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S'-convolvability with the Poisson kernel in the Euclidean case and the product domain case

by

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Abstract. We obtain real-variable and complex-variable formulas for the integral of an integrable distribution in the n-dimensional case. These formulas involve specific versions of the Cauchy kernel and the Poisson kernel, namely, the Euclidean version and the product domain version. We interpret the real-variable formulas as integrals of S'-convolutions. We characterize those tempered distribution that are S'-convolvable with the Poisson kernel in the Euclidean case and the product domain case. As an application of our results we prove that every integrable distribution on $\mathbb{R}^n$ has a harmonic extension to the upper half-space $\mathbb{R}^{n+1}_+$.

1. Introduction. The main purpose of this article is to study the convolvability of tempered distributions with the Poisson kernel. A motivation for our work is to develop complex-variable and real-variable representation formulas for the integral of an integrable distribution ([14, p. 243]) in various n-dimensional settings. The complex-variable representation formula in the one-dimensional case was obtained by W. Kierat and U. Skórnik in [10].

Several definitions of convolution of tempered distributions have been introduced by different authors (see [6]–[16]). We consider here the so called S'-convolution, a commutative operation that extends to appropriate pairs of tempered distributions the classical convolution of distributions as defined by L. Schwartz in [14], preserving the Fourier exchange formula $\mathcal{F}(T * S) = \mathcal{F}(T) \cdot \mathcal{F}(S)$. In view of this formula, it is not possible to use the classical definition of convolution because the Fourier transform of the Poisson kernel is not infinitely differentiable, so it does not belong to the space $\mathcal{O}_M$ defined by L. Schwartz in [14, p. 243].

When working in $\mathbb{R}^n$ with $n > 1$, it is natural to consider two versions of the Poisson kernel, namely, the Euclidean version and the product domain

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version. For each of them, we identify optimal spaces of distributions that are S'-convolvable with the appropriate Poisson kernel. To prove these results we obtain simple characterizations of the space of distributions involved in each case. As an application of our results we prove that every integrable distribution on $\mathbb{R}^n$ can be extended to a harmonic function in the upper half-space $\mathbb{R}^{n+1}_+$. In this regard, we extend a classical result of S. Bochner for integrable functions ([3], [4]), as well as a result of H. Bremermann for distributions with compact support in $\mathbb{R}$ ([5, p. 49]). We also consider the product of n upper half-planes, $\mathbb{R}^2_+ \times \ldots \times \mathbb{R}^2_+$, where the appropriate notion of harmonicity is the more restrictive notion of harmonicity in each upper half-plane. In this context we are able to obtain a result of H. Bremermann ([5, p. 152]).

The article is organized as follows: In Section 2 we include definitions and auxiliary results. In Section 3 we give real-variable and complex-variable formulas for the integral of an integrable distribution in the n-dimensional case, involving both Euclidean and product domain versions of the Poisson kernel, and in Section 4 we reinterpret the integrands in these formulas as S'-convolutions. In Section 5 we present our main results, namely, for each version of the Poisson kernel, we characterize those tempered distributions that are S'-convolvable with it. The proofs are based on appropriate characterizations of the weighted spaces of distributions relevant to each case. Finally, in Section 6 we apply our results to obtain harmonic extensions of integrable distributions.

The notation used in this article is standard. The symbols C_0^∞, S, C^∞, L^p, L^p_{loc}, D', S', E', etc., indicate the usual spaces of distributions or functions defined on $\mathbb{R}^n$, with complex values. The symbol $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ denotes the norm in the space L^p. When we need to emphasize that we are working in a particular setting, we write $D'(\mathbb{R})$, $S(\mathbb{R}^2)$, $\|\cdot\|_{L^p(K)}$, etc. Partial derivatives are denoted by ∂^α or $\partial^\alpha/\partial x^\alpha$, where α is a multi-index $(\alpha_1, \ldots, \alpha_n)$. We use the abbreviations $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. For a function g, we indicate by $\check{g}$ the function $x \mapsto g(-x)$. Given a distribution T, we write $\check{T}$ for the distribution $\varphi \mapsto (T, \check{\varphi})$, where φ is an appropriate test function. The Fourier transform is denoted by $\mathcal{F}$. The letter C denotes a positive constant that may change at different occurrences.

2. Preliminaries. We start by introducing the spaces of functions and distributions that we will use along this work ([14, p. 199]). Set

$$B = \{\varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \in C^\infty,\ \partial^\alpha \varphi \text{ is bounded for each multi-index } \alpha\}$$

endowed with the topology of uniform convergence on $\mathbb{R}^n$ of each derivative, and

$$\dot{B} = \{\varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \in C^\infty,\ \partial^\alpha \varphi \to 0 \text{ as } |x| \to \infty, \text{ for each multi-index } \alpha\}.$$

The space $\dot{B}$ is a closed subspace of B, and C_0^∞ is dense in $\dot{B}$.

D'_{L^1} will denote the strong dual of $\dot{B}$; D'_{L^1} is a subspace of D', the space of distributions. It can be proved ([14, p. 201]) that given $T \in D'_{L^1}$, we have the representation

$$T = \sum_\alpha \partial^\alpha f_\alpha \tag{1}$$

where the functions f_α belong to L^1 and the sum is finite. The distributions in D'_{L^1} are called *integrable*. As a consequence of (1), we have the inclusions $E' \subset D'_{L^1} \subset S'$.

The pointwise multiplication is well defined and continuous from $B \times B$ into B and from $\dot{B} \times B$ into $\dot{B}$. As a consequence, the space D'_{L^1} is closed under multiplication by functions in B.

Following [14, p. 203], we will consider in B an alternative notion of convergence:

A sequence $\{\varphi_j\}$ converges to φ if, for each multi-index α, $\sup_j \|\partial^\alpha \varphi_j\|_\infty < \infty$ and the sequence $\{\partial^\alpha \varphi_j\}$ converges to $\partial^\alpha \varphi$ uniformly on compact sets.

We denote by B_{c} the space B endowed with this notion of convergence. It can be proved that C_0^∞, and so $\dot{B}$, is dense in B_{c} ([14, p. 203]). Moreover, given a distribution T in D'_{L^1}, since T is well defined on C_0^∞ and continuous with respect to the topology of B_{c}, it can be uniquely extended to a continuous linear functional on B_{c}. In this sense we can say that D'_{L^1} and B_{c} are in duality. In fact, D'_{L^1} is also the dual of B_{c} ([14, p. 203]).

Y. Hirata and H. Ogata [7] defined the notion of S'-convolution in order to extend the validity of the Fourier exchange formula

$$\mathcal{F}(T * S) = \mathcal{F}(T) \cdot \mathcal{F}(S).$$

This notion has been studied and applied by many authors (see for instance [16], [15], [8], [6], [9], [11]–[13], [1], [2]). In particular, R. Shiraishi introduced in [16] an equivalent definition, which is the one we will use here.

Definition 1 ([16]). Given two tempered distributions T and S, we say that their S'-convolution exists if $T(\check{S} * \varphi) \in D'_{L^1}$ for each $\varphi \in S$. When the S'-convolution exists, the map

$$S \to \mathbb{C}, \qquad \varphi \mapsto (T(\check{S} * \varphi), 1)_{D'_{L^1}, B_{\mathrm{c}}},$$

is linear and continuous. Thus, it defines a tempered distribution which will be denoted by $T * S$.

In this definition, $T(\check{S} * \varphi)$ denotes the multiplicative product of the distribution T with the regularization $\check{S} * \varphi$. This product is well defined because the regularization is a C^∞ function of polynomial growth together with all its derivatives ([14, p. 248]).

It was proved by R. Shiraishi in [16] that $T * S$ exists if and only if $S * T$ exists, and they coincide. Moreover, this definition coincides with the definition given by L. Schwartz in all the cases in which Schwartz's definition is applicable.

3. The integral of an integrable distribution in the n-dimensional case. L. Schwartz defined in [14, p. 243] the integral of an integrable distribution as follows:

Definition 2 ([14]). Given $T \in D'_{L^1}(\mathbb{R}^n)$, the *integral* of T, denoted by $\int T$, is defined as

$$\int T = (T, 1)_{D'_{L^1}, B_c}. \tag{2}$$

This definition certainly coincides with the usual Lebesgue integral when T is a function in $L^1(\mathbb{R}^n)$. In general, given $T \in D'_{L^1}(\mathbb{R}^n)$ we can represent T as a finite sum,

$$T = \sum_\alpha \partial^\alpha f_\alpha \tag{3}$$

where $f_\alpha \in L^1(\mathbb{R}^n)$. Thus

$$(T, 1)_{D'_{L^1}, B_c} = (f_0, 1)_{L^1, L^\infty} = \int_{\mathbb{R}^n} f_0(x)\, dx.$$

W. Kierat and U. Skórnik obtained in [10] the following formula for the integral of an integrable distribution in one dimension:

Proposition 3 ([10]). *If* $T \in D'_{L^1}(\mathbb{R})$, *then*

$$(T, 1)_{D'_{L^1}, B_c} = \int_{-\infty}^{\infty} [\mathcal{C}(T)(x + iy) - \mathcal{C}(T)(x - iy)]\, dx \tag{4}$$

independently of $y > 0$. *Here* $\mathcal{C}(T)$ *is the Cauchy transform of* T, *defined for* $z \in \mathbb{C} \setminus \mathbb{R}$ *as*

$$\mathcal{C}(T)(z) = (T_t, C(t - z))_{D'_{L^1}, \dot{B}},$$

where $C(z) = 1/(2\pi i z)$ *is the one-dimensional Cauchy kernel.*

The right-hand side of (4) will be called the complex-variable version of the integral of T.

We can also give a real-variable version of (4). In fact, observe that

$$\frac{1}{2\pi i}\frac{1}{t - x - iy} - \frac{1}{2\pi i}\frac{1}{t - x + iy} = \frac{1}{\pi}\frac{y}{(t - x)^2 + y^2} = P_y(x - t) \tag{5}$$

for every $(x,y) \in \mathbb{R}^2_+$, where

$$P_y(x) = \frac{1}{y}\,\frac{1}{\pi((x/y)^2+1)}$$

is the one-dimensional Poisson kernel. Thus,

$$\mathcal{C}(T)(x+iy) - \mathcal{C}(T)(x-iy) = (T_t, P_y(x-t))_{D'_{L^1},\dot{B}}. \tag{6}$$

Since $\int_{-\infty}^{\infty} P_y(x)\,dx = 1$ for each $y>0$, we have

$$(T,1)_{D'_{L^1},B_c} = \Big(T_t, \int_{-\infty}^{\infty} P_y(x-t)\,dx\Big)_{D'_{L^1},B_c}.$$

We claim that we can exchange the integral with the action of the distribution T. Indeed, using (3), we can assume that $T = \partial^\alpha f$ for $f \in L^1(\mathbb{R})$. Thus,

$$\Big(T_t, \int_{-\infty}^{\infty} P_y(x-t)\,dx\Big)_{D'_{L^1},B_c} = (-1)^{|\alpha|}\int_{-\infty}^{\infty} f(t)\Big(\partial_t^\alpha \int_{-\infty}^{\infty} P_y(x-t)\,dx\Big)\,dt.$$

The function $t \mapsto P_y(x-t)$ is infinitely differentiable for each $x \in \mathbb{R}$, $y>0$. Moreover the function $x \mapsto \partial_t^\alpha P_y(x-t)$ is integrable on $\mathbb{R}$ for each $t \in \mathbb{R}$, $y>0$. Thus, we can exchange the derivative and integration to obtain

$$\Big(T_t, \int_{-\infty}^{\infty} P_y(x-t)\,dx\Big)_{D'_{L^1},B_c} = (-1)^{|\alpha|}\int_{-\infty}^{\infty} f(t)\Big(\int_{-\infty}^{\infty} \partial_t^\alpha P_y(x-t)\,dx\Big)\,dt.$$

Now the function $(x,t) \mapsto (-1)^{|\alpha|} f(t)\partial_t^\alpha P_y(x-t)$ is integrable on $\mathbb{R}^2$ for each $y>0$. So, the Fubini Theorem yields

$$\begin{aligned}
(-1)^{|\alpha|}&\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(t)\partial_t^\alpha P_y(x-t)\,dx\Big)\,dt\\
&= (-1)^{|\alpha|}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(t)\partial_t^\alpha P_y(x-t)\,dt\Big)\,dx\\
&= (-1)^{|\alpha|}\int_{-\infty}^{\infty} (f(t), \partial_t^\alpha P_y(x-t))_{L^1,L^\infty}\,dx\\
&= \int_{-\infty}^{\infty} (\partial_t^\alpha f(t), P_y(x-t))_{D'_{L^1},\dot{B}}\,dx.
\end{aligned}$$

Thus the claim is proved. We can then write

$$(T,1)_{D'_{L^1},B_c} = \int_{-\infty}^{\infty} (T_t, P_y(x-t))_{D'_{L^1},\dot{B}}\,dx. \tag{7}$$

The right-hand side of (7) is the real-variable version of the integral of T.

In the n-dimensional case, one should be able as well to recognize a complex-variable version and a real-variable version of the integral of a distribution $T \in D'_{L^1}$. For the real-variable version, one could choose different extensions of $P_y(x)$ to n dimensions, namely, the Euclidean version

$$P_y(x) = \frac{c(n)}{y^n} \frac{1}{(|x|^2/y^2 + 1)^{(n+1)/2}}, \tag{8}$$

where $c(n) = \Gamma((n+1)/2)/\pi^{(n+1)/2}$, $y > 0$, or any product domain version, for instance,

$$\mathcal{P}_{(y)}(x) = P_{y_1}(x_1) \dots P_{y_n}(x_n), \tag{9}$$

where $(y) > 0$, meaning that $y_1, \dots, y_n > 0$. Of course, depending on how we group the coordinates in $\mathbb{R}^n$, we obtain different versions, all of which give the same definition of $\int T$. For our choice (9), the product domain we are considering is the cartesian product $\mathbb{R}^2_+ \times \dots \times \mathbb{R}^2_+$ of n-copies of the upper half-plane.

In any case, the n-dimensional real-variable realization of (2) holds with a similar proof to the one-dimensional case. For the sake of completeness we now state this result for the kernels given by (8) and (9). The proof is the same as the one given in the one-dimensional case.

Proposition 4. *If $T \in D'_{L^1}$, then*

$$(T, 1)_{D'_{L^1}, B_c} = \int_{\mathbb{R}^n} (T_t, P_y(x-t))_{D'_{L^1}, \dot{B}}\, dx, \qquad y > 0, \tag{10}$$

and also

$$(T, 1)_{D'_{L^1}, B_c} = \int_{\mathbb{R}^n} (T_t, \mathcal{P}_{(y)}(x-t))_{D'_{L^1}, \dot{B}}\, dx, \ (y) > 0. \tag{11}$$

In order to obtain an n-dimensional complex-variable realization of (2), we need to select an appropriate n-dimensional version of the Cauchy kernel $C(z) = 1/(2\pi i z)$. Once again, we can consider the Euclidean case or the product domain case.

In the Euclidean case we will consider the kernel $K_n(z_1, \dots, z_n)$ defined as

$$K_n(z_1, \dots, z_n) = \frac{c(n)}{2i} \frac{\sum_{j=1}^n \overline{z}_j}{(\sum_{j=1}^n |z_j|^2)^{(n+1)/2}},$$

where the positive constant $c(n)$ is chosen so that

$$c(n) \int_{\mathbb{R}^n} \frac{du}{(|u|^2 + 1)^{(n+1)/2}} = 1.$$

The kernel K_n is a non-holomorphic version of the kernel C when $n > 1$ and it coincides with C when $n = 1$. Given $T \in D'_{L^1}(\mathbb{R}^n)$, we define the

n-dimensional Cauchy transform of T, $\mathcal{K}_n(T)$, as

$$\mathcal{K}_n(T)(z_1,\ldots,z_n) = (T_t, K_n(t_1-z_1,\ldots,t_n-z_n))_{D'_{L^1},\dot{B}}.$$

We have the following result:

PROPOSITION 5. *If $T \in D'_{L^1}(\mathbb{R}^n)$, then*

$$(12)\qquad (T,1)_{D'_{L^1},B_c} = \int_{\mathbb{R}^n} [\mathcal{K}_n(T)(x_1+iy,x_2,\ldots,x_n) - \mathcal{K}_n(T)(x_1-iy,x_2,\ldots,x_n)]\,dx_1\,dx_2\ldots dx_n$$

independently of $y>0$.

Proof. The proof is based on the fact that

$$c(n)\int_{\mathbb{R}^n} K_n(t_1-x_1-iy,t_2-x_2,\ldots,t_n-x_n) - K_n(t_1-x_1+iy,t_2-x_2,\ldots,t_n-x_n)\,dx_1\,dx_2\ldots dx_n = 1$$

independently of $t_1,\ldots,t_n\in\mathbb{R}$ and $y>0$. ∎

The integral $(T,1)_{D'_{L^1},B_c}$ can also be computed using the substitutions $z_j = x_j+iy$ for any fixed $j=1,\ldots,n$, $z_l = x_l$ for $l\neq j$.

Formula (12) will be adopted as the n-dimensional complex-variable version of the integral of T in the Euclidean case.

To obtain a product domain version, we will assume for clarity that our product domain is $\mathbb{R}\times\mathbb{R}$. The exact same techniques will apply to a Cartesian product with any number of factors, grouped in any way. Depending on the grouping, the formulas may look a little more complicated.

Fix $T\in D'_{L^1}(\mathbb{R}^2)$. We know that T has the representation (3), $T=\sum_\alpha \partial^\alpha f_\alpha$ where $f_\alpha\in L^1(\mathbb{R}^2)$. Using this representation, we can obtain a "Fubini Theorem" for integrable distributions. In fact,

$$\begin{aligned}(T,\varphi_1\otimes\varphi_2)_{D'_{L^1},\dot{B}} &= \sum_\alpha (\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2} f_{\alpha_1\alpha_2},\varphi_1\otimes\varphi_2)_{D'_{L^1},\dot{B}}\\ &= \sum_\alpha (-1)^{\alpha_1+\alpha_2}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f_{\alpha_1\alpha_2}(x_1,x_2)\partial^{\alpha_1}_{x_1}\varphi_1(x_1)\,dx_1\Big)\partial^{\alpha_2}_{x_2}\varphi_2(x_2)\,dx_2.\end{aligned}$$

We now observe that for each $x_2\in\mathbb{R}$, the integrable function $f_{\alpha_1\alpha_2}(\cdot,x_2)$ acts on $\partial^{\alpha_1}_{x_1}\varphi_1$ in the duality $(D'_{L^1}(\mathbb{R}),\dot{B}\,(\mathbb{R}))$. Likewise, the integrable function $(f_{\alpha_1\alpha_2}(\cdot,x_2),\partial^{\alpha_1}_{x_1}\varphi_1)_{D'_{L^1},\dot{B}}$ acts on $\partial^{\alpha_2}_{x_2}\varphi_2(x_2)$ in the same duality. Thus,

$$\begin{aligned}(T,\varphi_1\otimes\varphi_2)_{D'_{L^1},\dot{B}} &= \sum_\alpha (\partial^{\alpha_2}_{x_2}(\partial^{\alpha_1}_{x_1} f_{\alpha_1\alpha_2},\varphi_1)_{D'_{L^1},\dot{B}},\varphi_2)_{D'_{L^1},\dot{B}}\\ &=: ((T_{x_1},\varphi_1)_{D'_{L^1},\dot{B}},\varphi_2)_{D'_{L^1},\dot{B}}.\end{aligned}$$

As a consequence,

$$(13)\quad (T_t, \mathcal{P}_{(y)}(x-t))_{D'_{L^1},\dot{B}} = ((T_{t_1}, P_{y_1}(x_1-t_1))_{D'_{L^1},\dot{B}}, P_{y_2}(x_2-t_2))_{D'_{L^1},\dot{B}}.$$

Similarly,

$$(14)\quad (T_t, \mathcal{P}_{(y)}(x-t))_{D'_{L^1},\dot{B}} = ((T_{t_2}, P_{y_2}(x_2-t_2))_{D'_{L^1},\dot{B}}, P_{y_1}(x_1-t_1))_{D'_{L^1},\dot{B}}.$$

From (6) and (13), it follows that for every $T \in D'_{L^1}(\mathbb{R}^2)$ we can write

$$(15)\quad \begin{aligned}(T_t, \mathcal{P}_{(y)}(x-t))_{D'_{L^1},\dot{B}} &= \mathcal{C}_2[\mathcal{C}_1(T)(x_1+iy_1) - \mathcal{C}_1(T)(x_1-iy_1)](x_2+iy_2)\\ &\quad - \mathcal{C}_2[\mathcal{C}_1(T)(x_1+iy_1) - \mathcal{C}_1(T)(x_1-iy_1)](x_2-iy_2),\end{aligned}$$

where $\mathcal{C}_j$ denotes the action of the Cauchy transform $\mathcal{C}$ on the jth coordinate, $j = 1, 2$.

From (6) and (14) we can also write

$$(16)\quad \begin{aligned}(T_t, \mathcal{P}_{(y)}(x-t))_{D'_{L^1},\dot{B}} &= \mathcal{C}_1[\mathcal{C}_2(T)(x_2+iy_2) - \mathcal{C}_2(T)(x_2-iy_2)](x_1+iy_1)\\ &\quad - \mathcal{C}_1[\mathcal{C}_2(T)(x_2+iy_2) - \mathcal{C}_2(T)(x_2-iy_2)](x_1-iy_1).\end{aligned}$$

Thus, we can state the following result:

Proposition 6. *If $T \in D'_{L^1}(\mathbb{R}^2)$, then*

$$(17)\quad \begin{aligned}(T,1)_{D'_{L^1},B_c} &= \int_{\mathbb{R}^2} \{\mathcal{C}_2[\mathcal{C}_1(T)(x_1+iy_1) - \mathcal{C}_1(T)(x_1-iy_1)](x_2+iy_2)\\ &\quad - \mathcal{C}_2[\mathcal{C}_1(T)(x_1+iy_1) - \mathcal{C}_1(T)(x_1-iy_1)](x_2-iy_2)\}\,dx_1\,dx_2\\ &= \int_{\mathbb{R}^2} \{\mathcal{C}_1[\mathcal{C}_2(T)(x_2+iy_2) - \mathcal{C}_2(T)(x_2-iy_2)](x_1+iy_1)\\ &\quad - \mathcal{C}_1[\mathcal{C}_2(T)(x_2+iy_2) - \mathcal{C}_2(T)(x_2-iy_2)](x_1-iy_1)\}\,dx_1\,dx_2\end{aligned}$$

for any $y_1, y_2 > 0$.

We adopt (17) as the 2-dimensional complex-variable version of the integral of T in the product domain case.

Making repeated use of (5) we can see how the real-variable kernel $\mathcal{P}_{(y)}$ is related to the complex-variable kernel $\prod_{j=1}^n 1/(2\pi(x_j+iy_j))$, as well as how formulas (15) and (16) come about.

4. The integration formulas as integrals of S'-convolutions. It is possible to reinterpret the integrands in (10) and (11) as S'-convolutions. In fact, we have the following result:

PROPOSITION 7. *Given $T \in D'_{L^1}$, the distribution T is S'-convolvable with both kernels P_y and $\mathcal{P}_{(y)}$. Moreover, the S'-convolutions are given by*

$$(T_x, P_y(x-t))_{D'_{L^1}, \dot{B}} \tag{18}$$

and

$$(T_x, \mathcal{P}_{(y)}(x-t))_{D'_{L^1}, \dot{B}} \tag{19}$$

for each $y > 0$ and $(y) > 0$, respectively.

Proof. First consider the S'-convolution with P_y. Given $\varphi \in S$, it suffices to show that the classical convolution $P_y * \varphi$ is a function in B.

In fact, for each $y > 0$ the integral

$$\frac{c(n)}{y^n} \int_{\mathbb{R}^n} \left(\frac{|x-t|^2}{y^2} + 1\right)^{-(n+1)/2} \varphi(t)\, dt$$

defines a C^∞ function that we denote by $f(x)$. We want to show that $\partial^\alpha f$ is bounded on $\mathbb{R}^n$ for each n-tuple α. We have

$$\partial^\alpha f(x) = \frac{c(n)}{y^n} \int_{\mathbb{R}^n} \left(\frac{|x-t|^2}{y^2} + 1\right)^{-(n+1)/2} (\partial^\alpha \varphi)(t)\, dt.$$

We now use Peetre's inequality,

$$(|x-t|^2 + 1)^r \le 2^{|r|}(|x|^2 + 1)^r (|t|^2 + 1)^{|r|}$$

with $r = -(n+1)/2$, to obtain

$$|\partial^\alpha f(x)| \le \frac{c(n)}{y^n}\left(\frac{|x|^2}{y^2}+1\right)^{-(n+1)/2} \int_{\mathbb{R}^n} \left(\frac{|t|^2}{y^2}+1\right)^{(n+1)/2} |(\partial^\alpha \varphi)(t)|\, dt \tag{20}$$

for each $y > 0$. This estimate already shows that $f \in B$ for each $y > 0$. However, we can obtain an explicit dependence on y, by estimating the integral in (20). For $y > 0$ fixed, we write the integral as

$$\Big(\int_{|t|<y} + \int_{|t|\ge y}\Big)\left(\frac{|t|^2}{y^2} + 1\right)^{(n+1)/2} |(\partial^\alpha \varphi)(t)|\, dt = I_1 + I_2.$$

We estimate each term separately:

$$I_1 \le c(n) \int_{|t|<y} |(\partial^\alpha \varphi)(t)|\, dt \le c(n)\|\partial^\alpha \varphi\|_1,$$

$$I_2 \le c(n) \int_{|t|\ge y} (|t|/y)^{n+1} |(\partial^\alpha \varphi)(t)|\, dt \le \frac{c(n)}{y^{n+1}} \big\||t|^{n+1}\partial^\alpha \varphi\big\|_1.$$

Finally,

$$
(21)\qquad |\partial^\alpha f(x)| \le \frac{c(n)}{y^n}\left(\frac{|x|^2}{y^2}+1\right)^{-(n+1)/2} \times\left[\|\partial^\alpha\varphi\|_1 + \frac{1}{y^{n+1}}\left\||t|^{n+1}\partial^\alpha\varphi\right\|_1\right].
$$

Thus, the S'-convolution $T * P_y$ exists. We now show that $T * P_y$ is given by the integrable function $(T_x, P_y(x-t))_{D'_{L^1},\dot{B}}$. Estimates (20) or (21) show in fact that $P_y * \varphi$ belongs to $\dot{B}$. So, we can write

$$
\begin{aligned}
(T(P_y*\varphi),1)_{D'_{L^1},B_c} &= (T, P_y*\varphi)_{D'_{L^1},\dot{B}} = \Big(\sum_\alpha \partial^\alpha f_\alpha, P_y*\varphi\Big)_{D'_{L^1},\dot{B}}\\
&= \sum_\alpha (-1)^{|\alpha|}\int_{\mathbb{R}^n} f_\alpha(x)(\partial^\alpha P_y*\varphi)(x)\,dx\\
&= \sum_\alpha (-1)^{|\alpha|}\int_{\mathbb{R}^n\times\mathbb{R}^n} f_\alpha(x)\partial^\alpha P_y(x-t)\varphi(t)\,dt\,dx\\
&= \sum_\alpha \int_{\mathbb{R}^n} (\partial^\alpha f_\alpha(x), P_y(x-t))_{D'_{L^1},\dot{B}}\varphi(t)\,dt\\
&= \int_{\mathbb{R}^n} (T_x, P_y(x-t)_{D'_{L^1},\dot{B}}\varphi(t)\,dt.
\end{aligned}
$$

This concludes the proof of the claim. Similar calculations show that T is S'-convolvable with $\mathcal{P}_{(y)}$ for each $(y)>0$ and that the S'-convolution $T*\mathcal{P}_{(y)}$ is given by $(T_x, \mathcal{P}_{(y)})_{D'_{L^1},\dot{B}}$. ■

Doing the same work as in the proof of Proposition 7, we can show that the S'-convolution (18) defines a function $F(t,y)$ that belongs to C^∞ in the upper half-space $\mathbb{R}^{n+1}_+$. Moreover,

$$
\partial_t^\alpha\partial_y^k F(t,y) = \left(T_x, \partial_y^k\left[\frac{1}{y^n}\partial_t^\alpha\left(P\left(\frac{x-t}{y}\right)\right)\right]\right)_{D'_{L^1},\dot{B}},
$$

where $P(x) = c(n)(1+|x|^2)^{-(n+1)/2}$. For each $x\in\mathbb{R}^n$, the function $F_1(t,y) = (1/y^n)P((x-t)/y)$ satisfies the equation $(\partial_y^2 + \sum_{j=1}^n \partial_{t_j}^2)F_1 = 0$. So, $F(t,y)$ is harmonic in the upper half-space $\mathbb{R}^{n+1}_+$.

Likewise, the S'-convolution (19) defines a function $G(t,y)$ that belongs to C^∞ in the product domain $\mathbb{R}^2_+\times\ldots\times\mathbb{R}^2_+$. The function $G_1(t,y) = \prod_{j=1}^n (1/y_j)P((x_j-t_j)/y_j)$ satisfies the n equations $(\partial_{y_j}^2+\partial_{t_j}^2)G_1 = 0$ in the product domain $\mathbb{R}^2_+\times\ldots\times\mathbb{R}^2_+$. That is to say, $G(t,y)$ is harmonic on $\mathbb{R}^2_+$ in each pair of variables (t_j,y_j). A function with this property is called *n-harmonic* ([5, p. 148]). An n-harmonic function is harmonic on $\mathbb{R}^2_+\times\ldots\times\mathbb{R}^2_+$, although the converse is not true in general. For interesting connections with holomorphic functions in several variables, we refer again to [5].

5. Optimal spaces for the S'-convolution with the kernels P_y and $\mathcal{P}_{(y)}$. The fact that the functions $P_y * \varphi$ and $\mathcal{P}_{(y)} * \varphi$ not only belong to the space B, but also to $\dot{B}$, suggests that D'_{L^1} is not the largest class of tempered distributions that is S'-convolvable with either kernel. In this section we will characterize the classes of tempered distributions that are S'-convolvable with the kernels P_y and $\mathcal{P}_{(y)}$, respectively.

We first consider the S'-convolution with P_y.

Estimate (21) suggests that P_y might be S'-convolvable with distributions in weighted versions of the space D'_{L^1}. Such weighted spaces appeared naturally in the study made by L. Schwartz ([14, p. 214]) of Newtonian potentials of distributions, as well as in his paper [15]. For other occurrences of these spaces, see for instance [8], [9], [11]–[13], [1], [2].

Definition 8. Let $w(x) = (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^n$ and fix $\mu \in \mathbb{R}$. Then

$$w^\mu D'_{L^1} = \{T \in S' : w^{-\mu} T \in D'_{L^1}\}$$

with the topology induced by the map

$$w^\mu D'_{L^1} \to D'_{L^1}, \qquad T \mapsto w^{-\mu} T.$$

We observe that $w^\mu D'_{L^1}$ can also be defined as the space of those distributions $T \in D'$ such that $w^{-\mu} T \in D'_{L^1}$. In fact, if $w^{-\mu} T \in D'_{L^1}$ then T must be a tempered distribution.

We first obtain a representation of distributions in $w^\mu D'_{L^1}$ which is related to the representation obtained in [2] for the particular case $\mu = n$.

Proposition 9. *Given $T \in S'$, $\mu \in \mathbb{R}$, the following statements are equivalent*:

(i) $T \in w^\mu D'_{L^1}$.

(ii) $T = T_1 + |x|^\mu T_2$, *where $T_1 \in E'$, $T_2 \in D'_{L^1}$ and T_2 is zero in a neighborhood of zero.*

Proof. We first assume that (i) holds and we select a cut-off function $\theta \in C_0^\infty$ so that $0 \le \theta \le 1$, $\theta = 1$ for $|x| \le 1/2$, $\theta = 0$ for $|x| \ge 1$. Then

$$\begin{aligned} T &= \theta T + (1-\theta) T \\ &= \theta T + (1-\theta)\,\frac{(1+|x|^2)^{\mu/2}}{|x|^\mu}\,|x|^\mu (1+|x|^2)^{-\mu/2} T. \end{aligned}$$

Since $(1-\theta)(1+|x|^2)^{\mu/2}/|x|^\mu \in B$ and $(1+|x|^2)^{-\mu/2} T \in D'_{L^1}$, if we set $T_1 = \theta T$ and $T_2 = (1-\theta)|x|^{-\mu} T$, we obtain the representation stated in (ii).

The converse is quite direct, since $E' \subset w^\mu D'_{L^1}$ and $|x|^\mu T_2 \in w^\mu D'_{L^1}$ for any distribution $T \in D'_{L^1}$ that is zero near zero. ∎

The representation formula provided by Proposition 9 is only one of several possible representations. For example, given $T \in w^{\mu} D'_{L^1}$, we have, in the sense of S',

$$(1+|x|^2)^{-\mu/2} T = \sum_{\alpha} \partial^{\alpha} f_{\alpha}$$

or

$$T = \sum_{\alpha} (1+|x|^2)^{\mu/2} \partial^{\alpha} f_{\alpha} \tag{22}$$

where the sum is finite and $f_{\alpha} \in L^1$.

We are now ready to characterize those tempered distributions that are S'-convolvable with the kernel P_y.

THEOREM 10. *Given $T \in S'$, the following statements are equivalent*:

(i) $T \in w^{n+1} D'_{L^1}$.
(ii) T *is S'-convolvable with P_y for each $y > 0$.*

Proof. Assume that (i) holds. We need to show that for each $\varphi \in S$, the distribution $T(P_y * \varphi)$ belongs to D'_{L^1} for each $y > 0$. In fact, as a consequence of (21), we have $(1+|x|^2)^{(n+1)/2}(P_y * \varphi) \in B$, and so our claim is proved.

Conversely, fix $T \in S'$ so that $T(P_y * \varphi) \in D'_{L^1}$ for each $\varphi \in S$ and $y > 0$. We will prove that $T \in w^{n+1} D'_{L^1}$ by showing that T can be represented as in Proposition 9. We consider a cut-off function θ as in the proof of Proposition 9, and write

$$T = \theta T + (1-\theta) T.$$

We observe that $\theta T \in E'$. Now, we take $\varphi \in S$ such that for some $\varepsilon > 0$, we have $\varphi = 0$ for $|x| \geq \varepsilon$ and $\varphi > 0$ for $|x| < \varepsilon$. We claim that for an appropriate choice of $\varepsilon > 0$ there exists $C_n > 0$ so that

$$(P_y * \varphi)(x) \geq C_n \frac{y}{(|x|^2 + y^2)^{(n+1)/2}} \|\varphi\|_1 \tag{23}$$

for $|x| > 1/3$. In fact,

$$(P_y * \varphi)(x) = \int_{|t| \leq \varepsilon} \frac{c(n)}{y^n} \frac{\varphi(t)}{(1+|x-t|^2/y^2)^{(n+1)/2}} \, dt.$$

If, say, $0 < \varepsilon \leq 1/3$, we have for $|x| > 1/3$,

$$\frac{|x-t|}{y} \leq \frac{|x| + \varepsilon}{y} \leq 2 \frac{|x|}{y}$$

and therefore

$$(1+|x-t|^2/y^2)^{(n+1)/2} \leq (1+4|x|^2/y^2)^{(n+1)/2} \leq c(n) \frac{(|x|^2+y^2)^{(n+1)/2}}{y^{n+1}}.$$

Thus, estimate (23) holds. If we combine estimates (23) and (21), we conclude that for each $y > 0$, the function

$$\frac{1-\theta}{P_y * \varphi}\,(1+|x|^2)^{-(n+1)/2}$$

belongs to B and it is equal to zero near zero. Thus

$$(1-\theta)T = |x|^{n+1}\,\frac{(1+|x|^2)^{(n+1)/2}}{|x|^{n+1}}\,\frac{1-\theta}{P_y * \varphi}\,(1+|x|^2)^{-(n+1)/2}T(P_y * \varphi).$$

We conclude that the distribution T belongs to $w^{n+1}D'_{L^1}$. ∎

REMARK 11. Given $T \in w^{n+1}D'_{L^1}$, the S'-convolution $T * P_y$ is given by $(T_x, P_y(x-t))$, where the pairing is understood as

$$((1+|x|^2)^{-(n+1)/2}T_x, (1+|x|^2)^{(n+1)/2}P_y(x-t))_{D'_{L^1}, B_c} \tag{24}$$

for each $t \in \mathbb{R}^n$, $y > 0$.

A similar pairing was proposed in [15, p. 16] for the S'-convolution of the distribution p.v. $\frac{1}{x}$ with distributions in the space $(1+x^2)D'_{L^1}(\mathbb{R})$. Formula (24) can be proved in a similar way to the proof of Proposition 7, using this time the representation formula (22).

As in the case of D'_{L^1}, the function defined by (24) is a harmonic function of the variables t, y in the upper half-space $\mathbb{R}^{n+1}_+$.

We now turn to the kernel $\mathcal{P}_{(y)} = \prod_{i=1}^{n} P_{y_i}$. We want to characterize those tempered distributions that are S'-convolvable with $\mathcal{P}_{(y)}$. The relevant weighted D'_{L^1} space is introduced in the following definition.

DEFINITION 12. Let $w_j = (1+x_j^2)^{1/2}$, $j = 1, \ldots, n$. Then we set

$$w_1^2 \ldots w_n^2 D'_{L^1} = \{T \in S' : w_1^{-2} \ldots w_n^{-2} T \in D'_{L^1}\}$$

with the topology induced by the map

$$w_1^2 \ldots w_n^2 D'_{L^1} \to D'_{L^1}, \qquad T \mapsto w_1^{-2} \ldots w_n^{-2} T.$$

The space $w_1^2 \ldots w_n^2 D'_{L^1}$ can be viewed as a weighted space of distributions in the product domain $\mathbb{R} \times \ldots \times \mathbb{R}$. In the proposition that follows we summarize several inclusion properties of $w_1^2 \ldots w_n^2 D'_{L^1}$ with respect to relevant weighted spaces of distributions defined in $\mathbb{R}^n$.

PROPOSITION 13. *For $n \geq 2$, the following statements hold*:

(a) *The space $w^{n+1}D'_{L^1}$ is strictly contained in $w^{2n}D'_{L^1}$.*

(b) *The space $w_1^2 \ldots w_n^2 D'_{L^1}$ is strictly contained in $w^{2n}D'_{L^1}$.*

(c) *There exist distributions in $w^{n+1}D'_{L^1}$ that do not belong to the space $w_1^2 \ldots w_n^2 D'_{L^1}$.*

(d) *There exist distributions in $w_1^2 \ldots w_n^2 D'_{L^1}$ that do not belong to the space $w^{n+1}D'_{L^1}$.*

Proof. We first consider (a). Since $w^{-2n}w^{n+1} = w^{-n+1}$ belongs to B, we conclude easily that $w^{n+1}D'_{L^1} \subset w^{2n}D'_{L^1}$. To see that the inclusion is strict, we consider the tempered distribution T defined by the function $|x|^{n-\beta}$ for some $0 < \beta \le n-1$. Then $T \in w^{2n}D'_{L^1}$ because the function $|x|^{n-\beta}w^{-2n}$ is integrable on $\mathbb{R}^n$. We claim that $T \notin w^{n+1}D'_{L^1}$. Indeed, consider the sequence $\eta_j(x) = \eta(x/j)$, $j = 1, 2, \ldots$, where $\eta \in C_0^\infty$, $0 \le \eta \le 1$, $\eta(x) = 1$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| > 2$. It is quite simple to show that $\eta_j \to 1$ in $B_{\rm c}$, while for $j \ge 2$ we have

$$\begin{aligned}(w^{-(n+1)}T, \eta_j)_{S',S} &= \int_{\mathbb{R}^n} |x|^{n-\beta}\eta(x/j)\,\frac{dx}{(1+|x|^2)^{(n+1)/2}} \\ &\ge \int_{1<|x|<j} |x|^{n-\beta}\frac{dx}{(1+|x|^2)^{(n+1)/2}} \\ &\ge 2^{-(n+1)/2}\int_{1<|x|<j} \frac{dx}{|x|^{1+\beta}}.\end{aligned}$$

Since $\beta \le n-1$, this last integral goes to ∞ as $j \to \infty$. Hence, $w^{-n-1}T \notin D'_{L^1}$. This concludes the proof of (a).

To prove the inclusion in (b) we can use once again the fact that D'_{L^1} is closed under multiplication by functions in B. Thus, since $w^{-2n}w_1^2 \ldots w_n^2 \in B$, we have $w_1^2 \ldots w_n^2 D'_{L^1} \subset w^{2n}D'_{L^1}$. To see that the inclusion is strict, consider this time the distribution S defined as

$$S = \underbrace{\delta_0 \otimes \ldots \otimes \delta_0}_{n-1 \text{ times}} \otimes (1+x_n^2)^{\mu/2}$$

where $1 \le \mu < n$. We will first show that $w^{-n-1}S$ is continuous on C_0^∞ with respect to the topology of $\dot{B}$. Indeed, given $\varphi \in C_0^\infty$ we have

$$(w^{-(n+1)}S, \varphi)_{S',S} = ((1+x_n^2)^{-(n+1-\mu)/2}, \varphi(0, x_n))_{S',S}.$$

Since $\mu < n$, the function $(1+x_n^2)^{-(n+1-\mu)/2}$ is integrable on $\mathbb{R}$. Hence

$$|(w^{-(n+1)}S, \varphi)_{S',S}| \le C_n\|\varphi\|_\infty.$$

This shows that $w^{-(n+1)}S \in D'_{L^1}$ and thus $S \in w^{2n}D'_{L^1}$. On the other hand, if we consider the sequence $\beta_j(x) = \beta(x_1/j)\ldots\beta(x_n/j)$, $j = 1, 2, \ldots$, where β is the one-dimensional version of the cut-off function η used in the proof of (a), we find that $\beta_j \to 1$ in $B_{\rm c}$. However,

$$\begin{aligned}(w_1^{-2}\ldots w_n^{-2}S, \beta_j)_{S',S} &= \int_{-\infty}^{\infty} (1+x_n^2)^{-1+\mu/2}\beta(x_n/j)\,dx_n \\ &\ge \int_0^j (1+x_n^2)^{-1+\mu/2}\,dx_n\end{aligned}$$

and this integral goes to ∞ as $j \to \infty$ because $\mu \geq 1$. Hence $w_1^{-2} \dots w_n^{-2} S \notin D'_{L^1}$. This concludes the proof of (b).

Concerning the proof of (c), the distribution S considered in (b) provides a suitable example.

Finally, to prove (d), we observe that the function $w_1^2 \dots w_n^2 / w^{n+1}$ is not bounded along the diagonal $x_1 = \dots = x_n$ when $n \geq 2$. This suggests the following example:

We consider the tempered distribution U defined as

$$(U, \varphi)_{S',S} = \int_{-\infty}^{\infty} (1 + x_1^2)^{n-1} \varphi(x_1, \dots, x_1)\, dx_1.$$

We first show that $w_1^{-2} \dots w_n^{-2} U$ is continuous on C_0^∞ with the topology of $\dot{B}$. In fact, for $\varphi \in C_0^\infty$, we have

$$\begin{aligned}(w_1^{-2} \dots w_n^{-2} U, \varphi)_{S',S} &= (U, w_1^{-2} \dots w_n^{-2} \varphi)_{S',S} \\ &= \int_{-\infty}^{\infty} (1 + x_1^2)^{n-1} w_1^{-2} \dots w_1^{-2} \varphi(x_1, \dots, x_1)\, dx_1.\end{aligned}$$

Thus,

$$|(w_1^{-2} \dots w_n^{-2} U, \varphi)_{S',S}| \leq \|\varphi\|_\infty \int_{-\infty}^{\infty} \frac{dx_1}{1 + x_1^2}.$$

This shows that $U \in w_1^2 \dots w_n^2 D'_{L^1}$. On the other hand, if we consider the sequence $\{\beta_j\}$ introduced in (b), we see that $\beta_j \to 1$ in B_{c} but

$$\begin{aligned}(w^{-(n+1)} U, \beta_j)_{S',S} &= (U, w^{-(n+1)} \beta_j)_{S',S} \\ &= \int_{-\infty}^{\infty} (1 + x_1^2)^{n-1} (1 + n x_1^2)^{(-n-1)/2} \beta^n(x_1/j)\, dx_1 \\ &\geq n^{(-n-1)/2} \int_1^j (1 + x_1^2)^{(n-3)/2}\, dx_1.\end{aligned}$$

The last integral goes to ∞ as $j \to \infty$. Thus, $U \notin w^{-n-1} D'_{L^1}$, showing that (d) holds. ∎

We now obtain a characterization of the space $w_1^2 \dots w_n^2 D'_{L^1}$ by means of a representation very much in the spirit of the one in Proposition 9.

PROPOSITION 14. *Given $T \in S'$, the following statements are equivalent*:

(i) $T \in w_1^2 \dots w_n^2 D'_{L^1}$.

(ii) $T = T_0 + \sum x_{i_1}^2 \dots x_{i_k}^2 T_{i_1,\dots,i_k}$, *where* $T_0 \in E'$, $T_{i_1,\dots,i_k} \in D'_{L^1}$, *and the sum is taken over all the different k-tuples* $(i_1, \dots, i_k)$ *with* $1 \leq i_1 < \dots < i_k \leq n$, $1 \leq k \leq n$.

Proof. The proof of the implication (ii)⇒(i) is straightforward and we omit it.

For the converse, we consider a one-dimensional θ so that $\theta \in C_0^\infty(\mathbb{R})$, $0 \le \theta \le 1$, $\theta(x) = 1$ for $|x| < 1$ and $\theta(x) = 0$ for $|x| > 2$. If we set $\theta_j = \theta(x_j)$, we have

$$\begin{aligned} 1 &= (\theta_1 + (1-\theta_1)) \dots (\theta_n + (1-\theta_n)) \\ &= \theta_1 \dots \theta_n + \sum (1-\theta_{i_1}) \dots (1-\theta_{i_k}) \theta_{j_1} \dots \theta_{j_{n-k}} \end{aligned}$$

where the sum collects all the possible cross-products containing at least one factor of the form $1 - \theta_{i_l}$, without repetition. Thus,

$$\begin{aligned} T &= \theta_1 \dots \theta_n T \\ &\quad + \sum x_{i_1}^2 \dots x_{i_k}^2 \frac{1-\theta_{i_1}}{x_{i_1}^2} \dots \frac{1-\theta_{i_k}}{x_{i_k}^2} w_1^2 \dots w_n^2 \theta_{j_1} \dots \theta_{j_{n-k}} (w_1^2)^{-1} \dots (w_n^2)^{-1} T. \end{aligned}$$

We observe that the distribution $\theta_1 \dots \theta_n T$ belongs to E'. Moreover, the functions

$$w_{j_1}^2 \dots w_{j_{n-k}}^2 \theta_{j_1} \dots \theta_{j_{n-k}} \quad \text{and} \quad \frac{1-\theta_{i_1}}{x_{i_1}^2} \dots \frac{1-\theta_{i_k}}{x_{i_k}^2} w_{i_1}^2 \dots w_{i_k}^2$$

belong to B. Thus, the representation in (ii) holds. ■

Now, we are ready to characterize those tempered distributions that are S'-convolvable with the kernel $\mathcal{P}_{(y)}$ for each $(y) > 0$.

THEOREM 15. *Given $T \in S'$, the following statements are equivalent*:

(i) $T \in w_1^2 \dots w_n^2 D'_{L^1}$.
(ii) *T is S'-convolvable with $\mathcal{P}_{(y)}$ for each $(y) > 0$.*

Proof. To prove that (i)⇒(ii), we need to show that $T(\mathcal{P}_{(y)} * \varphi) \in D'_{L^1}$ for each $\varphi \in S$ and $(y) > 0$. For this purpose it suffices to prove that the function $(1+x_1^2) \dots (1+x_n^2)(\mathcal{P}_{(y)} * \varphi)$ belongs to B, which can be done pretty much repeating the steps followed in the proof of estimate (21).

We now prove that (ii)⇒(i). Fix $T \in S'$ so that $T(\mathcal{P}_{(y)} * \varphi)$ belongs to D'_{L^1} for each $(y) > 0$. We will show that T can be written as indicated in Proposition 14, by selecting an appropriate function φ. We first observe that if φ is of the form $\varphi_1(x_1) \dots \varphi_n(x_n)$, $\varphi_j \in S(\mathbb{R})$, then

$$\mathcal{P}_{(y)} * \varphi = \prod_{i=1}^n P_{y_i} * \varphi_i.$$

We will use functions $\varphi \in S$ of that form in what follows.

We can write, as in Proposition 14,

$$T = \theta_1 \dots \theta_n T + \sum (1-\theta_{i_1}) \dots (1-\theta_{i_k}) \theta_{j_1} \dots \theta_{j_{n-k}} T.$$

The distribution $\theta_1 \dots \theta_n T$ has compact support, so it belongs to D'_{L^1}. On the other hand, we can also write formally

$$(25)\qquad (1-\theta_{i_1})\dots(1-\theta_{i_k})\theta_{j_1}\dots\theta_{j_{n-k}}T = x_{i_1}^2 \dots x_{i_k}^2 \frac{1-\theta_{i_1}}{x_{i_1}^2(P_{y_{i_1}} * \alpha_{i_1})} \dots \frac{1-\theta_{i_k}}{x_{i_k}^2(P_{y_{i_k}} * \alpha_{i_k})} \times \frac{\theta_{j_1}}{P_{y_{j_1}} * \alpha_{j_1}} \dots \frac{\theta_{j_{n-k}}}{P_{y_{j_{n-k}}} * \alpha_{j_{n-k}}} T(\mathcal{P}_{(y)} * \varphi)$$

where $\alpha_l = \alpha(x_l)$, $\varphi = \alpha_1 \dots \alpha_n$, and α is a function to be chosen later.

Using the one-dimensional version of (23) we conclude that for an appropriate $\alpha \in C_0^\infty(\mathbb{R})$ with $\alpha = 0$ for $|x| \ge 1/3$, $\alpha > 0$ for $|x| < 1/3$, we have

$$(26)\qquad (P_{y_i} * \alpha)(x_i) \ge C \frac{y_i}{x_i^2 + y_i^2} \|\alpha\|_1 \quad \text{for } |x_i| > 1/3.$$

Moreover, we can also obtain an estimate from below for the convolution

$$(P_{y_i} * \alpha)(x_i) = \frac{1}{\pi y_i} \int_{-1/3}^{1/3} \frac{\alpha(t)}{1 + (x_i - t)^2/y_i^2}\, dt$$

for $|x_i| < 1$. In fact,

$$1 + \frac{(x_i - t)^2}{y_i^2} \le \frac{16}{9}\left(1 + \frac{1}{y_i^2}\right) \quad \text{for } |x_i| < 1,\ |t| < 1/3,\ y_i > 0.$$

So,

$$(27)\qquad (P_{y_i} * \alpha)(x_i) \ge \frac{9}{16} \frac{y_i}{1 + y_i^2} \|\alpha\|_1 \quad \text{for } |x_i| < 1,\ y_i > 0.$$

According to (26), (27), and (20), each of the ratios in (25) belongs to B. By hypothesis, $T(\mathcal{P}_{(y)} * \varphi)$ belongs to D'_{L^1}. Thus, we have showed that the distribution T can be represented as in Proposition 14. ∎

6. Applications to harmonic extensions of integrable distributions. In previous sections we have studied the S'-convolution of tempered distributions with appropriate Poisson kernels and we have characterized those tempered distributions that are S'-convolvable with the Euclidean version and the product domain version of the Poisson kernel. We also observed that in each case, the S'-convolution defined a function with appropriate harmonicity properties in the relevant domain. The purpose of this last section is to present some results about the boundary values of these functions.

Before stating the first result, we recall that D_{L^1} ([14, p. 199]) denotes the space of C^∞ functions that are integrable on $\mathbb{R}^n$ together with their derivatives of all orders.

PROPOSITION 16. *Given $T \in D'_{L^1}$, the S'-convolution $T * P_y$ converges to T in D'_{L^1} as $y \to 0^+$.*

Proof. According to Proposition 7, given $T \in D'_{L^1}$, the S'-convolution of T and P_y coincides with the regularization

$$(T_x, P_y(x-t))_{D'_{L^1}, \dot{B}} \tag{28}$$

as considered by L. Schwartz in [14] in several different settings. On the other hand, according to [14, p. 204], given $\theta \in D_{L^1}$, the map

$$T \mapsto (T_x, \theta(x-t))_{D'_{L^1}, \dot{B}}$$

is linear and continuous from D'_{L^1} into D_{L^1}. Moreover, if $\{\theta_a\}_{a\in A}$ is a net that converges to the distribution δ in D'_{L^1}, then the regularization $(T_x, \theta_a(x-t))_{D'_{L^1}, \dot{B}}$ converges to T in D'_{L^1}. So, to prove that (28) converges to T in D'_{L^1} as $y \to 0^+$, it suffices to show that P_y converges to δ in D'_{L^1} as $y \to 0^+$.

We know that D'_{L^1} is the dual of the space $\dot{B}$, which is a Fréchet space with respect to the family $\{s_m\}_{m=0}^{\infty}$ of seminorms given by

$$s_m(\varphi) = \sup_{0\le|\alpha|\le m} \|\partial^\alpha \varphi\|_\infty.$$

We consider in D'_{L^1} the strong dual topology. In this topology, convergence means uniform convergence over each bounded subset of $\dot{B}$. Recall that a subset $\mathcal{B}$ of $\dot{B}$ is bounded if for each $m = 0, 1, \ldots$ we have

$$\sup_{\varphi\in\mathcal{B}} s_m(\varphi) < \infty.$$

We now prove that P_y converges to δ in D'_{L^1} as $y \to 0^+$. In fact, given $\mathcal{B} \subset \dot{B}$ bounded and $\varphi \in \mathcal{B}$, we have

$$|(P_y, \varphi)_{D'_{L^1}, \dot{B}} - \varphi(0)| = \Big| \int_{\mathbb{R}^n} (\varphi(yu) - \varphi(0)) P(u)\, du \Big|.$$

For a fixed $M > 0$, we can estimate this last integral as

$$\Big(\int_{|u|>M} + \int_{|u|<M} \Big) |\varphi(yu) - \varphi(0)| P(u)\, du = I_M + J_M.$$

We have

$$I_M \le 2 s_0(\varphi) \int_{|u|>M} P(u)\, du.$$

Since the function φ belongs to a bounded subset $\mathcal{B}$ of $\dot{B}$ and P is an integrable function, given $\varepsilon > 0$ there exists $M_\varepsilon > 0$ so that

$$\sup_{\varphi\in\mathcal{B}} I_{M_\varepsilon} < \varepsilon. \tag{29}$$

To estimate J_{M_ε}, we write

$$\varphi(yu) - \varphi(0) = \int_0^1 (\nabla\varphi)(tyu) \bullet yu\,dt.$$

Thus, for $|u| < M_\varepsilon$ we have

$$|\varphi(yu) - \varphi(0)| \le n s_1(\varphi) y M_\varepsilon,$$

that is,

$$J_{M_\varepsilon} \le n s_1(\varphi) y M_\varepsilon.$$

Thus, there exists $\delta_\varepsilon > 0$ so that for $0 < y < \delta_\varepsilon$ we have

$$\sup_{\varphi\in\mathcal{B}} J_{M_\varepsilon} < \varepsilon. \tag{30}$$

From (29) and (30), we conclude that P_y converges to δ in D'_{L^1} as $y \to 0^+$. ∎

This result states that every integrable distribution on $\mathbb{R}^n$ has a harmonic extension to the upper half-space $\mathbb{R}^{n+1}_+$, extending the classical result of S. Bochner ([3], [4]) for integrable functions. It also extends a result of H. Bremermann for distributions with compact support in $\mathbb{R}$ ([5, p. 49]).

E. Stein and G. Weiss ([18]) have obtained an extension of Bochner's result to the space $\mathcal{M}$ of finite signed Borel measures in $\mathbb{R}^n$, with almost everywhere convergence at the boundary. Proposition 16 includes a version of this extension with convergence to the measure at the boundary, in the sense of the strong dual topology of D'_{L^1}. In fact, every finite signed Borel measure defines an integrable distribution. This follows from the observation that $\mathcal{M}$ is the dual of the space C_0 of continuous functions on $\mathbb{R}^n$ that vanish at ∞, equipped with the supremum norm. So the map

$$\varphi \mapsto (\mu, \varphi)_{\mathcal{M}, C_0^\infty} = \int_{\mathbb{R}^n} \varphi(x)\,d\mu(x)$$

is continuous on C_0^∞ with the topology of $\dot{B}$ because of the estimate

$$|(\mu, \varphi)_{\mathcal{M}, C_0^\infty}| \le \|\mu\|\,\|\varphi\|_\infty$$

where $\|\mu\|$ denotes the total variation of the measure μ.

Let us point out that the harmonic extension obtained in Proposition 16 is not unique. Indeed, if we add to $(T_x, P_y(x-t))_{D'_{L^1}, \dot{B}}$ any harmonic function on $\mathbb{R}^{n+1}_+$ that is zero for $y = 0$, then the resulting harmonic function is still an extension of the distribution T to the upper half-space. Of course, what we are observing is that the Dirichlet problem on an unbounded domain does not have a unique solution.

We now move on to the product domain case. Before stating the corresponding boundary value result, we remark that the notation $(y) \to (0)^+$ means that $y_j \to 0^+$ for each $j = 1, \ldots, n$.

PROPOSITION 17. *Given $T \in D'_{L^1}$, the S'-convolution $T * \mathcal{P}_{(y)}$ converges to T in D'_{L^1} as $(y) \to (0)^+$.*

Proof. According to Proposition 7, given $T \in D'_{L^1}$, the S'-convolution of T and $\mathcal{P}_{(y)}$ coincides with the regularization

$$(T_x, \mathcal{P}_{(y)}(x-t))_{D'_{L^1}, \dot{B}}.$$

Thus, it suffices to show that $\mathcal{P}_{(y)}$ converges to δ in D'_{L^1} as $(y) \to (0)^+$.

With the same notation as in Proposition 16, we can write

$$\begin{aligned} &|(\mathcal{P}_{(y)}, \varphi)_{D'_{L^1}, \dot{B}} - \varphi(0)| \\ &\quad = \Big| \int_{\mathbb{R}^n} (\varphi(y_1 u_1, \ldots, y_n u_n) - \varphi(0)) P(u_1) \ldots P(u_n)\, du_1 \ldots du_n \Big| \\ &\quad = \Big| \int_{|u|>M} + \int_{|u|<M} \Big|. \end{aligned}$$

The integral $\int_{|u|>M}$ can be estimated in the same way as in Proposition 16. To estimate $\int_{|u|<M}$, it is enough to notice that we can write

$$\begin{aligned} &\varphi(y_1 u_1, \ldots, y_n u_n) - \varphi(0, \ldots, 0) \\ &\quad = \varphi(y_1 u_1, \ldots, y_n u_n) - \varphi(0, y_2 u_2, \ldots, y_n u_n) \\ &\qquad + \varphi(0, y_2 u_2, \ldots, y_n u_n) - \varphi(0, 0, y_3 u_3, \ldots, y_n u_n) \\ &\qquad + \ldots + \varphi(0, \ldots, 0, y_n u_n) - \varphi(0, \ldots, 0) \\ &\quad = \int_0^1 \frac{\partial \varphi}{\partial x_1}(t_1 y_1 u_1, y_2 u_2, \ldots, y_n u_n) y_1 u_1\, dt_1 \\ &\qquad + \int_0^1 \frac{\partial \varphi}{\partial x_2}(0, t_2 y_2 u_2, y_3 u_3, \ldots, y_n u_n) y_2 u_2\, dt_2 \\ &\qquad + \ldots + \int_0^1 \frac{\partial \varphi}{\partial x_n}(0, \ldots, 0, t_n y_n u_n) y_n u_n\, dt_n. \end{aligned}$$

Thus, for M_ε as in Proposition 16 we can write

$$|\varphi(y_1 u_1, \ldots, y_n u_n) - \varphi(0, \ldots, 0)| \le n s_1(\varphi) M_\varepsilon \max_j y_j,$$

and the rest of the proof proceeds in the same way. ∎

Proposition 17 has been obtained by H. Bremermann with a different proof. In fact, the space he denotes by $\mathcal{O}_0$ is our space B_c, and so $\mathcal{O}'_0$ is the space D'_{L^1} of integrable distributions.

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