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Topological reflexivity of the spaces of (α,β)-derivations on operator algebras

by

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Abstract. We prove that the spaces of (α,β)-derivations on certain operator algebras are topologically reflexive in the weak operator topology. Characterizations of automorphisms and (α,β)-derivations on reflexive algebras are also given.

1. Introduction and preliminaries. The study of reflexive linear subspaces of the algebra $B(X)$ of all bounded linear operators on the Banach space X represents a very active research area in operator theory (see [8] for a beautiful general view). The originators of this research direction are Kadison and Larson. In [11], Kadison studied local derivations from a von Neumann algebra $\mathcal{R}$ into a dual $\mathcal{R}$-bimodule $\mathcal{M}$. A linear map from $\mathcal{R}$ into $\mathcal{M}$ is called a *local derivation* if it agrees with some derivation at each point in the algebra (the derivations possibly varying from point to point). This investigation was motivated by the study of the Hochschild cohomology of operator algebras. Independently, Larson and Sourour [13] studied local derivations of $B(X)$; they proved that every local derivation of $B(X)$ is a derivation. Since then, a considerable amount of work has been done concerning local derivations of various algebras. See, for example, [4, 5, 9, 10, 18]. In this paper, we will extend this research to a more general setting.

Let us now define our concept of topological reflexivity. Let X be a Banach space. We denote by $L(X)$ and $B(X)$ the algebras of all linear and all bounded linear operators on X respectively. Let $\mathcal{A}$ be a subalgebra of $B(X)$. Given two subsets $\mathcal{E} \subseteq L(\mathcal{A})$ and $\mathcal{F} \subseteq B(\mathcal{A})$, if τ is a vector topology on $\mathcal{A}$, we define

$$\operatorname{Loc}_\tau \mathcal{E} = \{\phi \in L(\mathcal{A}) : \phi(A) \in \overline{\mathcal{E}A}^{\tau},\ A \in \mathcal{A}\},$$
$$\operatorname{BLoc}_\tau \mathcal{F} = \{\phi \in B(\mathcal{A}) : \phi(A) \in \overline{\mathcal{F}A}^{\tau},\ A \in \mathcal{A}\}.$$

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In what follows $\tau \in \{\mathrm{d}, \mathrm{n}, \mathrm{s}, \mathrm{w}\}$, where $\mathrm{d}, \mathrm{n}, \mathrm{s}, \mathrm{w}$ denote the discrete, norm, strong and weak operator topology, respectively. If $\mathcal{E} = \mathrm{Loc}_\tau \mathcal{E}$ (resp. $\mathcal{F} = \mathrm{BLoc}_\tau \mathcal{F}$), we say that $\mathcal{E}$ is *topologically reflexive in* $L(\mathcal{A})$ (resp. $\mathcal{F}$ is *topologically reflexive in* $B(\mathcal{A})$) in the τ-topology. In particular, if $\mathcal{F} = \mathrm{BLoc}_\mathrm{d} \mathcal{F}$, we always say that $\mathcal{F}$ is *algebraically reflexive*, and if $\mathcal{F} = \mathrm{BLoc}_\mathrm{n} \mathcal{F}$, we say that $\mathcal{F}$ is *topologically reflexive*. Moreover, it is trivial that

$$\mathcal{E} \subseteq \mathrm{Loc}_\mathrm{d} \mathcal{E} \subseteq \mathrm{Loc}_\mathrm{n} \mathcal{E} \subseteq \mathrm{Loc}_\mathrm{s} \mathcal{E} \subseteq \mathrm{Loc}_\mathrm{w} \mathcal{E}$$

and

$$\mathcal{F} \subseteq \mathrm{BLoc}_\mathrm{d} \mathcal{F} \subseteq \mathrm{BLoc}_\mathrm{n} \mathcal{F} \subseteq \mathrm{BLoc}_\mathrm{s} \mathcal{F} \subseteq \mathrm{BLoc}_\mathrm{w} \mathcal{F}.$$

And it is obvious that topological reflexivity in the weak operator topology implies topological reflexivity in the discrete, norm and strong operator topologies.

In [1] Batty and Molnár investigated the topological reflexivity of the groups of *-automorphisms and surjective isometries of $B(H)$ in the strong operator topology, where H is a separable Hilbert space. Brešar and Šemrl [3] proved that the group of automorphisms of $B(H)$ is algebraically reflexive in $L(B(H))$ provided that H is a separable and infinite-dimensional Hilbert space, and in [15] Molnár showed that this group is topologically reflexive in $B(B(H))$.

Instead of automorphisms and surjective isometries, in the present paper we restrict our attention to the topological reflexivity of the spaces of (α, β)-derivations of operator algebras in a Banach space. Recall that a linear map δ of an algebra $\mathcal{A}$ into itself is called an (α, β)-*derivation* if there exist automorphisms α and β of $\mathcal{A}$ such that $\delta(AB) = \delta(A)\beta(B) + \alpha(A)\delta(B)$ for arbitrary A and B in $\mathcal{A}$ (see [2]). Obviously, derivations are $(1, 1)$-derivations where 1 is the identity on $\mathcal{A}$. We denote by $D_{(\alpha,\beta)}(\mathcal{A})$ and $BD_{(\alpha,\beta)}(\mathcal{A})$ the spaces of all (α, β)-derivations and continuous (α, β)-derivations on $\mathcal{A}$. We will show that the spaces of (α, β)-derivations on certain operator algebras are topologically reflexive in the weak operator topology.

Before proceeding, let us fix some notation. In what follows we denote by X a fixed complex Banach space. The usual notation $\mathrm{Lat}\, \mathcal{E}$ will stand for the lattice of invariant subspaces for a subset $\mathcal{E} \subseteq B(X)$, and $\mathrm{Alg}\, \mathcal{L}$ will denote the algebra of bounded linear operators leaving invariant every member of a family $\mathcal{L}$ of subspaces. $\mathcal{E}$ is *reflexive* if $\mathcal{E} = \mathrm{ref}\, \mathcal{E}$, where $\mathrm{ref}\, \mathcal{E} = \{T \in B(X) : Tx \in [\mathcal{E}x],\ x \in X\}$ and $[\cdot]$ denotes the norm closure.

For a lattice $\mathcal{L}$ of subspaces of X, if $N \in \mathcal{L}$, we denote $\bigvee\{M \in \mathcal{L} : N \not\subseteq M\}$ by N_- and $\bigwedge\{M \in \mathcal{L} : M \not\subseteq N\}$ by N_+.

For a subset $S \subseteq X$, $S^\perp = \{f \in X^* : f(S) = \{0\}\}$, where X^* is the dual space of X; if $x \in X$ and $f \in X^*$, the rank one operator $u \mapsto f(u)x$ is denoted by $x \otimes f$. If M is a subspace of X and $T \in B(X)$, the restriction of T to M is denoted by $T|_M$.

Let ϕ be an automorphism of an algebra $\mathcal{A}$. It is easy to see that ϕ preserves idempotents in both directions. Moreover, if $\mathcal{A}$ contains the identity I, and we let $P^\perp = I - P$ for every idempotent $P \in \mathcal{A}$, then $\phi(P)^\perp = \phi(P^\perp)$.

The following lemma will be used repeatedly.

LEMMA 1.1 ([14]). *Let $\mathcal{L}$ be a subspace lattice. Then $x \otimes f \in \operatorname{Alg}\mathcal{L}$ if and only if there exists an element $L \in \mathcal{L}$ such that $x \in L$ and $f \in (L_-)^\perp$.*

2. Topological reflexivity of the spaces of (α,β)-derivations of operator algebras. We begin with the following key lemma.

LEMMA 2.1. *Let $\mathcal{A}$ be a subalgebra of $B(X)$ containing the identity operator I, and α, β be automorphisms of $\mathcal{A}$. If $\delta \in \operatorname{Loc_w} D_{(\alpha,\beta)}(\mathcal{A})$, then*

$$\delta(PAQ) = \delta(PA)\beta(Q) + \alpha(P)\delta(AQ) - \alpha(P)\delta(A)\beta(Q)$$

for every $A \in \mathcal{A}$ and any idempotents P and Q in $\mathcal{A}$.

Proof. Given $A \in \mathcal{A}$ and two idempotents $P, Q \in \mathcal{A}$, there exists $\{\delta_n\} \subseteq D_{(\alpha,\beta)}(\mathcal{A})$ (depending on PAQ) such that $\{\delta_n(PAQ)\}$ converges to $\delta(PAQ)$ in the weak operator topology, i.e. for arbitrary $x \in X$ and $f \in X^*$,

$$f[(\delta_n(PAQ) - \delta(PAQ))x] \to 0 \quad (n \to \infty).$$

But

$$\delta_n(PAQ) = \delta_n(PA)\beta(Q) + \alpha(PA)\delta_n(Q) = \delta_n(PA)\beta(Q) + \alpha(P)\alpha(A)\delta_n(Q).$$

Thus, for any $x \in \operatorname{Ker}\beta(Q)$ and $f \in \{\operatorname{Rang}\alpha(P)\}^\perp$, we have

$$f[(\delta_n(PAQ) - \delta(PAQ))x] = -f(\delta(PAQ)x) \to 0 \quad (n \to \infty).$$

This shows that $f(\delta(PAQ)x) = 0$ for $x \in \operatorname{Ker}\beta(Q)$ and $f \in \{\operatorname{Rang}\alpha(P)\}^\perp$. Hence $\alpha(P)^\perp\delta(PAQ)\beta(Q)^\perp = 0$, or equivalently, $\alpha(P^\perp)\delta(PAQ)\beta(Q)^\perp = 0$.

Furthermore,

$$\begin{aligned}
\delta(PAQ)\beta(Q)^\perp &- \alpha(P)\delta(AQ)\beta(Q)^\perp \\
&= [\delta(PAQ) - \alpha(P)\delta(AQ)]\beta(Q)^\perp \\
&= [(\alpha(P)^\perp\delta(PAQ) + \alpha(P)\delta(PAQ)) \\
&\qquad - (\alpha(P)\delta(P^\perp AQ) + \alpha(P)\delta(PAQ))]\beta(Q)^\perp \\
&= \alpha(P)^\perp\delta(PAQ)\beta(Q)^\perp - \alpha(P)\delta(P^\perp AQ)\beta(Q)^\perp = 0,
\end{aligned}$$

that is,

$$\delta(PAQ)\beta(Q)^\perp = \alpha(P)\delta(AQ)\beta(Q)^\perp.$$

Similarly we have

$$\delta(PAQ^\perp)\beta(Q) = \alpha(P)\delta(AQ^\perp)\beta(Q).$$

Therefore we obtain

$$\begin{aligned}\delta(PAQ)-\alpha(P)\delta(AQ) &= (\delta(PAQ)-\alpha(P)\delta(AQ))\beta(Q)^{\perp}\\ &\quad+(\delta(PAQ)-\alpha(P)\delta(AQ))\beta(Q)\\ &= (\delta(PAQ)-\alpha(P)\delta(AQ))\beta(Q)\\ &= [(\delta(PA)-\delta(PAQ^{\perp}))\\ &\quad-(\alpha(P)\delta(A)-\alpha(P)\delta(AQ^{\perp}))]\beta(Q)\\ &= \delta(PA)\beta(Q)-\alpha(P)\delta(A)\beta(Q)\\ &\quad+(\alpha(P)\delta(AQ^{\perp})\beta(Q)-\delta(PAQ^{\perp})\beta(Q))\\ &= \delta(PA)\beta(Q)-\alpha(P)\delta(A)\beta(Q). \ \blacksquare\end{aligned}$$

Now we prove our first theorem.

THEOREM 2.2. *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H, and α,β be automorphisms of $\mathcal{M}$. Then $BD_{(\alpha,\beta)}(\mathcal{M})$ is topologically reflexive in $B(\mathcal{M})$ in the weak operator topology.*

Proof. Let $\delta\in \mathrm{BLoc_w}\, D_{(\alpha,\beta)}(\mathcal{M})$. It is not difficult to show that $\delta(I)=0$. By Lemma 2.1, we obtain $\delta(PQ)=\delta(P)\beta(Q)+\alpha(P)\delta(Q)$ for any idempotents P and Q in $\mathcal{M}$. Now the assertion follows easily from the fact that every automorphism of $\mathcal{M}$ is continuous and the linear span of all idempotents of $\mathcal{M}$ is norm dense in $\mathcal{M}$. ∎

In particular, we have Kadison's famous result.

COROLLARY 2.3 ([11]). *Every norm continuous local derivation on a von Neumann algebra is a derivation.*

By Lemma 2.1, we also have the following.

THEOREM 2.4. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra on an infinite-dimensional Hilbert space H, and α,β be automorphisms of $\mathcal{M}$. Then $D_{(\alpha,\beta)}(\mathcal{M})$ is topologically reflexive in $L(\mathcal{M})$ in the weak operator topology.*

Proof. Notice that every operator in a properly infinite von Neumann algebra on an infinite-dimensional Hilbert space H is the sum of five idempotents in $\mathcal{M}$ [17, Theorem 4]. ∎

Now we begin to investigate the topological reflexivity of the spaces of (α,β)-derivations of reflexive operator algebras on a Banach space.

LEMMA 2.5. *Let $\mathcal{A}\subseteq B(X)$ be an arbitrary reflexive algebra and $N\in \operatorname{Lat}\mathcal{A}$. If $N\not\subseteq N_-$, then*

(1) *for $f\in X^*$, if $x\otimes f\in\mathcal{A}$ for any $x\in N$, then $f\in(N_-)^{\perp}$;*
(2) *for $x\in X$, if $x\otimes f\in\mathcal{A}$ for any $f\in(N_-)^{\perp}$, then $x\in N$.*

Proof. (1) Suppose that $f \notin (N_-)^\perp$. Then there exists $y \in N_-$ such that $f(y) \neq 0$. Choose $x \in N \setminus N_-$. Then $x \otimes f \in \mathcal{A}$, hence $x \otimes f(N_-) \subseteq N_-$, and so $x \otimes f(y) = f(y)x \in N_-$, a contradiction.

(2) Choose $y \in N$ and $f \in (N_-)^\perp$ such that $f(y) \neq 0$. Since $x \otimes f \in \mathcal{A}$, we have $x \otimes f(N) \subseteq N$. Furthermore, $x \otimes f(y) = f(y)x \in N$, i.e. $x \in N$. ■

In what follows $\mathcal{B}$ will be a reflexive algebra on a Banach space X such that both $0_+ \neq 0$ and $X_- \neq X$ in $\operatorname{Lat}\mathcal{B}$.

LEMMA 2.6. *For $T \in \mathcal{B}$,*

(1) *if $RT = 0$ for every rank one operator $R \in \mathcal{B}$ of the form $x \otimes f$ with $x \in 0_+$ and $f \in X^*$, then $T = 0$;*

(2) *if $TR = 0$ for every rank one operator $R \in \mathcal{B}$ of the form $x \otimes f$ with $x \in X$ and $f \in (X_-)^\perp$, then $T = 0$.*

The proof is straightforward so we omit it.

LEMMA 2.7. *Given a nonzero $T \in \mathcal{B}$, the following statements are equivalent*:

(1) *T is of rank one*;

(2) *for all operators A and B in $\mathcal{B}$, $ATB = 0$ implies either $AT = 0$ or $TB = 0$;*

(3) *for all rank one operators R_1 and R_2 in $\mathcal{B}$, $R_1TR_2 = 0$ implies either $R_1T = 0$ or $TR_2 = 0$;*

(4) *for all rank one operators R_1 and R_2 in $\mathcal{B}$ of the forms $R_1 = x \otimes f$ with $x \in 0_+$ and $f \in X^*$, $R_2 = y \otimes g$ with $y \in X$ and $g \in (X_-)^\perp$, $R_1TR_2 = 0$ implies either $R_1T = 0$ or $TR_2 = 0$.*

Proof. We only show (4)⇒(1). It suffices to show that Tu and Tv are linearly dependent for arbitrary $u, v \in X$.

Since $T \neq 0$, by Lemma 2.6, there exists a rank one operator $R = x \otimes f \in \mathcal{B}$ with $x \in 0_+$ and $f \in X^*$ such that $RT \neq 0$. Then RTu and RTv are linearly dependent since RT is of rank one, and so there exist scalars λ and μ, not both zero, such that $RT(\lambda u + \mu v) = \lambda RTu + \mu RTv = 0$. For nonzero $g \in (X_-)^\perp$ we have $(\lambda u + \mu v) \otimes g \in \mathcal{B}$ and

$$RT((\lambda u + \mu v) \otimes g) = (RT(\lambda u + \mu v)) \otimes g = 0.$$

But $RT \neq 0$, so $T(\lambda u + \mu v) \otimes g = 0$, which in turn implies $\lambda(Tu) + \mu(Tv) = 0$, and so Tu and Tv are linearly dependent. ■

Hence we can easily obtain the following.

COROLLARY 2.8. *Every automorphism of $\mathcal{B}$ preserves rank one operators in both directions.*

For $x \in 0_+$ and $f \in (X_-)^\perp$, let $L_x = \{x \otimes h : h \in X^*\}$ and $R_f = \{u \otimes f : u \in X\}$. It is obvious that both L_x and R_f are subspaces of $\mathcal{B}$ consisting of rank one operators.

LEMMA 2.9. *Let $\phi : \mathcal{B} \to \mathcal{B}$ be an automorphism. Then*

(1) *for each $x \in 0_+$, there exists $y \in 0_+$ such that $\phi(L_x) = L_y$;*
(2) *for each $f \in (X_-)^\perp$, there exists $g \in (X_-)^\perp$ such that $\phi(R_f) = R_g$.*

Proof. We only prove (1).

For two arbitrary linearly independent h_1 and h_2 in X^*, using Corollary 2.8, let $\phi(x \otimes h_1) = y_1 \otimes g_1$ and $\phi(x \otimes h_2) = y_2 \otimes g_2$. Now $\phi(x \otimes (h_1 + h_2)) = y_1 \otimes g_1 + y_2 \otimes g_2$ has rank one, so either y_1 and y_2 are linearly dependent or g_1 and g_2 are, but they are not linearly dependent simultaneously. Hence either there exists $y \in X$ such that $\phi(L_x) \subseteq \{y \otimes h \in \mathcal{B} : h \in X^*\}$, or there is $g \in X^*$ such that $\phi(L_x) \subseteq \{u \otimes g \in \mathcal{B} : u \in X\}$.

For the case of $\phi(L_x) \subseteq \{y \otimes h \in \mathcal{B} : h \in X^*\}$, we have

CLAIM 1. $\phi(L_x) = \{y \otimes h \in \mathcal{B} : h \in X^*\}$.

Assume to the contrary that there exists $y \otimes h_0 \in \mathcal{B}$, but $y \otimes h_0 \notin \phi(L_x)$. Since ϕ preserves rank one operators in both directions, let $\phi(u \otimes g) = y \otimes h_0$. Then u and x are linearly independent. Choose $g_1 \in X^*$ such that g_1 and g are linearly independent and let $\phi(x \otimes g_1) = y \otimes h_1$. Then $\phi(x \otimes g_1 + u \otimes g)$ is of rank two since $x \otimes g_1 + u \otimes g$ is of rank two. However, $\phi(x \otimes g_1 + u \otimes g) = y \otimes h_1 + y \otimes h_0$ is of rank one, a contradiction.

CLAIM 2. $\phi(L_x) = \{y \otimes h : h \in X^*\}$.

Assume that $\phi(L_x)$ is a proper subspace of $\{y \otimes h : h \in X^*\}$. By Claim 1, there is $y \otimes h_2 \notin \mathcal{B}$. Choose nonzero $v \in 0_+$. Then $v \otimes h_2 \in \mathcal{B}$. Suppose $\phi(w \otimes k) = v \otimes h_2$. It follows that y and v are linearly independent. Let $\phi(x \otimes k) = y \otimes h_3$. Then h_3 and h_2 are also linearly independent. Hence $\phi(x \otimes k + w \otimes k) = y \otimes h_3 + v \otimes h_2$ has rank two. This leads to a contradiction with the fact that $\phi(x \otimes k + w \otimes k)$ is a rank one operator.

We have shown that $\phi(L_x) = \{y \otimes h : h \in X^*\} \subseteq \mathcal{B}$; by Lemma 2.5, $y \in 0_+$ and so $\phi(L_x) = L_y$.

For the case of $\phi(L_x) \subseteq \{u \otimes g \in \mathcal{B} : u \in X\}$, we can prove that $\phi(L_x) = R_g$ similarly.

We now show that $\phi(L_x) = R_g$ cannot occur. Otherwise, choose two linearly independent functionals h_1 and h_2 in X^* with $h_1(x) \neq 0$. Suppose $\phi(x \otimes h_1) = y_1 \otimes g$ and $\phi(x \otimes h_2) = y_2 \otimes g$. Then $\phi(x \otimes h_1 \cdot x \otimes h_2) = h_1(x)\phi(x \otimes h_2) = h_1(x) y_2 \otimes g$. On the other hand, $\phi(x \otimes h_1 \cdot x \otimes h_2) = y_1 \otimes g \cdot y_2 \otimes g = g(y_2) y_1 \otimes g$, hence y_1 and y_2 are linearly dependent, which contradicts the fact that h_1 and h_2 are linearly independent. ■

Now we are in a position to prove our main result.

THEOREM 2.10. *Let $\mathcal{B}$ be a reflexive algebra on a Banach space X such that both $0_+ \neq 0$ and $X_- \neq X$ in* $\operatorname{Lat}\mathcal{B}$. *Then $D_{(\alpha,\beta)}(\mathcal{B})$ is topologically reflexive in $L(\mathcal{B})$ in the weak operator topology.*

Proof. We only need to show that $\operatorname{Loc_w} D_{(\alpha,\beta)}(\mathcal{B}) \subseteq D_{(\alpha,\beta)}(\mathcal{B})$.

Let $\delta \in \operatorname{Loc_w} D_{(\alpha,\beta)}(\mathcal{B})$. Then $\delta(I) = 0$. By Lemma 2.1, $\delta(PQ) = \delta(P)\beta(Q) + \alpha(P)\delta(Q)$ for any idempotents $P, Q \in \mathcal{B}$.

To show that $\delta(AB) = \delta(A)\beta(B) + \alpha(A)\delta(B)$ for every $A, B \in \mathcal{B}$, we divide the proof into several steps.

STEP 1. *For any rank one operators $P, Q \in \mathcal{B}$, where $P = x \otimes f$ with $x \in 0_+$ and $f \in X^*$, $Q = y \otimes g$ with $y \in X$ and $g \in (X_-)^\perp$, we have*

$$\delta(PQ) = \delta(P)\beta(Q) + \alpha(P)\delta(Q).$$

Case 1. If $f(x) \neq 0$ and $g(y) \neq 0$, since both $P' = P/f(x)$ and $Q' = Q/g(y)$ are rank one idempotents, the conclusion follows by the linearity of δ.

Case 2. Suppose one of $f(x)$, $g(y)$ is 0, say $f(x) = 0$ and $g(y) \neq 0$. Choose $f' \in X^*$ such that $(f + f')(x) = f'(x) \neq 0$. Then by Case 1,

$$\begin{aligned}\delta(PQ) &= \delta(x \otimes (f + f') \cdot y \otimes g) - \delta(x \otimes f' \cdot y \otimes g)\\ &= \delta(x \otimes (f + f'))\beta(y \otimes g) + \alpha(x \otimes (f + f'))\delta(y \otimes g)\\ &\quad - \delta(x \otimes f')\beta(y \otimes g) - \alpha(x \otimes f')\delta(y \otimes g)\\ &= \delta(x \otimes f)\beta(y \otimes g) + \alpha(x \otimes f)\delta(y \otimes g)\\ &= \delta(P)\beta(Q) + \alpha(P)\delta(Q).\end{aligned}$$

Case 3. If both $f(x)$ and $g(y)$ are 0, choose $f' \in X^*$ and $y' \in X$ such that $(f + f')(x) = f'(x) \neq 0$ and $g(y + y') = g(y') \neq 0$. The conclusion follows by the same argument as in Case 2.

STEP 2. *For each $A \in \mathcal{B}$ and any rank one operator $Q \in \mathcal{B}$, we have*

$$\delta(AQ) = \delta(A)\beta(Q) + \alpha(A)\delta(Q)$$

where $Q = y \otimes g$ with $y \in X$ and $g \in (X_-)^\perp$.

For any rank one operator $P \in \mathcal{B}$ of the form $x \otimes f$ with $x \in 0_+$ and $f \in X^*$, by using an argument similar to that used in Step 1, we find that the assertion of Lemma 2.1 holds for each $A \in \mathcal{B}$ and any rank one operators $P, Q \in \mathcal{B}$, where $P = x \otimes f$ with $x \in 0_+$ and $f \in X^*$, $Q = y \otimes g$ with $y \in X$ and $g \in (X_-)^\perp$.

On the other hand, by Step 1,

$$\delta(PAQ) = \delta(PA \cdot Q) = \delta(PA)\beta(Q) + \alpha(PA)\delta(Q).$$

Hence

$$\alpha(P)\delta(AQ) = \alpha(P)\delta(A)\beta(Q) + \alpha(P)\alpha(A)\delta(Q),$$

and Lemmas 2.9 and 2.6 yield

$$\delta(AQ) = \delta(A)\beta(Q) + \alpha(A)\delta(Q).$$

STEP 3. *For any $A, B \in \mathcal{B}$, we have*

$$\delta(AB) = \delta(A)\beta(B) + \alpha(A)\delta(B),$$

i.e. δ is an (α, β)-derivation.

For each rank one operator $Q = y \otimes g \in \mathcal{A}$ with $y \in X$ and $g \in (X_-)^\perp$, by Step 2,

$$\delta(ABQ) = \delta(AB)\beta(Q) + \alpha(AB)\delta(Q).$$

On the other hand, also by Step 2,

$$\begin{aligned}\delta(ABQ) = \delta(A \cdot BQ) &= \delta(A)\beta(B)\beta(Q) + \alpha(A)\delta(BQ) \\ &= \delta(A)\beta(B)\beta(Q) + \alpha(A)\delta(B)\beta(Q) + \alpha(AB)\delta(Q).\end{aligned}$$

Thus $\delta(AB)\beta(Q) = \delta(A)\beta(B)\beta(Q) + \alpha(A)\delta(B)\beta(Q)$.

Again by Lemmas 2.9 and 2.6, we obtain the assertion. ∎

COROLLARY 2.10. *If $\mathcal{N}$ is a nest on X such that both $0_+ \neq 0$ and $X_- \neq X$ in $\mathcal{N}$, then the space of all (α, β)-derivations of the nest algebra $\operatorname{Alg}\mathcal{N}$ is topologically reflexive in $L(\operatorname{Alg}\mathcal{N})$ in the weak operator topology.*

For the spaces of (α, β)-derivations of standard algebras we have:

COROLLARY 2.11. *If $\mathcal{A}$ is a standard operator algebra in X with identity operator I, then the space of all (α, β)-derivations of $\mathcal{A}$ is topologically reflexive in $L(\mathcal{A})$ in the weak operator topology.*

In particular, we have

COROLLARY 2.12. *The set of all (α, β)-derivations of $B(X)$ is topologically reflexive in $L(B(X))$ in the weak operator topology.*

The following result is well known.

COROLLARY 2.13 ([13]). *Every local derivation of $B(X)$ is a derivation.*

3. Automorphisms and (α, β)-derivations of reflexive algebras. In this section $\mathcal{B}$ will be a reflexive algebra in a Banach space X such that both $0_+ \neq 0$ and $X_- \neq X$ in $\operatorname{Lat}\mathcal{B}$. The main purpose of this section is to characterise the (α, β)-derivations of $\mathcal{B}$. To this end, we need to characterise the automorphisms of $\mathcal{B}$ first. So we begin with

THEOREM 3.1. *Every automorphism of $\mathcal{B}$ is spatial.*

Proof. Let $\phi : \mathcal{B} \to \mathcal{B}$ be an automorphism. By Lemma 2.9, for each $x \in 0_+$ there exists $y_x \in 0_+$ such that $\phi(L_x) = L_{y_x}$.

Thus for arbitrary $x \otimes h \in L_x$ we have $\phi(x \otimes h) = y_x \otimes g_h$. Now we define two mappings $A_1 : 0_+ \to 0_+$ and $B_1^x : X^* \to X^*$ by $x \mapsto y_x$ and $h \mapsto g_h$

respectively. Then both A_1 and B_1^x are linear bijective maps since ϕ is an automorphism.

We claim that B_1^x is independent of x. To see this, for arbitrary $y \in 0_+$ and $h \in X^*$, suppose that $\phi(y \otimes h) = A_1 y \otimes B_1^y h$.

If x and y are linearly independent, let $\phi((x+y)\otimes h) = A_1(x+y)\otimes B_1^{x+y}h$. Then

$$A_1(x+y) \otimes B_1^{x+y}h = A_1 x \otimes B_1^x h + A_1 y \otimes B_1^y h$$

and so $A_1 x \otimes (B_1^x - B_1^{x+y})h = A_1 y \otimes (B_1^{x+y} - B_1^y)h$. Since x and y are linearly independent and A_1 is a linear bijective map, we obtain $B_1^x = B_1^{x+y} = B_1^y$.

If x and y are linearly dependent, suppose that $y = \lambda x$. We have $\phi(y \otimes h) = \lambda\phi(x \otimes h) = \lambda A_1 x \otimes B_1^x h$, thus $A_1 y \otimes B_1^y h = \lambda A_1 x \otimes B_1^x h$, and so $B_1^x h = B_1^y h$ for every $h \in X^*$.

Thus B_1^x is independent of x and we write simply B_1 instead of B_1^x. Then for every $x \otimes f$ with $x \in 0_+$ and $f \in X^*$ we have

$$\phi(x \otimes f) = A_1 x \otimes B_1 f.$$

Again by Lemma 2.9, for each $g \in (X_-)^\perp$ there is $h_g \in (X_-)^\perp$ such that $\phi(R_g) = R_{h_g}$, and so for arbitrary $y \otimes g \in R_g$ we have $\phi(y \otimes g) = u_y \otimes h_g$.

We can now define two linear bijective mappings $A_2^g : X \to X$ and $B_2 : (X_-)^\perp \to (X_-)^\perp$ by $y \mapsto u_y$ and $g \mapsto h_g$ respectively. Similarly we can verify that A_2^g is independent of g and so we denote it by A_2. Hence for any $y \otimes g$ with $y \in X$ and $g \in (X_-)^\perp$ we have

$$\phi(y \otimes g) = A_2 x \otimes B_2 g.$$

In particular, for arbitrary $x \otimes f$ with $x \in 0_+$ and $f \in (X_-)^\perp$ we have

$$\phi(x \otimes f) = A_1 x \otimes B_1 f = A_2 x \otimes B_2 f.$$

Thus there exists a nonzero scalar μ_x such that $A_2 x = \mu_x A_1 x$ and $B_1 f = \mu_x B_2 f$. By a similar argument we can show that μ_x is independent of x, and so we have a nonzero scalar μ such that $A_2 x = \mu A_1 x$ and $B_1 f = \mu B_2 f$. In other words, $A_2|_{0_+} = \mu A_1$ and $B_1|_{(X_-)^\perp} = \mu B_2$.

Let $A = A_2$ and $B = \mu^{-1} B_1$. Then

$$\phi(x \otimes f) = Ax \otimes Bf$$

where $x \in 0_+$ and $f \in X^*$ or $x \in X$ and $f \in (X_-)^\perp$.

For arbitrary $x \in X$, $f \in (X_-)^\perp$ and $T \in \mathcal{B}$, we have

$$\phi(Tx \otimes f) = \phi(T)\phi(x \otimes f) = \phi(T)Ax \otimes Bf.$$

On the other hand, $\phi(Tx \otimes f) = ATx \otimes Bf$, and we arrive at $\phi(T)Ax = ATx$ for any $x \in X$, that is, $\phi(T)A = AT$, and so $\phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}$.

It is straightforward to see that A has a closed graph and hence is bounded. ∎

THEOREM 3.2. *Let $\delta : \mathcal{B} \to \mathcal{B}$ be an (α, β)-derivation. Then there exists $A \in B(X)$ such that*

$$\delta(T) = A\beta(T) - \alpha(T)A, \quad \forall T \in \mathcal{B}.$$

Proof. By Theorem 3.1, suppose that there exists $B \in B(X)$ such that $\beta(T) = BTB^{-1}$ for all $T \in \mathcal{B}$. Choose $x_0 \in X$ and $f_0 \in (X_-)^\perp$ so that $f_0(x_0) = 1$. Then $x \otimes f_0 \in \mathcal{B}$ for every $x \in X$. Define a map $A : X \to X$ by

$$Ax = \delta(B^{-1}x \otimes f_0)Bx_0.$$

Then A is linear. For arbitrary $T \in \mathcal{B}$, we have

$$\delta(Tx \otimes f_0) = \delta(T)\beta(x \otimes f_0) + \alpha(T)\delta(x \otimes f_0).$$

Equivalently,

$$\delta(B^{-1}BTx \otimes f_0) = \delta(T)Bx \otimes f_0B^{-1} + \alpha(T)\delta(B^{-1}Bx \otimes f_0).$$

Furthermore,

$$\delta(B^{-1}BTx \otimes f_0)B = \delta(T)Bx \otimes f_0 + \alpha(T)\delta(B^{-1}Bx \otimes f_0)B.$$

Applying both operators in this equation to x_0, we obtain $ABTx = \delta(T)Bx + \alpha(T)ABx$. This is true for every $x \in X$, so $ABT = \delta(T)B + \alpha(T)AB$, hence $ABTB^{-1} = \delta(T) + \alpha(T)A$, i.e. $\delta(T) = A\beta(T) - \alpha(T)A$.

It is easy to verify that A has a closed graph and hence is bounded; we leave this to the reader. ■

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