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Preconditioners and Korovkin-type theorems for infinite-dimensional bounded linear operators via completely positive maps

by

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Abstract. The classical as well as noncommutative Korovkin-type theorems deal with the convergence of positive linear maps with respect to different modes of convergence, like norm or weak operator convergence etc. In this article, new versions of Korovkin-type theorems are proved using the notions of convergence induced by strong, weak and uniform eigenvalue clustering of matrix sequences with growing order. Such modes of convergence were originally considered for the special case of Toeplitz matrices and indeed the Korovkin-type approach, in the setting of preconditioning large linear systems with Toeplitz structure, is well known. Here we extend this finite-dimensional approach to the infinite-dimensional context of operators acting on separable Hilbert spaces. The asymptotics of these preconditioners are evaluated and analyzed using the concept of completely positive maps. It is observed that any two limit points, under Kadison's BW-topology, of the same sequence of preconditioners are equal modulo compact operators. Moreover, this indicates the role of preconditioners in the spectral approximation of bounded self-adjoint operators.

1. Introduction. The classical approximation theorem due to Korovkin [13] unified many approximation processes such as Bernstein polynomial approximation of continuous real functions. This discovery inspired several mathematicians to extend Korovkin's theorem in many ways and to several settings including function spaces, abstract Banach lattices, Banach algebras, Banach spaces, and so on. Such developments are referred to as Korovkin-type approximation theory (see [1], [2] and references therein). Noncommutative versions of Korovkin theorems can be found in many papers (see [2], [14], [15], [16], [21] and references therein, and refer to [17] for new perspectives). In most of these developments, the underlying modes of convergence have been the norm, strong or weak operator convergence of linear operators on a Hilbert space.

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This paper deals with noncommutative Korovkin-type theorems using the modes of convergence induced by strong, weak and uniform eigenvalue clustering of matrix sequences with growing order. Such notions have already been used for the special case of Toeplitz matrices in connection with the Frobenius optimal approximation of matrices of large size. This has been widely considered in the numerical linear algebra literature for the design of efficient solvers of complicated linear systems of large size (see [18], [20]). More specifically, the approximation is constrained on spaces of low complexity: as examples of high interest in several important applications (see [7], [11] and references therein), we may mention algebras of matrices associated with fast transforms like Fourier, trigonometric, Hartley, wavelet transforms ([12], [22]), or spaces with prescribed patterns of sparsity. In the context of general linear systems, accompanied with the minimization in Frobenius norm, these techniques were originally considered and studied by Huckle (see [10] and references therein), while the specific adaptation in the Toeplitz context began with the work of Tony Chan [8].

Recently, a unified structural analysis has been introduced by the third author in connection with Korovkin theory [18]. More precisely, the analysis of clustering of preconditioned systems which gives a measure for the approximation quality is reduced to a classical Korovkin test on a finite number of very elementary symbols associated with equally elementary Toeplitz matrices (Jordan matrices). Here we consider the same approach in an operator theory context. The Korovkin-type approach used in the finite-dimensional case, in the setting of preconditioning large linear systems with Toeplitz structure, is translated into the infinite-dimensional context of operators acting on separable Hilbert spaces.

This paper is structured as follows. In Section 2, basic definitions and theorems including the notion of complete positivity are given. A formulation of the problem and the new notions of convergence are introduced in Section 3, using the ideas from [18]. In Section 4, new versions of noncommutative Korovkin-type theorems are proved. The special cases of Toeplitz operators and Frobenius optimal maps are considered in the next section. We obtain more general and stronger versions of the developments of [18]. Finally, in the last section, we discuss the possible applications of the main results.

2. Preliminaries.

We begin with the classical Korovkin theorem.

THEOREM 2.1. *Let* $\{\Phi_n\}$ *be a sequence of positive linear maps on* $C[0,1]$. *If*

$$\Phi_n(f) \to f \quad \text{for every } f \text{ in the set } \{1, x, x^2\},$$

then

$$\varPhi_n(f) \to f \quad \textit{for every } f \textit{ in } C[0,1].$$

Here the convergence is the uniform convergence of sequences of functions. For the noncommutative versions of this theorem, the notions of *completely positive maps* and *Schwarz maps* are needed.

DEFINITION 2.2. Let $\mathbb{A}$ and $\mathbb{B}$ be C^*-algebras with identities $1_{\mathbb{A}}$ and $1_{\mathbb{B}}$ respectively, and $\varPhi : \mathbb{A} \to \mathbb{B}$ be a positive linear map such that $\varPhi(1_{\mathbb{A}}) \leq 1_{\mathbb{B}}$. For each positive integer n, let $\varPhi_n : M_n(\mathbb{A}) \to M_n(\mathbb{B})$ be defined as $\varPhi_n(a_{i,j}) = (\varPhi(a_{i,j}))$ for every matrix $(a_{i,j}) \in M_n(\mathbb{A})$. If $\varPhi_n$ is positive for each n, then $\varPhi$ is called a *completely positive map* (CP-map).

We now state a fundamental result, the Stinespring dilation theorem [19].

THEOREM 2.3. *Let $\mathbb{A}$ and $\mathbb{B}$ be C^*-algebras with identities $1_{\mathbb{A}}$ and $1_{\mathbb{B}}$ respectively. Let $\varPhi : \mathbb{A} \to \mathbb{B}$ be a completely positive linear map such that $\varPhi(1_{\mathbb{A}}) \leq 1_{\mathbb{B}}$. Assume that $\mathbb{B}$ is a subalgebra of $\mathbb{B}(\mathbb{H})$ for some Hilbert space $\mathbb{H}$. Then there exists a representation π of $\mathbb{A}$ on a Hilbert space $\mathbb{K}$ and a bounded linear map V from $\mathbb{H}$ to $\mathbb{K}$ such that $\varPhi(a) = V^*\pi(a)V$ for every $a \in \mathbb{A}$.*

REMARK. It is known that if either $\mathbb{A}$ or $\mathbb{B}$ is commutative, then every positive linear map is completely positive. Also the composition of two CP-maps is again a CP-map. Now if $\mathrm{CP}(\mathbb{A},\mathbb{B})$ denotes the class of all completely positive maps $\varPhi$ from $\mathbb{A}$ to $\mathbb{B}$ such that $\varPhi(1_{\mathbb{A}}) \leq 1_{\mathbb{B}}$ and $\mathbb{B}$ is a subalgebra of $B(\mathbb{H})$, then it is well known that $\mathrm{CP}(\mathbb{A},\mathbb{B})$ is compact and convex in Kadison's BW-topology [3].

Recall that any positive linear map $\varPhi : \mathbb{A} \to \mathbb{B}$ with $\varPhi(1_{\mathbb{A}}) \leq 1_{\mathbb{B}}$, satisfies the well known inequality of Kadison, namely,

$$\varPhi(a^2) \geq \varPhi(a)^2 \quad \text{for every } a \text{ such that } a = a^*.$$

DEFINITION 2.4. A positive linear map $\varPhi$ from a C^*-algebra $\mathbb{A}$ to a C^*-algebra $\mathbb{B}$ is called a *Schwarz map* if $\varPhi(a^*a) \geq \varPhi(a^*)\varPhi(a)$ for all a in $\mathbb{A}$.

REMARK. It can be easily seen that every completely positive map of norm less than 1 is a Schwarz map. Also, every Schwarz map is clearly positive and contractive. If the C^*-algebra is commutative, then every positive contractive map is a Schwarz map.

REMARK. In the case of an arbitrary C^*-algebra $\mathbb{A}$, a positive linear map $\varPsi$ with $\varPsi(1) \leq 1$ was called a *Jordan–Schwarz* map in [5], since it satisfies the inequality

$$\varPhi(a^* \circ a) \geq \varPhi(a^*) \circ \varPhi(a) \quad \text{for all } a \text{ in } \mathbb{A},$$

where $\circ$ is the Jordan product defined by $a \circ b = \frac{1}{2}(ab + ba)$.

Definition 2.5 ([5]). A $*$-closed and norm-closed subspace of a C^*-algebra $\mathbb{A}$, which is also closed with respect to the Jordan product, is called a *J^*-subalgebra* of $\mathbb{A}$.

Noncommutative versions of the classical Korovkin theorem have been obtained by various researchers for positive maps, Schwarz maps and CP-maps, in the settings of C^*-algebras and W^*-algebras. A short survey of these developments can be found in [17].

The concept of generalized Schwarz map was introduced by Uchiyama [21]. Below, the definition and an important inequality (Theorem 2.1 in [21]) are given, which will play a crucial role in the proof of new versions of Korovkin-type theorems.

Consider a binary operation $\circ$ in a C^*-algebra $\mathbb{A}$, satisfying the following conditions for all $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in \mathbb{A}$:

(1) $(\alpha x + \beta y) \circ z = \alpha(x \circ z) + \beta(y \circ z)$.
(2) $(x \circ y)^* = y^* \circ x^*$.
(3) $x^* \circ x \geq 0$.
(4) There is a real number M such that $\|x \circ y\| \leq M\|x\|\,\|y\|$.
(5) $(x \circ y) \circ z = x \circ (y \circ z)$.
(6) $x \circ y = y \circ x$ and $x \circ x = x^2$ if $x = x^*$.

Remark. Note that $\circ$ is bilinear and that the ordinary product satisfies (5) and the Jordan product satisfies (6). Conversely if $\circ$ satisfies (6), then $\circ$ is the Jordan product.

Definition 2.6. A linear map Φ on a C^*-algebra $\mathbb{A}$ is called a *generalized Schwarz map* with respect to the binary operation $\circ$ if Φ satisfies $\Phi(x^*) = \Phi(x)^*$ and $\Phi(x^*) \circ \Phi(x) \leq \Phi(x^* \circ x)$ for every $x \in \mathbb{A}$.

Remark. Note that a generalized Schwarz map Φ is not necessarily positive. However, under pointwise product in function spaces and with usual product of operators or matrices, all Schwarz maps are positive.

Theorem 2.7 ([21]). *Let Φ be a generalized Schwarz map on a C^*-algebra $\mathbb{A}$ with respect to $\circ$, and for $f, g \in \mathbb{A}$, let*

$$\begin{aligned} X &= \Phi(f^* \circ f) - \Phi(f)^* \circ \Phi(f) \geq 0, \\ Y &= \Phi(g^* \circ g) - \Phi(g)^* \circ \Phi(g) \geq 0, \\ Z &= \Phi(f^* \circ g) - \Phi(f)^* \circ \Phi(g). \end{aligned}$$

Then

$$|\phi(Z)| \leq |\phi(X)|^{1/2} |\phi(Y)|^{1/2} \quad \text{for all states } \phi \text{ on } \mathbb{A}. \tag{2.1}$$

In particular,

$$\tfrac{1}{2}\|Z\| \leq \|X\|^{1/2} \|Y\|^{1/2}. \tag{2.2}$$

REMARK. The above inequality holds for Schwarz maps with respect to the usual product and for contractive positive maps with respect to the Jordan product.

3. Formulation of the problem. The idea of preconditioners is used in the numerical linear algebra literature to replace complicated large linear systems by comparatively "simpler" ones. Here we introduce the notion of preconditioners in the setting of bounded linear operators on separable Hilbert spaces. We construct a sequence of completely positive maps using the notions in [18].

Let $\mathbb{H}$ be a complex separable Hilbert space and let $\{P_n\}$ be a sequence of orthogonal projections on $\mathbb{H}$ such that

$$\dim(P_n(\mathbb{H})) = n < \infty \quad \text{for each } n = 1, 2, \ldots,$$
$$\lim_{n\to\infty} P_n(x) = x \quad \text{for every } x \text{ in } \mathbb{H}.$$

Let $\{U_n\}$ be a sequence of unitary matrices over $\mathbb{C}$, where U_n is of order n for each n. For each $A \in \mathbb{B}(\mathbb{H})$, consider the truncations $A_n = P_n A P_n$, which can be regarded as $n \times n$ matrices in $M_n(\mathbb{C})$, by restricting the domain to the range of P_n. For each n, we define the commutative algebra M_{U_n} of matrices as follows:

$$M_{U_n} = \{A \in M_n(\mathbb{C}) : U_n^* A U_n \text{ complex diagonal}\}.$$

Recall that $M_n(\mathbb{C})$ is a Hilbert space with the Frobenius norm,

$$\|A\|_2^2 = \sum_{j,k=1}^{n} |A_{j,k}|^2$$

induced by the classical Frobenius scalar product,

$$\langle A, B\rangle = \operatorname{trace}(B^* A).$$

Observe that M_{U_n} is a closed convex set in $M_n(\mathbb{C})$ and hence, for each $A \in M_n(\mathbb{C})$, there exists a unique matrix $P_{U_n}(A)$ in M_{U_n} such that

$$\|A - X\|_2^2 \geq \|A - P_{U_n}(A)\|_2^2 \quad \text{for every } X \in M_{U_n}.$$

Now we recall the following two lemmas, which reveal some fundamental properties of $P_{U_n}(\cdot)$ for each n.

LEMMA 3.1 ([18]). *For $A, B \in M_n(\mathbb{C})$ and α, β complex numbers, we have*

$$P_{U_n}(A) = U_n \sigma(U_n^* A U_n) U_n^*, \tag{3.1}$$

where $\sigma(X)$ is the diagonal matrix having X_{ii} as the diagonal elements,

$$P_{U_n}(\alpha A + \beta B) = \alpha P_{U_n}(A) + \beta P_{U_n}(B), \tag{3.2}$$

$$P_{U_n}(A^*) = P_{U_n}(A)^*, \tag{3.3}$$

$$\operatorname{trace}(P_{U_n}(A)) = \operatorname{trace}(A), \tag{3.4}$$

$$\|P_{U_n}(A)\| = 1 \quad \text{(operator norm)}, \tag{3.5}$$

$$\|P_{U_n}(A)\|_F = 1 \quad \text{(Frobenius norm)}, \tag{3.6}$$

$$\|A - P_{U_n}(A)\|_F^2 = \|A\|_F^2 - \|P_{U_n}(A)\|_F^2. \tag{3.7}$$

LEMMA 3.2 ([9]). *If A is a Hermitian matrix, then the eigenvalues of $P_{U_n}(A)$ are contained in the closed interval $[\lambda_1(A), \lambda_n(A)]$, where $\lambda_j(A)$ are the eigenvalues of A arranged in nondecreasing order. Hence if A is positive definite, then $P_{U_n}(A)$ is positive definite as well.*

Now, we introduce a completely positive map on $\mathbb{B}(\mathbb{H})$ as follows.

DEFINITION 3.3. For each $A \in \mathbb{B}(\mathbb{H})$, $\Phi_n : \mathbb{B}(\mathbb{H}) \to M_n(\mathbb{C})$ is defined as

$$\Phi_n(A) = P_{U_n}(A_n),$$

where $P_{U_n}(A_n)$ is as in Lemma 3.1, for each positive integer n.

A straightforward but crucial implication of Lemma 3.1 is the following theorem.

THEOREM 3.4. *The maps Φ_n of Definition 3.3 are completely positive maps on $\mathbb{B}(\mathbb{H})$ such that:*

- $\|\Phi_n\| = 1$ *for each n.*
- Φ_n *is continuous in the strong operator topology for each n.*
- $\Phi_n(I) = I_n$ *for each n, where I is the identity operator on $\mathbb{H}$.*

Proof. From Lemma 3.2, it follows that $P_{U_n}(\cdot)$ is a positive linear map for each n. Since M_{U_n} is a commutative Banach algebra, $P_{U_n}(\cdot)$ is a completely positive map for each n. Hence Φ_n is a completely positive map, since it is the composition of CP-maps ($P_{U_n}(\cdot)$ and the map which sends A to P_nAP_n). Now, continuity in the strong operator topology follows easily from the definition. Moreover

$$\|\Phi_n\| = \sup_{\|A\|=1,\, A\in\mathbb{B}(\mathbb{H})} \|\Phi_n(A)\| = \sup_{\|A\|=1,\, A\in B(H)} \|P_{U_n}(A_n)\| = 1$$

by (3.5). The last item of the theorem follows easily from (3.1). ∎

3.1. Modified preconditioners. It is interesting to observe that the notion of preconditioners can be modified by replacing ‘diagonal transformation’ by ‘block diagonal transformation’. This can be done with the use of a ‘pinching function’, introduced in [6]. The details are as follows.

Let $\tilde{M}_{U_n} = \{A \in M_n(\mathbb{C}) : U_n^* A U_n$ is block diagonal$\}$, where the block diagonal is obtained for each A in $M_n(\mathbb{C})$ by applying the pinching function to A for each n (see [6] for the definition). To be more precise, let P_{n_k}, $k = 1, \dots, m_n$, be pairwise orthogonal orthogonal projections in $M_n(\mathbb{C})$ such that $\sum_{k=1}^{m_n} P_{n_k} = I_n$, the identity matrix. The modified preconditioner on $M_n(\mathbb{C})$ takes values

$$\Psi_n(A) = \sum_{k=1}^{m_n} P_{n_k} A P_{n_k} \quad \text{for every } A \in M_n(\mathbb{C}). \tag{3.8}$$

From Stinespring's theorem, it is clear that each Ψ_n is a CP-map. Now, if we define $P_{U_n}(A)$ in a similar way with M_{U_n} replaced by $\tilde{M}_{U_n}$, we can formulate an analogue of Lemma 3.1.

Lemma 3.5. *For $A,B \in M_n(\mathbb{C})$ and $\alpha, \beta \in \mathbb{C}$, we have*

$$\begin{gathered}
P_{U_n}(A) = U_n \Psi_n(U_n^* A U_n) U_n^*, \ \textit{where } \Psi_n \textit{ is as in } (3.8),\\
P_{U_n}(\alpha A + \beta B) = \alpha P_{U_n}(A) + \beta P_{U_n}(B),\\
P_{U_n}(A^*) = P_{U_n}(A)^*,\\
\operatorname{trace}(P_{U_n}(A)) = \operatorname{trace}(A),\\
\|P_{U_n}(A)\| = 1 \quad (\textit{operator norm}),\\
\|P_{U_n}(A)\|_F = 1 \quad (\textit{Frobenius norm}),\\
\|A - P_{U_n}(A)\|_F^2 = \|A\|_F^2 - \|P_{U_n}(A)\|_F^2.
\end{gathered}$$

We list some properties of the maps $\{\Psi_n\}$ as we did in Theorem 3.4.

Theorem 3.6. *The maps Ψ_n are completely positive maps on $\mathbb{B}(\mathbb{H})$ such that:*

- $\|\Psi_n\| = 1$ *for each n.*
- Ψ_n *is continuous in the strong operator topology.*
- $\Psi_n(I) = I_n$ *for each n, where I is the identity operator on $\mathbb{H}$.*

We may construct examples of modified preconditioners.

Example 3.7. Let U_n be unitaries in $M_n(\mathbb{C})$ as in Definition 3.8. For each positive integer n, let $\tilde{U}_n$ be unitaries in $\mathbb{B}(\mathbb{H})$ defined as $U_n \oplus (I - P_n)$. Observe that there are many interesting, concrete examples of unitaries U_n in [18]. For the sake of completeness, we recall some of them below.

Let $v = \{v_n\}_{n\in\mathbb{N}}$ with $v_n = (v_{nj})_{j\le n-1}$ be a sequence of trigonometric functions on an interval I. Let $S = \{S_n\}_{n\in\mathbb{N}}$ be a sequence of grids of n points on I, say $S_n = \{x_i^n : i = 0, 1, \dots, n-1\}$. Suppose that the generalized Vandermonde matrix

$$V_n = (v_{nj}(x_i^n))_{i,j=0}^{n-1}$$

is a unitary matrix. Then the algebra of the form M_{U_n} is a trigonometric algebra if $U_n = V_n^*$ with V_n a generalized trigonometric Vandermonde matrix.

We get examples of trigonometric algebras with the following choice of matrices U_n and grid S_n:

$$U_n = F_n = \left(\frac{1}{\sqrt{n}} e^{ijx_i^n}\right), \qquad i,j = 0,1,\dots,n-1,$$

$$S_n = \left\{x_i^n = \frac{2i\pi}{n} : i = 0,1,\dots,n-1\right\} \subset I = [-\pi,\pi];$$

$$U_n = G_n = \left(\sqrt{\frac{2}{n+1}} \sin(j+1)x_i^n\right), \qquad i,j = 0,1,\dots,n-1,$$

$$S_n = \left\{x_i^n = \frac{(i+1)\pi}{n+1} : i = 0,1,\dots,n-1\right\} \subset I = [0,\pi];$$

$$U_n = H_n = \left(\frac{1}{\sqrt{n}}[\sin(jx_i^n) + \cos(jx_i^n)]\right), \qquad i,j = 0,1,\dots,n-1,$$

$$S_n = \left\{x_i^n = \frac{2i\pi}{n} : i = 0,1,\dots,n-1\right\} \subset I = [-\pi,\pi].$$

3.2. Convergence of positive linear maps. We introduce different notions of convergence of sequences of positive linear maps in $B(\mathbb{H})$ in a *distributional* sense. We recall from [18] the definitions of different notions of convergence for preconditioners. To avoid confusion with the classical notions of strong, weak and operator norm convergence, we speak about *strong cluster convergence*, *weak cluster convergence*, and *uniform cluster convergence* to mean the strong, weak and uniform convergence respectively used in [18].

DEFINITION 3.8. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n\times n$ Hermitian matrices. We say that $A_n - B_n$ converges to 0 in the *strong cluster sense* if for any $\epsilon > 0$, there exist integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that all the singular values $\sigma_j(A_n - B_n)$ lie in the interval $[0,\epsilon)$ except for at most $N_{1,\epsilon}$ (independent of the size n) singular values, for all $n > N_{2,\epsilon}$.

If the number $N_{1,\epsilon}$ does not depend on ϵ, we say that $A_n - B_n$ converges to 0 in the *uniform cluster sense.* And if $N_{1,\epsilon}$ depends on ϵ, n and is $o(n)$, we say that $A_n - B_n$ converges to 0 in the *weak cluster sense.*

The following powerful lemma is due to Tyrtyshnikov [20, Lemma 3.1].

LEMMA 3.9. *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ Hermitian matrices. If $\|A_n - B_n\|_F^2 = O(1)$, then we have strong cluster convergence. If $\|A_n - B_n\|_F^2 = o(n)$, then the convergence is in the weak cluster sense.*

Using the above notions, we introduce the new notions of convergence of positive linear maps on $B(\mathbb{H})$.

DEFINITION 3.10. Let $\{\Phi_n\}$ be a sequence of positive linear maps on $B(\mathbb{H})$ and P_n be a sequence of projections on $\mathbb{H}$ with rank n that converges strongly to the identity. For a bounded self-adjoint operator A on $\mathbb{H}$, we say that $\{\Phi_n(A)\}$ converges to A in the *strong distribution sense* if the sequence of matrices $\{P_n\Phi_n(A)P_n - P_nAP_n\}$ converges to 0 in the strong cluster sense.

Similarly we say that $\{\Phi_n(A)\}$ converges to A in the *weak distribution sense* (*uniform distribution sense* respectively) if the sequence of matrices $\{P_n\Phi_n(A)P_n - P_nAP_n\}$ converges to 0 in the weak cluster sense (uniform cluster sense respectively).

Remark. The above definitions depend on the choice of P_n's, and make sense for non-self-adjoint operators/matrices also. However, for technical reasons we restrict to self-adjoint operators/matrices.

Remark. In the case of nets, the definitions are the same with convergence in terms of directed sets.

4. Korovkin-type theorems. In this section, we will prove Korovkin-type theorems for sequences of positive linear maps on $B(\mathbb{H})$ with modes of convergence in the distribution sense as introduced in the last section. The particular example of preconditioners is considered in the next section.

Consider a sequence $\{\Phi_n\}$ of CP-maps in $B(\mathbb{H})$ with $\|\Phi_n\| \le 1$. By the compactness of $\mathrm{CP}(\mathbb{B}(\mathbb{H}))$ in Kadison's BW-topology, $\{\Phi_n\}$ has limit points. Let Ω be the set of all limit points of $\{\Phi_n\}$. Next, we discuss some properties of the limit points Φ in Ω. The relation between $\Phi(A)$ and A for $A \in \mathbb{B}(\mathbb{H})$ is analyzed.

Lemma 4.1. *Let $\Phi \in \Omega$ and let $\{\Phi_{n_\alpha}\}$ be a subnet of $\{\Phi_n\}$ such that Φ_{n_α} converges to Φ in Kadison's BW-topology. Then for each m, the truncations $\Phi_{m,n_\alpha}(A) = P_m\Phi_{n_\alpha}(A)P_m$ converge uniformly in norm to $P_m\Phi(A)P_m$. That is,* $\lim_\alpha \|\Phi_{m,n_\alpha}(A) - P_m\Phi(A)P_m\| = 0$.

Proof. This follows immediately since P_m is of finite rank and therefore, on range(P_m), the weak, strong and operator norm topologies coincide. ∎

Remark. For each $A \in \mathbb{B}(\mathbb{H})$, note that $A_{n_\alpha} - \Phi_{n_\alpha}(A)$ converges in the strong operator topology to $A - \Phi(A)$. Hence,

$$P_mA_{n_\alpha}P_m - P_m\Phi_{n_\alpha}(A)P_m \underset{\alpha}{\to} P_mAP_m - P_m\Phi(A)P_m$$

in the norm topology for each m.

The above observations can be used to deduce the following result.

Theorem 4.2. *Let $A \in \mathbb{B}(\mathbb{H})$ be self-adjoint and suppose $\Phi_n(A)$ converges to A in the uniform distribution sense of Definition* 3.10. *Then $A - \Phi(A)$ is of finite rank.*

Proof. By assumption $P_{n_\alpha}\Phi_{n_\alpha}(A)P_{n_\alpha} - A_{n_\alpha}$ converges to 0 in the uniform cluster sense of Definition 3.8. Hence for each $\epsilon > 0$, there exist a β_ϵ in the directed set and N such that

$$\#\big(\sigma(A_{n_\alpha} - P_{n_\alpha}\Phi_{n_\alpha}(A)P_{n_\alpha}) \cap \mathbb{R} - (-\epsilon,\epsilon)\big) \leq N \quad \text{whenever } \alpha > \beta_\epsilon.$$

Therefore by the Cauchy interlacing theorem,

$$\#\big(\sigma(P_m(A_{n_\alpha} - P_{n_\alpha}\Phi_{n_\alpha}(A)P_{n_\alpha})P_m) \cap \mathbb{R} - (-\epsilon,\epsilon)\big) \leq N \quad \text{if } \alpha > \beta_\epsilon \text{ and } n_\alpha \geq m.$$

Now we take the limit over α; since we know by the above remark that $P_m(A_{n_\alpha} - P_{U_{n_\alpha}}(A_{n_\alpha}))P_m$ converges to $P_m(A - \Phi(A))P_m$ in the operator norm topology for every m, we obtain

$$\#\big(\sigma(P_m(A - \Phi(A))P_m) \cap \mathbb{R} - (-\epsilon,\epsilon)\big) \leq N \quad \text{for every } m.$$

Therefore $\mathbb{R} - (-\epsilon,\epsilon)$ contains no essential points of $A - \Phi(A)$ and hence by Arveson's Theorem [4, Theorem 2.3], it contains no essential spectral values of $A - \Phi(A)$. That is, the essential spectrum $\sigma_e(A - \Phi(A))$ is contained in the interval $(-\epsilon,\epsilon)$ for all $\epsilon > 0$. This implies that $\sigma_e(A - \Phi(A)) = \{0\}$. Hence $A - \Phi(A)$ is compact and it has at most N eigenvalues. Hence it is of finite rank by the spectral theorem. ■

The following corollary is an easy consequence of the above theorem.

COROLLARY 4.3. *Under the assumptions that $A \in \mathbb{B}(\mathbb{H})$ is self-adjoint and $\Phi_n(A)$ converges to A in the uniform distribution sense of Definition* 3.10, *the following results hold:*

- *A is compact if and only if $\Phi(A)$ is compact.*
- *A is Fredholm if and only if $\Phi(A)$ is Fredholm.*
- *A is Hilbert–Schimidt if and only if $\Phi(A)$ is Hilbert–Schimidt.*
- *A is of finite rank if and only if $\Phi(A)$ is of finite rank.*
- *A has a gap in the essential spectrum $\sigma_{\text{ess}}(A)$ of A if and only if $\sigma_{\text{ess}}(\Phi(A))$ has a gap.*

In the next theorem, we observe that if the mode of convergence is strong, then the change to preconditioners amounts to a compact perturbation.

THEOREM 4.4. *Let $A \in \mathbb{B}(\mathbb{H})$ be self-adjoint and suppose $\Phi_n(A)$ converges to A in the strong distribution sense of Definition* 3.10. *Then $A - \Phi(A)$ is compact.*

Proof. The argument is much the same as in the proof of Theorem 4.2, except that here the number of eigenvalues of $(A_{n_\alpha} - P_{n_\alpha}\Phi_{n_\alpha}(A)P_{n_\alpha})$ outside $(-\epsilon,\epsilon)$ is not bounded by a constant, but by a number $N_{1,\epsilon}$, which depends on ϵ. Hence we can conclude that $A - \Phi(A)$ is compact, and can have countably many eigenvalues. ■

The following results are easy consequences of the above theorem:

Corollary 4.5. *Assume that $A \in \mathbb{B}(\mathbb{H})$ is self-adjoint and $\Phi_n(A)$ converges to A in the strong distribution sense of Definition* 3.10. *Then*

- *A is compact if and only if $\Phi(A)$ is compact.*
- *A is Fredholm if and only if $\Phi(A)$ is Fredholm.*
- *A has a gap in the essential spectrum $\sigma_{\mathrm{ess}}(A)$ of A if and only if $\sigma_{\mathrm{ess}}(\Phi(A))$ has a gap.*

Now we prove the noncommutative analogue of the remainder estimate in the classical Korovkin-type theorems, as proved in [18].

Lemma 4.6. *Let $\{A_1, \dots, A_m\}$ be a finite set of operators in $\mathbb{B}(\mathbb{H})$ and Φ_n be a sequence of positive linear Schwarz maps on $\mathbb{B}(\mathbb{H})$ such that, for every n, $\|\Phi_n\| \le 1$ and $\|\Phi_n(A) - A\| = O(\theta_n)$ for every A in the set $D = \{A_1, \dots, A_m, \sum_{k=1}^m A_k A_k^*\}$, where $\theta_n \to 0$ as $n \to \infty$. Then $\|\Phi_n(A) - A\| = O(\theta_n)$ for every A in the algebra generated by $\{A_1, \dots, A_m\}$.*

Proof. By linearity, we have

$$\Phi_n\Big(\sum_{k=1}^m A_k A_k^*\Big) = \sum_{k=1}^m \Phi_n(A_k A_k^*).$$

Also by adding and subtracting the term $\sum_{k=1}^m \Phi_n(A_k)\Phi_n(A_k)^*$,

$$\begin{aligned}\Phi_n\Big(\sum_{k=1}^m A_k A_k^*\Big) - \sum_{k=1}^m A_k A_k^* &= \Big[\Phi_n\Big(\sum_{k=1}^m A_k A_k^*\Big) - \sum_{k=1}^m \Phi_n(A_k)\Phi_n(A_k)^*\Big] \\ &\quad + \Big[\sum_{k=1}^m \Phi_n(A_k)\Phi_n(A_k)^* - \sum_{k=1}^m A_k A_k^*\Big].\end{aligned}$$

The norm of the left side above as well as of the last term of the right side are $O(\theta_n)$. The first term of the right side is

$$\sum_{k=1}^m [\Phi_n(A_k A_k^*) - \Phi_n(A_k)\Phi_n(A_k)^*].$$

Hence its norm is $O(\theta_n)$. But each summand $\Phi_n(A_k A_k^*) - \Phi_n(A_k)\Phi_n(A_k)^*$ is a nonnegative operator by the Schwarz inequality for positive linear maps. Therefore the norm of each summand is $O(\theta_n)$. Also since each Φ_n is a Schwarz map, by applying inequality (2.2) to the maps Φ_n for each n and the operators A_k and A_l, we get

$$\|\Phi_n(A_k^* A_l) - \Phi_n(A_k)^*\Phi_n(A_l)\| = O(\theta_n). \tag{4.1}$$

Also, we can estimate $\|{A_k}^* A_l - \Phi_n(A_k)^* \Phi_n(A_l)\|$ as follows:

$$\begin{aligned}
&\|A_k^* A_l - \Phi_n(A_k)^* \Phi_n(A_l)\| \\
&\qquad = \|(A_k - \Phi_n(A_k) + \Phi_n(A_k))^* A_l - \Phi_n(A_k)^* (\Phi_n(A_l) - A_l + A_l)\| \\
&\qquad \le \|(A_k - \Phi_n(A_k))^* A_l\| + \|\Phi_n(A_k)^* (\Phi_n(A_l) - A_l)\|.
\end{aligned}$$

Now each of the terms on the right side is $O(\theta_n)$, by the assumption on A_k, A_l and since $\|\Phi_n\| \le 1$. Therefore

$$\|A_k^* A_l - \Phi_n(A_k)^* \Phi_n(A_l)\| = O(\theta_n). \tag{4.2}$$

Also we have

$$\begin{aligned}
&\|\Phi_n(A_k^* A_l) - A_k^* A_l\| \\
&\qquad = \|\Phi_n(A_k^* A_l) - \Phi_n(A_k)^* \Phi_n(A_l) + \Phi_n(A_k)^* \Phi_n(A_l) - A_k^* A_l\|.
\end{aligned}$$

Applying (4.1) and (4.2) in the above identity, we get

$$\|\Phi_n(A_k^* A_l) - A_k^* A_l\| = O(\theta_n).$$

Thus the assertion is proved for every operator of the form $A_k^* A_l$ and hence in the algebra generated by $\{A_1, \ldots, A_m\}$. ∎

Before proving more general Korovkin-type theorems, we prove the following lemma.

Lemma 4.7. *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ Hermitian matrices such that $\{A_n - B_n\}$ converges to 0 in the strong (weak respectively) cluster sense. Assume that $\{B_n\}$ is positive definite and invertible such that there exists a $\delta > 0$ with*

$$B_n \ge \delta I_n > 0 \quad \text{for all } n.$$

Then for a given $\epsilon > 0$, there exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that all eigenvalues of $B_n^{-1} A_n$ lie in the interval $(1 - \epsilon, 1 + \epsilon)$ except possibly for $N_{1,\epsilon} = O(1)$ ($N_{1,\epsilon} = o(n)$ respectively) eigenvalues for every $n > N_{2,\epsilon}$.

Proof. First we observe that, since $\{A_n - B_n\}$ converges to 0 in the strong (weak respectively) cluster sense, by definition, for any given $\epsilon > 0$, there exist integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that all eigenvalues of $A_n - B_n$ lie in $(-\epsilon, \epsilon)$ except for at most $N_{1,\epsilon} = O(1)$ ($N_{1,\epsilon} = o(n)$ respectively) eigenvalues whenever $n \ge N_{2,\epsilon}$. Hence by the spectral theorem there exist orthogonal projections P_n and Q_n whose ranges are orthogonal such that

$$\operatorname{rank}(P_n) + \operatorname{rank}(Q_n) = n, \quad \operatorname{rank}(Q_n) \le N_{1,\epsilon}, \quad \|P_n(A_n - B_n)P_n\| < \varepsilon,$$

and

$$A_n - B_n = P_n(A_n - B_n)P_n + Q_n(A_n - B_n)Q_n.$$

Hence for $\epsilon_1 = \epsilon\delta > 0$, there exist natural numbers $N_{1,\epsilon}, N_{2,\epsilon}$ such that there exists a decomposition

$$A_n - B_n = R_n + N_n \quad \text{for all } n \geq N_{2,\epsilon}, \tag{4.3}$$

where the rank of R_n is bounded above by $N_{1,\epsilon}$ and $\|N_n\| \leq \epsilon_1$. Now let β be an eigenvalue of $B_n^{-1}A_n$ with x being the associated eigenvector of norm one, so

$$B_n^{-1}A_n(x) = \beta x.$$

Hence

$$(A_n - B_n)(x) = (\beta - 1)B_n(x),$$

which implies that

$$\langle (A_n - B_n)(x), x\rangle = (\beta - 1)\langle B_n(x), x\rangle,$$

so

$$\beta - 1 = \frac{\langle (A_n - B_n)(x), x\rangle}{\langle B_n(x), x\rangle}.$$

Now from the decomposition (4.3), we have

$$\beta - 1 = \frac{\langle (R_n + N_n)(x), x\rangle}{\langle B_n(x), x\rangle} = \frac{\langle R_n(x), x\rangle}{\langle B_n(x), x\rangle} + \frac{\langle N_n(x), x\rangle}{\langle B_n(x), x\rangle}.$$

Since $\|N_n\| \leq \epsilon_1$ and $B_n \geq \delta I_n > 0$, the second term in the last sum is less than $\epsilon_1/\delta = \epsilon$. Also since the rank of R_n is bounded above by $N_{1,\epsilon} = O(1)$ ($o(n)$ respectively), there are only at most $N_{1,\epsilon}$ linearly independent vectors x for which $R_n(x) \neq 0$, by the rank-nullity theorem. Hence, except for at most $N_{1,\epsilon} = O(1)$ ($o(n)$ respectively) eigenvalues,

$$|\beta - 1| \leq \epsilon.$$

This means that all eigenvalues of $B_n^{-1}A_n$ lie in $(1-\epsilon, 1+\epsilon)$ except possibly for $N_{1,\epsilon} = O(1)$ ($o(n)$ respectively) eigenvalues. ∎

Now we prove our main result of this section, a noncommutative Korovkin-type theorem. Here $\circ$ denotes the Jordan product of operators or matrices.

THEOREM 4.8. *Let $\{A_1, \ldots, A_m\}$ be a finite set of self-adjoint operators on $\mathbb{H}$, and Φ_n be a sequence of contractive positive maps on $B(\mathbb{H})$ such that $\Phi_n(A)$ converges to A in the strong (weak respectively) distribution sense for A in $\{A_1, \ldots, A_m, A_1^2, \ldots, A_m^2\}$. In addition, assume that the difference $P_n(A_k^2)P_n - (P_n(A_k)P_n)^2$ converges to the 0 matrix in the strong (weak respectively) cluster sense, for each k. Then $\Phi_n(A)$ converges to A in the strong (weak respectively) distribution sense for all A in the J^*-subalgebra $\mathbb{A}$ generated by $\{A_1, \ldots, A_m\}$.*

Proof. First we consider the following sequences of Hermitian matrices:

$$X_n = P_n\Phi_n(A_k^2)P_n - (P_n\Phi_n(A_k)P_n)^2 \geq 0,$$
$$Y_n = P_n\Phi_n(A_l^2)P_n - (P_n\Phi_n(A_l)P_n)^2 \geq 0,$$
$$Z_n = P_n\Phi_n(A_k \circ A_l)P_n - (P_n\Phi_n(A_k)P_n) \circ (P_n\Phi_n(A_l)P_n).$$

These sequences are norm bounded, in particular we have

$$\|Y_n\| < \gamma < \infty \quad \text{for all } n, \text{ for some } \gamma > 0. \tag{4.4}$$

Also if we write

$$\begin{aligned} X_n &= P_n\Phi_n(A_k^2)P_n - (P_n\Phi_n(A_k)P_n)^2 \\ &= [P_n\Phi_n(A_k^2)P_n - P_n(A_k^2)P_n] + [P_n(A_k^2)P_n - (P_n(A_k)P_n)^2] \\ &\quad + [(P_n(A_k)P_n)^2 - (P_n\Phi_n(A_k)P_n)^2], \end{aligned}$$

the first two terms on the right converge to 0 in the strong (weak respectively) cluster sense by assumption. Also since $P_n\Phi_n(A_k)P_n - P_n(A_k)P_n = R_n + N_n$, where R_n and N_n are as in the proof of Lemma 4.7, we have

$$P_n\Phi_n(A_k)P_n - P_n(A_k)P_n = R_n + N_n,$$

so

$$(P_n\Phi_n(A_k)P_n - R_n)^2 = (P_n(A_k)P_n + N_n)^2.$$

From the above identity, we can deduce that $(P_n(A_k)P_n)^2 - (P_n\Phi_n(A_k)P_n)^2 = R_n' + N_n'$, where R_n' has bounded rank and N_n' has small norm as required for the convergence in the strong (weak respectively) cluster sense to 0. Hence the third term also converges to the 0 matrix in the strong (weak respectively) cluster sense. Therefore X_n converges to the 0 matrix in the strong (weak respectively) cluster sense.

Now for each fixed x with $\|x\| = 1$, if we consider the state ϕ_x on $B(\mathbb{H})$ defined as

$$\phi_x(A) = \langle A(x), x\rangle,$$

then by (2.1) applied to the contractive positive maps $P_n\Phi_n(\cdot)P_n$, we get

$$|\langle Z_n(x), x\rangle| \leq |\langle X_n(x), x\rangle|^{1/2} |\langle Y_n(x), x\rangle|^{1/2}. \tag{4.5}$$

Now let $\delta > 0$ be given and $\epsilon = \delta^2/\gamma$. As in the proof of Lemma 4.7, there exist integers $N_{1,\epsilon} = O(1)$ ($o(n)$ respectively) and $N_{2,\epsilon}$ such that

$$X_n = N_n + R_n \quad \text{for all } n > N_{2,\epsilon},$$

where $\|N_n\| < \epsilon$ and the rank of R_n is less than $N_{1,\epsilon} = O(1)$ ($o(n)$ respectively). Applying this and (4.4) in (4.5), we get

$$|\langle Z_n(x), x\rangle| \leq \sqrt{\gamma}\,[|\langle N_n(x), x\rangle|^{1/2} + |\langle R_n(x), x\rangle|^{1/2}] \quad \text{for all } n > N_{2,\epsilon}.$$

Since the rank of R_n is bounded above by $N_{1,\epsilon} = O(1)$ ($o(n)$ respectively), there are only at most $N_{1,\epsilon}$ linearly independent vectors x for which

$R_n(x)\neq 0$, by the rank-nullity theorem. Hence $|\langle Z_n(x), x\rangle| \leq \delta$ except for at most $N_{1,\epsilon} = O(1)$ ($o(n)$ respectively) linearly independent vectors x. Therefore all eigenvalues of Z_n, except for possibly $N_{1,\epsilon} = O(1)$ ($o(n)$ respectively), lie in the interval $(-\delta, \delta)$, whenever $n > N_{2,\epsilon}$. Since $\delta > 0$ was arbitrary, Z_n converges to the 0 matrix in the strong (weak respectively) cluster sense.

Now consider

$$\begin{aligned}
&P_n\Phi_n(A_k \circ A_l)P_n - P_n(A_k \circ A_l)P_n \\
&\quad = [P_n\Phi_n(A_k \circ A_l)P_n - (P_n\Phi_n(A_k)P_n) \circ (P_n\Phi_n(A_l)P_n)] \\
&\qquad + [(P_n\Phi_n(A_k)P_n) \circ (P_n\Phi_n(A_l)P_n) - (P_n(A_k)P_n) \circ (P_n(A_l)P_n)] \\
&\qquad + [(P_n(A_k)P_n) \circ (P_n(A_l)P_n) - P_n(A_k \circ A_l)P_n].
\end{aligned}$$

The first term on the right hand side is Z_n and the last term is also in the form of Z_n for the positive contractive maps $P_n(\cdot)P_n$ on $B(\mathbb{H})$. Therefore both these terms converge to the 0 matrix in the strong (weak respectively) cluster sense. By simple computation, the same can be proved for the middle term. Hence the conclusion is proved for operators of the form $A_k \circ A_l$.

The same proof can be repeated for operators of the form $A_j \circ (A_k \circ A_l)$, using the boundedness of $A_k \circ A_l$ and the convergence assumption on A_j in the strong (weak respectively) cluster sense. Continuing like this inductively, we see that the assertion is true for any operator which is a polynomial in $\{A_1, \ldots, A_m\}$ with respect to the Jordan product.

Now for $A \in \mathbb{A}$ and $\epsilon > 0$, let T be a polynomial in $\{A_1, \ldots, A_m\}$, with respect to the Jordan product, such that

$$\|A - T\| < \epsilon/3, \qquad \|\Phi_n(A) - \Phi_n(T)\| < \epsilon/3.$$

Write

$$\begin{aligned}
P_n\Phi_n(A)P_n - P_nAP_n = [P_n\Phi_n(A)P_n - P_n\Phi_n(T)P_n] + [P_n\Phi_n(T)P_n - P_nTP_n] \\
+ [P_nTP_n - P_nAP_n].
\end{aligned}$$

The norm of the sum of the first and third terms is less than $2\epsilon/3$. The middle term $P_n\Phi_n(T)P_n - P_nTP_n$ can be split into a term with norm less than $\epsilon/3$ and a term with constant rank independent of n (or of order $o(n)$ respectively) since T is a polynomial in $\{A_1, \ldots, A_m\}$, with respect to the Jordan product. Thus the sequence of matrices $P_n\Phi_n(A)P_n - P_nAP_n$ converges to 0 in the strong (weak respectively) cluster sense. ∎

REMARK. Note that even if A_k and A_l are self-adjoint, their composition need not be self-adjoint. But the Jordan product of two self-adjoint elements is self-adjoint. The proof of the above theorem uses this fact.

5. Korovkin-type theory for Toeplitz operators. In this section, we consider the case of the Toeplitz operator $A = A(f)$, where $f \in C[0, 2\pi]$ and $\mathbb{H} = L^2[0, 2\pi]$. We get stronger versions of some of the results in [18]. First we recall the Korovkin-type results in [18]. The notation $A_n(f)$ is used to denote the finite Toeplitz matrix with symbol f.

THEOREM 5.1 ([18]). *Let f be a continuous periodic real-valued function. Then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in the strong cluster sense if $P_{U_n}(A_n(p)) - A_n(p)$ converges to 0 in the strong cluster sense for all trigonometric polynomials p.*

THEOREM 5.2 ([18]). *Let f be a continuous periodic real-valued function. Then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in the weak cluster sense if $P_{U_n}(A_n(p)) - A_n(p)$ converges to 0 for all trigonometric polynomials p.*

Before proving the general results, we prove the following lemma, the remainder estimate version of the classical Korovkin theorem as proved in [18], which is used to obtain more general versions of Theorems 5.1 and 5.2. This lemma is the commutative version of Lemma 4.6.

LEMMA 5.3. *Let $\{g_1, \ldots, g_m\}$ be a finite set of continuous periodic functions and Φ_n be a sequence of positive linear maps on $C[0, 2\pi]$ such that, for every n, $\|\Phi_n\| \le 1$ and*

$$\Phi_n(g) = g + O(\theta_n) \quad \text{for every } g \in D = \Big\{g_1, \ldots, g_m, \sum_{k=1}^{m} g_k g_k^*\Big\},$$

where $\theta_n \to 0$ as $n \to \infty$. Then $\Phi_n(g) = g + O(\theta_n)$ for every g in the algebra generated by $\{g_1, \ldots, g_m\}$.

Proof. The proof is obtained by replacing functions by operators, in the proof of Lemma 4.6. Using linearity of Φ_n's, we write

$$\begin{aligned}\Phi_n\Big(\sum_{k=1}^{m} g_k g_k^*\Big) - \sum_{k=1}^{m} g_k g_k^* &= \Big(\sum_{k=1}^{m} \Phi_n(g_k g_k^*) - \sum_{k=1}^{m} \Phi_n(g_k)\Phi_n(g_k)^*\Big) \\ &\quad + \Big(\sum_{k=1}^{m} \Phi_n(g_k)\Phi_n(g_k)^* - \sum_{k=1}^{m} g_k g_k^*\Big).\end{aligned}$$

The left side above as well as the last term of the right side are $O(\theta_n)$. Hence the first term of the right side,

$$\sum_{k=1}^{n} [\Phi_n(g_k g_k^*) - \Phi_n(g_k)\Phi_n(g_k)^*],$$

is $O(\theta_n)$. But each summand $\Phi_n(g_k g_k^*) - \Phi_n(g_k)\Phi_n(g_k)^*$ is nonnegative by the Schwarz inequality for positive linear maps. Therefore each summand is $O(\theta_n)$. Also since every positive contractive map in a commutative C^*-algebra is a Schwarz map, each Φ_n is a Schwarz map. Therefore by applying inequality (2.2) to the maps Φ_n for each n and the functions g_k, g_l, we get

$$\Phi_n(g_k^* g_l) - \Phi_n(g_k)^* \Phi_n(g_l) = O(\theta_n).$$

Also, we observe that

$$\Phi_n(g_k)^* \Phi_n(g_l) - g_k^* g_l = (g_k^* + O(\theta_n))(g_l + O(\theta_n)) - g_k^* g_l = O(\theta_n).$$

Using the above two identities, we deduce that

$$\Phi_n(g_k^* g_l) - g_k^* g_l = [\Phi_n(g_k^* g_l) - \Phi_n(g_k)^* \Phi_n(g_l)] + [\Phi_n(g_k)^* \Phi_n(g_l) - g_k^* g_l] = O(\theta_n).$$

Therefore the assertion is proved for every function of the form $g_k^* g_l$ and hence in the algebra generated by $\{g_1, \ldots, g_m\}$. ∎

Now we prove some general versions of Theorems 5.1 and 5.2. The technique of the proof is the same as in Theorem 4.8. Still we provide all the details.

THEOREM 5.4. *Let $\{g_1, \ldots, g_m\}$ be a finite set of real-valued continuous 2π-periodic functions such that $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in the strong cluster sense for every f in $\{g_1, \ldots, g_m, g_1^2, \ldots, g_m^2\}$. Then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in the strong cluster sense for all f in the C^*-algebra $\mathbb{A}$ generated by $\{g_1, \ldots, g_m\}$.*

Proof. For any $k, l = 1, \ldots, m$, set.

$$\begin{aligned} X_n &= P_{U_n}(A_n(g_k^2)) - P_{U_n}(A_n(g_k))^2 \geq 0, \\ Y_n &= P_{U_n}(A_n(g_l^2)) - P_{U_n}(A_n(g_l))^2 \geq 0, \\ Z_n &= P_{U_n}(A_n(g_k^* \circ g_l)) - P_{U_n}(A_n(g_k))^* \circ P_{U_n}(A_n(g_l)). \end{aligned}$$

(Here $\circ$ denotes the usual pointwise product in the case of scalar-valued functions and matrix product in the case of matrices.) Observe that X_n, Y_n and Z_n are all Hermitian matrices of order n. It is clear that all the above sequences of matrices are norm bounded; in particular, for all n,

$$\|Y_n\| < \gamma < \infty. \tag{5.1}$$

Also if we write

$$\begin{aligned} X_n = \Phi_n(g_k^2) - \Phi_n(g_k)^2 &= [\Phi_n(g_k^2) - A_n(g_k^2)] + [A_n(g_k^2) - A_n(g_k)^2] \\ &\quad + [A_n(g_k)^2 - \Phi_n(g_k)^2], \end{aligned}$$

the first term on the right hand side converges to 0 in the strong cluster sense by assumption. The second term is

$$A_n(g_k^2) - A_n(g_k)^2 = P_n H(g_k)^2 P_n + Q_n H(g_k)^2 Q_n, \tag{5.2}$$

where Q_n's are projections and $H(g_k)$ is the Hankel operator, which is compact, since the symbols are continuous. This equality is due to Widom [23, p. 2]. Hence $A_n(g_k^2) - A_n(g_k)^2$ can be written as the sum of sequences of matrices that are truncations of compact operators. But the complement of any neighborhood of 0 contains only finitely many eigenvalues of a compact operator. Also the truncations of a compact operator on a separable Hilbert space converge to the operator in norm. Therefore we conclude that $A_n(g_k^2) - A_n(g_k)^2$ converges to the 0 matrix in the strong cluster sense.

Since $\Phi_n(g_k) - A_n(g_k) = R_n + N_n$, where R_n and N_n are sequences of matrices with the properties mentioned before, the third term can be written as

$$A_n(g_k)^2 - \Phi_n(g_k)^2 = A_n(g_k)^2 - [A_n(g_k) + R_n + N_n]^2 = R'_n + N'_n,$$

where R'_n and N'_n are sequences of matrices with bounded rank and small norm respectively. Hence the third term also converges to the 0 matrix in the strong cluster sense. Therefore X_n converges to the 0 matrix in the strong cluster sense.

By similar arguments to the proof of Theorem 4.8, we conclude that Z_n converges to the 0 matrix in the strong cluster sense.

Now consider

$$\begin{aligned} P_{U_n}(A_n(g_k \circ g_l)) - A_n(g_k \circ g_l) \\ &= [P_{U_n}(A_n(g_k \circ g_l)) - P_{U_n}(A_n(g_k))P_{U_n}(A_n(g_l))] \\ &\quad + [P_{U_n}(A_n(g_k))P_{U_n}(A_n(g_l)) - A_n(g_k)A_n(g_l)] \\ &\quad + [A_n(g_k)A_n(g_l) - A_n(g_k \circ g_l)]. \end{aligned}$$

By similar arguments, we see that each term on the right hand side converges to the 0 matrix in the strong cluster sense. Hence the assertion of the theorem is proved for all functions of the form $g_k g_l$. Hence it is true for any function in the algebra generated by $\{g_1, \ldots, g_m\}$.

Now for $f \in \mathbb{A}$ and $\epsilon > 0$, let g be a function in the algebra generated by $\{g_1, \ldots, g_m\}$ such that

$$\|A_n(f) - A_n(g)\| < \epsilon/3, \qquad \|P_{U_n}(A_n(g)) - P_{U_n}(A_n(f))\| < \epsilon/3.$$

Write

$$\begin{aligned} A_n(f) - P_{U_n}(A_n(f)) &= [A_n(f) - A_n(g)] + [A_n(g) - P_{U_n}(A_n(g))] \\ &\quad + [P_{U_n}(A_n(g)) - P_{U_n}(A_n(f))]. \end{aligned}$$

The norm of the sum of the first and third terms is less than $2\epsilon/3$. The middle term $A_n(g) - P_{U_n}(A_n(g))$ can be split into a term with norm less than $\epsilon/3$ and a term with constant rank independent of n since g is in the algebra generated by $\{g_1, \ldots, g_m\}$. ∎

Corollary 5.5. *If $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in the strong cluster sense for all f in $\{1, x, x^2\}$, then it does so for all f in $C[0, 2\pi]$.*

COROLLARY 5.6. *Under the assumption of Theorem* 5.4, *if* $f \in \mathbb{A}$ *is strictly positive, then for any* $\epsilon > 0$ *and* n *large enough, the matrices* $P_{U_n}(A_n(f))^{-1}(A_n(f))$ *have eigenvalues in* $(1-\epsilon, 1+\epsilon)$ *except for at most* $N_\epsilon = O(1)$ *outliers.*

Proof. Since $f \in \mathbb{A}$ is strictly positive, $A_n(f)$ is positive definite. This implies that $P_{U_n}(A_n(f))$ is positive definite. Hence the proof is completed by invoking Lemma 4.7. ∎

Now we prove the exact analogue of Theorem 5.4 in the case of convergence in the weak cluster sense. The proof is more or less the same apart from some obvious modifications. However, all the details are provided.

THEOREM 5.7. *Let* $\{g_1, \ldots, g_m\}$ *be a finite set of real-valued continuous* 2π*-periodic functions such that* $P_{U_n}(A_n(f)) - A_n(f)$ *converges to* 0 *in the weak cluster sense for* f *in* $\{g_1, \ldots, g_m, g_1^2, \ldots, g_m^2\}$. *Then* $P_{U_n}(A_n(f)) - A_n(f)$ *converges to* 0 *in the weak cluster sense for all* f *in the* C^**-algebra* $\mathbb{A}$ *generated by* $\{g_1, \ldots, g_m\}$.

Proof. The proof is the same as that of Theorem 5.4, except that the splitting of terms must be as the sum of one with small norm and the other of order $o(n)$. We give the details below. Applying (2.2) with $\Phi_n = P_{U_n}(A_n(\cdot))$ and X_n, Y_n, Z_n as in the proof of Theorem 5.4, if we write

$$\begin{aligned} X_n = \Phi_n(g_k^2) - \Phi_n(g_k)^2 = [\Phi_n(g_k^2) - A_n(g_k^2)] + [A_n(g_k^2) - A_n(g_k)^2] \\ + [A_n(g_k)^2 - \Phi_n(g_k)^2], \end{aligned}$$

the first term on the right hand side converges to 0 in the weak cluster sense by assumption. The second term, $A_n(g_k^2) - A_n(g_k)^2$, converges to the 0 matrix in the strong cluster sense by the same argument in the proof of Theorem 5.4, and hence it converges in the weak cluster sense. By a simple computation, we find that the third term also converges to the 0 matrix in the weak cluster sense. Hence X_n converges to the 0 matrix in the weak cluster sense.

By similar arguments to the proof of Theorem 4.8, we conclude that Z_n converges to the 0 matrix in the weak cluster sense.

Now consider

$$\begin{aligned} &P_{U_n}(A_n(g_k \circ g_l)) - A_n(g_k \circ g_l) \\ &\quad = [P_{U_n}(A_n(g_k \circ g_l)) - P_{U_n}(A_n(g_k))P_{U_n}(A_n(g_l))] \\ &\qquad + [P_{U_n}(A_n(g_k))P_{U_n}(A_n(g_l)) - A_n(g_k)A_n(g_l)] \\ &\qquad + [A_n(g_k)A_n(g_l) - A_n(g_k \circ g_l)]. \end{aligned}$$

By similar arguments, each term on the right hand side converges to the 0 matrix in the weak cluster sense. Hence the conclusion is proved for all functions of the form $g_k g_l$. Hence it is true for any function in the algebra generated by $\{g_1, \dots, g_m\}$.

Now for $f \in \mathbb{A}$ and $\epsilon > 0$, let g be a function in the algebra generated by $\{g_1, \dots, g_m\}$ such that

$$\|A_n(f) - A_n(g)\| < \epsilon/3, \qquad \|P_{U_n}(A_n(g)) - P_{U_n}(A_n(f))\| < \epsilon/3.$$

Write

$$\begin{aligned} A_n(f) - P_{U_n}(A_n(f)) = [A_n(f) - A_n(g)] + [A_n(g) - P_{U_n}(A_n(g))] \\ + [P_{U_n}(A_n(g)) - P_{U_n}(A_n(f))]. \end{aligned}$$

The norm of the sum of the first and third terms is less than $2\epsilon/3$. The middle term $A_n(g) - P_{U_n}(A_n(g))$ can be split into a term with norm less than $\epsilon/3$ and a term of order $o(n)$, since g is in the algebra generated by $\{g_1, \dots, g_m\}$. ∎

COROLLARY 5.8. *With the hypotheses of Theorem* 5.7, *if* $f \in \mathbb{A}$ *is positive, then for any* $\epsilon > 0$ *and* n *large enough, the matrices*

$$P_{U_n}(A_n(f))^{-1}(A_n(f))$$

have eigenvalues in $(1 - \epsilon, 1 + \epsilon)$ *except at most* $N_\epsilon = o(n)$ *outliers.*

Proof. This follows easily from Lemma 4.7. ∎

REMARK. It is to be noted that Theorems 5.4 and 5.7 and their corollaries are much stronger than the corresponding theorems in [18], where it has been assumed that the convergence takes place on the algebra generated by a test set. Here it is assumed that the convergence takes place only on the test set, as in the classical Korovkin-type theorems. However, it is not clear whether the assumption of convergence on g_k^2 for each k can be replaced by convergence of $\sum_{k=1}^{n} g_k^2$ as in the usual case.

5.1. LPO sequences. It can also be observed that similar stronger versions of Theorems 5.3 and 5.4 of [18] are valid. First we recall some of the necessary preliminaries from [18].

The behavior of the eigenvalues of $P_{U_n}(A_n(f))$ has been studied in [18] when U_n is the sequence of generalized Vandermonde matrices (Example 3.7). Recall that the jth row of U_n is a vector of trigonometric functions calculated at the grid point $x_j^{(n)}$. From Lemma 3.1, it follows that the jth eigenvalue λ_j of $P_{U_n}(A_n(f))$ is $\sigma(U_n A_n(f) U_n^*)_{j,j}$. Thus λ_j is the value of a trigonometric function at $x = x_j^{(n)}$. Now we consider the function $[L_n[U_n](f)](x)$ obtained by replacing $x_j^{(n)}$ by x in $[0, 2\pi]$ in the expression of λ_j. To make it precise, let $v(x)$ denote the trigonometric function whose values at the grid points $\{x_j^{(n)}\}$ form the jth generic row of U_n^*. We define the linear operator $L_n[U_n]$ on $C[0, 2\pi]$ as follows:

$$L_n[U_n](f) = v(x) A_n(f) v^*(x). \tag{5.3}$$

$L_n[U_n](f)$ is the continuous expression of the diagonal elements of $U_nA_n(f)U_n^*$. It is clear that $L_n[U_n]$ is a sequence of completely positive linear maps on $C[0,2\pi]$ of norm less than or equal to 1.

We end this section with the proof of two results, which are stronger versions of Theorems 5.3 and 5.4 of [18].

THEOREM 5.9. *Let $L_n[U_n](g) = g + \epsilon_n(g)$ for every g in the finite set $\{g_1,\dots,g_m,\sum_{k=1}^m g_k^2\}$, where each g_k is a real-valued, continuous function and $\epsilon_n(g)$ converges uniformly to 0. Then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in the weak cluster sense for all f in the C^*-algebra $\mathbb{A}$ generated by $\{g_1,\dots,g_m\}$.*

Proof. First we observe that $L_n[U_n](g) = g + \epsilon_n(g)$ for every g in the algebra generated by $\{g_1,\dots,g_m\}$, by Lemma 5.3. Also we have

$$0 \le \|A_n(f_l) - P_{U_n}(A_n(f_l))\|_F^2 = \|A_n(f_l)\|_F^2 - \|P_{U_n}A_n(f_l)\|_F^2 \tag{5.4}$$

for every function f_l in the algebra generated by $\{g_1,\dots,g_m\}$. Here $\|(\cdot)\|_F$ denotes the Frobenius norm of matrices. Also since

$$L_n[U_n](f_l) = \lambda_i(P_{U_n}(A_n(f_l))) = f_l(x_i^n) + \epsilon_n(f_l),$$

for every l, where $\lambda_i(P_{U_n}(A_n(\cdot)))$ are the eigenvalues of $P_{U_n}(A_n(\cdot))$, we get

$$\|P_{U_n}(A_n(f_l))\|_F^2 = \sum_{i=1}^n \lambda_i^2(P_{U_n}(A_n(f_l))) = \sum_{i=1}^n [(f_l + \varepsilon_n(f_l))(x_i^n)]^2.$$

Hence

$$\|P_{U_n}(A_n(f_l))\|_F^2 = \sum_{i=1}^n f_l^2(x_i^n) + o(n).$$

Since $\{x_i^{(n)}\}$ is quasiuniformly distributed (see [18] for definition), by Lemma 5.1 in [18] we get

$$\sum_{i=0}^{n-1} [f_l(x_i^{(n)} + \varepsilon_n(f_l)(x_i^{(n)})^2] = \frac{n}{2\pi}\int_0^{2\pi} f_l^2 + o(n). \tag{5.5}$$

Also

$$\|A_n(f_l)\|_F^2 = \sum_{i=1}^n \lambda_i(A_n(f_l))^2,$$

for every l, and hence by the Szegö–Tyrtyshnikov Theorem 5.1 in [18], we find

$$\|A_n(f_l)\|_F^2 = \frac{n}{2\pi}\int_0^{2\pi} f_l^2 + o(n). \tag{5.6}$$

Now, from (5.4)–(5.6) we get

$$\|A_n(f_l) - P_{U_n}(A_n(f_l))\|_F^2 = o(n)$$

for every function f_l in the algebra generated by $\{g_1, \ldots, g_m\}$. Therefore by Tyrtyshnikov's Lemma 3.9, $P_{U_n}(A_n(f_l)) - A_n(f_l)$ converges to 0 in the weak cluster sense. Hence by Theorem 5.7, $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in the weak cluster sense for every f in the C^*-algebra $\mathbb{A}$ generated by $\{g_1, \ldots, g_m\}$. ■

Theorem 5.10. *With the assumptions in Theorem* 5.9, *if* $\epsilon_n(g) = O(1/n)$ *for* g *in the finite set* $\{g_1, \ldots, g_m, \sum_{k=1}^m g_k^2\}$ *and if the "grid point algebra" is uniformly distributed, then the convergence is in the strong cluster sense, provided the test functions in the set* $\{g_1, \ldots, g_m\}$ *are Lipschitz continuous and belong to the Krein algebra.*

Proof. The proof can be obtained by replacing the polynomials p by $\{g_1, \ldots, g_n\}$ in the proof of Theorem 5.4 in [18]. The idea is to replace the $o(n)$ term by constants in (5.5) and (5.6). For (5.5), we use the hypothesis $\epsilon_n(g) = O(1/n)$ and that the "grid point algebra" is uniformly distributed. For (5.6), we use Widom's theorem ([18, Theorem 5.2] or see [24]). Thus we attain

$$\|A_n(f_l) - P_{U_n}(A_n(f_l))\|_F^2 = O(1).$$

This completes the proof due to Lemma 3.9. ■

6. Discussion of the main results. In this section, we discuss the future possibilities and possible applications of the theory developed.

1. In Theorems 4.2 and 4.4, we considered one of the limit points Φ of Φ_n's in Kadison's BW-topology. Here $\Phi_n(A)$ is a preconditioner for the truncation A_n for each n. But it is not clear whether $\Phi(A)$ is a preconditioner of A for at least one limit point ϕ of Φ_n. Theorems 4.2 and 4.4 imply that, for any two limit points ϕ, ψ of the sequence $\Phi_n(\cdot)$, $\phi(A)$ is a compact perturbation of $\psi(A)$. Hence $\phi(A)$ and $\psi(A)$ have the same essential spectrum by Weyl's theorem.

2. Lemma 4.6 and Theorem 4.8 are of theoretical interest, since they share the same spirit of the classical Korovkin theorem. Consequently, the test on a finite number of elements guarantees the assertion on the whole C^*-algebra generated by these elements. Theorem 4.8 partly answers the following question. Suppose the usage of preconditioners works for a finite number of self-adjoint operators on $\mathbb{H}$. Does it work for any operator in the C^*-algebra generated by these operators?

3. The results of Section 5 are stronger versions of the results in [18]. Theorems 5.4 and 5.7 concern arbitrary continuous functions, while in [18], trigonometric polynomials were considered. Also the assumptions are reduced to a finite number of elements as in the classical Korovkin theorem. As the third author observed in [18], we expect that Theorems 5.4, 5.7, 5.9

and 5.10 can be used to obtain new preconditioners belonging to different algebras. Corollaries 5.6 and 5.8 are expected to be useful in deriving and analyzing good preconditioners for the conjugate gradient method.

4. Finally, we observe that the modified version (equation (3.8)) of preconditioners obtained using the pinching function is closer to the operator in the Frobenius norm. However, the computation of preconditioners constructed through a diagonal transformation is simpler.

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