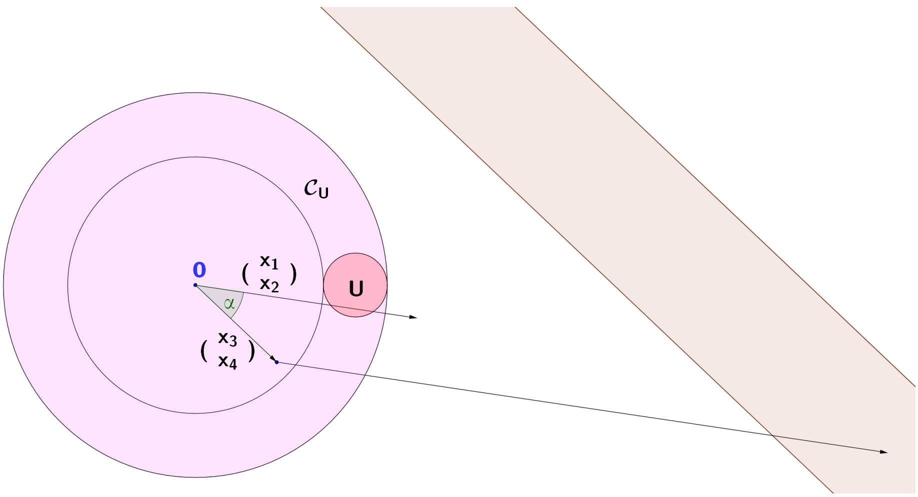
$x = (x_1, x_2, x_3, x_4)$ and $y = (x_3, x_4, 0, 0)$, hence (2.3) and (2.4) give

$$\begin{cases} \lambda_{n_i}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (\mu_{n_i} + n_i\lambda_{n_i})\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(U), & (2.5) \\ \lambda_{n_i}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(V). & (2.6) \end{cases}$$

Now, the vectors $\binom{x_1}{x_2}$ and $\binom{x_3}{x_4}$ are linearly independent. Indeed, suppose they are linearly dependent. Then upon taking appropriate linear combinations and replacing x, we can write $x = (0, 0, x_3, x_4)$, $y = (x_3, x_4, 0, 0)$ and

$$\begin{cases} (\mu_{n_i} + n_i\lambda_{n_i})\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(U), \\ \lambda_{n_i}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(V). \end{cases}$$

If one chooses two non-empty open sets U and V such that no straight line passing through the origin intersects both U and V, then we have a contradiction.



Thus $\binom{x_1}{x_2}$ and $\binom{x_3}{x_4}$ are linearly independent and let α denote the angle between them, so $|\sin(\alpha)| > 0$. Choose

$$0 < a < |\sin(\alpha)|(r - \varepsilon)\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\| \quad \text{and} \quad U = B\left(\begin{pmatrix} a/2 \\ 0 \end{pmatrix}, a/4\right)$$

and let $\mathcal{C}_U$ denote the annulus obtained by rotating the ball U about the origin. With a little computation, one may notice that the set $\mathbb{R}\binom{x_3}{x_4} +$

$[r-\varepsilon, r+\varepsilon]\binom{x_1}{x_2}$ does not intersect $\mathcal{C}_U$ contradicting (2.5) as shown in the figure. So $\left(\begin{smallmatrix} R_\theta & R_\theta \\ 0 & R_\theta \end{smallmatrix}\right)$ is not 2-supercyclic. ∎

REMARK 2.3. One can easily notice that the above matrix is 3-supercyclic if π and θ are linearly independent over $\mathbb{Q}$.

REMARK 2.4. The previous proof has two parts depending on the "shape" of the basis. Actually, we will constantly make distinctions according to the basis' shape.

2.2. A leading example. We will give an example to show that we need more tools if we want to go further in a precise manner. The next result proves that the supercyclic constants cannot be improved for the two matrices given in Example 2.1, i.e. the first operator is not $(N-1)$-supercyclic and the second one is not N-supercyclic. Moreover, in the following, T is a direct sum of rotations' multiples, each acting on $\mathbb{R}^2$. Hence, when one usually considers a vector component, we consider a vector bi-component instead, meaning that for the next result the natural way to define a vector is not as being in $\mathbb{R}^{2N}$ but rather in $(\mathbb{R}^2)^N$. In the following, the kth *bi-component* of a vector $(x_1, \dots, x_{2N})$ is the vector on which the kth rotation matrix acts, i.e. the vector (x_{2k-1}, x_{2k}).

PROPOSITION 2.5. *Let $N \geq 2$. Then for every choice of $a_1, \dots, a_N \in \mathbb{R}$ and $\theta_1, \dots, \theta_N \in \mathbb{R}$, the matrix*

$$R_N := \begin{pmatrix} a_1 R_{\theta_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_N R_{\theta_N} \end{pmatrix}$$

is not $(N-1)$-supercyclic on $\mathbb{R}^{2N}$.

Proof. First, upon reordering blocks in R_N and taking a scalar multiple, one may suppose $0 < |a_1| \leq \cdots \leq |a_{N-1}| \leq a_N = 1$. Indeed, the nullity of one of the a_i implies that R_N does not have dense range and is not $(N-1)$-supercyclic.

We are going to prove by induction that R_N is not $(N-1)$-supercyclic on $\mathbb{R}^{2N}$.

For $N = 2$, the result follows from Herzog's result [8].

Suppose that for every $2 \leq k < N$, every $\theta_1, \dots, \theta_k$ and every $0 < |a_1| \leq \cdots \leq |a_{k-1}| \leq a_k = 1$ no matrix of the form R_k is $(k-1)$-supercyclic. Let us prove it also for R_N. Assume to the contrary that R_N is $(N-1)$-supercyclic and let $M = \operatorname{span}\{x^1, \dots, x^{N-1}\}$ be an $(N-1)$-supercyclic subspace for R_N. Define $x^i_{N+1} := x^i$ for every $1 \leq i \leq N-1$.

We argue that for every $k \in \{1, \ldots, N\}$, M is spanned by a family of vectors $\{x_k^1, \ldots, x_k^{N-1}\}$ such that if we define $p_{N+1} := 0$ and

$$p_k := \sup\{j \in \{1, \ldots, N-1\} : \text{the } k\text{th bi-component of } x_k^j \text{ is non-null}\}$$

then we have the following extra properties:

(a) if $p_{k+1} \neq N-1$ then $p_k \in \{p_{k+1}+1, p_{k+1}+2\}$, and if $k \neq N$ then $x_k^j \neq x_{k+1}^j$ for every $1 \leq j \leq p_{k+1}$,
(b) if $p_k = p_{k+1}+2$, then $x_k^{p_k-1} = \binom{0}{1}$ and $x_k^{p_k} = \binom{1}{0}$,
(c) for every $k \leq l \leq N$ and every $p_l < j \leq N-1$, the lth bi-component of the vector x_k^j is null.

We are going to prove this by decreasing induction on $k \in \{1, \ldots, N\}$. Let us begin with the case $k = N$.

Upon taking appropriate linear combinations of basis elements of M and reordering, one may assume that we have a basis $x_N^1, \ldots, x_N^{N-1}$ of M such that the last bi-component is non-zero either for x_N^1 (i.e. $p_N = 1$) or for x_N^1 and x_N^2 (i.e. $p_N = 2$) and is null for the other basis vectors. Moreover, in this last case, one may also require them to be $\binom{0}{1}$ and $\binom{1}{0}$ as in the proof of Proposition 2.2. One may easily notice that the assertion is satisfied for $k = N$.

Assume that the assertion is true for $N, \ldots, k+1$; let us check it for k.

Define $x_k^j = x_{k+1}^j$ for every $1 \leq j \leq p_{k+1}$. Upon taking appropriate linear combinations of the vectors $x_{k+1}^{p_{k+1}+1}, \ldots, x_{k+1}^{N-1}$ and reordering one may get $N-1-p_{k+1}$ vectors $x_k^{p_{k+1}+1}, \ldots, x_k^{N-1}$ with $\operatorname{span}\{x_k^1, \ldots, x_k^{N-1}\} = M$ satisfying one of the following three conditions:

$\triangleright$ The kth bi-component of the vectors $x_k^{p_{k+1}+1}, \ldots, x_k^{N-1}$ is null, i.e. $p_k = p_{k+1}$. But this yields a contradiction: indeed, as R_N is $(N-1)$-supercyclic there exists a strictly increasing sequence $(n_i)_{i \in \mathbb{N}}$ and $N-1$ real sequences $(\lambda_1^{(n_i)})_{i \in \mathbb{N}}, \ldots, (\lambda_{N-1}^{(n_i)})_{i \in \mathbb{N}}$ such that

$$\begin{pmatrix} a_1^{n_i} R_{\theta_1}^{n_i} \left(\sum_{j=1}^{N-1} \lambda_j^{(n_i)} \begin{pmatrix} x_k^j(1) \\ x_k^j(2) \end{pmatrix} \right) \\ \vdots \\ a_k^{n_i} R_{\theta_k}^{n_i} \left(\sum_{j=1}^{p_k} \lambda_j^{(n_i)} \begin{pmatrix} x_k^j(2k-1) \\ x_k^j(2k) \end{pmatrix} \right) \\ \vdots \\ a_N^{n_i} R_{\theta_N}^{n_i} \left(\sum_{j=1}^{p_N} \lambda_j^{(n_i)} \begin{pmatrix} x_k^j(2N-1) \\ x_k^j(2N) \end{pmatrix} \right) \end{pmatrix} \xrightarrow[i \to \infty]{} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}. \tag{2.7}$$

The last bi-component above implies that for every $1 \le j \le p_N$, $a_N^{n_i}\lambda_j^{(n_i)} \to 0$ as $i \to \infty$ because R_N is an isometry, $1 \le p_N \le 2$, and if $p_N = 2$ then

$$\sum_{j=1}^{p_N} \lambda_j^{(n_i)} \begin{pmatrix} x_k^j(2N-1) \\ x_k^j(2N) \end{pmatrix} = \begin{pmatrix} \lambda_2^{(n_i)} \\ \lambda_1^{(n_i)} \end{pmatrix}$$

by induction hypothesis. Step by step, following the same idea, we can prove in the same way that for every $1 \le j \le p_{k+1}$, $a_{k+1}\lambda_j^{(n_i)} \to 0$ as $i \to \infty$ because $|a_{k+1}| \le \cdots \le |a_N|$. Moreover if $p_j = N-1$ for some $j \in \{k+1, \ldots, N\}$ then we conclude at this step that for every $1 \le j \le N-1$, $a_k^{n_i}\lambda_j^{(n_i)} \to 0$ as $i \to \infty$, contradicting (2.7). Since $p_k = p_{k+1}$ and $|a_k| \le |a_{k+1}|$, for every $1 \le j \le p_k$, $a_k^{n_i}\lambda_j^{(n_i)} \to 0$ as $i \to \infty$, which contradicts (2.7).

$\triangleright$ The kth bi-component of the vectors $x_k^{p_{k+1}+2}, \ldots, x_k^{N-1}$ is null but not for $x_k^{p_{k+1}+1}$. Then $p_k = p_{k+1} + 1$ and for every $p_k < j \le N-1$, the kth bi-component of the vector x_k^j is null by construction and for every $k+1 \le l \le N$ and every $p_l < j \le N-1$, the lth bi-component of x_k^j is also null because $p_k > p_{k+1} > \cdots > p_N$ and the family $\{x_k^{p_{k+1}+1}, \ldots, x_k^{N-1}\}$ is obtained by taking linear combinations of the vectors $x_{k+1}^{p_{k+1}+1}, \ldots, x_{k+1}^{N-1}$ whose lth bi-component is null by induction hypothesis.

$\triangleright$ The kth bi-component of the vectors $x_k^{p_{k+1}+3}, \ldots, x_k^{N-1}$ is null but not for $x_k^{p_{k+1}+1}$ and $x_k^{p_{k+1}+2}$, and these two components can be chosen to be $\binom{0}{1}$ and $\binom{1}{0}$. Here, $p_k = p_{k+1} + 2$ and we conclude as above.

This ends the induction process.

Let us denote by $y^1, \ldots, y^{N-1}$ the vectors $x_1^1, \ldots, x_1^{N-1}$ obtained thanks to the induction process. We proved that the sequence $(p_{N+1-k})_{0 \le k \le N}$ is increasing until it reaches $N-1$ and is constant afterwards and $p_{N+1} = 0$, hence $p_2 = N-1$. This remark now permits us to conclude. Indeed, as M is an $(N-1)$-supercyclic subspace for R_N, there exist a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ and $N-1$ real sequences $(\lambda_1^{(n_i)})_{i\in\mathbb{N}}, \ldots, (\lambda_{N-1}^{(n_i)})_{i\in\mathbb{N}}$ such that

$$\begin{pmatrix} a_1^{n_i} R_{\theta_1}^{n_i} \left(\sum_{j=1}^{N-1} \lambda_j^{(n_i)} \begin{pmatrix} x_1^j \\ x_2^j \end{pmatrix} \right) \\ a_2^{n_i} R_{\theta_2}^{n_i} \left(\sum_{j=1}^{N-1} \lambda_j^{(n_i)} \begin{pmatrix} x_3^j \\ x_4^j \end{pmatrix} \right) \\ \vdots \\ a_N^{n_i} R_{\theta_N}^{n_i} \left(\sum_{j=1}^{p_N} \lambda_j^{(n_i)} \begin{pmatrix} x_{2N-1}^j \\ x_{2N}^j \end{pmatrix} \right) \end{pmatrix} \xrightarrow[i\to\infty]{} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}. \tag{2.8}$$

By a similar reasoning to the induction process with $p_k = p_{k+1}$, we observe that for every $1 \le j \le p_2 = N-1$, $a_2^{n_i}\lambda_j^{(n_i)} \to 0$ as $i \to \infty$, and since $|a_1| \le |a_2|$, for every $1 \le j \le N-1$, $a_1^{n_i}\lambda_j^{(n_i)} \to 0$ as $i \to \infty$, which contradicts (2.8). ∎

The key in the proof is the adaptation of the basis to the shape of R_N and we are going to make constant use of this method in what follows. This motivated us to detail this method in the next subsection.

2.3. Basis reduction. Let $m, N \in \mathbb{N}$, T be a linear operator on $\mathbb{R}^N$, $\{x^1, \dots, x^m\}$ be a linearly independent family in $\mathbb{R}^N$ and M be the subspace spanned by this family. Using the Jordan real decomposition one may suppose that

$$T = \begin{pmatrix} a_1\mathcal{B}_1 & 0 & \cdots & 0 \\ 0 & a_2\mathcal{B}_2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_\gamma\mathcal{B}_\gamma \end{pmatrix},$$

where

$$\mathcal{B}_i = \begin{pmatrix} \mathcal{A}_i & \mathcal{A}_i & 0 & \cdots & 0 \\ 0 & \mathcal{A}_i & \mathcal{A}_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & \mathcal{A}_i \\ 0 & \cdots & & 0 & \mathcal{A}_i \end{pmatrix}$$

is a classical or real Jordan block for any $1 \le i \le \gamma$ with $\mathcal{A}_i = 1$ or $\mathcal{A}_i = R_{\theta_i}$ respectively, and γ is the number of Jordan blocks in the decomposition of T. Define $\tau_i = 1$ when $\mathcal{B}_i$ is classical and $\tau_i = 2$ when $\mathcal{B}_i$ is real, and take also ρ_i such that $\tau_i\rho_i$ is $\mathcal{B}_i$'s size; we will call ρ_i the *relative size* of the block $\mathcal{B}_i$. If $\mathcal{B}_i$ is a classical Jordan block, then its relative size is just its size, while if $\mathcal{B}_i$ is a real Jordan block, then its relative size is its size divided by 2. We will also denote by $\rho := \sum_{i=1}^{\gamma}\rho_i$ the relative size of the matrix of T. Observe that with these notations, $N = \sum_{i=1}^{\gamma}\rho_i\tau_i$.

NOTATION. For the sake of clarity, we introduce a new notation before stating the next theorem. Let T be a linear operator on $\mathbb{R}^N$ in the above Jordan form and $x \in \mathbb{R}^N$. We define, for $1 \le i \le \rho$,

$$\chi_i(x) = \begin{cases} x_{\sum_{l=1}^{p-1}\tau_l\rho_l + i - \sum_{l=1}^{p-1}\rho_l} & \text{if } \tau_p = 1, \\ \begin{pmatrix} x_{\sum_{l=1}^{p-1}\tau_l\rho_l + 2(i-\sum_{l=1}^{p-1}\rho_l)-1} \\ x_{\sum_{l=1}^{p-1}\tau_l\rho_l + 2(i-\sum_{l=1}^{p-1}\rho_l)} \end{pmatrix} & \text{if } \tau_p = 2, \end{cases}$$

where p is the unique natural number satisfying $\sum_{l=1}^{p-1}\rho_l < i \le \sum_{l=1}^{p}\rho_l$. Roughly speaking, p is the number of the block $\mathcal{B}_p$ of T which acts on $\chi_i(x)$. This probably seems a bit complicated at first sight but the underlying idea is natural: the operator T is seen as almost a "sum" of operators $\mathcal{A}_i$ acting on either $\mathbb{R}$ or $\mathbb{R}^2$. Then it is natural to consider the vectors T is acting on, as a direct sum of vectors that the operators $\mathcal{A}_i$ are acting on. To summarise, on some parts (classical) T acts like an operator on $\mathbb{R}$, and on the others (real) it acts as on $\mathbb{R}^2$, thus $\chi_i(x)$ may be either a scalar or a vector of size 2.

Let us explain this on an example. Consider the operator on $\mathbb{R}^7$ defined by

$$T=\begin{pmatrix} a\mathcal{B}_1 & 0 & 0\\ 0 & b\mathcal{B}_2 & 0\\ 0 & 0 & c\mathcal{B}_3\end{pmatrix}=\begin{pmatrix} a & 0 & 0 & 0 & 0\\ 0 & bR_\theta & bR_\theta & 0 & 0\\ 0 & 0 & bR_\theta & 0 & 0\\ 0 & 0 & 0 & c & 0\\ 0 & 0 & 0 & 0 & c\end{pmatrix}.$$

Then we have $\tau_1=1$, $\tau_2=2$, $\tau_3=1$, $\rho_1=1$, $\rho_2=2$, $\rho_3=2$ and we shall decompose $x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ as

$$x=\begin{pmatrix}\chi_1(x)\\ \chi_2(x)\\ \chi_3(x)\\ \chi_4(x)\\ \chi_5(x)\end{pmatrix}$$

with $\chi_1(x)=x_1$, $\chi_2(x)=\binom{x_2}{x_3}$, $\chi_3(x)=\binom{x_4}{x_5}$, $\chi_4(x)=x_6$, $\chi_5(x)=x_7$.

Let us state the awaited theorem which is the main tool to prove the results announced at the beginning of the article.

Theorem 2.6. *Let T be a linear operator on $\mathbb{R}^N$ in Jordan form. Let also M be an m-dimensional subspace. Then there exist a basis $\{y^1,\dots,y^m\}$ of M, a non-decreasing sequence $(\kappa_i)_{i\in\mathbb{Z}_+}$ of integers and a sequence $(\Lambda_i)_{i\in\mathbb{Z}_+}\subset\mathbb{R}\cup\mathbb{R}^2$ of sets satisfying:*

(a) $\kappa_0=1$, $\Lambda_0=\{\chi_\rho(y^j):\kappa_0\le j\le m\}$.

(b) *For every $i\in\mathbb{Z}_+$,*

$$\kappa_{i+1}=\kappa_i+\dim(\operatorname{span}\{\Lambda_i\}),\quad \Lambda_{i+1}=\{\chi_{\rho-(i+1)}(y^j):\kappa_{i+1}\le j\le m\}.$$

(c) *For every $i\in\{0,\dots,\rho-1\}$, $\{\chi_{\rho-i}(y^j):\kappa_i\le j<\kappa_{i+1}\}$ is either empty or linearly independent.*

(d) $\kappa_\rho=m+1$.

(e) *For every $p\in\{1,\dots,\rho\}$ and $j\in\{\kappa_p,\dots,m\}$, $\chi_{\rho-p+1}(y^j)=0$ or $\binom{0}{0}$.*

Proof. We want to construct a basis of M adapted to the decomposition of T. Of course, this reduction heavily depends on T. Let $x^1, \dots, x^m$ be any basis of M. We are going to define an increasing sequence $(\kappa_p)_{p\in\mathbb{Z}_+}$ of natural numbers and a sequence $(\Lambda'_p)_{p\in\mathbb{Z}_+}$ of sets. For every step of the reduction, the sequence of natural numbers marks the vector number up to which the reduction has been completed, and the sequence of sets contains the part of the vectors that we have to reduce on the next step.

First define $\kappa_0 = 1$ and $\Lambda'_0 = \{\chi_\rho(x^i) : \kappa_0 \le i \le m\}$. By definition, Λ'_0 is either a subset of $\mathbb{R}$ or of $\mathbb{R}^2$, so $\dim(\mathrm{span}\{\Lambda'_0\}) = 0, 1$ or 2.

• If $\dim(\mathrm{span}\{\Lambda'_0\}) = 0$, then $\|\chi_\rho(x^i)\| = 0$ for any $1 \le i \le m$, and we set $\kappa_1 := \kappa_0$ and $x_1^j := x^j$ for every $\kappa_0 \le j \le m$.

• If $\dim(\mathrm{span}\{\Lambda'_0\}) = 1$, upon taking proper linear combinations of $x^1, \dots, x^m$ and reordering, one may obtain a new basis $x_1^1, \dots, x_1^m$ of M with $\|\chi_\rho(x_1^1)\| = 1$ and $\|\chi_\rho(x_1^i)\| = 0$ for any $\kappa_0 + 1 \le i \le m$ and set $\kappa_1 := \kappa_0 + 1$.

• If $\dim(\mathrm{span}\{\Lambda'_0\}) = 2$, upon taking proper linear combinations of $x^1, \dots, x^m$ and reordering, one may obtain a new basis $x_1^1, \dots, x_1^m$ of M with $\chi_\rho(x_1^1) = \binom{0}{1}$, $\chi_\rho(x_1^2) = \binom{1}{0}$ and $\|\chi_\rho(x_1^i)\| = 0$ for $\kappa_0 + 2 \le i \le m$ and set $\kappa_1 := \kappa_0 + 2$.

Then set also $\Lambda'_1 = \{\chi_{\rho-1}(x_1^i) : \kappa_1 \le i \le m\}$. Thus the dimension of $\mathrm{span}\{\Lambda'_1\}$ is either $0, 1$ or 2. Define $x_2^i = x_1^i$ for every $1 \le i < \kappa_1$. Upon taking appropriate linear combinations of the vectors $x_1^{\kappa_1}, \dots, x_1^m$ and reordering one may get $m - \kappa_1 + 1$ vectors $x_2^{\kappa_1}, \dots, x_2^m$ with $\mathrm{span}\{x_2^1, \dots, x_2^m\} = M$ satisfying one of the following three conditions:

• If $\dim(\mathrm{span}\{\Lambda'_1\}) = 0$, then $\|\chi_{\rho-1}(x_2^i)\| = 0$ for any $\kappa_1 \le i \le m$, and we set $\kappa_2 := \kappa_1$.

• If $\dim(\mathrm{span}\{\Lambda'_1\}) = 1$, then $\|\chi_{\rho-1}(x_2^{\kappa_1})\| = 1$ and $\|\chi_{\rho-1}(x_2^i)\| = 0$ for any $\kappa_1 + 1 \le i \le m$, and we set $\kappa_2 := \kappa_1 + 1$.

• If $\dim(\mathrm{span}\{\Lambda'_1\}) = 2$, then $\chi_{\rho-1}(x_2^{\kappa_1}) = \binom{0}{1}$, $\chi_{\rho-1}(x_2^{\kappa_1+1}) = \binom{1}{0}$ and $\|\chi_{\rho-1}(x_2^i)\| = 0$ for $\kappa_1 + 2 \le i \le m$, and we set $\kappa_2 := \kappa_1 + 2$.

Suppose that this construction has been carried out until we obtain $x_k^1, \dots, x_k^m$. Then set $\Lambda'_k = \{\chi_{\rho-k}(x_k^i) : \kappa_k \le i \le m\}$, thus $\dim(\mathrm{span}\{\Lambda'_k\}) = 0, 1$ or 2. Define $x_{k+1}^i = x_k^i$ for every $1 \le i < \kappa_k$. Upon taking appropriate linear combinations of the vectors $x^{\kappa_k}, \dots, x^m$ and reordering one may get $m - \kappa_k + 1$ vectors $x_{k+1}^{\kappa_k}, \dots, x_{k+1}^m$ with $\mathrm{span}\{x_{k+1}^1, \dots, x_{k+1}^m\} = M$ satisfying one of the following three conditions:

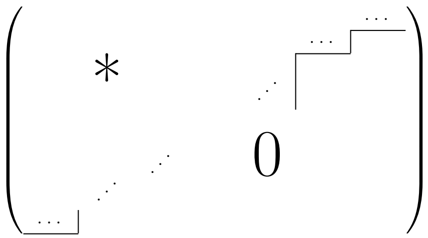
• If $\dim(\mathrm{span}\{\Lambda'_k\}) = 0$, then $\|\chi_{\rho-k}(x_{k+1}^i)\| = 0$ for any $\kappa_k \le i \le m$, and we set $\kappa_{k+1} := \kappa_k$.

• If $\dim(\mathrm{span}\{\Lambda'_k\}) = 1$, then $\|\chi_{\rho-k}(x_{k+1}^{\kappa_k})\| = 1$ and $\|\chi_{\rho-k}(x_{k+1}^i)\| = 0$ for any $\kappa_k + 1 \le i \le m$, and we set $\kappa_{k+1} := \kappa_k + 1$.

• If $\dim(\operatorname{span}\{\Lambda'_k\}) = 2$, then $\chi_{\rho-k}(x_{k+1}^{\kappa_k}) = \binom{0}{1}$, $\chi_{\rho-k}(x_{k+1}^{\kappa_1+1}) = \binom{1}{0}$ and $\|\chi_{\rho-k}(x_{k+1}^{i})\| = 0$ for $\kappa_k + 2 \le i \le m$, and we set $\kappa_{k+1} := \kappa_k + 2$.

As a consequence, step by step we finally get a basis $(y^1, \ldots, y^m) := (x_\rho^1, \ldots, x_\rho^m)$ of M and we set $\kappa_q := \kappa_\rho$ for every $q > \rho$. We set $\Lambda_0 = \{\chi_\rho(y^j) : \kappa_0 \le j \le m\}$ and $\Lambda_i = \{\chi_{\rho-i}(y^j) : \kappa_i \le j \le m\}$. Thus (a) is satisfied by definition. It suffices to remark that then $\dim(\operatorname{span}\{\Lambda_i\}) = \dim(\operatorname{span}\{\Lambda'_i\})$ to check (b), (c) and (e). Moreover, (d) is also satisfied as $(y^1, \ldots, y^m)$ form a basis of M, so y^m is non-zero. Hence $\Lambda_\rho = \emptyset$ ($\Leftrightarrow \kappa_\rho = m + 1$). ■

Remark 2.7. The reduced basis we have described in the previous theorem has the following inverse staircase shape:



We keep this notation for the rest of this paper. The reader needs to have in mind this notation when we decompose an operator in Jordan form or when we reduce a basis. When we need to refer to Theorem 2.6, we will say that some basis has been reduced with respect to an operator.

Notation. From now on, we will need to work with several vectors $x^1, \ldots, x^m$. For this reason, we replace the heavy notation $\chi_i(x^j)$ we introduced before Theorem 2.6 by a shorter one, χ_i^j.

2.4. Primary matrices

Definition 2.8. Let $\rho, N \in \mathbb{N}$. An operator T on $\mathbb{R}^N$ is said to be *primary* of order ρ when $T = \bigoplus_{i=1}^{\rho} \mathcal{A}_i$ with $\mathcal{A}_i = 1$ or R_{θ_i} with $\theta_i \in \mathbb{R}$.

Remark 2.9. One can see at first glance that if T is primary of order ρ on $\mathbb{R}^N$, then $\rho \in [\![\lfloor (N+1)/2 \rfloor, N]\!]$. Moreover, ρ is the relative size of T.

We begin our study from primary matrices. Even though the next result is a partial generalisation of Proposition 2.5, their proofs are independent. Moreover, its proof puts forward some useful ideas.

Proposition 2.10. *Let $\rho \in \mathbb{N}$. There is no $(\rho - 1)$-supercyclic primary matrix of order ρ.*

Proof. Let $T = \bigoplus_{i=1}^{\rho} \mathcal{A}_i$ be a primary matrix of order ρ. Following the notation we introduced before, $\rho_i = 1$ for every $1 \le i \le \rho$. Now suppose, in order to obtain a contradiction, that T is $(\rho - 1)$-supercyclic. Let

$M = \operatorname{span}\{x^1, \dots, x^{\rho-1}\}$ be a $(\rho-1)$-supercyclic subspace for T and then reduce the basis of M by using Theorem 2.6.

First, note that $\kappa_p \neq \kappa_{p+1}$ for any $p < \rho$. Indeed, assume otherwise and let $p < \rho$ be the smallest integer such that $\kappa_p = \kappa_{p+1}$. This implies $\dim(\operatorname{span}\{\Lambda_p\}) = 0$ and thus $\|\chi^j_{\rho-p}\| = 0$ for any $\kappa_p \le j \le \rho - 1$. But, for all $i \in \mathbb{N}$ and all real sequences $(\lambda_j)_{1\le j\le\rho-1}$,

$$T^i\Big(\sum_{j=1}^{\rho-1}\lambda_j x^j\Big) = \begin{cases} \mathcal{A}_1^i\Big(\sum_{j=1}^{\rho-1}\lambda_j\chi_1^j\Big) & (L_1), \\ \vdots & \vdots \\ \mathcal{A}_{\rho-p}^i\Big(\sum_{j=1}^{\kappa_p-1}\lambda_j\chi_{\rho-p}^j\Big) & (L_{\rho-p}), \\ \mathcal{A}_{\rho-p+1}^i\Big(\sum_{j=1}^{\kappa_p-1}\lambda_j\chi_{\rho-p+1}^j\Big) & (L_{\rho-p+1}), \\ \vdots & \vdots \\ \mathcal{A}_{\rho}^i\Big(\sum_{j=1}^{\kappa_1-1}\lambda_j\chi_{\rho}^j\Big) & (L_{\rho}). \end{cases}$$

Clearly, if $p = 0$ then $(L_\rho) = 0$ and M fails to be $(\rho-1)$-supercyclic for T. In the following, we assume $p > 0$ without loss of generality. Then, by $(\rho-1)$-supercyclicity of M, there exist $(n_i)_{i\in\mathbb{N}}$ and $\rho - 1$ real sequences $(\lambda_1^{(n_i)})_{i\in\mathbb{N}}, \dots, (\lambda_{\rho-1}^{(n_i)})_{i\in\mathbb{N}}$ such that for any $j \in [\![1,\rho]\!] \setminus \{\rho - p\}$,

$$(L_j) \xrightarrow[i\to\infty]{} 0 \quad \text{and} \quad (L_{\rho-p}) \xrightarrow[i\to\infty]{} Y \quad \text{with } \|Y\| = 1.$$

We shall prove that for any $1 \le j < \kappa_p$, $\lambda_j^{(n_i)} \to 0$ as $i \to \infty$. Such an integer j belongs to a unique interval $[\kappa_q, \kappa_{q+1}[$ and we shall prove this property by induction on q.

If $1 \le j < \kappa_1$, then $\{\chi_\rho^j\}_{1\le j<\kappa_1} \neq \emptyset$ is a linearly independent family and, $\mathcal{A}_\rho$ being an isometry, (L_ρ) gives $\sum_{j=1}^{\kappa_1-1}\lambda_j^{(n_i)}\chi_\rho^j \to 0$ as $i \to \infty$, hence $\lambda_j^{(n_i)} \to 0$ as $i \to \infty$ for every $1 \le j < \kappa_1$.

We assume that the assertion is true for $1 \le q < p$. We have to prove it for $q + 1$ too. Since $(L_{\rho-q})$ converges to 0 and $\mathcal{A}_{\rho-q}$ is an isometry, we have $\sum_{j=1}^{\kappa_{q+1}-1}\lambda_j^{(n_i)}\chi_{\rho-q}^j \to 0$ as $i \to \infty$. The recurrence hypothesis implies $\lambda_j^{(n_i)} \to 0$ as $i \to \infty$ for any $1 \le j < \kappa_q$. Hence $\sum_{j=\kappa_q}^{\kappa_{q+1}-1}\lambda_j^{(n_i)}\chi_{\rho-q}^j \to 0$ as $i \to \infty$. However, $\{\chi_{\rho-q}^j\}_{\kappa_q\le j<\kappa_{q+1}} \neq \emptyset$ is a linearly independent family by the reduction properties, and because $\kappa_q \neq \kappa_{q+1}$ we have $\lambda_j^{(n_i)} \to 0$ as $i \to \infty$ for every $1 \le j < \kappa_{q+1}$. This ends the induction step.

Thus, for any $1 \leq j < \kappa_p$, $\lambda_j^{(n_i)} \to 0$ as $i \to \infty$. Considering these limits in $(L_{\rho-p})$ and the fact that $\mathcal{A}_{\rho-p}$ is an isometry yields

$$\mathcal{A}_{\rho-p}^{i}\Big(\sum_{j=1}^{\kappa_p-1} \lambda_j^{(n_i)} \chi_{\rho-p}^{j}\Big) \xrightarrow[i\to\infty]{} 0.$$

But this contradicts the convergence of $(L_{\rho-p})$ to some unit vector. Hence $\kappa_p \neq \kappa_{p+1}$ for every $p < \rho$.

Considering that $\kappa_0 = 1$ and that the sequence $(\kappa_p)_{0\leq p\leq \rho}$ is increasing, one obtains $\kappa_{\rho-1} \geq \rho$, hence by Theorem 2.6 $\kappa_\rho = \kappa_{\rho-1}$. This contradiction proves that T is not $(\rho - 1)$-supercyclic. ∎

2.5. For a single real Jordan block. The aim of this section is to generalise Proposition 2.2 to the case of a real Jordan block of arbitrary dimension. The following two lemmas are useful to express in another way the iterates of a subspace by a real Jordan block.

LEMMA 2.11. *Define*

$$\Delta_n(i) := \binom{i}{n} - \sum_{k=1}^{n-1} \Delta_k(i)\binom{i}{n-k} \qquad \textit{for any } n, i \geq 0.$$

Then Δ_n is a polynomial in i of degree n and its leading coefficient is $(-1)^{n+1}/n!$.

Proof. We argue by induction on $n \geq 0$. The lemma is obviously true for $n = 1$ because $\Delta_1(i) = i$.

Assume that we have verified the assertion for $1 \leq k < n$. Let us prove it for $k = n$. Denote by δ_n the leading coefficient of Δ_n. The leading coefficient in i of $\binom{i}{k}$ is $1/k!$ for all $k \in \mathbb{Z}_+$. Combining this with the induction hypothesis, one gets

$$\delta_n = \frac{1}{n!} - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{(n-k)!k!} = \frac{1}{n!}\Big(1 - \sum_{k=1}^{n-1}(-1)^{k+1}\binom{n}{k}\Big).$$

Now, it is easy to check that

$$\sum_{k=1}^{n-1}\binom{n}{k}(-1)^{k+1} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

This yields

$$\delta_n = \begin{cases} 1/n! & \text{if } n \text{ is odd,} \\ -1/n! & \text{if } n \text{ is even.} \end{cases}$$

This ends the induction and the proof of the lemma. ∎

In order to fully understand the interest of introducing the sequence Δ_n, we also need the following lemma:

LEMMA 2.12. *Let $i, n \in \mathbb{N}$ with $i \geq n$ and let $(u_k)_{1 \leq k \leq n}$ be a sequence of real numbers. For every $1 \leq k \leq n$ define*

$$L_k := \sum_{j=0}^{n-k} \binom{i}{j} u_{k+j}.$$

Then

$$L_k = u_k + \sum_{j=1}^{n-k} \Delta_j(i) L_{k+j}.$$

Proof. Once again, we use induction on n. For $n = 1$, the result is straightforward since $L_1 = \sum_{j=0}^{1-1} \binom{i}{j} u_{1+j} = u_1$.

Now, assume that the assertion is true for any natural number strictly smaller than n and let us prove it for n. For $1 \leq k < n$, set $\mathcal{L}_k = \sum_{j=0}^{n-k-1} \binom{i}{j} u_{k+j} = L_k - \binom{i}{n-k} u_n$. Then the induction hypothesis gives

$$\mathcal{L}_k = u_k + \sum_{j=1}^{n-k-1} \Delta_j(i) \mathcal{L}_{k+j}$$

and so for $1 \leq k \leq n-1$,

$$L_k = \mathcal{L}_k + \binom{i}{n-k} u_n = u_k + \sum_{j=1}^{n-k-1} \Delta_j(i) \mathcal{L}_{k+j} + \binom{i}{n-k} u_n.$$

Finally, using the definition of $\mathcal{L}_k$ and $\Delta_{n-k}(i)$ and noting that $L_n = u_n$, we obtain

$$\begin{aligned} L_k &= u_k + \sum_{j=1}^{n-k-1} \Delta_j(i) \Big(L_{k+j} - \binom{i}{n-k-j} u_n \Big) + \binom{i}{n-k} u_n \\ &= u_k + \sum_{j=1}^{n-k-1} \Delta_j(i) L_{k+j} + \bigg(\binom{i}{n-k} - \sum_{j=1}^{n-k-1} \Delta_j(i) \binom{i}{n-k-j} \bigg) u_n \\ &= u_k + \sum_{j=1}^{n-k-1} \Delta_j(i) L_{k+j} + \Delta_{n-k}(i) u_n \ = u_k + \sum_{j=1}^{n-k} \Delta_j(i) L_{k+j}. \ \blacksquare \end{aligned}$$

Here comes a generalisation of Proposition 2.2; its proof is of importance to understanding the mechanisms involved in later proofs.

PROPOSITION 2.13. *Let $N > 1$ and $\theta \in \mathbb{R}$. Then the operator defined by*

$$J_N := \begin{pmatrix} R_\theta & R_\theta & 0 & \cdots & 0 \\ 0 & R_\theta & R_\theta & 0 \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & R_\theta \\ 0 & \cdots & 0 & 0 & R_\theta \end{pmatrix}$$

is not N-supercyclic on $\mathbb{R}^{2N}$.

Proof. As we already noticed, Proposition 2.2 proves the case $N = 2$. Let then $N \geq 3$ and assume to the contrary that J_N is N-supercyclic. Let also $M = \operatorname{span}\{x^1, \dots, x^N\}$ be an N-supercyclic subspace whose basis $x^1, \dots, x^N$ is reduced. Then $\kappa_N = N + 1$ as pointed out in Theorem 2.6. Moreover, Proposition 2.2 claims that J_2 is not 2-supercyclic, thus the vectors

$$\begin{pmatrix} \chi^1_{N-1} \\ \chi^1_N \end{pmatrix}, \quad \begin{pmatrix} \chi^2_{N-1} \\ \chi^2_N \end{pmatrix}, \quad \begin{pmatrix} \chi^3_{N-1} \\ \chi^3_N \end{pmatrix}$$

span a subspace of dimension 3, yielding $\kappa_2 \geq 4$. In addition, N-supercyclicity of M implies the existence of a sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers and of $((\lambda_1^{(n_i)})_{i\in\mathbb{N}}, \dots, (\lambda_N^{(n_i)})_{i\in\mathbb{N}}) \in (\mathbb{R}^{\mathbb{N}})^N$ such that

$$T^{n_i}\Big(\sum_{j=1}^N \lambda_j^{(n_i)} x^j\Big) = \begin{pmatrix} R_\theta^{n_i} \displaystyle\sum_{k=0}^{N-1} \binom{n_i}{k} \Big(\sum_{j=1}^N \lambda_j^{(n_i)} \begin{pmatrix} x^j_{2k+1} \\ x^j_{2(k+1)} \end{pmatrix}\Big) \\ R_\theta^{n_i} \displaystyle\sum_{k=0}^{N-2} \binom{n_i}{k} \Big(\sum_{j=1}^N \lambda_j^{(n_i)} \begin{pmatrix} x^j_{2(k+1)+1} \\ x^j_{2(k+2)} \end{pmatrix}\Big) \\ \vdots \\ R_\theta^{n_i} \displaystyle\sum_{j=1}^N \lambda_j^{(n_i)} \begin{pmatrix} x^j_{2N-1} \\ x^j_{2N} \end{pmatrix} \end{pmatrix}$$

$$\xrightarrow[i\to\infty]{} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}.$$

Denote by $(L_1), \dots, (L_N)$ the lines appearing in $T^{n_i}(\sum_{j=1}^N \lambda_j^{(n_i)} x^j)$ above, and define

$$(u_k) = \sum_{j=1}^N \lambda_j^{(n_i)} \begin{pmatrix} x^j_{2k-1} \\ x^j_{2k} \end{pmatrix}.$$

We remark that the (L_k)'s and (u_k)'s depend on i. Then, using Lemma 2.12 the preceding identity implies

$$
(2.9) \qquad \begin{cases} \left\|(u_1) + \sum_{j=1}^{N-1} \Delta_j(n_i)(L_{j+1})\right\| \xrightarrow[i\to\infty]{} 0, \\ \qquad \vdots \\ \left\|(u_k) + \sum_{j=1}^{N-k} \Delta_j(n_i)(L_{j+k})\right\| \xrightarrow[i\to\infty]{} 0, \\ \qquad \vdots \\ \|(u_N)\| \xrightarrow[i\to\infty]{} 1, \end{cases}
$$

where Δ_j is defined in Lemma 2.11.

We now come to the key point of the proof: we prove by induction on k that $\lambda_j^{(n_i)}/\Delta_k(n_i) \to 0$ as $i \to \infty$, for every $1 \le k \le N-1$ and every $1 \le j \le \kappa_k - 1$.

If $k = 1$, then we divide (L_N) by $\Delta_1(n_i)$ and take the limit:

$$
\left\| \sum_{j=\kappa_0}^{\kappa_1 - 1} \frac{\lambda_j^{(n_i)}}{\Delta_1(n_i)} \chi_N^j \right\| \xrightarrow[i\to\infty]{} 0.
$$

In addition, the fact that $\{\chi_N^j\}_{\kappa_0 \le j \le \kappa_1 - 1}$ is linearly independent (and not empty since $\kappa_2 \ge 4$) leads to

$$
\frac{\lambda_j^{(n_i)}}{\Delta_1(n_i)} \xrightarrow[i\to\infty]{} 0 \qquad \text{for any } \kappa_0 \le j \le \kappa_1 - 1.
$$

We now assume that the assertion is true for any natural number smaller than k and we prove it for $k+1$. First, we divide (L_{N-k}) by $\Delta_{k+1}(n_i)$ and take the limit:

$$
\left\| \sum_{j=\kappa_0}^{\kappa_k - 1} \frac{\lambda_j^{(n_i)}}{\Delta_{k+1}(n_i)} \chi_{N-k}^j + \sum_{j=\kappa_k}^{\kappa_{k+1} - 1} \frac{\lambda_j^{(n_i)}}{\Delta_{k+1}(n_i)} \chi_{N-k}^j + \sum_{j=1}^{k} \frac{\Delta_j(n_i)}{\Delta_{k+1}(n_i)} (L_{N-k+j}) \right\| \xrightarrow{i\to\infty} 0.
$$

The induction hypothesis shows that the leftmost sum above converges to 0. Moreover, as the sequence given by the jth line (L_j) is bounded for all $1 \le j \le N$, and as Lemma 2.11 gives $\deg(\Delta_{k+1}) > \deg(\Delta_j)$ for every $1 \le j \le k$, the rightmost sum also converges to 0. This yields

$$
\left\| \sum_{j=\kappa_k}^{\kappa_{k+1} - 1} \frac{\lambda_j^{(n_i)}}{\Delta_{k+1}(n_i)} \chi_{N-k}^j \right\| \xrightarrow[i\to\infty]{} 0.
$$

Moreover, $\{\chi_{N-k}^j\}_{\kappa_k \le j \le \kappa_{k+1} - 1}$ is either linearly independent or empty, and taking this into account in the line above and using Lemma 2.11, we conclude that

$$\frac{\lambda_j^{(n_i)}}{\Delta_{k+1}(n_i)} \xrightarrow[i\to\infty]{} 0 \qquad \text{for any } \kappa_k \le j \le \kappa_{k+1}-1.$$

So by induction hypothesis and as $\deg(\Delta_{k+1}) > \deg(\Delta_j)$ for every $1 \le j \le k$, we conclude that for every $1 \le j \le \kappa_{k+1}-1$, $\lambda_j^{(n_i)}/\Delta_{k+1}(n_i) \to 0$ as $i \to \infty$.

We now come back to the proof of the proposition. As we claimed before, $\kappa_2 \ge 4$ and $\kappa_N = N+1$. It follows that there exists $2 \le p \le N-1$ such that $\kappa_p = \kappa_{p+1}$. Divide then (L_{N-p}) by $\Delta_p(n_i)$ to get

$$\Bigg\| \sum_{j=\kappa_0}^{\kappa_{p+1}-1} \frac{\lambda_j^{(n_i)}}{\Delta_p(n_i)} \chi_{N-k}^j + \sum_{j=1}^{p} \frac{\Delta_j(n_i)}{\Delta_p(n_i)} (L_{N-p+j}) \Bigg\| \xrightarrow[i\to\infty]{} 0.$$

The left sum above tends to 0 because $\lambda_j^{(n_i)}/\Delta_p(n_i) \to 0$ as $i \to \infty$ for any $1 \le j \le \kappa_p - 1$. Furthermore, combine Lemma 2.11 and the boundedness of the sequence given by the kth line (L_k) to deal with the second sum:

$$\Bigg\| \frac{(-1)^{p+1}}{p!} (L_N) \Bigg\| \xrightarrow[i\to\infty]{} 0.$$

This contradicts the convergence $\|(L_N)\| \to 1$ as $i \to \infty$ given in (2.9). So, J_N is not N-supercyclic. ∎

2.6. Sum of Jordan blocks with different moduli. Later, we will need to be able to distinguish the behaviour of blocks with different moduli. The main idea is that all the coefficients we introduced in older blocks do not have significant influence in the new block. The following lemma is a technical tool towards this idea.

LEMMA 2.14. *Let $h \in \mathbb{Z}_+$ and $\gamma, m, N \in \mathbb{N}$ with $h < m$. Let also $T = a\mathcal{C}$ be an operator on $\mathbb{R}^N$ with $0 < |a| < 1$ and $\mathcal{C} = \bigoplus_{i=1}^{\gamma} \mathcal{B}_i$, $\mathcal{B}_i$ being a Jordan block of modulus 1 with $\mathcal{A}_i = 1$ or R_{θ_i}. Let M be an $(m-h)$-dimensional subspace and $x^1, \dots, x^m \in \mathbb{R}^N$ where $x^{h+1}, \dots, x^m$ denotes a reduced basis of M (adapted to T via Theorem 2.6), and let also $0 < |a| < |b| \le 1$. Assume that there exist a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ and m real sequences $(\lambda_1^{(n_i)})_{i\in\mathbb{N}}, \dots, (\lambda_m^{(n_i)})_{i\in\mathbb{N}}$ and $q \in \mathbb{Z}_+$ such that $b^{n_i}\lambda_j^{(n_i)}/n_i^q \to 0$ as $i \to \infty$ for any $1 \le j \le h$ and $T^{n_i}(\sum_{j=1}^m \lambda_j^{(n_i)} x^j) \to 0$ as $i \to \infty$. Then there exists $q' \in \mathbb{Z}_+$ satisfying $a^{n_i}\lambda_j^{(n_i)}/n_i^{q'} \to 0$ as $i \to \infty$ for any $1 \le j \le m$.*

Proof. Denote as usual by $\tau_i\rho_i$ the size of the block $\mathcal{B}_i$ with $\tau_i = 1, 2$ and $\rho = \sum_{i=1}^{\gamma} \rho_i$. Then reducing T and keeping in mind that the last $m-h$ vectors from $\{x_1, \dots, x_m\}$ are reduced, one may express $T^{n_i}(\sum_{j=1}^m \lambda_j^{(n_i)} x^j)$

as follows:

$$\begin{cases} a^{n_i}\mathcal{A}_1^{n_i}\Big(\sum_{j=1}^{m}\lambda_j^{(n_i)}\chi_1^j+\sum_{j=1}^{\rho_1-1}\binom{n_i}{j}\sum_{g=1}^{h+\kappa_{\rho-j}-1}\lambda_g^{(n_i)}\chi_{1+j}^g\Big) & (L_1),\\ \vdots & \vdots\\ a^{n_i}\mathcal{A}_1^{n_i}\Big(\sum_{j=1}^{h+\kappa_{\rho+1-\rho_1}-1}\lambda_j^{(n_i)}\chi_{\rho_1}^j\Big) & (L_{\rho_1}),\\ \vdots & \vdots\\ a^{n_i}\mathcal{A}_\gamma^{n_i}\Big(\sum_{j=1}^{h+\kappa_{\rho_\gamma}-1}\lambda_j^{(n_i)}\chi_{\rho+1-\rho_\gamma}^j & \\ \qquad +\sum_{j=1}^{\rho_\gamma-1}\binom{n_i}{j}\sum_{g=1}^{h+\kappa_{\rho_\gamma-j}-1}\lambda_g^{(n_i)}\chi_{\rho-\rho_\gamma+1+j}^g\Big) & (L_{\rho-\rho_\gamma+1}),\\ \vdots & \vdots\\ a^{n_i}\mathcal{A}_\gamma^{n_i}\Big(\sum_{j=1}^{h+\kappa_1-1}\lambda_j^{(n_i)}\chi_\rho^j\Big) & (L_\rho). \end{cases}$$

We are going to make constant distinction between the first h vectors and the last $m-h$ vectors; one has to keep in mind that the common notations for a reduced basis only refer to a reduction on the last vectors.

Let us prove the lemma by decreasing induction on $l \in \{1,\dots,\rho\}$. Our assertion is that for every $1 \le l \le \rho$, there exists $q' \in \mathbb{Z}_+$ such that for any $1 \le j \le h+\kappa_{\rho+1-l}-1$, $a^{n_i}\lambda_j^{(n_i)}/n_i^{q'} \to 0$ as $i \to \infty$.

We begin by proving the assertion with $l=\rho$. Observing that $\mathcal{A}_\gamma$ is an isometry, (L_ρ) gives

$$a^{n_i}\Big(\sum_{j=1}^{h}\lambda_j^{(n_i)}\chi_\rho^j+\sum_{j=h+\kappa_0}^{h+\kappa_1-1}\lambda_j^{(n_i)}\chi_\rho^j\Big)\xrightarrow[i\to\infty]{}0. \tag{2.10}$$

According to the assumptions, for any $1 \le j \le h$, $b^{n_i}\lambda_j^{(n_i)}/n_i^q \to 0$ as $i \to \infty$. Recalling that $0<|a|<|b|\le 1$, we deduce $a^{n_i}\lambda_j^{(n_i)} \to 0$ as $i\to\infty$ for any $1 \le j \le h$. Substitute this result into (2.10):

$$a^{n_i}\sum_{j=h+\kappa_0}^{h+\kappa_1-1}\lambda_j^{(n_i)}\chi_\rho^j\xrightarrow[i\to\infty]{}0.$$

Moreover, $\{\chi_\rho^j\}_{j=h+\kappa_0}^{h+\kappa_1-1}$ is linearly independent or empty, so for every $1 \le j \le h+\kappa_1-1$, $a^{n_i}\lambda_j^{(n_i)} \to 0$ as $i \to \infty$.

Assume that $l \in \{1, \ldots, \rho\}$ and that the assertion is true for natural numbers strictly greater than l and smaller than ρ. By induction hypothesis, there exists $q' \in \mathbb{Z}_+$ such that

$$\frac{a^{n_i}\lambda_j^{(n_i)}}{n_i^{q'}} \xrightarrow[i\to\infty]{} 0 \quad \text{for any } 1 \le j \le h + \kappa_{\rho-l} - 1.$$

If $\kappa_{\rho-l} < \kappa_{\rho+1-l}$, then (L_l) gives

$$\begin{aligned}(2.11)\qquad a^{n_i}\mathcal{A}_f^{n_i}\Bigg(&\sum_{j=1}^{h+\kappa_{\rho-l}-1}\lambda_j^{(n_i)}\chi_l^j + \sum_{j=h+\kappa_{\rho-l}}^{h+\kappa_{\rho+1-l}-1}\lambda_j^{(n_i)}\chi_l^j \\ &+ \sum_{j=1}^{d}\binom{n_i}{j}\sum_{g=1}^{h+\kappa_{\rho+1-l-j}-1}\lambda_g^{(n_i)}\chi_{l+j}^g\Bigg) \xrightarrow[i\to\infty]{} 0\end{aligned}$$

with $f \in [\![1, \gamma]\!]$ and $d \in [\![0, \rho_f - 1]\!]$. Note that $\mathcal{A}_f$ is an isometry and divide the preceding equation by $n_i^{q'}$; then the first sum tends to 0 by induction hypothesis:

$$\frac{1}{n_i^{q'}}a^{n_i}\Bigg(\sum_{j=h+\kappa_{\rho-l}}^{h+\kappa_{\rho+1-l}-1}\lambda_j^{(n_i)}\chi_l^j + \sum_{j=1}^{d}\binom{n_i}{j}\sum_{g=1}^{h+\kappa_{\rho+1-l-j}-1}\lambda_g^{(n_i)}\chi_{l+j}^g\Bigg) \xrightarrow[i\to\infty]{} 0.$$

Now dividing the last equation by n_i^d, then taking into account the boundedness of $\left(\binom{n_i}{j}/n_i^d\right)_{i\in\mathbb{N}}$ for any $1 \le j \le d$ and the induction hypothesis for the last sum, we obtain

$$\frac{1}{n_i^{q'+d}}a^{n_i}\Bigg(\sum_{j=h+\kappa_{\rho-l}}^{h+\kappa_{\rho+1-l}-1}\lambda_j^{(n_i)}\chi_l^j\Bigg) \xrightarrow[i\to\infty]{} 0.$$

Hence, as $\kappa_{\rho-l} < \kappa_{\rho+1-l}$, $\{\chi_l^j\}_{j=h+\kappa_{\rho-l}}^{h+\kappa_{\rho+1-l}-1}$ is linearly independent, and therefore

$$\frac{a^{n_i}\lambda_j^{(n_i)}}{n_i^{q'+d}} \xrightarrow[i\to\infty]{} 0 \quad \text{for any } 1 \le j \le h + \kappa_{\rho+1-l} - 1.$$

If $\kappa_{\rho-l} = \kappa_{\rho+1-l}$, then the proof is the same but the first sum is missing in (2.11).

Hence there exists $q' \in \mathbb{Z}_+$ such that for all $j \in \{1, \ldots, h + \kappa_\rho - 1\}$, $a^{n_i}\lambda_j^{(n_i)}/n_i^{q'} \to 0$ as $i \to \infty$. This ends the induction and also the proof of the lemma because Theorem 2.6 implies $\kappa_\rho = m - h + 1$, thus $h + \kappa_\rho - 1 = m$. ∎

2.7. Sum of Jordan blocks with the same modulus. The lemma below deals with the growth of coefficients as before but in the case of Jordan blocks with the same modulus. It actually depends on the relative size of the two biggest blocks. The proof is close to the one of Lemma 2.14 but is more technical.

LEMMA 2.15. *Let $\gamma, m, N \in \mathbb{N}$ and $\gamma \geq 2$. Let $T = \bigoplus_{i=1}^{\gamma} \mathcal{B}_i$ be an operator on $\mathbb{R}^N$, where $\mathcal{B}_i$ is a Jordan block of modulus one and $\mathcal{A}_i = 1$ or R_{θ_i}. Moreover, assume that one of the following conditions holds:*

$$\rho_1 \geq \cdots \geq \rho_\gamma \tag{2.12}$$

$$\rho_2 \geq \cdots \geq \rho_\gamma \quad \text{and} \quad \rho_1 = \rho_2 - 1. \tag{2.13}$$

Let M be an m-dimensional subspace and let $x^1, \dots, x^m \in \mathbb{R}^N$ denote a reduced basis of M via Theorem 2.6 for which there exist a strictly increasing sequence of natural numbers $(n_i)_{i\in\mathbb{N}}$ and real sequences $(\lambda_j^{(n_i)})_{i\in\mathbb{N},1\leq j\leq m}$ such that

$$T^{n_i}\Big(\sum_{j=1}^{m} \lambda_j^{(n_i)} x^j\Big) \underset{i\to\infty}{\longrightarrow} (\underbrace{1,\dots,1}_{\rho_1 \text{ times}}, 0, \dots, 0).$$

Then $\lambda_j^{(n_i)}/n_i^{\rho_1} \to 0$ as $i \to \infty$ for any $1 \leq j \leq m$.

Proof. Denote by $\tau_i\rho_i$ the size of the block $\mathcal{B}_i$ with $\tau_i = 1$ or 2 and $\rho = \sum_{i=1}^{\gamma} \rho_i$. Then the following natural expression of $T^{n_i}(\sum_{j=1}^{m} \lambda_j^{(n_i)} x^j)$:

$$\begin{cases}
\mathcal{A}_1^{n_i}\Big(\sum_{j=1}^{m} \lambda_j^{(n_i)} \chi_1^j + \sum_{j=1}^{\rho_1 - 1} \binom{n_i}{j} \sum_{g=1}^{\kappa_{\rho-j}-1} \lambda_g^{(n_i)} \chi_{1+j}^g\Big) & (L_1), \\
\vdots & \vdots \\
\mathcal{A}_1^{n_i}\Big(\sum_{j=1}^{\kappa_{\rho+1-\rho_1}-1} \lambda_j^{(n_i)} \chi_{\rho_1}^j\Big) & (L_{\rho_1}), \\
\vdots & \vdots \\
\mathcal{A}_\gamma^{n_i}\Big(\sum_{j=1}^{\kappa_{\rho_\gamma}-1} \lambda_j^{(n_i)} \chi_{\rho+1-\rho_\gamma}^j + \sum_{j=1}^{\rho_\gamma - 1} \binom{n_i}{j} \sum_{g=1}^{\kappa_{\rho_\gamma - j}-1} \lambda_g^{(n_i)} \chi_{\rho-\rho_\gamma+1+j}^g\Big) & (L_{\rho-\rho_\gamma+1}), \\
\vdots & \vdots \\
\mathcal{A}_\gamma^{n_i}\Big(\sum_{j=1}^{\kappa_1 - 1} \lambda_j^{(n_i)} \chi_\rho^j\Big) & (L_\rho),
\end{cases}$$

becomes

$$\begin{cases} \mathcal{A}_1^{n_i}\Big(\sum_{j=1}^{m}\lambda_j^{(n_i)}\chi_1^j+\sum_{j=1}^{\rho_1-1}\Delta_j(n_i)(L_{1+j})\Big) & (L_1),\\ \vdots & \vdots\\ \mathcal{A}_1^{n_i}\Big(\sum_{j=1}^{\kappa_{\rho+1-\rho_1}-1}\lambda_j^{(n_i)}\chi_{\rho_1}^j\Big) & (L_{\rho_1}),\\ \vdots & \vdots\\ \mathcal{A}_\gamma^{n_i}\Big(\sum_{j=1}^{\kappa_{\rho_\gamma}-1}\lambda_j^{(n_i)}\chi_{\rho+1-\rho_\gamma}^j+\sum_{j=1}^{\rho_\gamma-1}\Delta_j(n_i)(L_{\rho-\rho_\gamma+1+j})\Big) & (L_{\varrho-\varrho_\gamma+1})\\ \vdots & \vdots\\ \mathcal{A}_\gamma^{n_i}\Big(\sum_{j=1}^{\kappa_1-1}\lambda_j^{(n_i)}\chi_\rho^j\Big) & (L_\rho) \end{cases}$$

with the help of Lemmas 2.11 and 2.12. Moreover, observe that every $l\in\{1,\dots,\rho\}$ can be written as $l=\rho-\sum_{i=\gamma-j+1}^{\gamma}\rho_i-d_l$ with $j\in[\![0,\gamma-1]\!]$ and $d_l\in[\![0,\rho_{\gamma-j}-1]\!]$ in a unique way.

Thus for every $l\in\{1,\dots,\rho\}$ we can define

$$\delta_l=\begin{cases} d_l+1 & \text{if } j=0,\\ \max(d_l+1,\rho_{\gamma-j+1}) & \text{if } j\neq 0.\end{cases}$$

Roughly speaking, δ_l is the relative size of the biggest Jordan block of T under the lth line (inclusive) of T. Moreover, we need to explain that a line has to be understood as a normal line in a classical Jordan block but as two lines for a real Jordan block. In addition, denote

$$\nu_l=\begin{cases} \delta_l & \text{if (2.12) holds},\\ \delta_l-1 & \text{if (2.13) holds}.\end{cases}$$

We are going to prove that for any $1\le j\le\kappa_{\rho+1-l}-1$, $\lambda_j^{(n_i)}/n_i^{\nu_l}\to 0$ as $i\to\infty$ by decreasing induction on $l\in\{1,\dots,\rho\}$.

Consider first $l=\rho$; then $\mathcal{A}_\gamma$ being an isometry, (L_ρ) gives

$$\sum_{j=1}^{\kappa_1-1}\lambda_j^{(n_i)}\chi_\rho^j\xrightarrow[i\to\infty]{}0. \tag{2.14}$$

By independence of the family $\{\chi_\rho^j\}_{j=\kappa_0}^{\kappa_1-1}$, we conclude that for any $1\le j\le\kappa_1-1$, $\lambda_j^{(n_i)}\to 0$ as $i\to\infty$.

Now we assume that the assertion is satisfied for $l+1,\dots,\rho$ and we prove it for l. The induction hypothesis yields

$$\frac{\lambda_j^{(n_i)}}{n_i^{\nu_{l+1}}} \xrightarrow[i\to\infty]{} 0 \qquad \text{for any } 1 \le j \le \kappa_{\rho-l} - 1.$$

If $\kappa_{\rho-l} < \kappa_{\rho+1-l}$, then remember (L_l):

$$\Big\|\mathcal{A}_f^{n_i}\Big(\sum_{j=1}^{\kappa_{\rho-l}-1} \lambda_j^{(n_i)}\chi_l^j + \sum_{j=\kappa_{\rho-l}}^{\kappa_{\rho+1-l}-1} \lambda_j^{(n_i)}\chi_l^j + \sum_{j=1}^{d_l} \varDelta_j(n_i)(L_{l+j})\Big)\Big\| \xrightarrow[i\to\infty]{} 0 \text{ or } 1$$

where $f \in [\![1,\gamma]\!]$ and $d_l \in [\![0,\rho_f - 1]\!]$. Keep in mind that A_f is an isometry and divide the preceding line by $n_i^{\nu_{l+1}}$; then by induction the first sum tends to zero and we get

$$\frac{1}{n_i^{\nu_{l+1}}}\Big(\sum_{j=\kappa_{\rho-l}}^{\kappa_{\rho+1-l}-1} \lambda_j^{(n_i)}\chi_l^j + \sum_{j=1}^{d_l} \varDelta_j(n_i)(L_{l+j})\Big) \xrightarrow[i\to\infty]{} 0. \tag{2.15}$$

It then suffices to use Lemma 2.11 and to compare ν_{l+1} and d_l to obtain:

- If (2.12) holds:

$$\begin{cases} \dfrac{1}{n_i^{\nu_{l+1}}}\Big(\displaystyle\sum_{j=\kappa_{\rho-l}}^{\kappa_{\rho+1-l}-1} \lambda_j^{(n_i)}\chi_l^j\Big) \xrightarrow[i\to\infty]{} 0 \\ \qquad\qquad\qquad\qquad \text{if } d_l < \nu_{l+1}, \text{ i.e. } \delta_l = \delta_{l+1}, \\ \dfrac{1}{n_i^{\nu_{l+1}}}\Big(\displaystyle\sum_{j=\kappa_{\rho-l}}^{\kappa_{\rho+1-l}-1} \lambda_j^{(n_i)}\chi_l^j\Big) + \dfrac{(-1)^{d_l+1}}{d_l!}(L_{l+d_l}) \xrightarrow[i\to\infty]{} 0 \\ \qquad\qquad\qquad\qquad \text{if } d_l = \nu_{l+1}, \text{ i.e. } \delta_l = \delta_{l+1}+1. \end{cases}$$

Since $\kappa_{\rho-l} < \kappa_{\rho+1-l}$, the family $\{\chi_l^j\}_{j=\kappa_{\rho-l}}^{\kappa_{\rho+1-l}-1}$ is linearly independent and we deduce that for any $1 \le j \le \kappa_{\rho+1-l} - 1$,

$$\frac{\lambda_j^{(n_i)}}{n_i^{\delta_l}} \xrightarrow[i\to\infty]{} 0.$$

- If (2.13) holds:

$$\begin{cases} \dfrac{1}{n_i^{\nu_{l+1}}}\Big(\displaystyle\sum_{j=\kappa_{\rho-l}}^{\kappa_{\rho+1-l}-1} \lambda_j^{(n_i)}\chi_l^j\Big) \xrightarrow[i\to\infty]{} 0 & \\ \qquad\qquad \text{if } d_l < \nu_{l+1},\ \text{hence } \delta_l = \delta_{l+1}, & \\ \dfrac{1}{n_i^{\nu_{l+1}}}\Big(\displaystyle\sum_{j=\kappa_{\rho-l}}^{\kappa_{\rho+1-l}-1} \lambda_j^{(n_i)}\chi_l^j\Big) + \dfrac{(-1)^{d_l+1}}{d_l!}(L_{l+d_l}) \xrightarrow[i\to\infty]{} 0 & \\ \qquad\qquad \text{if } d_l = \nu_{l+1},\ \text{hence } \delta_l = \delta_{l+1}, & \\ \dfrac{1}{n_i^{\nu_{l+1}+1}}\Big(\displaystyle\sum_{j=\kappa_{\rho-l}}^{\kappa_{\rho+1-l}-1} \lambda_j^{(n_i)}\chi_l^j\Big) + \dfrac{(-1)^{d_l+1}}{d_l!}(L_{l+d_l}) \xrightarrow[i\to\infty]{} 0 & \\ \qquad\qquad \text{if } d_l = \nu_{l+1}+1, \text{ i.e. } \delta_l = \delta_{l+1}+1. & \end{cases}$$

However, in the second and third cases above, we are not working on the block $\mathcal{B}_1$ yet because the condition $\rho_1 = \rho_2 - 1$ is not compatible with $d_l = \delta_{l+1} - 1$ or $d_l = \delta_{l+1}$ given by the second and third cases. Thus, $(L_{l+d_l}) \to 0$ as $i \to \infty$. Moreover, $\kappa_{\rho-l} < \kappa_{\rho+1-l}$ yields the independence of the family $\{\chi_l^j\}_{j=\kappa_{\rho-l}}^{\kappa_{\rho+1-l}-1}$. Hence, we deduce that for any $1 \le j \le \kappa_{\rho+1-l} - 1$,

$$\frac{\lambda_j^{(n_i)}}{n_i^{\delta_l - 1}} \xrightarrow[i\to\infty]{} 0.$$

If $\kappa_{\rho-l} = \kappa_{\rho+1-l}$, then the proof is the same, except that in (2.15) the first term is missing.

Thus for every $1 \le j \le \kappa_\rho - 1$, $\lambda_j^{(n_i)}/n_i^{\nu_1} \to 0$ as $i \to \infty$. This ends the induction process.

Then, by Theorem 2.6, $\kappa_\rho - 1 = m$ and noticing that $\nu_1 = \rho_1$ in case (2.12) and $\nu_1 = \rho_2 - 1$ in case (2.13), we end the proof of the lemma. ∎

From this lemma, we deduce a general result for operators which are given as a direct sum of Jordan blocks with the same modulus.

THEOREM 2.16. *Let T be an operator on $\mathbb{R}^N$ which is a direct sum of Jordan blocks of modulus* 1. *Then T is not* $(\rho - 1)$*-supercyclic.*

Proof. Let us call $D := \sum_{i=1}^{\gamma}(\rho_i - 1)$ the *degree* of T.

The proof is by induction on D. If $D = 0$, then T is a primary matrix of order ρ and Proposition 2.10 states that T is not $(\rho - 1)$-supercyclic.

Assume that the assertion is true from 0 to $D - 1$, and let us prove it for D.

Suppose that T is $(\rho - 1)$-supercyclic. Without loss of generality, we can assume that $T = \bigoplus_{i=1}^{\gamma} \mathcal{B}_i$, where $\mathcal{B}_i$ is a Jordan block of modulus one and $\rho_1 \ge \cdots \ge \rho_\gamma$. Then T is clearly not $(\rho - 1)$-supercyclic whenever it contains only one block, by either Proposition 2.13 if the block is real or [3] if it is a classical Jordan block. We can also assume that $\rho_1 > 1$ thanks to Proposition 2.10.

Thus one can write

$$T = \left(\begin{array}{c|cccc|ccc} \mathcal{A}_1 & \mathcal{A}_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 0 & \multicolumn{4}{c|}{\mathcal{C}_1} & 0 & \cdots & 0 \\ \cline{1-5} 0 & \cdots & \cdots & 0 & 0 & \mathcal{B}_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \mathcal{B}_\gamma \end{array}\right)$$

with

$$\mathcal{B}_1 = \left(\begin{array}{c|cccc} \mathcal{A}_1 & \mathcal{A}_1 & 0 & \cdots & 0 \\ \hline 0 & \multicolumn{4}{c}{\mathcal{C}_1} \end{array}\right).$$

Denote also by S the diagonal block matrix which is the direct sum of $\mathcal{C}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\gamma$. As T is $(\rho-1)$-supercyclic, let $M = \operatorname{span}\{x^1, \ldots, x^{\rho-1}\}$ be a $(\rho-1)$-supercyclic subspace for T where the basis is reduced via Theorem 2.6. According to the induction hypothesis, S is not $(\rho-2)$-supercyclic, and thus there exists $p < \rho$ such that $\kappa_p = \rho$. Indeed, suppose $\kappa_{\rho-1} < \rho$; this means that $\chi_j^{\rho-1} = \binom{0}{0}$ for every $2 \le j \le \rho$. Thus, $d := \dim(\operatorname{span}\{(\chi_j^1)_{2\le j\le\rho}, \ldots, (\chi_j^{\rho-1})_{2\le j\le\rho}\}) \le \rho-2$. Hence, as M is $(\rho-1)$-supercyclic for T, it follows that S is d-supercyclic. But this would contradict the fact that S is not $(\rho-2)$-supercyclic.

Since T is $(\rho-1)$-supercyclic, there exist a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers and real sequences $(\lambda_j^{(n_i)})_{i\in\mathbb{N},\,1\le j\le\rho-1}$ such that

$$T^{n_i}\Big(\sum_{j=1}^{\rho-1} \lambda_j^{(n_i)} x^j\Big) \xrightarrow[i\to\infty]{} (\underbrace{1, \ldots, 1}_{\rho_1 \text{ times}}, 0, \ldots, 0).$$

Then there are two options: either $\rho_1 - 1 \ge \rho_2$ or $\rho_1 = \rho_2$. In both cases, Lemma 2.15 applied to $T = S$ leads to $\lambda_j^{(n_i)}/n_i^{\rho_1-1} \to 0$ as $i \to \infty$ for any $1 \le j \le \rho - 1$ because there exists $p < \rho$ such that $\kappa_p = \rho$. On the other hand, applying Lemma 2.12 to the first line of $T^{n_i}(\sum_{j=1}^{\rho-1} \lambda_j^{(n_i)} x^j)$ and taking the limit provides

$$\Big\|\mathcal{A}_1^{n_i}\Big(\sum_{j=1}^{\rho-1} \lambda_j^{(n_i)} \chi_1^j + \sum_{j=1}^{\rho_1-1} \Delta_j(n_i)(L_{1+j})\Big)\Big\| \xrightarrow[i\to\infty]{} 1.$$

Now, dividing by $n_i^{\rho_1-1}$ and recalling that A_1 is an isometry yields

$$\sum_{j=1}^{\rho-1} \frac{\lambda_j^{(n_i)}}{n_i^{\rho_1-1}} \chi_1^j + \sum_{j=1}^{\rho_1-1} \frac{\Delta_j(n_i)}{n_i^{\rho_1-1}}(L_{1+j}) \xrightarrow[i\to\infty]{} 0.$$

Using both $\lambda_j^{(n_i)}/n_i^{\rho_1-1} \to 0$ as $i \to \infty$ and Lemma 2.11, one may observe that all the terms in the previous equation tend to 0 apart from the term $j = \rho_1 - 1$ in the last sum, so

$$\frac{(-1)^{\rho_1}}{(\rho_1-1)!} \underset{i\to\infty}{\longrightarrow} 0,$$

which is absurd. Thus T is not $(\rho-1)$-supercyclic and this ends the induction process. ■

2.8. General matrix. The next theorem reduces the study of m-supercyclic operators on $\mathbb{R}^N$ to that of operators which are direct sums of Jordan blocks of modulus one.

THEOREM 2.17. *Let $T = \bigoplus_{i=1}^{\gamma} a_i\mathcal{C}_i$ where $|a_1| < \cdots < |a_\gamma| \le 1$, and $\mathcal{C}_i$ is a direct sum of Jordan blocks of modulus one for any $1 \le i \le \gamma$. Assume that for any $1 \le i \le \gamma$, $\mathcal{C}_i$ is m_i-supercyclic and m_i is optimal. Then T is not $(\sum_{i=1}^{\gamma} m_i - 1)$-supercyclic.*

Proof. Let $T_p := \bigoplus_{i=\gamma+1-p}^{\gamma} a_i\mathcal{C}_i$ and let $t(p)$ denote this matrix's size. We may prove by induction that for any $1 \le p \le \gamma$, T_p is not $(\sum_{i=\gamma+1-p}^{\gamma} m_i-1)$-supercyclic. Actually, we prove a little more:

- For any $1 \le p \le \gamma$, T_p is not $(\sum_{i=\gamma+1-p}^{\gamma} m_i-1)$-supercyclic. Moreover, for any b-supercyclic subspace with reduced basis $M=\operatorname{span}\{x^1,\dots,x^b\}$ and with $\sum_{i=\gamma+1-p}^{\gamma} m_i \le b \le t(p)$, if $T_p^{n_i}(\sum_{j=1}^{b} \lambda_j^{(n_i)} x^j) \to 0$ as $i \to \infty$, then there exists $q \in \mathbb{Z}_+$ such that
$$\frac{a_{\gamma+1-p}^{n_i}\lambda_j^{(n_i)}}{n_i^q} \underset{i\to\infty}{\longrightarrow} 0 \quad \text{for any } 1 \le j \le b.$$

Assume that $p=1$; then $T_p = a_\gamma\mathcal{C}_\gamma$ and by definition, T_p is m_γ-supercyclic and m_γ is the minimum supercyclic constant. Let $M = \operatorname{span}\{x^1,\dots,x^b\}$ be a b-supercyclic subspace with reduced basis and $m_\gamma \le b \le t(1)$. Let also $T_p^{n_i}(\sum_{j=1}^{b} \lambda_j^{(n_i)} x^j) \to 0$ as $i \to \infty$. Applying Lemma 2.14 with $h = 0$, $\mathcal{C} = \mathcal{C}_\gamma$, $a = a_\gamma$, $m = b$, and $N = t(p)$, we see that there exists $q \in \mathbb{Z}_+$ such that $a_\gamma^{n_i}\lambda_j^{(n_i)}/n_i^q \to 0$ as $i \to \infty$ for any $1 \le j \le b$.

Assume the assertion is true for integers smaller than p, and let us prove it for p. Write $T_p = a_{\gamma+1-p}\mathcal{C}_{\gamma+1-p} \oplus T_{p-1}$ and $\mathcal{C}_{\gamma+1-p} = \bigoplus_{i=1}^{t} \mathcal{B}_i$ where $\mathcal{B}_i$ is a Jordan block of modulus one and of size $\tau_i\rho_i$ with $\tau_i = 1$ or 2, and define $\rho := \sum_{i=1}^{t} \rho_i$.

To obtain a contradiction, assume that $k = \sum_{i=\gamma+1-p}^{\gamma} m_i-1$ and let $M = \operatorname{span}\{x^1,\dots,x^k\}$ be a k-supercyclic subspace with reduced basis given by Theorem 2.6. Then for any $1 \le i \le k$, decompose $x^i = y^i \oplus z^i$ relative to the direct sum decomposition of T_p stated above. A straightforward use of the induction hypothesis provides $h := \dim(\operatorname{span}\{z^1,\dots,z^k\}) \ge \sum_{i=\gamma+2-p}^{\gamma} m_i$.

Furthermore, we can show that $\dim(\operatorname{span}\{y^{h+1},\dots,y^k\}) \geq m_{\gamma+1-p}$. Indeed, it suffices to prove that $\operatorname{span}\{y^{h+1},\dots,y^k\}$ is supercyclic for $a_{\gamma+1-p}\mathcal{C}_{\gamma+1-p}$. Take any u belonging to the domain of $\mathcal{C}_{\gamma+1-p}$. Then there exist $(n_i)_{i\in\mathbb{N}}$ and $(\lambda_j^{(n_i)})_{i\in\mathbb{N},\,1\leq j\leq k}$ so that $T_p^{n_i}(\sum_{j=1}^k \lambda_j^{(n_i)} x^j) \to u \oplus 0$ as $i\to\infty$ by k-supercyclicity of T_p. Moreover the induction hypothesis implies that for any $1\leq j\leq h$, there exists $q\in\mathbb{Z}_+$ such that $a_{\gamma+1-(p-1)}^{n_i}\lambda_j^{(n_i)}/n_i^q \to 0$ as $i\to\infty$ for any $1\leq j\leq h$, and since $a_{\gamma+1-p} < a_{\gamma+2-p}$, we obtain

$$a_{\gamma+1-p}^{n_i}\lambda_j^{(n_i)}P(n_i) \xrightarrow[i\to\infty]{} 0 \quad \text{for any polynomial } P. \tag{2.16}$$

Now we come back to $T_p^{n_i}(\sum_{j=1}^k \lambda_j^{(n_i)} x^j) \to u\oplus 0$ as $i\to\infty$; projecting onto the first components, and separating the sum, we get

(2.17)

$$(a_{\gamma+1-p}\mathcal{C}_{\gamma+1-p})^{n_i}\Big(\sum_{j=1}^h \lambda_j^{(n_i)} y_j\Big) + (a_{\gamma+1-p}\mathcal{C}_{\gamma+1-p})^{n_i}\Big(\sum_{j=h+1}^k \lambda_j^{(n_i)} y_j\Big) \xrightarrow[i\to\infty]{} u.$$

Then, we remark that $(a_{\gamma+1-p}\mathcal{C}_{\gamma+1-p})^{n_i}(\sum_{j=1}^h \lambda_j^{(n_i)} y_j)$ can be expressed as

$$\begin{cases} a_{\gamma+1-p}^{n_i}\mathcal{A}_1^{n_i}\Big(\sum_{j=1}^h \lambda_j^{(n_i)}\chi_1^j + \sum_{j=1}^{\rho_1-1}\binom{n_i}{j}\sum_{g=1}^h \lambda_g^{(n_i)}\chi_{1+j}^g\Big) & (L_1),\\ \vdots & \vdots\\ a_{\gamma+1-p}^{n_i}\mathcal{A}_1^{n_i}\Big(\sum_{j=1}^h \lambda_j^{(n_i)}\chi_{\rho_1}^j\Big) & (L_{\rho_1}),\\ \vdots & \vdots\\ a_{\gamma+1-p}^{n_i}\mathcal{A}_t^{n_i}\Big(\sum_{j=1}^h \lambda_j^{(n_i)}\chi_{r+1-\rho_t}^j + \sum_{j=1}^{\rho_t-1}\binom{n_i}{j}\sum_{g=1}^h \lambda_g^{(n_i)}\chi_{r+1-\rho_t+j}^g\Big) & (L_{r+1-\rho_t}),\\ \vdots & \vdots\\ a_{\gamma+1-p}^{n_i}\mathcal{A}_t^{n_i}\Big(\sum_{j=1}^h \lambda_j^{(n_i)}\chi_r^j\Big) & (L_r). \end{cases}$$

Since $\mathcal{A}_j$ is an isometry, (2.16) shows that for any $1\leq j\leq r$, $(L_j)\to 0$ as $i\to\infty$. Hence, the first sum in (2.17) converges to 0, leading to

$$(a_{\gamma+1-p}\mathcal{C}_{\gamma+1-p})^{n_i}\Big(\sum_{j=h+1}^k \lambda_j^{(n_i)} y_j\Big) \xrightarrow[i\to\infty]{} u.$$

Thus, $\operatorname{span}\{y^{h+1},\dots,y^k\}$ is supercyclic for $a_{\gamma+1-p}\mathcal{C}_{\gamma+1-p}$ and the dimension of the space $\operatorname{span}\{y^{h+1},\dots,y^k\}$ is greater than or equal to $m_{\gamma+1-p}$.

Then, as the basis is reduced by Theorem 2.6, we have $x^j = y^j \oplus 0$ for any $h+1 \leq j \leq k$, thus $k = \dim(\mathrm{span}\{x^1, \ldots, x^k\}) \geq \sum_{i=\gamma+2-p}^{\gamma} m_i + m_{\gamma+1-p} = k+1$. This contradiction proves that T_p is not $(\sum_{i=\gamma+1-p}^{\gamma} m_i - 1)$-supercyclic.

Let us now focus on the second part of the assertion. For this purpose, let $M = \mathrm{span}\{x^1, \ldots, x^b\}$ be a b-supercyclic subspace whose basis is reduced and $\sum_{i=\gamma+1-p}^{\gamma} m_i \leq b \leq t(p)$. Let also $T_p^{n_i}(\sum_{j=1}^{b} \lambda_j^{(n_i)} x^j) \to 0$ as $i \to \infty$.

For every $1 \leq i \leq b$, decompose $x^i = y^i \oplus z^i$ relative to the direct sum decomposition of T_p. Then we just have to invoke Lemma 2.14 ($a_{\gamma+1-p} = a$, $a_{\gamma+2-p} = b$, $b = m$, $t(p) - t(p-1) = N$, $h =$ number of non-zero vectors among $z^1, \ldots, z^b$, $x^i = y^i$, $\gamma =$ number of blocks in $\mathcal{C}_{\gamma+1-p}$, $\mathcal{C}_{\gamma+1-p} = \mathcal{C}$, $a_{\gamma+1-p}\mathcal{C}_{\gamma+1-p} = T$) and the result follows: there exists $q \in \mathbb{Z}_+$ such that $a_{\gamma+1-p}^{n_i} \lambda_j^{(n_i)} / n_i^q \to 0$ as $i \to \infty$ for any $1 \leq j \leq b$.

This completes the proof of the induction and of the theorem. ∎

We are now ready to state global results on supercyclicity for operators on $\mathbb{R}^N$. These results follow straightforwardly from Theorems 2.17 and 2.16 and generalise Hilden and Wallen's and Herzog's results for supercyclic operators.

COROLLARY 2.18. *Let $N \geq 2$ and T be an operator on $\mathbb{R}^N$. Then T is not $(\rho - 1)$-supercyclic.*

Proof. Without loss of generality, one may assume that T is in Jordan form and also upon reordering blocks and considering a multiple of T instead of T itself, one may assume that the sequence of moduli satisfies $|a_1| \leq \cdots \leq |a_\gamma| \leq 1$. As a consequence, one may realise T as a direct sum of matrices $S_1, \ldots, S_t$, where S_1 contains all Jordan blocks with the smallest modulus and so on. Let $\rho^{(j)}$ denote the relative size of the matrix S_j, $j = 1, \ldots, t$. First, Theorem 2.16 implies that no matrix S_j is $(\rho^{(j)} - 1)$-supercyclic. Then, one uses Theorem 2.17 to come back to T, hence T is not $(\sum_{j=1}^{t} \rho^{(j)} - 1)$-supercyclic. Next, one just has to recall the definition of ρ and of $\rho^{(j)}$ providing $\sum_{j=1}^{t} \rho^{(j)} = \rho$. This proves the corollary. ∎

A direct application of the preceding corollary yields a more concrete result:

COROLLARY 2.19. *Let $N \geq 2$. There is no $(\lfloor (N+1)/2 \rfloor - 1)$-supercyclic operator on $\mathbb{R}^N$. Moreover, there always exists an $(\lfloor (N+1)/2 \rfloor)$-supercyclic operator on $\mathbb{R}^N$.*

Proof. The first part follows from Corollary 2.18. Indeed, if N is even, the lowest relative size of a matrix is $N/2$, and it is $(N+1)/2$ if N is odd. Thus the relative size of a matrix cannot be lower than $\lfloor (N+1)/2 \rfloor$ on $\mathbb{R}^N$. For the second part, see Example 2.1. ∎

Question. Does there exist a theorem similar to Theorem 2.17 in the case of a direct sum of Jordan blocks of modulus one?

Question. Does there exist a $(2N-2)$-supercyclic real Jordan block on $\mathbb{R}^{2N}$? If so, what is the best supercyclic constant for a real Jordan block on $\mathbb{R}^{2N}$?

3. Strong n-supercyclicity. The aim of this section is to study the existence of strong n-supercyclic operators in $\mathbb{R}^N$. Of course, this is interesting only if $n \leq N$. Bourdon, Feldman and Shapiro [3] answer this question for the complex case. Indeed, they prove that n-supercyclicity cannot occur non-trivially in finite complex dimensions, and thus strong n-supercyclicity cannot either. However, in the real setting, we noticed in the previous section that n-supercyclicity can occur and thus the question about strong n-supercyclicity is still open. For this, we need the following proposition from [4]. It provides a more concrete definition of strongly n-supercyclic operators:

Proposition 3.1 ([4, Proposition 1.13]). *Let X be a completely separable Baire vector space. The following are equivalent:*

(i) *T is strongly n-supercyclic.*
(ii) *There exists an n-dimensional subspace L such that for every $i \in \mathbb{Z}_+$, $T^i(L)$ is n-dimensional and $\mathcal{B} := \bigcup_{i=1}^{\infty} \pi_n^{-1}(\tilde{T}^i(L))$ is dense in X^n.*
(iii) *There exists an n-dimensional subspace L such that for every $i \in \mathbb{Z}_+$, $T^i(L)$ is n-dimensional and $\mathcal{E} := \bigcup_{i=1}^{\infty} (T^i(L) \times \cdots \times T^i(L))$ is dense in X^n.*

Remark 3.2. Moreover, from the definition of strong n-supercyclicity or using the previous proposition, one may observe that if T is a strongly n-supercyclic operator on $\mathbb{R}^N$ with $n \leq N$, then T is bijective.

We turn to the case of strongly n-supercyclic operators on a real finite-dimensional vector space. Our first result is interesting and provides a partial answer to the above question:

Proposition 3.3. *Let $n < N$. An operator T on $\mathbb{R}^N$ is strongly n-supercyclic if and only if $(T^{-1})^*$ is strongly $(N-n)$-supercyclic, and each strong n-supercyclic subspace for T is orthogonal to all strongly $(N-n)$-supercyclic subspaces of $(T^{-1})^*$.*

This duality property is very useful if one combines it with Corollary 2.19; in this way we get:

Corollary 3.4. *For any $N \geq 1$ and any $1 \leq n \leq 2N+1$ there is no strongly n-supercyclic operator on $\mathbb{R}^{2N+1}$.*

For any $N \geq 2$ and any $1 \leq n \leq 2N$, $n \neq N$, there is no strongly n-supercyclic operator on $\mathbb{R}^{2N}$.

This corollary enables one to give many examples of n-supercyclic operators that are not strongly n-supercyclic and thus answers the question of Shkarin [13] by proving that n-supercyclicity and strong n-supercyclicity are not equivalent.

EXAMPLE 3.5 (A 2-supercyclic operator that is not strongly 2-supercyclic). Any rotation on $\mathbb{R}^3$ around any one-dimensional subspace and with angle linearly independent of π in $\mathbb{Q}$ is 2-supercyclic but not strongly 2-supercyclic.

We now turn to the proof of Proposition 3.3. We need the following two well-known lemmas:

LEMMA 3.6. *Let M be a subspace of $\mathbb{R}^N$, and let T be an automorphism on $\mathbb{R}^N$. Then $(T^i(M))^\perp = (T^{-i})^*(M^\perp)$ for any $i \in \mathbb{N}$.*

LEMMA 3.7. *Let $\Phi : \mathbb{P}_n(\mathbb{R}^N) \to \mathbb{P}_{N-n}(\mathbb{R}^N)$ be defined by the formula $\Phi(M) = M^\perp$ for every $M \in \mathbb{P}_n(\mathbb{R}^N)$. Then Φ is a homeomorphism.*

The first lemma is classical, and the second one may be found in [11]. Then the proof of the above proposition is straightforward:

Proof of Proposition 3.3. The combination of Lemmas 3.6 and 3.7 when M is a strongly n-supercyclic subspace for T implies that the set

$$\Phi(\{T^i(M)\}_{i\in\mathbb{N}}) = \{(T^{-i})^*(M^\perp)\}_{i\in\mathbb{N}}$$

is dense in $\mathbb{P}_{N-n}(\mathbb{R}^N)$. ∎

3.1. Strongly 2-supercyclic operators on $\mathbb{R}^4$. The idea is to prove by induction that there is no strongly n-supercyclic operator on $\mathbb{R}^N$ for $N \geq 3$ and $1 \leq n < N$. The first step is to prove this for N small. This is already done for $N = 3$ in Corollary 3.4; we now focus on the case $N = 4$. We begin by characterising 2-supercyclic subspaces for a direct sum of rotations; we prove that none of them is strongly 2-supercyclic.

PROPOSITION 3.8. *Let R be a direct sum of two rotations,*

$$R = \left(\begin{array}{c|c} R_{\theta_1} & 0 \\ \hline 0 & R_{\theta_2} \end{array}\right)$$

with $\{\theta_1, \theta_2, \pi\}$ linearly independent over $\mathbb{Q}$. Then

$$\mathcal{ES}_2(R) = \left\{ \operatorname{span}\left\{ \begin{pmatrix} x_1 \\ x_2 \\ ay_1 \\ ay_2 \end{pmatrix}, \begin{pmatrix} bx_1 \\ bx_2 \\ y_1 \\ y_2 \end{pmatrix} \right\} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}, ab \neq 1 \right\}.$$

Proof. First, let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ ay_1 \\ ay_2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} bx_1 \\ bx_2 \\ y_1 \\ y_2 \end{pmatrix}$$

satisfy the above conditions; we are going to prove that the span of these two vectors is 2-supercyclic for R.

Note that $x - ay$ is non-zero but its third and fourth components are null, and $y - bx$ is also non-zero but its first two components are null.

Let U and V be two non-empty open sets in $\mathbb{R}^2$. As θ_1, θ_2, π are linearly independent over $\mathbb{Q}$, there exist $i \in \mathbb{N}$ and $c_1, c_2 \in \mathbb{R}$ such that

$$c_1(1-ab)R^i_{\theta_1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in U \quad \text{and} \quad c_2(1-ab)R^i_{\theta_2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in V.$$

Thus $R^i(c_1(x-ay)+c_2(-bx+y)) \in U \times V$, hence $\operatorname{span}\{x, y\}$ is a 2-supercyclic subspace for R.

Now for the converse, suppose that there exists a two-dimensional subspace $M = \operatorname{span}\{x, y\}$ which is 2-supercyclic for R and which does not satisfy the conditions stated in the proposition. Hence, either $\binom{x_1}{x_2}, \binom{y_1}{y_2}$ are linearly independent, or we can assume that $\binom{y_1}{y_2} = \binom{0}{0}$. Set $\mathcal{C}_1 := \{z \in \mathbb{R}^2 : \|z\| < 1\}$ and define $\mathcal{C}_2 := \{z \in \mathbb{R}^2 : \|z\| > t\}$ where t is a positive real number to be choosen later. Then, as M is a 2-supercyclic subspace for R, $\{R^i(\lambda x + \mu y)\}_{i \in \mathbb{N},\, (\lambda,\mu) \in \mathbb{R}^2}$ is dense in $\mathbb{R}^4$. As a result, there exist $i \in \mathbb{N}$ and $(\lambda, \mu) \in \mathbb{R}^2$ such that $R^i(\lambda x + \mu y) \in \mathcal{C}_1 \times \mathcal{C}_2$. This is equivalent to

$$\begin{pmatrix} \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ \lambda\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \mu\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \end{pmatrix} \in R^{-i}_{\theta_1}(\mathcal{C}_1) \times R^{-i}_{\theta_2}(\mathcal{C}_2) = \mathcal{C}_1 \times \mathcal{C}_2.$$

Define

$$\Gamma_1 := \left\{(\lambda, \mu) \in \mathbb{R}^2 : \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{C}_1\right\}.$$

Since $\mathcal{C}_1$ is bounded and either $\binom{x_1}{x_2}, \binom{y_1}{y_2}$ are linearly independent, or $\binom{y_1}{y_2} = \binom{0}{0}$, one may deduce that Γ_1 is bounded too. Note that

$$\Omega := \left\{\lambda\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \mu\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} : (\lambda, \mu) \in \Gamma_1\right\}$$

is then also bounded. We now define t to be an upper bound for Ω. Then one deduces that $\mathcal{C}_2 \cap \Omega = \emptyset$. This contradicts the fact that M is a 2-supercyclic subspace for R. So M satisfies the proposition's conditions. ∎

COROLLARY 3.9. *R is not strongly 2-supercyclic on $\mathbb{R}^4$.*

Proof. Assume otherwise; then any strongly 2-supercyclic subspace is given by Proposition 3.8. Thus if $x, y \in \mathbb{R}^4$ span a strongly 2-supercyclic subspace for R, then

$$R^i(\lambda x + \mu y) = \begin{pmatrix} (\lambda + b\mu) R^i_{\theta_1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ (\lambda a + \mu) R^i_{\theta_2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{pmatrix}.$$

Moreover, according to Proposition 3.1, for any two non-empty open sets $U_1, U_2 \subset \mathbb{R}^2$, there exist $i \in \mathbb{N}$ and $\lambda, \mu, \alpha, \beta \in \mathbb{R}$ such that

$$(\lambda + b\mu) R^i_{\theta_1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in U_1 \quad \text{and} \quad (\alpha + b\beta) R^i_{\theta_1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in U_2.$$

But this cannot happen if we choose U_1 and U_2 such that no straight line passing through the origin intersects both U_1 and U_2. Therefore, R is not strongly 2-supercyclic. ∎

We are now going to deal with two other different-shaped operators on $\mathbb{R}^4$.

PROPOSITION 3.10. *The operators*

$$\left(\begin{array}{c|c} A & A \\ \hline 0 & A \end{array}\right) \quad \textit{and} \quad \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

with

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}, \qquad (a, b, c, d) \in \mathbb{R}^4,$$

are not strongly 2-supercyclic.

Proof. Without loss of generality, one may assume $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ because strongly 2-supercyclic operators have dense range.

If $R = \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$, then R is not strongly 2-supercyclic by Proposition 2.2.

Now assume that $R = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is strongly 2-supercyclic. Upon considering a scalar multiple, relabelling and rearranging blocks, one can suppose

$$R = \left(\begin{array}{c|c} R_\theta & 0 \\ \hline 0 & C \end{array}\right) \quad \text{with} \quad C = \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \quad \text{and} \quad c^2 + d^2 \le 1.$$

Let $M = \operatorname{span}\{x, y\}$ be a strongly 2-supercyclic subspace for R.

If $c^2+d^2=1$, then Corollary 3.9 implies R is not strongly 2-supercyclic.

If $c^2+d^2<1$, since M is strongly 2-supercyclic for R, by Proposition 3.1, for any non-empty open sets U_1,U_2,V_1,V_2 in $\mathbb{R}^2$ there exist $i\in\mathbb{N}$ and $(\lambda,\mu,\alpha,\beta)\in\mathbb{R}^4$ such that

$$\begin{cases} R^i(\lambda x+\mu y)\in U_1\times U_2\\ R^i(\alpha x+\beta y)\in V_1\times V_2\end{cases} \Leftrightarrow \begin{cases} \lambda\begin{pmatrix}x_1\\x_2\end{pmatrix}+\mu\begin{pmatrix}y_1\\y_2\end{pmatrix}\in R_\theta^{-i}(U_1),\\ \alpha\begin{pmatrix}x_1\\x_2\end{pmatrix}+\beta\begin{pmatrix}y_1\\y_2\end{pmatrix}\in R_\theta^{-i}(V_1),\\ \lambda\begin{pmatrix}x_3\\x_4\end{pmatrix}+\mu\begin{pmatrix}y_3\\y_4\end{pmatrix}\in C^{-i}(U_2),\\ \alpha\begin{pmatrix}x_3\\x_4\end{pmatrix}+\beta\begin{pmatrix}y_3\\y_4\end{pmatrix}\in C^{-i}(V_2).\end{cases}$$

From this, we deduce that $\binom{x_1}{x_2},\binom{y_1}{y_2}$ are linearly independent. Indeed, if not, then we choose U_1 and V_1 such that no straight line passing through the origin intersects both U_1,V_1 to obtain a contradiction with the above equivalence.

Let $U_1=\{z\in\mathbb{R}^2:\|z\|<1\}$ and $U_2=\{z\in\mathbb{R}^2:\|z\|>t\}$ with t to be defined later. Set also

$$\varGamma=\left\{(\lambda,\mu)\in\mathbb{R}^2:\lambda\begin{pmatrix}x_1\\x_2\end{pmatrix}+\mu\begin{pmatrix}y_1\\y_2\end{pmatrix}\in U_1\right\}.$$

Since $\binom{x_1}{x_2},\binom{y_1}{y_2}$ are linearly independent and U_1 is a bounded set, so is $\varGamma$, then

$$\varOmega=\left\{\lambda\begin{pmatrix}x_3\\x_4\end{pmatrix}+\mu\begin{pmatrix}y_3\\y_4\end{pmatrix}:(\lambda,\mu)\in\varGamma\right\}$$

is obviously bounded too and we define t as an upper bound for $\varOmega$.

On the other hand, we have

$$C^{-1}=\frac{1}{c^2+d^2}\begin{pmatrix}c & d\\ -d & c\end{pmatrix}=\frac{1}{\sqrt{c^2+d^2}}R_\varphi,$$

thus $C^{-i}(U_2)\subseteq U_2$ for any $i\in\mathbb{N}$. Hence, $\varOmega\cap\bigcup_{i\in\mathbb{N}}C^{-i}(U_2)=\emptyset$, contradicting the strong 2-supercyclicity of R. ∎

3.2. General result. We are now going to prove by induction on the space dimension that there is no non-trivial strongly n-supercyclic operator. The following proposition inspired by Bourdon, Feldman and Shapiro [3] is useful for the induction step:

PROPOSITION 3.11. *Let X be a Hausdorff topological vector space. Let also $T : X \to X$ be a bounded operator and K a closedinvariant subspace for T. If T is strongly n-supercyclic, then so is the quotient map $T_K : X/K \to X/K$.*

The proof is a simple verification using the characterisation of strong n-supercyclicity given in Proposition 3.1.

THEOREM 3.12. *For any $N \geq 3$ and $1 \leq n < N$, there is no strongly n-supercyclic operator on $\mathbb{R}^N$.*

Proof. We give a proof by induction on N. First, according to Corollary 3.4 and Herzog's result [9], there is no strongly n-supercyclic operator on $\mathbb{R}^3$ with $n = 1, 2$.

So, let $N \geq 4$ and $n < N$. We want to prove that there is no strongly n-supercyclic operator on $\mathbb{R}^N$. By Corollary 3.4, we can suppose that $N \geq 4$ is even and $n = N/2$. Let R be an operator on $\mathbb{R}^N$; using the Jordan real decomposition one may suppose

$$R = \begin{pmatrix} \boxed{J_1} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \boxed{J_q} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \boxed{\mathcal{J}_1} & 0 & 0 \\ 0 & \cdots & \cdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{\mathcal{J}_r} \end{pmatrix},$$

where J_i is a classical Jordan block:

$$J_i = \begin{pmatrix} \mu_i & \mu_i & 0 & 0 \\ 0 & \ddots & \ddots & \cdots \\ \vdots & 0 & \ddots & \mu_i \\ 0 & \cdots & 0 & \mu_i \end{pmatrix},$$

and $\mathcal{J}_i$ is a real Jordan block:

$$\mathcal{J}_i = \begin{pmatrix} \mathcal{A}_i & \mathcal{A}_i & 0 & 0 \\ 0 & \ddots & \ddots & \cdots \\ \vdots & 0 & \ddots & \mathcal{A}_i \\ 0 & \cdots & 0 & \mathcal{A}_i \end{pmatrix}.$$

For contradiction, assume that M is a strongly $N/2$-supercyclic subspace for R. Then we have to consider two cases: either $q = 0$ or $q \neq 0$.

If $q \neq 0$, then $K = \text{span}\{(1, 0, \dots, 0)\}$ is R-invariant. Consider the quotient $\mathbb{R}^N/K$ and apply Proposition 3.11 to deduce that R_K is strongly $N/2$-supercyclic on $\mathbb{R}^{N-1}$. In addition, as $N \geq 4$ we have $1 \leq N/2 < N - 1$; but this contradicts Corollary 3.4 because $N - 1$ is odd.

If $q = 0$, there are two cases: either $N = 4$ or $N \geq 6$.

If $N = 4$, then Corollary 3.9 and Proposition 3.10 imply that R is not strongly 2-supercyclic.

If $N \geq 6$, notice that $K = \text{span}\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0)\}$ is R-invariant. One may consider the quotient by K, and apply Proposition 3.11 to deduce that R_K is strongly $N/2$-supercyclic on $\mathbb{R}^{N-2}$. Moreover, $1 \leq N/2 \neq (N - 2)/2 = N/2 - 1 < N - 2$. This contradicts the induction hypothesis. These contradictions prove that there is no strongly n-supercyclic operator on $\mathbb{R}^N$ with $1 \leq n < N$. ∎

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