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Conditions equivalent to C^* independence

by

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be mutually commuting unital C^* subalgebras of $\mathcal{B}(\mathcal{H})$. It is shown that $\mathcal{A}$ and $\mathcal{B}$ are C^* independent if and only if for all natural numbers n, m, for all n-tuples $A = (A_1, \ldots, A_n)$ of doubly commuting nonzero operators of $\mathcal{A}$ and m-tuples $B = (B_1, \ldots, B_m)$ of doubly commuting nonzero operators of $\mathcal{B}$,

$$\mathrm{Sp}(A, B) = \mathrm{Sp}(A) \times \mathrm{Sp}(B),$$

where Sp denotes the joint Taylor spectrum.

1. Introduction. Independence, one of the most important concepts in classical probability theory, has appeared in many forms in quantum context, and recently in the theory of free probability [15]. Various notions of independence have been studied [4, 6, 8, 9, 10, 12].

In quantum context, observables are represented by self-adjoint operators. To study the problem of quantum measurements, R. Haag and D. Kastler [7] introduced statistical independence. The statistical independence of $\mathcal{A}$ and $\mathcal{B}$ in the category of C^* algebras is called C^* independence, which can be defined as follows: for any states ϕ_1 on $\mathcal{A}$ and ϕ_2 on $\mathcal{B}$, there is a state ϕ on $C^*(\mathcal{A}, \mathcal{B})$ such that $\phi|_{\mathcal{A}} = \phi_1$ and $\phi|_{\mathcal{B}} = \phi_2$, where $C^*(\mathcal{A}, \mathcal{B})$ denotes the unital C^* algebra generated by $\mathcal{A}$ and $\mathcal{B}$.

H. Roos [11] showed that if $\mathcal{A}$ and $\mathcal{B}$ are mutually commuting C^* subalgebras of a C^* algebra, then $\mathcal{A}$ and $\mathcal{B}$ are C^* independent if and only if for any $A \in \mathcal{A}, B \in \mathcal{B}$, $A \neq 0$ & $B \neq 0$ implies that $AB \neq 0$. M. Florig and S. Summers [5] studied the relation between C^* independence and W^* independence. L. J. Bunce and J. Hamhalter [1] gave some equivalent conditions in terms of faithfulness of states on C^* algebras.

In this short paper, we study the C^* independence by using the joint spectrum and establish some equivalent characterizations.

THEOREM 1.1 (Main Theorem). *Let $\mathcal{A}$ and $\mathcal{B}$ be mutually commuting unital C^* algebras on a Hilbert space $\mathcal{H}$. Then $\mathcal{A}$ and $\mathcal{B}$ are C^* independent*

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if and only if for all natural numbers n, m, for all n-tuples $A = (A_1, \dots, A_n)$ of doubly commuting (i.e. $A_iA_j = A_jA_i, A_i^*A_j = A_jA_i^*$) nonzero operators of $\mathcal{A}$ and m-tuples $B = (B_1, \dots, B_m)$ of doubly commuting nonzero operators of $\mathcal{B}$,

$$\mathrm{Sp}(A, B) = \mathrm{Sp}(A) \times \mathrm{Sp}(B),$$

where Sp *denotes the joint Taylor spectrum.*

Throughout this paper, the symbol $\mathcal{H}$ will be used to denote an infinite-dimensional complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators acting on $\mathcal{H}$. Let $C(X)$ be the set of all continuous complex-valued functions on the topological space X. By $\mathrm{Sp}(T)$ we denote the joint Taylor spectrum of an n-tuple $T = (T_1, \dots, T_n)$ of commuting operators on $\mathcal{H}$. For a single operator, $A > 0$ means $A \geq 0$ and $A \neq 0$.

2. Proof of the main theorem

Definition 2.1 ([13]). Let $T = (T_1, \dots, T_n)$ be an n-tuple of commuting operators on $\mathcal{H}$. Then $\mathrm{Sp}(T)$ consists of all points $\lambda = (\lambda_1, \dots, \lambda_n)$ in $\mathbb{C}^n$ such that the Koszul complex $K_*(T - \lambda, \mathcal{H})$ of the operators $(T_1 - \lambda_1, \dots, T_n - \lambda_n)$ is not exact.

J. L. Taylor [14] showed $\mathrm{Sp}(T_1, \dots, T_n)$ is a nonempty compact set in $\mathbb{C}^n$. It is a well known theorem by Gelfand and Naimark that if $\mathcal{A}$ is a unital commutative C^* algebra and $M_{\mathcal{A}}$ is its maximal ideal space, then $\mathcal{A}$ is isometrically $*$-isomorphic to $C(M_{\mathcal{A}})$. Let $T = (T_1, \dots, T_n)$ be an n-tuple of commuting normal operators, and denote by $M_{C^*(T)}$ the maximal ideal space of $C^*(T)$. Then the common joint spectrum [16] of $T = (T_1, \dots, T_n)$ is the set

$$\sigma_{C^*(T)}(T_1, \dots, T_n) = \{(f(T_1), \dots, f(T_n)) \in \mathbb{C}^n;\ f \in M_{C^*(T)}\}.$$

In this case, M. Chō and M. Takaguchi [2] showed that

$$\mathrm{Sp}(T_1, \dots, T_n) = \sigma_{C^*(T)}(T_1, \dots, T_n).$$

Thus there is an isometrical $*$-isomorphism γ from $C^*(T_1, \dots, T_n)$ onto the algebra $C(\mathrm{Sp}(T_1, \dots, T_n))$, which satisfies

$$I \mapsto 1; \quad T_i \mapsto z_i; \quad T_i^* \mapsto z_i^*; \quad h_i(T_i)h_j(T_j) \mapsto h_i(z_i)h_j(z_j),$$

where $h_i(z_i)$ (resp. $h_j(z_j)$) is a continuous function on $\mathrm{Sp}(T_1, \dots, T_n)$.

Lemma 2.2 ([9]). *Let $\mathcal{A}$ and $\mathcal{B}$ be commuting unital C^* algebras on $\mathcal{H}$. Then $\mathcal{A}$ and $\mathcal{B}$ are C^* independent if and only if $0 < A \in \mathcal{A}$ & $0 < B \in \mathcal{B}$ implies that $AB > 0$.*

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be mutually commuting unital C^* algebras on $\mathcal{H}$. Then the following statements are equivalent:*

(1) *$\mathcal{A}$ and $\mathcal{B}$ are C^* independent.*

(2) *For all natural numbers n, m, for all n-tuples $A = (A_1, \ldots, A_n)$ of commuting nonzero positive operators (i.e. $A_i > 0$, $i = 1, \ldots, n$) of $\mathcal{A}$ and m-tuples $B = (B_1, \ldots, B_m)$ of commuting nonzero positive operators of $\mathcal{B}$,*

$$\sigma_{C^*(A,B)}(A, B) = \sigma_{C^*(A)}(A) \times \sigma_{C^*(B)}(B).$$

(3) *For all natural numbers n, m, for all n-tuples $A = (A_1, \ldots, A_n)$ of commuting nonzero positive operators of $\mathcal{A}$ and m-tuples $B = (B_1, \ldots, B_m)$ of commuting nonzero positive operators of $\mathcal{B}$,*

$$\mathrm{Sp}(A, B) = \mathrm{Sp}(A) \times \mathrm{Sp}(B).$$

Proof. (2)$\Leftrightarrow$(3). This follows by M. Chō and M. Takaguchi's theorem.

(1)$\Leftrightarrow$(3). By Lemma 2.2, we need only show that

$$[0 < A \in \mathcal{A} \ \&\ 0 < B \in \mathcal{B} \text{ implies } AB > 0] \ \Leftrightarrow\ \text{(3) holds.}$$

$\Rightarrow$ Suppose there exist natural numbers n, m and nonzero positive operator tuples $A^0 = (A_1^0, \ldots, A_n^0)$, $B^0 = (B_1^0, \ldots, B_m^0)$ such that

$$\mathrm{Sp}(A^0, B^0) \neq \mathrm{Sp}(A^0) \times \mathrm{Sp}(B^0).$$

Then there is a point $P_0 = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m) \in \mathbb{C}^{n+m}$ which satisfies

$$(\lambda_1, \ldots, \lambda_n) \in \mathrm{Sp}(A^0), \qquad (\mu_1, \ldots, \mu_m) \in \mathrm{Sp}(B^0), \qquad P_0 \notin \mathrm{Sp}(A^0, B^0).$$

Since $\mathrm{Sp}(A^0, B^0)$ is compact, there is a $\delta > 0$ such that

$$B(P_0, \delta) \cap \mathrm{Sp}(A^0, B^0) = \emptyset,$$

where $B(P_0, \delta) = \{z : |z - P_0| < \delta\}$. Take $\delta' = \delta/(n+m)$. Then

$$B(P_0, \delta) \supseteq (U_1 \times \cdots \times U_n \times V_1 \times \cdots \times V_m),$$

so

$$\mathrm{Sp}(A^0, B^0) \cap (U_1 \times \cdots \times U_n \times V_1 \times \cdots \times V_m) = \emptyset,$$

where $U_i = \{z_i : |z_i - \lambda_i| < \delta'\}$ and $V_j = \{w_j : |w_j - \mu_j| < \delta'\}$.

Let

$$f_i(z_i) = \begin{cases} \delta' - |z_i - \lambda_i| & \text{if } z_i \in U_i, \\ 0 & \text{if } z_i \notin U_i. \end{cases}$$

$$g_j(w_j) = \begin{cases} \delta' - |w_j - \mu_j| & \text{if } w_j \in V_j, \\ 0 & \text{if } w_j \notin V_j. \end{cases}$$

Notice that $f_i(z_i), g_j(w_j)$ are continuous everywhere and

$$\prod_{i=1}^{n} f_i(z_i)|_{\mathrm{Sp}(A^0)} > 0, \quad \prod_{j=1}^{m} g_j(w_j)|_{\mathrm{Sp}(B^0)} > 0 \quad \text{and}$$
$$\prod_{i=1}^{n}\prod_{j=1}^{m} f_i(z_i)g_j(w_j)|_{\mathrm{Sp}(A^0,B^0)} = 0.$$

By the continuous functional calculus,

$$0 < \prod_{i=1}^{n} f_i(A_i^0) \in \mathcal{A}, \quad 0 < \prod_{j=1}^{m} g_j(B_j^0) \in \mathcal{B} \quad \text{and}$$
$$\prod_{i=1}^{n}\prod_{j=1}^{m} f_i(A_i^0)g_j(B_j^0) = 0,$$

which is a contradiction.

$\Leftarrow$ Let $n = m = 1$, and let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be any nonzero positive operators. Then

$$\mathrm{Sp}(A, B) = \sigma(A) \times \sigma(B).$$

Thus

$$\sigma(AB) = \sigma(A) \cdot \sigma(B).$$

It follows by the commutativity of $\mathcal{A}$ and $\mathcal{B}$ that AB is a positive operator. Moreover,

$$\begin{aligned}\|AB\| &= \max\{|\lambda|;\ \lambda \in \sigma(AB)\}\\ &= \max\{|a \cdot b|;\ a \in \sigma(A),\ b \in \sigma(B)\} = \|A\| \cdot \|B\| > 0,\end{aligned}$$

so $AB > 0$, which completes the proof.

LEMMA 2.4 ([3]). *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of doubly commuting operators on $\mathcal{H}$. Then $0 \notin \mathrm{Sp}(T)$ if and only if $\sum_{i=1}^{n} (T_i)^{f(i)}(T_i^*)^{f(i)}$ is invertible for all $f : \{1, \ldots, n\} \to \{1, *\}$.*

Proof of the main theorem.

$\Leftarrow$ This easily follows by Lemma 2.3.

$\Rightarrow$ Let n, m be any natural numbers, let $A = (A_1, \ldots, A_n)$ be an n-tuple of doubly commuting nonzero operators and $B = (B_1, \ldots, B_m)$ be an m-tuple of doubly commuting nonzero operators. Then

$$\mathrm{Sp}(A, B) \subseteq \mathrm{Sp}(A) \times \mathrm{Sp}(B).$$

It suffices to prove the reverse inclusion.

If $(0,0)\in \mathrm{Sp}(A)\times\mathrm{Sp}(B)$, then $A=(A_1,\dots,A_n)$ and $B=(B_1,\dots,B_m)$ are not invertible. By Lemma 2.4, there exist $u(i)$ and $v(j)$ such that

$$0\in\sigma\Big(\sum_{i=1}^{n}(A_i)^{u(i)}(A_i^*)^{u(i)}\Big)\quad\text{and}\quad 0\in\sigma\Big(\sum_{j=1}^{m}(B_j)^{v(j)}(B_j^*)^{v(j)}\Big).$$

Without loss of generality, assume $0\in\sigma(\sum_{i=1}^{n}A_iA_i^*)$ and $0\in\sigma(\sum_{j=1}^{m}B_jB_j^*)$. Notice that $\sum_{i=1}^{n}A_iA_i^*\in\mathcal{A}$ is a nonzero positive operator and $\sum_{j=1}^{m}B_jB_j^*\in\mathcal{B}$ is a nonzero positive operator. Then by Lemma 2.3,

$$\mathrm{Sp}\Big(\sum_{i=1}^{n}A_iA_i^*,\sum_{j=1}^{m}B_jB_j^*\Big)=\sigma\Big(\sum_{i=1}^{n}A_iA_i^*\Big)\times\sigma\Big(\sum_{j=1}^{m}B_jB_j^*\Big).$$

Thus

$$\sigma\Big(\sum_{i=1}^{n}A_iA_i^*+\sum_{j=1}^{m}B_jB_j^*\Big)=\sigma\Big(\sum_{i=1}^{n}A_iA_i^*\Big)+\sigma\Big(\sum_{j=1}^{m}B_jB_j^*\Big),$$

so

$$0\in\sigma\Big(\sum_{i=1}^{n}A_iA_i^*+\sum_{j=1}^{m}B_jB_j^*\Big),$$

and hence, by Lemma 2.4, $(0,0)\in\mathrm{Sp}(A,B)$.

If $(\lambda_1,\dots,\lambda_n,\mu_1,\dots,\mu_m)\in\mathrm{Sp}(A)\times\mathrm{Sp}(B)$, then

$$0\in\mathrm{Sp}(A_1-\lambda_1,\dots,A_n-\lambda_n)\quad\text{and}\quad 0\in\mathrm{Sp}(B_1-\mu_1,\dots,B_m-\mu_m).$$

By the proof above,

$$(0,0)\in\mathrm{Sp}(A_1-\lambda_1,\dots,A_n-\lambda_n,B_1-\mu_1,\dots,B_m-\mu_m),$$

that is,

$$(\lambda_1,\dots,\lambda_n,\mu_1,\dots,\mu_m)\in\mathrm{Sp}(A,B),$$

which completes the proof.

Corollary 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be mutually commuting unital C^* algebras on $\mathcal{H}$. Then the following statements are equivalent:*

(1) *$\mathcal{A}$ and $\mathcal{B}$ are C^* independent.*
(2) *For any $A\in\mathcal{A}$ and $B\in\mathcal{B}$, $A>0$ & $B>0$ implies $AB>0$.*
(3) *For all natural numbers n,m, for all n-tuples $A=(A_1,\dots,A_n)$ of commuting nonzero positive operators of $\mathcal{A}$ and m-tuples $B=(B_1,\dots,B_m)$ of commuting nonzero positive operators of $\mathcal{B}$,*

$$\mathrm{Sp}(A,B)=\mathrm{Sp}(A)\times\mathrm{Sp}(B).$$

(4) *There exist natural numbers n,m such that for all n-tuples $A=(A_1,\dots,A_n)$ of commuting nonzero positive operators of $\mathcal{A}$ and m-tuples $B=(B_1,\dots,B_m)$ of commuting nonzero positive operators of $\mathcal{B}$,*

$$\mathrm{Sp}(A,B)=\mathrm{Sp}(A)\times\mathrm{Sp}(B).$$

(5) *For all natural numbers* n, m*, for all* n*-tuples* $A = (A_1, \ldots, A_n)$ *of doubly commuting nonzero operators of* $\mathcal{A}$ *and* m*-tuples* $B = (B_1, \ldots, B_m)$ *of doubly commuting nonzero operators of* $\mathcal{B}$*,*

$$\mathrm{Sp}(A, B) = \mathrm{Sp}(A) \times \mathrm{Sp}(B).$$

(6) *There exist natural numbers* n, m*, such that for all* n*-tuples* $A = (A_1, \ldots, A_n)$ *of doubly commuting nonzero operators of* $\mathcal{A}$ *and* m*-tuples* $B = (B_1, \ldots, B_m)$ *of doubly commuting nonzero operators of* $\mathcal{B}$*,*

$$\mathrm{Sp}(A, B) = \mathrm{Sp}(A) \times \mathrm{Sp}(B).$$

Indeed, we have the following implications:

$$\begin{tikzcd}
(1) \arrow[r, Leftrightarrow] & (2) \arrow[r, Leftrightarrow] & (3) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (5) \arrow[d, Rightarrow] \\
& & (4) \arrow[ul, Rightarrow] & (6) \arrow[l, Rightarrow]
\end{tikzcd}$$

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