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$\mathcal{U}$-filters and uniform compactification

by

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Abstract. We show that the uniform compactification of a uniform space $(X,\mathcal{U})$ can be considered as a space of filters on X. We apply these filters to study the $\mathcal{LUC}$-compactification of a topological group.

1. Introduction. The purpose of this paper is to represent the spectrum $\Delta(\mathcal{UC}(X))$ of the C^*-algebra $\mathcal{UC}(X)$ of all bounded, uniformly continuous, complex-valued functions on a uniform space $(X,\mathcal{U})$ as a space of $\mathcal{U}$-ultrafilters. The spectrum $\Delta(\mathcal{UC}(X))$ is known as the *uniform compactification* or the *Samuel compactification* of X (see [I]). The characteristic property of the compact, Hausdorff topological space $\Delta(\mathcal{UC}(X))$ is that the C^*-algebra of all bounded, continuous, complex-valued functions on $\Delta(\mathcal{UC}(X))$ is isometrically $*$-isomorphic to the C^*-algebra $\mathcal{UC}(X)$. The consideration of the Stone–Čech compactification βY of a discrete topological space Y as the space of all ultrafilters on Y has been the main tool in analyzing the properties of βY (see [CN]). The Stone–Čech compactification of a completely regular topological space can be considered as the space of all z-ultrafilters (see [GJ]). The uniform compactification and the Stone–Čech compactification of a discrete group are the same, and ultrafilters have been the main tool in analyzing the algebraic structure of βG (see [HS]).

The $\mathcal{U}$-ultrafilters have a similar role in the study of the uniform compactification of a uniform space and the $\mathcal{LUC}$-compactification of a topological group as the ultrafilters have in the study of βY and βG. In fact, using $\mathcal{U}$-ultrafilters (together with some other machinery developed in the author's PhD thesis [A]), among many results we have already characterized the C^*-algebra of complex-valued functions on a locally compact, topological group G which corresponds to the universal semigroup compactification of

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G with respect to the property that the semigroup compactification contains a right zero element (see [AF]).

In 1997, M. Koçak and D. Strauss considered the uniform compactification of a uniform space $(X,\mathcal{U})$ as the space of so-called near ultrafilters (see [KS]), leading to many interesting results. A near ultrafilter on X need not be a filter on X in the ordinary sense of the word. The approach we present in this paper differs from the one given in [KS] in an essential way, since we do use filters. As far as we are aware, our approach is the first one using filters. Our approach has some further advantages in describing the properties of $\Delta(\mathcal{UC}(X))$. For example, we obtain a bijective correspondence between the non-empty, closed subsets of $\Delta(\mathcal{UC}(X))$ and the $\mathcal{U}$-filters on X, and so we obtain a full analogue with the Stone–Čech compactification of a discrete space. Also, in Section 5 we describe the uniform compactification of a subspace of X using $\mathcal{U}$-ultrafilters. Filters have also a central role in [BP], where the local topological structure of the $\mathcal{LUC}$-compactification of a locally compact, topological group G is studied. However, in [BP] the $\mathcal{LUC}$-compactification is viewed as a quotient of the Stone–Čech compactification βG_d, where G_d denotes the group G with the discrete topology, instead of a space of filters on G.

The paper is organized as follows: In Section 2, we gather some basic definitions and results that we will use throughout this paper. In Section 3, we introduce $\mathcal{U}$-filters and $\mathcal{U}$-ultrafilters on a uniform space $(X,\mathcal{U})$ and we give some properties of $\mathcal{U}$-ultrafilters. In Section 4, we introduce the topological space γX consisting of all $\mathcal{U}$-ultrafilters on X and we show that γX is the uniform compactification of X. In Section 5, we describe the uniform compactification of a subspace of X using $\mathcal{U}$-ultrafilters. In Section 6, we describe the semigroup operation of the $\mathcal{LUC}$-compactification of a topological group G in terms of $\mathcal{U}$-ultrafilters. Furthermore, we describe those elements of the $\mathcal{LUC}$-compactification which are in the minimal ideal of the $\mathcal{LUC}$-compactification or in the closure of the minimal ideal.

2. Preliminaries. We first remind the reader of some basic definitions.

We denote by $\mathbb{N}$ the set of all positive integers, that is, $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let X be a non-empty set. We denote by $\mathcal{P}(X)$ the set of all subsets of X. A *filter* on X is a non-empty set $\varphi \subseteq \mathcal{P}(X)$ with the following properties:

(i) If $A, B \in \varphi$, then $A \cap B \in \varphi$.
(ii) If $A \in \varphi$ and $A \subseteq B \subseteq X$, then $B \in \varphi$.
(iii) $\emptyset \notin \varphi$.

A *filter base* on X is a non-empty set $\mathcal{A} \subseteq \mathcal{P}(X)$ such that $\emptyset \notin \mathcal{A}$ and, for all sets $A, B \in \mathcal{A}$, there exists a set $C \in \mathcal{A}$ such that $C \subseteq A \cap B$. If $\mathcal{A}$ is a filter base on X, then the filter φ on X *generated* by $\mathcal{A}$ is

$$\varphi = \{A \subseteq X : \text{there exists some set } B \in \mathcal{A} \text{ such that } B \subseteq A\}.$$

Indeed, it is easy to verify that φ is the smallest filter (with respect to inclusion) on X which contains the set $\mathcal{A}$. If φ is a filter on X, then a set $\mathcal{B} \subseteq \mathcal{P}(X)$ is a *filter base* for φ if and only if $\mathcal{B} \subseteq \varphi$ and, for every set $A \in \varphi$, there exists a set $B \in \mathcal{B}$ such that $B \subseteq A$. In particular, if $\mathcal{A}$ is a filter base on X and φ is the filter on X generated by $\mathcal{A}$, then $\mathcal{A}$ is a filter base for φ.

Let X be a non-empty set. If $U, V \subseteq X \times X$, then

$$U^{-1} = \{(x, y) : (y, x) \in U\}$$

and

$$U \circ V = \{(x, z) : (x, y) \in V \text{ and } (y, z) \in U \text{ for some } y \in X\}.$$

The set U is *symmetric* if $U = U^{-1}$. We denote $U \circ U$ by U^2. We denote the *diagonal* $\{(x, x) : x \in X\}$ of $X \times X$ by Δ_X. A *uniform structure* on X is a filter $\mathcal{U}$ on $X \times X$ with the following properties:

(i) If $U \in \mathcal{U}$, then $\Delta_X \subseteq U$.
(ii) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.
(iii) If $U \in \mathcal{U}$, then there exists a set $V \in \mathcal{U}$ such that $V^2 \subseteq U$.

A *uniform space* is a pair $(X, \mathcal{U})$, where X is a non-empty set and $\mathcal{U}$ is a uniform structure on X.

Let $(X, \mathcal{U})$ be a uniform space. If $A \subseteq X$ and $U \in \mathcal{U}$, then

$$U[A] = \{y \in X : \text{there exists } x \in A \text{ such that } (x, y) \in U\}.$$

If $A = \{x\}$ for some $x \in X$, then we denote $U[A]$ simply by $U[x]$.

A uniform structure $\mathcal{U}$ on a set X determines a topology on X. In this topology, a neighborhood base of a point $x \in X$ is given by the sets $U[x]$, where $U \in \mathcal{U}$. We denote the C^*-algebra of all bounded, continuous, complex-valued functions on X by $\mathcal{C}(X)$. In this paper, we assume that all the topological spaces considered are Hausdorff.

A topological space X is *uniformizable* if and only if there exists a uniform structure $\mathcal{U}$ on X such that the topology determined by $\mathcal{U}$ is the original topology of X. Every topological group G is uniformizable. In fact, there are two natural uniform structures on G, both of which determine the topology of G. Let e denote the identity of G and let $\mathcal{N}_e$ denote the filter of all neighborhoods of e in G. For every set $U \in \mathcal{N}_e$, define

$$U_R = \{(x, y) \in G \times G : xy^{-1} \in U\}, \qquad U_L = \{(x, y) \in G \times G : x^{-1}y \in U\}.$$

The *right uniform structure* and the *left uniform structure* on G are generated by the sets $\{U_R : U \in \mathcal{N}_e\}$ and $\{U_L : U \in \mathcal{N}_e\}$, respectively. A detailed treatment of uniform structures on topological groups is given in [RD].

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces. A function $f : X \to Y$ is *uniformly continuous* if and only if, for every set $V \in \mathcal{V}$, there exists a set $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ for every $(x,y) \in U$. We denote the C^*-algebra of all bounded, uniformly continuous, complex-valued functions on X by $\mathcal{UC}(X)$.

Finally, we recall the concept of a semigroup compactification of a topological group. A general treatment of semigroup compactifications is given in [BJ]. A *right topological semigroup* is a semigroup X with a topology such that, for every $y \in X$, the mapping $x \mapsto xy$ from X to X is continuous. Let G be a topological group. A *semigroup compactification* of G is a pair (ψ, X) such that X is a compact, right topological semigroup, $\psi : G \to X$ is a continuous homomorphism, $\psi(G)$ is dense in X, and $x \mapsto \psi(s)x$ is a continuous mapping from X to X for every $s \in G$.

3. $\mathcal{U}$-filters.

For the rest of this paper, let $(X,\mathcal{U})$ be a uniform space. For every $x \in X$, we denote the filter of all neighborhoods of x in X by $\mathcal{N}_x$. For every subset A of X, we denote the interior of A in X by A°.

In this section, we introduce the main object of this paper, namely $\mathcal{U}$-filters and $\mathcal{U}$-ultrafilters on X, and we give some properties of $\mathcal{U}$-ultrafilters.

Definition 3.1. A *$\mathcal{U}$-family* on X is a non-empty set $\mathcal{A} \subseteq \mathcal{P}(X)$ such that, for every set $A \in \mathcal{A}$, there exist sets $B \in \mathcal{A}$ and $U \in \mathcal{U}$ such that $U[B] \subseteq A$. A *$\mathcal{U}$-filter* on X is a filter φ on X such that φ is a $\mathcal{U}$-family on X.

Remark 3.2.

(i) If $\Delta_X \in \mathcal{U}$, then every filter φ on X is a $\mathcal{U}$-filter on X.
(ii) Let $\mathcal{A}$ be a non-empty subset of $\mathcal{P}(X)$. Property (iii) of a uniform structure implies that $\{U[A] : U \in \mathcal{U},\ A \in \mathcal{A}\}$ is a $\mathcal{U}$-family on X.
(iii) Let $\mathcal{A}$ be a $\mathcal{U}$-family on X. Let $A \in \mathcal{A}$ and pick sets $B \in \mathcal{A}$ and $U \in \mathcal{U}$ such that $U[B] \subseteq A$. Then $V[B] \subseteq A$ for every set $V \in \mathcal{U}$ such that $V \subseteq U$. So, if necessary, we may assume that the given set $U \in \mathcal{U}$ is symmetric, or satisfies $U^2[B] \subseteq A$, etc.
(iv) If $\mathcal{A}$ is a filter base and a $\mathcal{U}$-family on X, then the filter φ on X generated by $\mathcal{A}$ is a $\mathcal{U}$-filter on X.
(v) If φ is a $\mathcal{U}$-filter on X and $A \subseteq X$, then $A \in \varphi$ if and only if $A^\circ \in \varphi$.

We will use the above observations with no further mention.

Every $\mathcal{U}$-filter φ on X satisfies $A^\circ \in \varphi$ for every $A \in \varphi$, but we should point out that the converse does not hold. That is, if φ is a filter on X such that $A^\circ \in \varphi$ for every $A \in \varphi$, then φ is not necessarily a $\mathcal{U}$-filter on X. For example, let $X = \mathbb{R}$, let $A =]0,1[$, and let φ be the filter on $\mathbb{R}$ generated by $\{A\}$. Since A is an open subset of $\mathbb{R}$, the filter φ has the property that $B^\circ \in \varphi$ for every $B \in \varphi$. However, φ is not a $\mathcal{U}$-family on $\mathbb{R}$.

Zorn's Lemma implies that every $\mathcal{U}$-filter on X is contained in some maximal (with respect to inclusion) $\mathcal{U}$-filter on X.

DEFINITION 3.3. A *$\mathcal{U}$-ultrafilter* on X is a $\mathcal{U}$-filter on X which is not properly contained in any other $\mathcal{U}$-filter on X.

The following simple fact about $\mathcal{U}$-ultrafilters is very useful: If p and q are $\mathcal{U}$-ultrafilters on X, then $p = q$ if and only if $p \subseteq q$.

THEOREM 3.4. *If $x \in X$, then $\mathcal{N}_x$ is a $\mathcal{U}$-ultrafilter on X.*

Proof. By Remark 3.2(ii), $\mathcal{N}_x$ is a $\mathcal{U}$-filter on X, and so we only need to show that $\mathcal{N}_x$ is a maximal $\mathcal{U}$-filter on X. Suppose that $\mathcal{N}_x$ is properly contained in some $\mathcal{U}$-filter φ on X. Pick some set $A \in \varphi \setminus \mathcal{N}_x$. Pick a set $B \in \varphi$ and a symmetric set $U \in \mathcal{U}$ such that $U^2[B] \subseteq A$. Now, $x \notin U[B]$, and so $U[x] \cap B = \emptyset$, in contradiction with $U[x], B \in \varphi$. ∎

Recall that a non-empty subset $\mathcal{A}$ of $\mathcal{P}(X)$ has the *finite intersection property* if and only if $\bigcap_{k=1}^{n} A_k \neq \emptyset$ whenever $A_1, \ldots, A_n \in \mathcal{A}$ for some $n \in \mathbb{N}$. We leave the proofs of the following two lemmas to the reader.

LEMMA 3.5. *If $\mathcal{A}$ is a $\mathcal{U}$-family on X such that $\mathcal{A}$ has the finite intersection property, then there exists a $\mathcal{U}$-ultrafilter p on X such that $\mathcal{A} \subseteq p$.*

LEMMA 3.6. *Let φ be a $\mathcal{U}$-filter on X. If A is a subset of X such that $U[A] \cap B \neq \emptyset$ for every $U \in \mathcal{U}$ and every $B \in \varphi$, then there exists a $\mathcal{U}$-ultrafilter p on X such that $\varphi \cup \{U[A] : U \in \mathcal{U}\} \subseteq p$.*

The following theorem follows easily from the previous two lemmas, and so we omit the proof. The reader may compare the following theorem to [HS, Theorem 3.6].

THEOREM 3.7. *If $\varphi \subseteq \mathcal{P}(X)$, then the following statements are equivalent:*

(i) *φ is a $\mathcal{U}$-ultrafilter on X.*
(ii) *φ is a $\mathcal{U}$-filter on X and, if a set $A \subseteq X$ satisfies $U[A] \notin \varphi$ for some $U \in \mathcal{U}$, then there exists a set $B \in \varphi$ such that $A \cap B = \emptyset$.*
(iii) *φ is a maximal (with respect to inclusion) subset of $\mathcal{P}(X)$ such that φ is a $\mathcal{U}$-family on X and φ has the finite intersection property.*
(iv) *φ is a $\mathcal{U}$-filter on X and, if $A_1, \ldots A_n \subseteq X$ for some $n \in \mathbb{N}$ satisfy $\bigcup_{k=1}^{n} A_k \in \varphi$, then there exists $k \in \{1, \ldots, n\}$ such that $U[A_k] \in \varphi$ for every $U \in \mathcal{U}$.*
(v) *φ is a $\mathcal{U}$-filter on X and, for every set $A \subseteq X$, either $U[A] \in \varphi$ for every $U \in \mathcal{U}$ or $U[X \setminus A] \in \varphi$ for every $U \in \mathcal{U}$.*

We finish this section with some remarks concerning the previous theorem. In statement (ii), it is not enough to assume that the set $A \subseteq X$ satisfies

$A \notin \varphi$ to conclude that there exists a set $B \in \varphi$ such that $A \cap B = \emptyset$. For example, let $X = \mathbb{R}$ and consider the $\mathcal{U}$-ultrafilter $\varphi = \mathcal{N}_0$ on X. Put $A = \{0\}$. Then $A \notin \varphi$ but $A \cap B \neq \emptyset$ for every $B \in \varphi$. In statement (iv), we cannot conclude that $A_k \in \varphi$ for some $k \in \{1, \ldots, n\}$. For example, let $X = \mathbb{R}$, $A_1 = \mathbb{Q}$, and $A_2 = \mathbb{R} \setminus \mathbb{Q}$. This also shows that the two given statements in statement (v) are not exclusive, that is, it may happen that $U[A] \in \varphi$ for every $U \in \mathcal{U}$ and $U[X \setminus A] \in \varphi$ for every $U \in \mathcal{U}$.

4. The topological space γX. In this section, we define a topology on the set of all $\mathcal{U}$-ultrafilters on X and we establish some properties of the resulting space.

DEFINITION 4.1. Define $\gamma X = \{p : p \text{ is a } \mathcal{U}\text{-ultrafilter on } X\}$. For every subset A of X, define $\widehat{A} = \{p \in \gamma X : A \in p\}$. For every $\mathcal{U}$-filter φ on X, define $\widehat{\varphi} = \{p \in \gamma X : \varphi \subseteq p\}$.

Observe that $\widehat{A} = \widehat{A^\circ}$ for every subset A of X.

THEOREM 4.2. *If φ and ψ are $\mathcal{U}$-filters on X, then the following statements hold:*

(i) $\widehat{\varphi} = \bigcap_{A \in \varphi} \widehat{A}$.
(ii) $\varphi = \bigcap_{p \in \widehat{\varphi}} p$.
(iii) $\varphi \subseteq \psi$ *if and only if* $\widehat{\psi} \subseteq \widehat{\varphi}$.

Proof. (i) This is obvious.

(ii) The inclusion $\varphi \subseteq \bigcap_{p \in \widehat{\varphi}} p$ is obvious, so suppose that A is a subset of X such that $A \notin \varphi$. Then $\mathcal{A} = \varphi \cup \{U[X \setminus A] : U \in \mathcal{U}\}$ is a $\mathcal{U}$-family on X such that $\mathcal{A}$ has the finite intersection property. By Lemma 3.5, there exists an element $p \in \widehat{\varphi}$ such that $U[X \setminus A] \in p$ for every $U \in \mathcal{U}$. Now, it is enough to show that $A \notin p$. If $A \in p$, then there exists a set $C \in p$ and a symmetric set $U \in \mathcal{U}$ such that $U[C] \subseteq A$. But then $U[X \setminus A] \cap C = \emptyset$, contrary to $U[X \setminus A], C \in p$.

(iii) This follows from statement (ii). ■

The set $\{\widehat{A} : A \subseteq X\}$ is a base for a topology on γX, and we equip γX with this topology. We denote the closure of a subset Y of γX in γX by $\overline{Y}$ with one exception: If $A \subseteq X$, then we denote the closure of $\widehat{A}$ in γX by $\mathrm{cl}_{\gamma X}(\widehat{A})$ instead of $\overline{\widehat{A}}$.

We proceed to show that γX is the uniform compactification of X. The following definition is reasonable by Theorem 3.4.

DEFINITION 4.3. The *natural embedding* $e : X \to \gamma X$ is given by $e(x) = \mathcal{N}_x$ for every $x \in X$.

Note that, if $A \subseteq X$ and $x \in X$, then $e(x) \in \widehat{A}$ if and only if $x \in A^\circ$.

THEOREM 4.4. *The mapping $e : X \to \gamma X$ is an embedding and $e(X)$ is dense in γX.*

Proof. Since X is Hausdorff, the mapping e is injective. If $A \subseteq X$, then $e^{-1}(\widehat{A}) = A^\circ$, and so e is continuous. If A is an open subset of X, then $e(A) = \widehat{A} \cap e(X)$, and so e is an embedding. If A is a subset of X such that $\widehat{A} \neq \emptyset$, then $e(x) \in \widehat{A}$ for every $x \in A^\circ$, and so $e(X)$ is dense in γX. ■

COROLLARY 4.5. *If X is compact, then $e : X \to \gamma X$ is a homeomorphism.*

By Theorem 4.4, we may (and will) consider X as a subspace of γX. So, for every $x \in X$, we denote $e(x)$ simply by x. In particular, if $A \subseteq X$, then we denote $\overline{e(A)}$ by $\overline{A}$.

We leave the proofs of the following three statements to the reader.

LEMMA 4.6. *Let $A \subseteq X$ and let $p \in \gamma X$. The following statements are equivalent:*

(i) $p \in \overline{A}$.
(ii) *$A \cap B \neq \emptyset$ for every $B \in p$.*
(iii) *$U[A] \in p$ for every $U \in \mathcal{U}$.*

In particular, $p \in \overline{A}$ for every $A \in p$.

COROLLARY 4.7. *If $A, B \subseteq X$, then $U[A] \cap U[B] \neq \emptyset$ for every $U \in \mathcal{U}$ if and only if $\overline{A} \cap \overline{B} \neq \emptyset$.*

LEMMA 4.8. *If $A \subseteq X$, then the following statements hold:*

(i) $\widehat{X \setminus A} = \gamma X \setminus \overline{A}$.
(ii) *If A is an open subset of X, then $\mathrm{cl}_{\gamma X}(\widehat{A}) = \overline{A}$.*

The next theorem and its proof were suggested to us by the referee. Instead of giving direct proofs to Theorems 4.10 and 4.11 below, we can use the following correspondence between $\mathcal{U}$-ultrafilters and near ultrafilters and apply the existing results in [KS]. As in [KS], $\widetilde{X}$ denotes the space of all near ultrafilters.

THEOREM 4.9. *Let $p \in \gamma X$ and let $\xi \in \widetilde{X}$. The following statements hold:*

(i) *The set $\mathcal{B} = \{U[A] : U \in \mathcal{U},\ A \in \xi\}$ is a filter base on X. The filter $u(\xi)$ on X generated by $\mathcal{B}$ is a $\mathcal{U}$-ultrafilter.*
(ii) *The set $n(p) = \{A \subseteq X : U[A] \in p$ for every $U \in \mathcal{U}\}$ is a near ultrafilter on X.*
(iii) *The mapping $p \mapsto n(p)$ from γX to $\widetilde{X}$ is a homeomorphism.*

Proof. (i) Clearly, $u(\xi)$ is a $\mathcal{U}$-filter on X. To see that $u(\xi)$ is a $\mathcal{U}$-ultrafilter on X, suppose that φ is a $\mathcal{U}$-filter on X which properly contains

$u(\xi)$. Pick some set $A \in \varphi \setminus u(\xi)$. Pick sets $B \in \varphi$ and $U \in \mathcal{U}$ such that $U[B] \subseteq A$. Now, $B \notin \xi$, and so there exist $n \in \mathbb{N}$ and sets $A_1, \dots, A_n \in \xi$ and $V \in \mathcal{U}$ such that $B \cap \bigcap_{k=1}^{n} V[A_k] = \emptyset$. Since $B \in \varphi$ and $\bigcap_{k=1}^{n} V[A_k] \in u(\xi)$, this contradicts our assumption.

(ii) Clearly, $n(p)$ has the near finite intersection property. To see that $n(p)$ is a near ultrafilter on X, let A be a subset of X such that $A \notin n(p)$. Pick a symmetric set $U \in \mathcal{U}$ such that $U^3[A] \notin p$. By Lemma 3.6, there exists a set $B \in p$ such that $U^2[A] \cap B = \emptyset$. Since $p \subseteq n(p)$, the set $\{A\} \cup n(p)$ does not have the near finite intersection property, thus proving the claim.

(iii) The fact that the given mapping is bijective follows from the equality $u(n(p)) = p$ for every $p \in \gamma X$. The inclusion $u(n(p)) \subseteq p$ is obvious, so let $A \in p$. Pick sets $B \in p$ and $U \in \mathcal{U}$ such that $U[B] \subseteq A$. Now, $B \in n(p)$, so $U[B] \in u(n(p))$, and so $A \in u(n(p))$, as required.

If $A \subseteq X$ and $p \in \gamma X$, then $n(p) \in \mathcal{C}_A$ (see [KS, p. 97]) if and only if $p \in \overline{A}$ by Lemma 4.6. Statement (i) of Lemma 4.8 shows that the sets $\overline{A}$, where $A \subseteq X$, form a base for the closed sets of γX. Therefore, the given mapping is a homeomorphism. ∎

The following two theorems follow immediately from statement (iii) of the previous theorem, [KS, Theorem 6], and [KS, Theorem 8].

THEOREM 4.10. *The space γX is a compact, Hausdorff space.*

In the following theorem, we denote the natural embeddings from X to γX and from Y to γY by e_X and e_Y, respectively.

THEOREM 4.11. *If $(Y, \mathcal{V})$ is a uniform space and $f : X \to Y$ is a uniformly continuous function, then there exists a unique continuous function $\widehat{f} : \gamma X \to \gamma Y$ such that $\widehat{f} \circ e_X = e_Y \circ f$.*

If $f \in \mathcal{UC}(X)$, then Theorem 4.11 and Corollary 4.5 imply that f extends to γX. We denote this extension by $\widehat{f}$.

The next theorem shows that γX is the uniform compactification of X. It can be established using the correspondence suggested by the referee and [KS, Theorem 10], where the Stone–Weierstrass Theorem was necessary in the proof. Since our arguments do not use the Stone–Weierstrass Theorem but rely on the properties of $\mathcal{U}$-filters, we feel that they are worth presenting.

THEOREM 4.12. *The mapping $f \mapsto \widehat{f}$ is an isometric $*$-isomorphism from $\mathcal{UC}(X)$ to $\mathcal{C}(\gamma X)$.*

Proof. We show that the given mapping is surjective and leave the rest of the proof to reader. Let $g \in \mathcal{C}(\gamma X)$ and let f denote the restriction of g to X. Now, $\widehat{f} = g$, and so it is enough to show that $f \in \mathcal{UC}(X)$. Let $r > 0$. For every $p \in \gamma X$, pick an open subset A_p of X such that $A_p \in p$

and $|\widehat{f}(q) - \widehat{f}(q')| \le r$ for all $q, q' \in \widehat{A}_p$. For every $p \in \gamma X$, pick an open subset B_p of X and a set $U_p \in \mathcal{U}$ such that $B_p \in p$ and $U_p[B_p] \subseteq A_p$. Pick $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \gamma X$ such that $\gamma X \subseteq \bigcup_{k=1}^n \widehat{B}_{p_k}$. Put $U = \bigcap_{k=1}^n U_{p_k}$ and let $(x, y) \in U$. Pick $k \in \{1, \ldots, n\}$ such that $x \in B_{p_k}$. Now, $y \in U[B_{p_k}]$, so $x, y \in A_{p_k}$, and hence $|f(x) - f(y)| \le r$, thus finishing the proof. ∎

We finish this section by showing that $\mathcal{U}$-filters can be used to describe the topology of γX.

DEFINITION 4.13. Define $\overline{\varphi} = \bigcap_{A \in \varphi} \overline{A}$ for every $\mathcal{U}$-filter φ on X.

Note that $\overline{\varphi}$ is a non-empty, closed subset of γX.

THEOREM 4.14. *If φ is a $\mathcal{U}$-filter on X, then $\widehat{\varphi} = \overline{\varphi}$.*

Proof. The inclusion $\widehat{\varphi} \subseteq \overline{\varphi}$ is obvious, so let $p \in \overline{\varphi}$. Let $A \in \varphi$ and pick sets $B \in \varphi$ and $U \in \mathcal{U}$ such that $U[B] \subseteq A$. Since $p \in \overline{B}$, we have $U[B] \in p$ by Lemma 4.6, and so $A \in p$. Therefore, $\varphi \subseteq p$, as required. ∎

THEOREM 4.15. *If C is a non-empty, closed subset of γX, then there exists a unique $\mathcal{U}$-filter φ on X such that $\widehat{\varphi} = C$.*

Proof. Let C be a non-empty, closed subset of γX. Put $\varphi = \bigcap_{p \in C} p$. Clearly, φ is a filter on X. Let us first show that φ is a $\mathcal{U}$-family on X, hence, a $\mathcal{U}$-filter on X. Let $A \in \varphi$. If $p \in C$, then $A \in p$, and so there exist sets $B_p \in p$ and $U_p \in \mathcal{U}$ such that $U_p[B_p] \subseteq A$. Now, $\{\widehat{B}_p : p \in C\}$ is an open cover of C, and so there exist $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in C$ such that $C \subseteq \bigcup_{k=1}^n \widehat{B}_{p_k}$. Put $U = \bigcap_{k=1}^n U_{p_k}$ and $B = \bigcup_{k=1}^n B_{p_k}$. Then $B \in p$ for every $p \in C$, and so $B \in \varphi$. Since $U[B] \subseteq A$, the set φ is a $\mathcal{U}$-family on X.

Next, let us show that $\widehat{\varphi} = C$. The inclusion $C \subseteq \widehat{\varphi}$ is obvious, so suppose that $q \in \gamma X \setminus C$. Then there exists an open subset A of X such that $A \in q$ and $\widehat{A} \cap C = \emptyset$. For every $p \in C$, there exists a set $B_p \in p$ such that $A \cap B_p = \emptyset$. As above, there exist $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in C$ such that $B := \bigcup_{k=1}^n B_{p_k} \in p$ for every $p \in C$, and so $B \in \varphi$. Now, $A \cap B = \emptyset$, and so $q \notin \widehat{\varphi}$, as required.

The uniqueness of φ follows from Theorem 4.2(iii). ∎

5. Uniform compactifications of subspaces. In this section, we use $\mathcal{U}$-filters on X to describe the uniform compactification of a subspace of X.

If Y is a subspace of X, then Y is uniformizable. Indeed, the induced uniform structure $\mathcal{U}_Y = \{(Y \times Y) \cap U : U \in \mathcal{U}\}$ on Y determines the relative topology of Y. We always assume that the uniform structure on a subspace of X is given by the induced uniform structure.

DEFINITION 5.1. Let φ be a $\mathcal{U}$-filter on X. A non-empty subset Y of X is a *φ-subset* of X if and only if $A \cap Y \neq \emptyset$ for every $A \in \varphi$. If Y is a φ-subset of X, define $\varphi_Y = \{A \cap Y : A \in \varphi\}$.

Note that if φ is a $\mathcal{U}$-filter on X, and Y is a φ-subset of X, then φ_Y is a filter on Y.

We use the following observation repeatedly in the following proofs: Let Y be a subspace of X, and let $A \subseteq Y$, let $U \in \mathcal{U}$, and $U_Y = (Y \times Y) \cap U$. Then $U_Y[A] = U[A] \cap Y$.

Lemma 5.2. *Let φ be a $\mathcal{U}$-filter on X and let $Y \subseteq X$. Then Y is a φ-subset of X if and only if $\overline{Y} \cap \overline{\varphi} \neq \emptyset$. In particular, if $p \in \gamma X$, then Y is a p-subset of X if and only if $p \in \overline{Y}$.*

Proof. If Y is a φ-subset of X, then $\bigcap_{B \in \varphi_Y} \overline{B} = \bigcap_{A \in \varphi} \overline{A \cap Y} \neq \emptyset$. Clearly, $\bigcap_{A \in \varphi} \overline{Y \cap A} \subseteq \overline{Y} \cap \overline{\varphi}$, and so $\overline{Y} \cap \overline{\varphi} \neq \emptyset$. Suppose now that Y is not a φ-subset of X. Pick a set $B \in \varphi$ and a symmetric set $U \in \mathcal{U}$ such that $U^2[B] \cap Y = \emptyset$. Then $\overline{B} \cap \overline{Y} = \emptyset$ by Corollary 4.7. Since $B \in \varphi$, we have $\overline{\varphi} \subseteq \overline{B}$, and so $\overline{Y} \cap \overline{\varphi} = \emptyset$. ∎

Theorem 5.3. *Let φ be a $\mathcal{U}$-filter on X and let Y be a φ-subset of X. The following statements hold:*

(i) *φ_Y is a $\mathcal{U}_Y$-filter on Y.*
(ii) *If φ is a $\mathcal{U}$-ultrafilter on X, then φ_Y is a $\mathcal{U}_Y$-ultrafilter on Y.*

Proof. (i) This is obvious.

(ii) Suppose that $\varphi = p$ for some $p \in \gamma X$. We apply statement (ii) of Theorem 3.7 to show that p_Y is a $\mathcal{U}_Y$-ultrafilter on Y. So, suppose that $B \subseteq Y$ satisfies $U_Y[B] \notin p_Y$ for some $U_Y \in \mathcal{U}_Y$. Pick $U \in \mathcal{U}$ such that $U_Y = (Y \times Y) \cap U$. Now, $U[B] \notin p$, and so there exists a set $C \in p$ such that $B \cap C = \emptyset$. Since $C \cap Y \in p_Y$, the statement follows. ∎

Theorem 5.4. *Let Y be a subspace of X and let φ be a $\mathcal{U}_Y$-filter on Y. The following statements hold:*

(i) *$\mathcal{A} = \{U[A] : U \in \mathcal{U},\ A \in \varphi\}$ is a $\mathcal{U}$-family and a filter base on X.*
(ii) *If ψ is the $\mathcal{U}$-filter on X generated by $\mathcal{A}$, then Y is a ψ-subset of X and $\psi_Y = \varphi$.*
(iii) *If φ is a $\mathcal{U}_Y$-ultrafilter on Y, then ψ is a $\mathcal{U}$-ultrafilter on X and $\psi \in \overline{Y}$.*

Proof. (i) This is obvious.

(ii) Let ψ be the $\mathcal{U}$-filter on X generated by $\mathcal{A}$. First, $U[Y] \in \psi$ for every $U \in \mathcal{U}$, so Y is a p-subset of X for every $p \in \widehat{\psi}$ by Lemma 4.6, and so Y is a ψ-subset of X by Theorem 4.2(ii). The inclusion $\psi_Y \subseteq \varphi$ is obvious, so let $A \in \varphi$. Pick sets $B \in \varphi$ and $U_Y \in \mathcal{U}_Y$ such that $U_Y[B] \subseteq A$. Pick a set $U \in \mathcal{U}$ such that $U_Y = (Y \times Y) \cap U$. Then $U[B] \cap Y \subseteq A$. Here, $U[B] \in \psi$, and so $A \in \psi_Y$, as required.

(iii) Suppose that φ is a $\mathcal{U}_Y$-ultrafilter on Y. We apply statement (iv) of Theorem 3.7 to show that ψ is a $\mathcal{U}$-ultrafilter on X. So, suppose that

$A_1, \ldots, A_n \subseteq X$ for some $n \in \mathbb{N}$ satisfy $\bigcup_{k=1}^{n} A_k \in \psi$. Pick $k \in \{1, \ldots, n\}$ such that $U_Y[A_k \cap Y] \in \varphi$ for every $U_Y \in \mathcal{U}_Y$. Now, $U^2[A_k] \in \psi$ for every $U \in \mathcal{U}$, as required. Since $U[Y] \in \psi$ for every $U \in \mathcal{U}$, we have $\psi \in \overline{Y}$. ∎

The assertion of the next theorem that γY and $\overline{Y}$ are homeomorphic is also given in [KS].

THEOREM 5.5. *If Y is a subspace of X, then the mapping $F : \overline{Y} \to \gamma Y$ given by $F(p) = p_Y$ is a homeomorphism.*

Proof. First, F is well-defined by Lemma 5.2 and Theorem 5.3(ii). Also, F is surjective by Theorem 5.4, and so it is enough to show that F is injective and continuous. We leave the details to the reader. ∎

In the following corollary, we put $\mathrm{cl}_{\gamma Y}(\varphi_Y) = \bigcap_{A \in \varphi_Y} \mathrm{cl}_{\gamma Y}(A)$, where $\mathrm{cl}_{\gamma Y}(A)$ denotes the closure of a subset A of Y in γY.

COROLLARY 5.6. *Let Y be a subspace of X and let φ be a $\mathcal{U}$-filter on X such that Y is a φ-subset of X. Then $\overline{Y} \cap \overline{\varphi}$ is homeomorphic to the subspace $\mathrm{cl}_{\gamma Y}(\varphi_Y)$ of γY.*

Proof. We claim that the mapping $F : \overline{Y} \cap \overline{\varphi} \to \mathrm{cl}_{\gamma Y}(\varphi_Y)$ given by $F(p) = p_Y$ is a homeomorphism. First, if $p \in \overline{Y} \cap \overline{\varphi}$, then $\varphi \subseteq p$ by Theorem 4.14, and so $\varphi_Y \subseteq p_Y$. Therefore, $F(p) \in \mathrm{cl}_{\gamma Y}(\varphi_Y)$, again by Theorem 4.14. By Theorem 5.5, we need only show that F is surjective. Let $q \in \mathrm{cl}_{\gamma Y}(\varphi_Y)$. By Theorem 5.5, there exists $p \in \overline{Y}$ such that $p_Y = q$. Hence, we only need to show that $p \in \overline{\varphi}$.

Suppose that $p \notin \overline{\varphi}$. Pick a set $A \in \varphi$ such that $p \notin \overline{A}$. By Lemma 4.6, there exists a set $U \in \mathcal{U}$ such that $U[A] \notin p$. By Theorem 3.7(ii), there exists a set $B \in p$ such that $A \cap B = \emptyset$. But now, φ_Y and p_Y contain disjoint elements, and so $p_Y \notin \mathrm{cl}_{\gamma Y}(\varphi_Y)$, a contradiction. ∎

We now obtain the following result, originally due to M. Katětov. The norm in the following corollary is the supremum-norm.

COROLLARY 5.7. *Let Y be a subspace of X. If a function $f : Y \to \mathbb{R}$ is bounded and uniformly continuous, then there exists a real-valued function $F \in \mathcal{UC}(X)$ such that $f(y) = F(y)$ for every $y \in Y$ and $\|f\| = \|F\|$.*

Proof. Let $f : Y \to \mathbb{R}$ be as above. By Theorems 4.11 and 5.5, there exists a continuous function $g : \overline{Y} \to \mathbb{R}$ which extends f such that $\|f\| = \|g\|$. The statement follows from Tietze's Theorem and Theorem 4.12. ∎

REMARK 5.8. Recall that a subset Y of X is *uniformly discrete* if and only if there exists a set $U \in \mathcal{U}$ such that $U[x] \cap U[y] = \emptyset$ for all distinct $x, y \in Y$. In this case, γY is the Stone–Čech compactification βY of Y. Theorem 5.5 implies the following statements. Theorem 5.9 is a well-known result and Theorem 5.10 is given in [KS].

THEOREM 5.9. *If Y is a uniformly discrete subset of X, then $\overline{Y}$ is homeomorphic to the Stone–Čech compactification βY of Y.*

THEOREM 5.10. *If X is not totally bounded, then γX contains a homeomorphic copy of $\beta\mathbb{N}$.*

6. $\mathcal{LUC}$-compactification of a topological group. Throughout this section, let G be a (not necessarily locally compact) topological group and let e denote the identity of G. We consider the right uniform structure on G and we denote the space γG by $G^{\mathcal{LUC}}$.

The proof of the following theorem uses similar arguments to those in the proofs of [KS, Theorem 14] and [KS, Theorem 17].

THEOREM 6.1. *The group operation of G can be extended to $G^{\mathcal{LUC}}$ in such a way that $G^{\mathcal{LUC}}$ is a semigroup compactification of G. Furthermore, this operation on $G^{\mathcal{LUC}}$ is jointly continuous on $G \times G^{\mathcal{LUC}}$.*

We denote the product (given by the previous theorem) of $p, q \in G^{\mathcal{LUC}}$ simply by pq. For every $s \in G$ and for every $q \in G^{\mathcal{LUC}}$, define the mappings $\lambda_s : G^{\mathcal{LUC}} \to G^{\mathcal{LUC}}$ and $\rho_q : G^{\mathcal{LUC}} \to G^{\mathcal{LUC}}$ by $\lambda_s(p) = sp$ and $\rho_q(p) = pq$ for every $p \in G^{\mathcal{LUC}}$. We recall that these mappings are continuous.

DEFINITION 6.2. For every subset A of G and for every $p \in G^{\mathcal{LUC}}$, define

$$\Omega_p(A) = \{s \in G : s^{-1}A \in p\}.$$

LEMMA 6.3. *Let $A \subseteq G$, let $s \in G$, and let $p, q \in G^{\mathcal{LUC}}$. The following statements hold:*

(i) *$A \in sq$ if and only if $s^{-1}A \in q$.*
(ii) *$\Omega_p(A)$ is an open subset of G.*
(iii) *If $p \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{s^{-1}A})$, then $sp \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{A})$.*
(iv) *If $A \in pq$, then $\Omega_q(A) \in p$.*
(v) *If $p \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{\Omega_q(A)})$, then $pq \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{A})$.*

Proof. (i) This is obvious.

(ii) If $s \in \Omega_p(A)$, then there exists an open subset B of G such that $s \in B$ and $\rho_p(\widehat{B}) \subseteq \widehat{A}$. Now, $B \subseteq \Omega_p(A)$, thus proving the claim.

(iii) Suppose that $p \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{s^{-1}A})$. If $B \in sp$, then $\widehat{s^{-1}B^\circ} \cap \widehat{s^{-1}A^\circ} \neq \emptyset$, so $s^{-1}B^\circ \cap s^{-1}A^\circ \neq \emptyset$, and so $B^\circ \cap A^\circ \neq \emptyset$. Therefore, $\widehat{B} \cap \widehat{A} \neq \emptyset$, thus proving the claim.

(iv) Suppose that $A \in pq$. Pick an open subset B of G such that $B \in p$ and $\rho_q(\widehat{B}) \subseteq \widehat{A}$. Now, $B \subseteq \Omega_q(A)$, and so $\Omega_q(A) \in p$.

(v) Suppose that $pq \notin \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{A})$. Pick an open subset B of G such that $B \in pq$ and $A^\circ \cap B = \emptyset$. Now, $\Omega_q(A) = \Omega_q(A^\circ)$. Also, it is easy to verify that $\Omega_q(C \cap D) = \Omega_q(C) \cap \Omega_q(D)$ for all subsets C and D of G. Since

$B \in pq$, we have $\Omega_q(B) \in p$ by statement (iv). Since $A^\circ \cap B = \emptyset$, we have $\Omega_q(A) \cap \Omega_q(B) = \emptyset$. Therefore, $p \notin \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{\Omega_q(A)})$. ∎

We finish this paper by characterizing those points of $G^{\mathcal{LUC}}$ which are in the minimal ideal of $G^{\mathcal{LUC}}$ or in its closure. We denote the minimal ideal by K. In what follows, we apply the fact that K is the union of all minimal left ideals of $G^{\mathcal{LUC}}$ (see [HS, p. 34]).

Analogues of the theorems given below are given in [TR] for more general semigroup compactifications. However, the approach given in [TR] does not use filters, but certain equivalence classes of z-filters instead.

Definition 6.4. A subset A of G is *syndetic* if and only if there exists a finite subset F of G such that $G = \bigcup_{s\in F} s^{-1}A$.

Theorem 6.5. *If $p \in G^{\mathcal{LUC}}$, then the following statements are equivalent:*

(i) *$p \in K$.*
(ii) *If $A \in p$, then $\Omega_p(A)$ is syndetic.*
(iii) *If $q \in G^{\mathcal{LUC}}$, then $p \in G^{\mathcal{LUC}}qp$.*

Proof. (i)⇒(ii). Suppose that $p \in K$. Let $A \in p$. Let L be the minimal left ideal of $G^{\mathcal{LUC}}$ such that $p \in L$. If $q \in L$, then $L = G^{\mathcal{LUC}}q = \overline{Gq}$, and so $\widehat{A} \cap Gq \neq \emptyset$. Pick some $s \in G$ such that $sq \in \widehat{A}$. Then $s^{-1}A \in q$ by Lemma 6.3(i). So, for every element $q \in L$, there exists an element $s \in G$ such that $s^{-1}A \in q$. Hence, $L \subseteq \bigcup_{s\in F} \widehat{s^{-1}A}$ for some finite subset F of G.

To see that $\Omega_p(A)$ is syndetic, let $t \in G$. Since $tp \in L$, there exists some $s \in F$ such that $s^{-1}A \in tp$. Now, $t^{-1}s^{-1}A = (st)^{-1}A \in p$, so $st \in \Omega_p(A)$, and so $t \in s^{-1}\Omega_p(A)$ for some $s \in F$.

(ii)⇒(iii). Suppose that the set $\Omega_p(A)$ is syndetic for every set $A \in p$. Suppose also that there exists some element $q \in G^{\mathcal{LUC}}$ such that $p \notin G^{\mathcal{LUC}}qp$. Since $G^{\mathcal{LUC}}$ is a regular topological space, there exists a set $A \in p$ such that $\mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{A}) \cap G^{\mathcal{LUC}}pq = \emptyset$. By assumption, there exists a finite subset F of G such that $G = \bigcup_{t\in F} t^{-1}\Omega_p(A)$. Pick $t \in F$ such that $q \in \overline{t^{-1}\Omega_p(A)}$. By Lemmas 6.3(ii) and 4.8(ii), we have $q \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{t^{-1}\Omega_p(A)})$, and so $tq \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{\Omega_p(A)})$ by Lemma 6.3(iii). Therefore, $tqp \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{A})$ by Lemma 6.3(v), a contradiction.

(iii) This is obvious. ∎

Definition 6.6. A subset A of G is *piecewise syndetic* if and only if there exists a finite subset F of G such that the set $\{s^{-1}(\bigcup_{t\in F} t^{-1}A) : s \in G\}$ has the finite intersection property.

Theorem 6.7. *Let $A \subseteq G$. The following statements hold:*

(i) *If $\widehat{A} \cap K \neq \emptyset$, then A is piecewise syndetic.*
(ii) *If A is open and piecewise syndetic, then $\mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{A}) \cap K \neq \emptyset$.*

Proof. (i) Suppose that $p \in \widehat{A} \cap K$. By Theorem 6.5, there exists a finite subset F of G such that $G = \bigcup_{t\in F} t^{-1}\Omega_p(A)$. Let $s \in G$ and pick $t \in F$ such that $s \in t^{-1}\Omega_p(A)$. Then $ts \in \Omega_p(A)$, and so $(ts)^{-1}A = s^{-1}(t^{-1}A) \in p$. Therefore, $s^{-1}(\bigcup_{t\in F} t^{-1}A) \in p$ for every $s \in G$, and so A is piecewise syndetic.

(ii) Suppose that A is open and piecewise syndetic. Pick a finite subset F of G such that the set $\{s^{-1}(\bigcup_{t\in F} t^{-1}A) : s \in G\}$ has the finite intersection property. Put $B = \bigcup_{t\in F} t^{-1}A$ and $\mathcal{A} = \{Us^{-1}B : U \in \mathcal{N}_e,\ s \in G\}$. Here, B is an open subset of G and $\mathcal{A}$ is a $\mathcal{U}$-family on G such that $\mathcal{A}$ has the finite intersection property. By Lemma 3.5, there exists an element $p \in G^{\mathcal{LUC}}$ such that $\mathcal{A} \subseteq p$.

For a while, fix $s \in G$. Since $Us^{-1}B \in p$ for every $U \in \mathcal{N}_e$, we have $p \in \overline{s^{-1}B}$ by Lemma 4.6, so $p \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{s^{-1}B})$ by Lemma 4.8(ii), and hence $sp \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{B})$ by Lemma 6.3(iii). Since $s \in G$ was arbitrary, we obtain $G^{\mathcal{LUC}}p \subseteq \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{B})$. Here, $G^{\mathcal{LUC}}p$ is a left ideal of $G^{\mathcal{LUC}}$, and so we may pick some element $q \in K \cap G^{\mathcal{LUC}}p$. Since $q \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{B})$, we have $B \cap C \neq \emptyset$ for every $C \in q$. By the definition of the set B, we may assume that there exists an element $t \in F$ such that $t^{-1}A \cap C \neq \emptyset$ for every $C \in q$. Now, $q \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{t^{-1}A})$ by Lemmas 4.6 and 4.8(ii), and so $tq \in \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{A})$ by Lemma 6.3(iii). Since $q \in K$, we have $tq \in K$, and so $K \cap \mathrm{cl}_{G^{\mathcal{LUC}}}(\widehat{A}) \neq \emptyset$. ■

The following corollary is now obvious.

Corollary 6.8. *If $p \in G^{\mathcal{LUC}}$, then $p \in \overline{K}$ if and only if every set $A \in p$ is piecewise syndetic.*

Definition 6.9. A subset A of G is *central* if and only if there exists an idempotent $e \in K$ such that $A \in e$.

Theorem 6.10. *Let $A \subseteq G$. The implications* (i)⇒(ii)⇒(iii)⇒(iv) *hold for the following statements:*

(i) *$\widehat{A} \cap K \neq \emptyset$.*
(ii) *The set $\{s \in G : s^{-1}A$ is central$\}$ is syndetic.*
(iii) *There exists some $s \in G$ such that $s^{-1}A$ is central.*
(iv) *A is piecewise syndetic.*

Proof. (i)⇒(ii). Suppose that $p \in \widehat{A} \cap K$. Let L be the minimal left ideal of $G^{\mathcal{LUC}}$ such that $p \in L$. Pick an idempotent $e \in L$. Now, $A \in p = pe$, so $\Omega_e(A) \in p$ by Lemma 6.3(iv), and so there exists some element $s \in G$ such that $s^{-1}A \in e$. Since $e \in K$, we have $G = \bigcup_{t\in F} t^{-1}\Omega_e(s^{-1}A)$ for some finite subset F of G by Theorem 6.5. Let $B = \{s \in G : s^{-1}A \text{ is central}\}$. Now, it is enough to show that $G = \bigcup_{u\in sF} u^{-1}B$. But if $v \in G$, then there exists an element $t \in F$ such that $tv \in \Omega_e(s^{-1}A)$, so $(tv)^{-1}s^{-1}A = (stv)^{-1}A \in e$, and thus $stv \in B$. Hence, $v \in t^{-1}s^{-1}B = (st)^{-1}B$, as required.

(ii)⇒(iii). A syndetic subset of G is not empty.

(iii)⇒(iv). Suppose that there exists an element $s \in G$ such that $s^{-1}A$ is central. Pick an idempotent $e \in K$ such that $s^{-1}A \in e$, that is, $A \in se$. Since $se \in K$, the set A is piecewise syndetic by Theorem 6.7(i). ∎

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