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$\mathcal{F}$-bases with brackets and with individual brackets in Banach spaces

by

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Abstract. We provide a partial answer to the question of Vladimir Kadets whether given an $\mathcal{F}$-basis of a Banach space X, with respect to some filter $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$, the coordinate functionals are continuous. The answer is positive if the character of $\mathcal{F}$ is less than $\mathfrak{p}$. In this case every $\mathcal{F}$-basis is an M-basis with brackets which are determined by an element of $\mathcal{F}$.

1. Introduction. Given any filter $\mathcal{F}$ of subsets of $\mathbb{N}$ and a Banach space X we say that a sequence $(e_n)_{n=1}^{\infty}$ is an *$\mathcal{F}$-basis* for X if for each $x \in X$ there is a unique sequence $(a_n)_{n=1}^{\infty}$ of scalars such that

$$x = \mathcal{F}\text{-}\lim_{n\to\infty} \sum_{k=1}^{n} a_k e_k$$

in the norm topology of X (i.e. for each $\varepsilon > 0$ there is a set $A \in \mathcal{F}$ such that $\|x - \sum_{k=1}^{n} a_k e_k\| < \varepsilon$ for $n \in A$). In that case we define the coordinate functionals by $e_n^*(x) = a_n$ and the partial sum projections by $S_n(x) = \sum_{k=1}^{n} e_k^*(x) e_k$ for $n \in \mathbb{N}$. Of course, all these maps are linear.

The present paper is motivated by a question posed by V. Kadets during the 4th conference *Integration, Vector Measures and Related Topics*; he asked whether it is true in general that the e_n^* are continuous. Let us note that continuity of the coordinate functionals has been usually included in the definition of $\mathcal{F}$-basis (cf. [1], [3]). Of particular interest is the case where $\mathcal{F} = \mathcal{F}_{\mathrm{st}}$ is the filter of statistical convergence defined by

$$\mathcal{F}_{\mathrm{st}} = \left\{ A \subset \mathbb{N} \colon \lim_{n\to\infty} \frac{1}{n} |A \cap \{1, \ldots, n\}| = 1 \right\}.$$

For any filter $\mathcal{F}$ of subsets of $\mathbb{N}$ let $\chi(\mathcal{F})$ stand for its *character*, that is, the minimal cardinality of a subfamily of $\mathcal{F}$ which generates $\mathcal{F}$:

$$\chi(\mathcal{F}) = \min\{|\mathcal{B}| \colon \mathcal{B} \subset \mathcal{F},\ \forall_{A\in\mathcal{F}}\, \exists_{B\in\mathcal{B}}\ B \subseteq A\}.$$

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We will show that the answer to Kadets' question is positive when the character of $\mathcal{F}$ is less than $\mathfrak{p}$, the *pseudointersection number*, which is the least cardinal number κ such that $P(\kappa^+)$ is false, where $P(\kappa)$ is the following statement:

$P(\kappa)$: If $\mathscr{A}$ is a family of subsets of $\mathbb{N}$ such that $|\mathscr{A}| < \kappa$ and $A_1 \cap \cdots \cap A_k$ is infinite for any $A_1, \ldots, A_k \in \mathscr{A}$, then there is an infinite set $B \subset \mathbb{N}$ such that $B \setminus A$ is finite for each $A \in \mathscr{A}$.

It is known that $\omega_1 \le \mathfrak{p} \le \mathfrak{c}$ and that $\mathfrak{p} = \mathfrak{c}$ provided we assume Martin's axiom (this is known as Booth's lemma; cf. [2, Theorem 11C]). We thus obtain continuity of the coordinate functionals associated with $\mathcal{F}$-bases for which $\chi(\mathcal{F}) \le \omega$ (i.e. $\mathcal{F}$ is countably generated) and, under Martin's axiom, for which $\chi(\mathcal{F}) < \mathfrak{c}$. In fact, we will see (Theorem 1) that any $\mathcal{F}$-basis $(e_n)_{n=1}^{\infty}$ of a Banach space X, with $\chi(\mathcal{F}) < \mathfrak{p}$, is an *$M$-basis with brackets* (cf. [4]), that is, there is a sequence $n_1 < n_2 < \cdots$ of natural numbers such that for each $x \in X$ we have

$$x = \lim_{k\to\infty} \sum_{j=1}^{n_k} e_j^*(x) e_j.$$

Of course, every such basis generates a finite-dimensional Schauder decomposition of X. The inequality $\chi(\mathcal{F}) < \mathfrak{p}$ also implies that $\{n_1 < n_2 < \cdots\}$ may be required to be a member of $\mathcal{F}$.

My first proof of continuity of the coordinate functionals worked for countably generated filters and it was D. H. Fremlin who indicated that the argument should go through for some models where the Baire Category Theorem is valid for uncountably many meagre sets. This led me to the condition $\chi(\mathcal{F}) < \mathfrak{p}$.

In Section 3 we introduce the notion of *$\mathcal{F}$-basis with individual brackets*, analogous to the one of *M-basis with individual brackets* (see definitions therein), and we show that many $\mathcal{F}$-bases which arise naturally from Schauder bases belong to this class (Theorem 2).

It should be mentioned that since the statistical filter $\mathcal{F}_{\mathrm{st}}$ is *tall* (i.e. every infinite subset of $\mathbb{N}$ contains an infinite subset belonging to the dual ideal of $\mathcal{F}_{\mathrm{st}}$), we have $\chi(\mathcal{F}_{\mathrm{st}}) \ge \mathfrak{p}$, so the question posed by Kadets remains unanswered in the case $\mathcal{F} = \mathcal{F}_{\mathrm{st}}$.

2. Continuity of coordinate functionals. Hereinafter $\mathcal{F}$ stands for a filter of subsets of $\mathbb{N}$ and $(e_n)_{n=1}^{\infty}$ is an $\mathcal{F}$-basis of a Banach space X unless otherwise stated. The coordinate functionals and the partial sum projections corresponding to $(e_n)_{n=1}^{\infty}$ will be denoted by e_n^* and S_n.

PROPOSITION 1. *For every $A \in \mathcal{F}$ the space*

$$X_A = \Big\{x \in X : \sup_{\nu \in A} \|S_\nu(x)\| < \infty\Big\},$$

equipped with the norm $\|\cdot\|_A$ defined by

$$\|x\|_A = \sup_{\nu \in A} \|S_\nu(x)\|,$$

is a Banach space.

Proof. First, observe that $\|\cdot\|_A$ is indeed a norm on X_A. Homogeneity and the triangle inequality are trivial. Moreover, if $\|x\|_A = 0$ then for each $\varepsilon > 0$ one may find $B \in \mathcal{F}$ such that $\|S_\nu(x) - x\| < \varepsilon$ for $\nu \in B$, but then for each $\nu \in A \cap B$, which is non-empty as an element of $\mathcal{F}$, we have $S_\nu(x) = 0$. Hence $\|x\| < \varepsilon$ and consequently $x = 0$.

Now, assume $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $(X_A, \|\cdot\|_A)$. Then for every $\varepsilon > 0$ one may find $m \in \mathbb{N}$ such that

$$\|S_\nu(x_m - x_n)\| < \varepsilon/3 \quad \text{for each } n \geq m \text{ and } \nu \in A.$$

We may choose ν in such a way that

$$\|S_\nu(x_m) - x_m\| < \varepsilon/3 \quad \text{and} \quad \|S_\nu(x_n) - x_n\| < \varepsilon/3.$$

These three inequalities give $\|x_m - x_n\| < \varepsilon$, which shows that $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $(X_A, \|\cdot\|)$. Therefore, there exists x_0 in the $\|\cdot\|$-closure of X_A such that

$$\lim_{n\to\infty} \|x_n - x_0\| = 0. \tag{1}$$

Similarly, for every $\nu \in A$ and $m, n \in \mathbb{N}$ we have

$$\|S_\nu(x_m) - S_\nu(x_n)\| = \|S_\nu(x_m - x_n)\| \leq \|x_m - x_n\|_A,$$

which shows that $(S_\nu(x_n))_{n=1}^\infty$ is a Cauchy sequence in $(X, \|\cdot\|)$, and each of its elements lies in $\operatorname{span}\{e_j\}_{j\leq\nu}$. Hence, there is $y_\nu \in \operatorname{span}\{e_j\}_{j\leq\nu}$ such that

$$\lim_{n\to\infty} \|S_\nu(x_n) - y_\nu\| = 0. \tag{2}$$

For every $j \in \mathbb{N}$ denote $\alpha_j = e_j^*(y_\nu)$ for any $\nu \in A$, $j \leq \nu$. This definition does not depend on the choice of ν. Indeed, if $k, \ell \in A$ satisfy $j \leq k \leq \ell$, then the continuity of e_j^* on the finite-dimensional subspace $\operatorname{span}\{e_i\}_{i\leq\ell}$ gives

$$e_j^*(y_k) = e_j^*\Big(\lim_{n\to\infty} S_k(x_n)\Big) = \lim_{n\to\infty} e_j^*(S_k(x_n)) = \lim_{n\to\infty} e_j^*(S_\ell(x_n)) = e_j^*(y_\ell).$$

We shall show that

$$x_0 = \mathcal{F}\text{-}\sum_{n=1}^\infty \alpha_n e_n,$$

thus, in particular, $S_\nu(x_0) = y_\nu$ for every $\nu \in A$. To this end fix any $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that for each $n \geq m$ we have $\|S_\nu(x_m) - S_\nu(x_n)\| < \varepsilon/3$

(for any $\nu \in A$) and $\|x_m - x_n\| < \varepsilon/3$. Now, let $B \in \mathcal{F}$ be such that for each $\nu \in B$ we have $\|S_\nu(x_m) - x_m\| < \varepsilon/3$. Then $A \cap B \in \mathcal{F}$ and for every $\nu \in A \cap B$ we get

$$\begin{aligned}\|y_\nu - x_0\| &= \Big\|\lim_{n\to\infty} S_\nu(x_n) - \lim_{n\to\infty} x_n\Big\| \\ &\le \lim_{n\to\infty} \|S_\nu(x_m) - S_\nu(x_n)\| + \|S_\nu(x_m) - x_m\| \\ &\quad + \lim_{n\to\infty} \|x_m - x_n\| \le \varepsilon,\end{aligned}$$

in view of (1) and (2). This shows that

$$x_0 = \mathcal{F}\text{-}\lim_{\substack{\nu\to\infty\\ \nu\in A}} y_\nu.$$

Moreover, a similar estimate, for an arbitrary $\nu \in A$ and $m \in \mathbb{N}$ chosen as above, yields

$$\begin{aligned}\|y_\nu\| &\le \|x_0\| + \frac{1}{3}\varepsilon + \|S_\nu(x_m)\| + \|x_m\| + \frac{1}{3}\varepsilon \\ &\le \frac{2}{3}\varepsilon + \|x_0\| + \|x_m\|_A + \|x_m\|,\end{aligned}$$

which implies

$$\sup_{\nu\in A} \|S_\nu(x_0)\| = \sup_{\nu\in A} \|y_\nu\| < \infty,$$

thus $x_0 \in X_A$. Now, for any $n \in \mathbb{N}$ we have

$$\begin{aligned}\|x_n - x_0\|_A &= \sup_{\nu\in A} \|S_\nu(x_n) - S_\nu(x_0)\| = \sup_{\nu\in A} \Big\|S_\nu(x_n) - \lim_{m\to\infty} S_\nu(x_m)\Big\| \\ &\le \limsup_{m\to\infty} \sup_{\nu\in A} \|S_\nu(x_n) - S_\nu(x_m)\|,\end{aligned}$$

which shows that $\lim_{n\to\infty} \|x_n - x_0\|_A = 0$, and consequently $(X_A, \|\cdot\|_A)$ is a Banach space. ∎

Proposition 2. *If $\chi(\mathcal{F}) < \mathfrak{p}$ then there exists a set $A \in \mathcal{F}$ such that $X_A = X$.*

Proof. For any $A \in \mathcal{F}$ the identity mapping $i_A\colon (X_A, \|\cdot\|_A) \to (X, \|\cdot\|)$ is continuous, since $\|\cdot\|_A \ge \|\cdot\|$. By Proposition 1 and the Open Mapping Theorem, either i_A is surjective, or its image X_A is a meagre subset of $(X, \|\cdot\|)$.

Let $\{A_\alpha\}_{\alpha<\chi(\mathcal{F})} \subset \mathcal{F}$ be a family generating $\mathcal{F}$. For every $x \in X$ there exists a set $B \in \mathcal{F}$ such that $\sup_{n\in B} \|S_n(x)\| < \infty$, so $x \in X_{A_\alpha}$ for some $\alpha < \chi(F)$. Therefore,

$$X = \bigcup_{\alpha<\chi(\mathcal{F})} X_{A_\alpha},$$

and since the Baire Category Theorem is valid for less than $\mathfrak{p}$ meagre sets in any Polish space (cf. [2, §22C]), not all the subspaces X_{A_α} may be meagre in $(X, \|\cdot\|)$. Consequently, there is a set $A \in \mathcal{F}$ with $X_A = X$. ∎

EXAMPLE 1. A slight modification of [1, Example 1] shows that in general one cannot expect that $X_A = X$ for some $A \in \mathcal{F}$. Namely, let $(e_n)_{n=1}^{\infty}$ be the canonical basis of ℓ_2 with the coordinate functionals $(e_n^*)_{n=1}^{\infty}$. Put also $x_n = \sum_{i=1}^{n} e_i$. Then, as shown in [1], $(x_n)_{n=1}^{\infty}$ is an $\mathcal{F}_{\mathrm{st}}$-basis of ℓ_2 with the coordinate functionals given by $x_n^* = e_n^* - e_{n+1}^*$. They are, of course, continuous, but for any increasing sequence $n_1 < n_2 < \cdots$ of natural numbers we will define an element $x = \sum_{k=1}^{\infty} a_k e_k$ of ℓ_2 such that $\sup_{k\in\mathbb{N}} \|S_{n_k}(x)\| = \infty$.

To this end choose an increasing subsequence $(m_j)_{j=1}^{\infty}$ of $(n_j)_{j=1}^{\infty}$ with $m_j > j^4$ and put

$$a_k = \begin{cases} 1/\sqrt[4]{k} & \text{if there is } j \in \mathbb{N} \text{ such that } k = m_j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Repeating the argument from [1, Example 1] we get our claim, which shows that in this case $(\ell_2)_A \subsetneq \ell_2$ for every infinite set $A \subset \mathbb{N}$ (not only for every $A \in \mathcal{F}_{\mathrm{st}}$).

THEOREM 1. *If $\chi(\mathcal{F}) < \mathfrak{p}$ then any $\mathcal{F}$-basis is an M-basis with brackets and all the coordinate functionals are continuous. Moreover, the equality*

$$x = \lim_{k\to\infty} \sum_{j=1}^{n_k} e_j^*(x) e_j \tag{3}$$

holds true for each $x \in X$, where the sequence $n_1 < n_2 < \cdots$ may be chosen in such a way that $\{n_1, n_2, \ldots\} \in \mathcal{F}$.

Proof. We may assume that $\mathcal{F}$ does not contain any finite sets, since otherwise X would be finite-dimensional.

Let $A \in \mathcal{F}$ satisfy $X_A = X$. Applying the Open Mapping Theorem to the operator $i_A\colon (X_A, \|\cdot\|_A) \to (X, \|\cdot\|)$ we infer that the inverse operator i_A^{-1} is bounded, i.e. there is a constant $K < \infty$ such that $\|S_\nu(x)\| \le K\|x\|$ for all $x \in X$ and $\nu \in A$. This easily implies that all the coordinate functionals are continuous.

Indeed, fix any $j \in \mathbb{N}$ and suppose, in search of a contradiction, that there is a sequence $(x_n)_{n=1}^{\infty}$ of elements of X such that $\|x_n\| = 1$ for $n \in \mathbb{N}$ and $e_j^*(x_n) \to \infty$. Pick any $\nu \in A$, $\nu \ge j$. Obviously, $e_1, \ldots, e_\nu$ are linearly independent and since the finite-dimensional subspace $\operatorname{span}\{e_i\}_{i\le\nu,\, i\ne j}$ is closed, we infer that

$$\delta := \inf\bigl\{\|e_j + y\| : y \in \operatorname{span}\{e_i\}_{i\le\nu,\, i\ne j}\bigr\} > 0.$$

Since

$$S_\nu(x_n) = e_j^*(x_n)e_j + \sum_{i=1, i\ne j}^{\nu} e_i^*(x_n)e_i,$$

we have

$$\|S_\nu(x_n)\| \ge \delta \cdot |e_j^*(x_n)| \xrightarrow[n\to\infty]{} \infty,$$

which contradicts the continuity of S_ν.

Now, in order to show that $(e_n)_{n=1}^\infty$ is an M-basis with brackets, one may simply use the definition of $\mathfrak{p}$ to produce an infinite set $B \subset \mathbb{N}$ such that $|B \setminus A_\alpha| < \omega$ for every A_α in some fixed (centered) family $(A_\alpha)_{\alpha<\chi(\mathcal{F})}$ generating $\mathcal{F}$. Then $S_\nu(x) \to x$ for every $x \in X$ as $\nu \in B$, $\nu \to \infty$. However, there is no reason why B should be an element of $\mathcal{F}$. Instead one may use the following argument for which I am grateful to Vladimir Kadets.

Observe that $(\mathrm{Id}_X - S_\nu)_{\nu\in A}$ is a uniformly bounded sequence of operators which converges to 0 on the dense subspace of X spanned by the set $\{e_n\}_{n=1}^\infty$. Let $\{n_1 < n_2 < \cdots\}$ be an enumeration of A. Then equality (3) holds for every $x \in X$. Since e_n^*'s are all continuous, the coefficients of every such expansion are uniquely determined, hence the basis in question is in fact an M-basis with brackets. ■

3. $\mathcal{F}$-bases with individual brackets. In view of Theorem 1, the inequality $\chi(\mathcal{F}) < \mathfrak{p}$ implies that for all $x \in X$ one may find a common set $A \in \mathcal{F}$ such that $S_\nu(x)$ converge to x as $\nu \in A$ and $\nu \to \infty$, whereas Example 1 shows that this is not possible in general. These two facts motivate the following definition.

Definition. A sequence $(e_n)_{n=1}^\infty$ of elements of a Banach space X is called an *$\mathcal{F}$-basis with individual brackets* if it is an $\mathcal{F}$-basis of X and for each $x \in X$ there is a set $A \in \mathcal{F}$ (possibly depending on x) such that

$$\lim_{\substack{\nu\to\infty\\ \nu\in A}} \|S_\nu(x) - x\| = 0.$$

This notion is similar to that of M-basis with individual brackets, which was considered by Kadets [4]. Recall that $(e_n)_{n=1}^\infty \subset X$ is called an *M-basis with individual brackets* if there is a sequence $(e_n^*)_{n=1}^\infty$ of functionals such that $(e_n, e_n^*)_{n=1}^\infty$ is a Markushevich basis (i.e. a biorthogonal system with $\overline{\mathrm{span}}\,\{e_n\}_{n=1}^\infty = X$ and $\overline{\mathrm{span}}^{w^*}\{e_n^*\}_{n=1}^\infty = X^*$) and for each $x \in X$ there exists a sequence $n_1 < n_2 < \cdots$ of natural numbers for which (3) holds true.

Kadets [4] showed that the space ℓ_2 admits an M-basis with individual brackets which is not an M-basis with brackets. The basis exhibited by Kadets was in fact an $\mathcal{F}_\mathrm{s}$-basis with a *summable filter* $\mathcal{F}_\mathrm{s}$ given by

$$\mathcal{F}_\mathrm{s} = \biggl\{A \subset \mathbb{N}\colon \sum_{n\in\mathbb{N}\setminus A} \biggl((n+1)\sum_{k=1}^n \frac{1}{k}\biggr)^{-1} < \infty\biggr\}.$$

Since $\mathcal{F}_\mathrm{s}$ is a tall filter, we have $\chi(\mathcal{F}_\mathrm{s}) \geq \mathfrak{p}$, which, in light of Theorem 1, is not accidental.

Obviously, if $\mathcal{F}$ is a *P-filter* (i.e. for every countable family $\mathcal{G} \subset \mathcal{F}$ there is a set $A \in \mathcal{F}$ such that $|A \setminus B| < \omega$ for each $B \in \mathcal{G}$) then every $\mathcal{F}$-basis is an $\mathcal{F}$-basis with individual brackets. The $\mathcal{F}$-bases exhibited in [1] and [4] are also examples of $\mathcal{F}$-bases with individual brackets. Those constructions may be generalised in the following way.

Let X be a Banach space with a Schauder basis $(f_n)_{n=1}^\infty$ and let $(\gamma_n)_{n=1}^\infty$ be a sequence of non-zero scalars such that the series $\sum_{n=1}^\infty \gamma_n f_n$ diverges. We put

$$e_n = \sum_{j=1}^n \gamma_j f_j \quad \text{and} \quad e_n^* = \frac{1}{\gamma_n} f_n^* - \frac{1}{\gamma_{n+1}} f_{n+1}^* \qquad \text{for } n \in \mathbb{N}.$$

Then it may be easily checked that $(e_n, e_n^*)_{n=1}^\infty$ is a Markushevich basis of X (the fact that $(e_n^*)_{n=1}^\infty$ is a total subset of X^* follows from our supposition on the series $\sum_{n=1}^\infty \gamma_n f_n$). Let $(S_n)_{n=1}^\infty$ and $(T_n)_{n=1}^\infty$ be the partial sum projections corresponding to $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$, respectively. Then

$$\begin{aligned} S_n(x) &= \sum_{j=1}^n e_j^*(x) e_j = \sum_{j=1}^n \left(\frac{1}{\gamma_j} f_j^*(x) - \frac{1}{\gamma_{j+1}} f_{j+1}^*(x) \right) \sum_{k=1}^j \gamma_k f_k \\ &= \sum_{k=1}^n \sum_{j=k}^n \gamma_k \left(\frac{1}{\gamma_j} f_j^*(x) - \frac{1}{\gamma_{j+1}} f_{j+1}^*(x) \right) f_k \\ &= \sum_{k=1}^n \left(f_k^*(x) - \frac{\gamma_k}{\gamma_{n+1}} f_{n+1}^*(x) \right) f_k = T_n(x) - \frac{f_{n+1}^*(x)}{\gamma_{n+1}} \sum_{j=1}^n \gamma_j f_j, \end{aligned}$$

whence

$$\|S_n(x) - T_n(x)\| = \frac{|f_{n+1}^*(x)|}{|\gamma_{n+1}|} \Big\| \sum_{j=1}^n \gamma_j f_j \Big\| \quad \text{for } x \in X \text{ and } n \in \mathbb{N}.$$

Consequently, $(e_n)_{n=1}^\infty$ is an $\mathcal{F}$-basis of X, where $\mathcal{F}$ is the filter generated by the sets of the form

$$\left\{ n \in \mathbb{N} \colon \frac{|f_{n+1}^*(x)|}{|\gamma_{n+1}|} \Big\| \sum_{j=1}^n \gamma_j f_j \Big\| < 1 \right\} \quad (x \in X),$$

provided only that the intersection of any finite number of these sets is infinite (this condition, jointly with the fact that $(e_n, e_n^*)_{n=1}^\infty$ is biorthogonal, guarantes that every expansion with respect to $(e_n)_{n=1}^\infty$ is unique).

The reason why all the $\mathcal{F}$-bases arising in this manner are $\mathcal{F}$-bases with individual brackets is that for any $x \in X$ and $n \in \mathbb{N}$ the difference $S_n(x) - T_n(x)$ involves only the $(n+1)$st coordinate of x with respect to the basis $(f_n)_{n=1}^\infty$. We shall see that this is a special case of a more general result.

To formulate the announced result we need a piece of notation. Namely, if $(f_n, f_n^*)_{n=1}^\infty$ is a Schauder basis of a Banach space X and $T \colon X \to X$ is a finite-rank operator which may be written as

$$T(x)=\sum_{j=1}^{k} f_{n_j}^*(x)x_j \quad (x\in X)$$

with some non-zero $x_1,\dots,x_k\in X$ and some natural numbers $n_1<\cdots<n_k$, then we write $\mathrm{supp}_{(f_n)}(T)$ for the set $\{n_1,\dots,n_k\}$. If $A,B\subset\mathbb{N}$ are finite then we write $A<B$ provided $\max A<\min B$.

Theorem 2. *Let $(e_n)_{n=1}^\infty$ be an $\mathcal{F}$-basis of a Banach space X with partial sum projections $(S_n)_{n=1}^\infty$. Suppose that there is a Schauder basis $(f_n)_{n=1}^\infty$ of X with partial sum projections $(T_n)_{n=1}^\infty$ such that for some set $\{n_1<n_2<\cdots\}\in\mathcal{F}$ we have*

$$\mathrm{supp}_{(f_n)}(S_{n_1}-T_{n_1})<\mathrm{supp}_{(f_n)}(S_{n_2}-T_{n_2})<\cdots.$$

Then $(e_n)_{n=1}^\infty$ is an $\mathcal{F}$-basis with individual brackets.

Proof. By the definition of $\mathcal{F}$-basis, the set

$$D_x:=\{n\in\mathbb{N}\colon \|S_n(x)-T_n(x)\|<1\}$$

belongs to $\mathcal{F}$ for each $x\in X$.

Fix any $x\in X$. We shall find $y\in X$ such that for arbitrarily large $M>0$ the inequality

$$\|S_{n_j}(y)-T_{n_j}(y)\|\geq M\|S_{n_j}(x)-T_{n_j}(x)\| \tag{4}$$

holds true for all but finitely many $j\in\mathbb{N}$. Then setting $A=\{n_1,n_2,\dots\}\cap D_y$ yields a set $A\in\mathcal{F}$ for which

$$\lim_{\substack{\nu\to\infty\\ \nu\in A}}\|S_\nu(x)-T_\nu(x)\|=0,$$

which implies that $x\in\widetilde{X}_A$.

Define a sequence $1\leq r_1<r_2<\cdots$ by

$$r_j=\max\mathrm{supp}_{(f_n)}(S_{n_j}-T_{n_j}).$$

Then for each $j\in\mathbb{N}$ we have

$$S_{n_j}(z)-T_{n_j}(z)=\sum_{i=r_{j-1}+1}^{r_j} f_i^*(z)x_{i,j}\quad (z\in X) \tag{5}$$

for some $x_{i,j}\in X$ (we put $r_0=0$). We claim that there exists a sequence $1\leq\nu_1<\nu_2<\cdots$ of natural numbers such that if we define a sequence $(\lambda_n)_{n=1}^\infty$ by saying that $\lambda_n=k$ if and only if $r_{\nu_{k-1}}<n\leq r_{\nu_k}$ (where $\nu_0=0$), then the series

$$y:=\sum_{n=1}^{\infty}\lambda_n f_n^*(x)f_n \tag{6}$$

converges in $(X, \|\cdot\|)$. Indeed, we may define $(\nu_j)_{j=1}^{\infty}$ inductively by first choosing $\nu_1 \geq 1$ such that for any $\nu_1 \leq p \leq q$ we have

$$\Big\| \sum_{n=p}^{q} f_n^*(x) f_n \Big\| < 2^{-3}$$

and, after defining $1 \leq \nu_1 < \cdots < \nu_{j-1}$, we pick $\nu_j > \nu_{j-1}$ such that for any $\nu_j \leq p \leq q$ we have

$$\| \sum_{n=p}^{q} f_n^*(x) f_n \Big\| < (j+1)^{-3}.$$

Now, if $(\lambda_n)_{n=1}^{\infty}$ is defined as above, then for any $\varepsilon > 0$ we may find $k \in \mathbb{N}$ so large that $\sum_{j \geq k} j^{-2} < \varepsilon$. Then for any $m > \nu_{k-1}$ we have

$$\begin{aligned} \Big\| \sum_{n=\nu_{k-1}+1}^{m} \lambda_n f_n^*(x) f_n \Big\| &\leq \sum_{j=k}^{\infty} \Big\| \sum_{n=\nu_{j-1}+1}^{\nu_j} \lambda_n f_n^*(x) f_n \Big\| \\ &= \sum_{j=k}^{\infty} j \cdot \Big\| \sum_{n=\nu_{j-1}+1}^{\nu_j} f_n^*(x) f_n \Big\| < \sum_{j=k}^{\infty} j^{-2} < \varepsilon, \end{aligned}$$

which shows that the series given by (6) converges.

Now, fix any $j \in \mathbb{N}$. There is a unique $k \in \mathbb{N}$ such that $(r_{j-1}, r_j] \subset (r_{\nu_{k-1}}, r_{\nu_k}]$ and for any $r_{j-1} < i \leq r_j$ we have $r_{\nu_{k-1}} < i \leq r_{\nu_k}$. Hence for any such i we have $\lambda_i = k$. Then, by (6), we get $f_i^*(y) = \lambda_i f_i^*(x) = k f_i^*(x)$ for $r_{j-1} < i \leq r_j$, so formula (5) yields

$$\|S_{n_j}(y) - T_{n_j}(y)\| = k \|S_{n_j}(x) - T_{n_j}(x)\|.$$

Therefore, inequality (4) is valid whenever j satisfies

$$(r_{j-1}, r_j] \subset \bigcup_{k \geq M} (r_{\nu_{k-1}}, r_{\nu_k}],$$

which is true for all but finitely many $j \in \mathbb{N}$. ∎

In view of Theorem 2, it is tempting to ask whether there exists any $\mathcal{F}$-basis that is not an $\mathcal{F}$-basis with individual brackets.

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