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Characterization of associate spaces of weighted Lorentz spaces with applications

by

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Abstract. We characterize associate spaces of weighted Lorentz spaces $G\Gamma(p,m,w)$ and present some applications of this result including necessary and sufficient conditions for a Sobolev-type embedding into L^∞.

1. Introduction and main results. Let $(\mathcal{R},\mu)$ be a σ-finite non-atomic measure space with $b=\mu(\mathcal{R})\in(0,\infty]$. We denote by $\mathfrak{M}(\mathcal{R})$ the set of all μ-measurable functions on $\mathcal{R}$ whose values belong to $[-\infty,\infty]$. We also define $\mathfrak{M}_+(\mathcal{R})=\{g\in\mathfrak{M}(\mathcal{R})\colon g\ge 0\}$, and $\mathfrak{M}_0(\mathcal{R})=\{g\in\mathfrak{M}(\mathcal{R})\colon g \text{ is finite a.e. in } \mathcal{R}\}$.

The function space $G\Gamma(p,m,w)(\mathcal{R})$ (denoted simply by $G\Gamma(p,m,w)$ when no confusion can arise), introduced and studied in [FR2] and [FRZ], is defined as the collection of all functions $g\in\mathfrak{M}(\mathcal{R},\mu)$ such that

$$\|g\|_{G\Gamma(p,m,w)}=\Big(\int_0^b w(t)\Big(\int_0^t g^*(s)^p\,ds\Big)^{m/p}dt\Big)^{1/m}<\infty,$$

where $m,p\in(0,\infty)$, w is a weight (that is, a positive measurable function) on $(0,b)$, and g^* is the *non-increasing rearrangement* of g, given by

$$g^*(t)=\sup\{\lambda\in\mathbb{R}\colon \mu(\{x\in\mathcal{R}\colon |g(x)|>\lambda\})>t\}\quad\text{ for } t\in(0,b).$$

We also define the *maximal non-increasing rearrangement* of g by

$$g^{**}(t)=\frac{1}{t}\int_0^t g^*(s)\,ds\quad\text{ for } t\in(0,b),$$

and we note that the estimate

$$g^*(t)\le g^{**}(t)\tag{1.1}$$

holds universally for every $g\in\mathfrak{M}(\mathcal{R})$ and every $t\in(0,b)$.

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Our main goal is to give a precise and easily-computable characterization of the norm in the *associate space* (sometimes also called the *Köthe dual*) of the space $G\Gamma(p,m,w)$. The associate space $G\Gamma(p,m,w)'$ of $G\Gamma(p,m,w)$ is defined as the collection of all functions $g \in \mathfrak{M}(\mathcal{R})$ such that

$$\|g\|_{G\Gamma(p,m,w)'} = \sup_{\|f\|_{G\Gamma(p,m,w)}\le 1} \int_0^b f^*(t)g^*(t)\,dt < \infty. \tag{1.2}$$

Such a result is of interest for a number of reasons. In general, an associate space is a key thing to know about any Banach function space (see definitions below). Moreover, the spaces $G\Gamma(p,m,w)$ cover several types of important function spaces and have plenty of applications. For example, if $b=\infty$, $p=1$, $m>1$ and $w(t)=t^{-m}v(t)$, $t\in(0,\infty)$, where v is another weight on $(0,b)$, then $G\Gamma(p,m,w)$ reduces to the space $\Gamma^m(v)$, whose norm is

$$\|g\|_{\Gamma^m(v)} = \Big(\int_0^\infty g^{**}(t)^m v(t)\,dt\Big)^{1/m}.$$

This space was introduced by Sawyer [Sa] who used it to describe the behavior of classical operators on Lorentz spaces and observed, among other results, that, under certain restrictions on the parameters involved, this space is the associate space of the space $\Lambda^{m'}(\tilde v)$, introduced by Lorentz [L], where $m'=m/(m-1)$, $\tilde v$ is an appropriate weight, and the norm in $\Lambda^{m'}(\tilde v)$ is given by

$$\|g\|_{\Lambda^{m'}(\tilde v)} = \Big(\int_0^\infty g^{*}(t)^{m'} \tilde v(t)\,dt\Big)^{1/m'}.$$

The spaces of type Λ and Γ have been extensively investigated during the last 25 years under the common label *classical Lorentz spaces*, and an avalanche of papers by many authors devoted to their detailed study is available nowadays.

Another important example is obtained when $b=1$, $m=1$, $p\in(1,\infty)$ and $w(t)=t^{-1}\big(\log\frac{2}{t}\big)^{-1/p}$, $t\in(0,1)$. In this case $G\Gamma(p,m,w)$ coincides with the so-called *small Lebesgue space*, first studied by Fiorenza [F]. He proved that this space is the associate space of the so-called *grand Lebesgue space*, introduced in [IS] in connection with integrability properties of Jacobians. It was shown later by Fiorenza and Karadzhov [FK] that the norm in the small Lebesgue space can be equivalently written in the form of the norm in the $G\Gamma(p,m,w)$ space with the above-mentioned parameters and weight. For further results in this direction, see also [FR1, FR2]. Our characterization of the associate space of $G\Gamma(p,m,w)$ thus gives a new description of the grand Lebesgue space.

In [FR2] and [FRZ] the authors studied the associate spaces of the spaces $G\Gamma(p,m,w)$, but obtained only an upper bound for $\|g\|_{G\Gamma(p,m,w)'}$, moreover under the restriction that $\mu(R) < \infty$ and either $p \neq 1$ [FR2, Theorem 6] or $m \le p$ [FRZ, Theorem 3.2].

We are going to give a complete general characterization of the associate space of $G\Gamma(p,m,w)$ without any restrictions on the parameters involved. However, it is reasonable to adopt a general assumption that p, m and w are such that

$$\int_0^t w(s)s^{m/p}\,ds + \int_t^b w(s)\,ds < \infty \qquad \text{for every } t \in (0,b), \tag{1.3}$$

because if this requirement is not satisfied, then the "space" $G\Gamma(p,m,w)$ contains only the zero function. Under the assumption (1.3), we denote

$$u(t) = \int_0^t w(s)s^{m/p}\,ds + t^{m/p}\int_t^b w(s)\,ds, \qquad t \in (0,b). \tag{1.4}$$

The principal background tool in the proofs will be the duality results of [GP] and [Si]. It will be useful, in accordance with the terminology used in the first-mentioned paper, to call a weight w *non-degenerate* (with respect to the power function $t^{m/p}$) if (1.3) is satisfied and moreover

$$\int_0^t w(s)\,ds = \int_t^b w(s)s^{m/p}\,ds = \infty \qquad \text{for every } t \in (0,b). \tag{1.5}$$

We do not restrict our results here to non-degenerate weights, but we shall see that the characterizing conditions for degenerate weights are different from those concerning non-degenerate ones.

We shall now formulate our main theorem. Here and throughout, the symbol $\approx$ means that the two sides are bounded by each other up to multiplicative constants independent of appropriate quantities. As usual, for $p \in (1,\infty)$, we write $p' = p/(p-1)$. Throughout the paper, we use the convention $0 \cdot \infty = 0$. Another convention we use is that $b/2 = \infty$ when $b = \infty$.

Theorem 1.1. *Assume that* $0 < m, p < \infty$. *Let* w *be a weight on* $(0,b)$ *such that* (1.3) *is satisfied. Let* u *be defined by* (1.4).

(i) *Let* $0 < m \le 1$ *and* $0 < p \le 1$. *Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b/2)} g^{**}(t)\frac{t}{u(t)^{1/m}}.$$

(ii) *Let* $0 < m \le 1$ *and* $1 < p < \infty$. *Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b/2)} \Big(\int_t^b g^{**}(s)^{p'}\,ds\Big)^{1/p'}\frac{t^{1/p}}{u(t)^{1/m}}.$$

(iii) *Let* $1 < m < \infty$, $0 < p \le 1$ *and let* (1.5) *be satisfied. Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \bigg(\int_0^{b/2} g^{**}(t)^{m'} \frac{t^{m'+m/p-1}\int_0^t w(s)s^{m/p}\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt\bigg)^{1/m'}.$$

(iv) *Let* $1 < m < \infty$, $0 < p \le 1$ *and let either* $\int_0^b w(s)\,ds < \infty$ *or* $\int_0^b w(s)s^{m/p}\,ds < \infty$ *or both. Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \bigg(\int_0^{b/2} g^{**}(t)^{m'} \frac{t^{m'+m/p-1}\int_0^t w(s)s^{m/p}\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt\bigg)^{1/m'} + \frac{\limsup_{t\to 0_+} g^{**}(t)}{\big(\int_0^b w(s)\,ds\big)^{1/m}} + \frac{\int_0^b g^*(s)\,ds}{\big(\int_0^b w(s)s^{m/p}\,ds\big)^{1/m}}.$$

(v) *Let* $1 < m < \infty$, $1 < p < \infty$ *and let* (1.5) *be satisfied. Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \bigg(\int_0^{b/2} \Big(\int_t^b g^{**}(s)^{p'}\,ds\Big)^{m'/p'} \frac{t^{m'/p+m/p-1}\int_0^t w(s)s^{m/p}\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt\bigg)^{1/m'}.$$

(vi) *Let* $1 < m < \infty$, $1 < p < \infty$ *and let either* $\int_0^b w(s)\,ds < \infty$ *or* $\int_0^b w(s)s^{m/p}\,ds < \infty$ *or both. Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \bigg(\int_0^{b/2} \Big(\int_t^b g^{**}(s)^{p'}\,ds\Big)^{m'/p'} \frac{t^{m'/p+m/p-1}\int_0^t w(s)s^{m/p}\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt\bigg)^{1/m'} + \frac{\big(\int_0^b g^{**}(s)^{p'}\,ds\big)^{1/p'}}{\big(\int_0^b w(s)\,ds\big)^{1/m}} + \frac{\int_0^b g^*(s)\,ds}{\big(\int_0^b w(s)s^{m/p}\,ds\big)^{1/m}}.$$

For the proof of Theorem 1.1 we will develop a simple but powerful argument based on combination of results from [GP] and [Si] with an elementary inequality involving rearrangements, contained in the next result.

THEOREM 1.2. *Assume that* $1 < p < \infty$. *Let* $g \in L^1_{\mathrm{loc}}(\mathcal{R},\mu)$. *Then*

$$g^{**}(t) + \bigg(\frac{1}{t}\int_t^b g^{**}(s)^{p'-1}g^*(s)\,ds\bigg)^{1/p'} \approx \bigg(\frac{1}{t}\int_t^b g^{**}(s)^{p'}\,ds\bigg)^{1/p'} \tag{1.6}$$

for every $t \in (0, b/2)$.

We shall now turn our attention to an application of Theorem 1.1 to Sobolev-type embeddings which was first pointed out in [FRZ].

A (quasi-)normed linear space X is said to be (continuously) *embedded* into another such space Y, and denoted by $X \hookrightarrow Y$, if $X \subset Y$ and the identity operator is bounded from X to Y.

Let Ω be a bounded open connected set (a domain) in $\mathbb{R}^n$, where $n \in \mathbb{N}$, $n \geq 2$. We say that Ω is a *John domain* if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parameterized by arclength, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \geq cr \quad \text{for } r \in [0,l],$$

where $\partial\Omega$ is the boundary of Ω. The class of John domains is known to include some other families of domains that are considered classical, such as domains having Lipschitz boundary or domains having the cone property. John domains arise in connection with the study of holomorphic dynamical systems and quasiconformal mappings, and they are known to support Sobolev inequalities with the same exponents as the standard Sobolev ones (see [Bo, HK, KM, CPS1]). Being a John domain is a necessary condition for a Sobolev inequality to hold on simply connected open sets in $\mathbb{R}^2$ and on more general higher-dimensional domains (see [BK]).

For $k \in \mathbb{N}$, the Sobolev space $W^k G\Gamma(p,m,w)(\Omega)$ is defined as the collection of all weakly-differentiable functions u defined on Ω such that $|\nabla^j u| \in G\Gamma(p,m,w)(\Omega)$ for every $j \in \mathbb{N} \cup \{0\}$, $j \leq k$, where $\nabla^j u$ is the jth gradient of u, $\nabla^0 u = u$ and $|\cdot|$ is the Euclidean norm. The space $W^k G\Gamma(p,m,w)(\Omega)$, endowed with the functional

$$\|u\|_{W^k G\Gamma(p,m,w)(\Omega)} = \sum_{j=0}^{k} \big\| |\nabla^j u| \big\|_{G\Gamma(p,m,w)(\Omega)},$$

is a Banach space.

It was proved in [FRZ, Lemma 1.4] that the condition

$$t^{-1/n'} \in G\Gamma(p,m,w)'(0,b) \tag{1.7}$$

is sufficient for the Sobolev embedding

$$W^1 G\Gamma(p,m,w)(\Omega) \hookrightarrow L^\infty(\Omega), \tag{1.8}$$

where $b = |\Omega|$. Embeddings of type (1.8) are known to have a number of applications, for example they are intimately connected with the question whether the Sobolev space is a Banach algebra (cf. e.g. [A, C, CPS2]). Our aim is to point out that, as can be deduced from our results, (1.7) is in fact not only sufficient, but also *necessary*, for (1.8) to hold. Furthermore, we shall include Sobolev embeddings of any order.

However, before we can state this result, we first need to know for which parameters p, m, w the space $G\Gamma(p,m,w)$ satisfies the axioms of rearrangement-invariant Banach function space. We say that X is a *Banach* *function space* over a σ-finite measure space $(\mathcal{R},\mu)$ if for all non-negative μ-measurable real functions f, g and $\{f_j\}_{j\in\mathbb{N}}$ on $\mathcal{R}$ and every $\lambda\geq 0$, the following properties hold:

- (P1) $\|f\|_X=0$ if and only if $f=0$ a.e.; $\|\lambda f\|_X=\lambda\|f\|_X$; $\|f+g\|_X\leq\|f\|_X+\|g\|_X$;
- (P2) $f\leq g$ a.e. implies $\|f\|_X\leq\|g\|_X$;
- (P3) $f_j\nearrow f$ a.e. implies $\|f_j\|_X\nearrow\|f\|_X$;
- (P4) for every $E\subset\mathcal{R}$ with $\mu(E)<\infty$ one has $\|\chi_E\|_X<\infty$;
- (P5) for every $E\subset\mathcal{R}$ with $\mu(E)<\infty$ one has $\int_E f(x)\,d\mu\leq C_E\|f\|_X$ for some constant C_E independent of f.

We say that X is a *rearrangement-invariant Banach function space* if (P1)–(P5) are satisfied and moreover $\|f\|_X=\|g\|_X$ whenever $f^*=g^*$ on $(0,b)$. Here and throughout, χ_E denotes the characteristic function of E.

We shall now state a necessary and sufficient condition for the space $G\Gamma(p,m,w)$ to be a rearrangement-invariant Banach function space. In view of applications, we restrict ourselves to the case $1\leq p,m<\infty$. We note that the result is known for certain particular cases. We omit the details but we refer the reader to [FR2, Theorem 5].

THEOREM 1.3. *Suppose that $1\leq p,m<\infty$ and let w be a weight on $(0,b)$. Then the space $G\Gamma(p,m,w)$ is a Banach function space if and only if*

$$\int_0^b w(t)\min\{1,t^{m/p}\}\,dt<\infty. \tag{1.9}$$

Now we are in a position to characterize a higher-order Sobolev embedding. The results are collected in the following theorem. It will be useful to recall that Ω is a bounded domain, therefore $b<\infty$.

THEOREM 1.4. *Let $n\in\mathbb{N}$, $n\geq 2$. Let $\Omega\subset\mathbb{R}^n$ be a John domain and let $b=|\Omega|$. Let $1\leq m,p<\infty$ and let w be a weight on $(0,b)$ such that*

$$\int_0^b w(t)t^{m/p}\,dt<\infty. \tag{1.10}$$

Let $k\in\mathbb{N}$. Then the Sobolev embedding

$$W^kG\Gamma(p,m,w)(\Omega)\hookrightarrow L^\infty(\Omega) \tag{1.11}$$

holds if and only if either $k\geq n$, or $k\leq n-1$ and one of the following conditions is satisfied:

(i) *$m=1$, $1\leq p<n/k$ and*

$$\sup_{t\in(0,b/2)}\frac{t^{k/n}}{\int_0^t w(s)s^{1/p}\,ds+t^{1/p}\int_t^b w(s)\,ds}<\infty;$$

(ii) $m=1$, $p=n/k$ *and*

$$\sup_{t\in(0,b/2)} \frac{t^{k/n}\big(\log\frac{b}{t}\big)^{1-k/n}}{\int_0^t w(s)s^{1/p}\,ds+t^{1/p}\int_t^b w(s)\,ds}<\infty;$$

(iii) $m=1$, $n/k<p<\infty$ *and*

$$\sup_{t\in(0,b/2)} \frac{t^{1/p}}{\int_0^t w(s)s^{1/p}\,ds+t^{1/p}\int_t^b w(s)\,ds}<\infty;$$

(iv) $1<m<\infty$, $1\le p<n/k$, $\int_0^b w(t)\,dt=\infty$ *and*

$$\int_0^{b/2} \frac{t^{m'k/n+m/p-1}\int_0^t w(s)s^{m/p}\,ds\int_t^b w(s)\,ds}{\big(\int_0^t w(s)s^{m/p}\,ds+t^{m/p}\int_t^b w(s)\,ds\big)^{m'+1}}\,dt<\infty;$$

(v) $1<m<\infty$, $p=n/k$, $\int_0^b w(t)\,dt=\infty$ *and*

$$\int_0^{b/2} \frac{t^{m'k/n+mk/n-1}\big(\log\frac{b}{t}\big)^{m'(1-k/n)}\int_0^t w(s)s^{mk/n}\,ds\int_t^b w(s)\,ds}{\big(\int_0^t w(s)s^{mk/n}\,ds+t^{mk/n}\int_t^b w(s)\,ds\big)^{m'+1}}\,dt<\infty;$$

(vi) $1<m<\infty$, $n/k<p<\infty$ *and*

$$\int_0^{b/2} \frac{t^{m'/p+m/p-1}\int_0^t w(s)s^{m/p}\,ds\int_t^b w(s)\,ds}{\big(\int_0^t w(s)s^{m/p}\,ds+t^{m/p}\int_t^b w(s)\,ds\big)^{m'+1}}\,dt<\infty.$$

Using the results of [CPS1] one can obtain sufficient conditions for the Sobolev embedding (1.11) also for domains with worse boundary than just John domains, as long as a lower bound for their isoperimetric function is known. In many customary cases, such conditions will also be necessary in a certain broader sense. We recall that the *perimeter* of a measurable set E in Ω is given by

$$P(E,\Omega)=\mathcal{H}^{n-1}(\Omega\cap\partial^M E),$$

where $\partial^M E$ denotes the essential boundary of E, in the sense of geometric measure theory [M, Z]. The *isoperimetric function* $I_\Omega:[0,1]\to[0,\infty]$ of Ω is then given by

$$I_\Omega(s)=\inf\{P(E,\Omega): E\subset\Omega,\ s\le|E|\le 1/2\}\quad \text{if } s\in[0,1/2],$$

and $I_\Omega(s)=I_\Omega(1-s)$ if $s\in(1/2,1]$. We omit the details.

In our last application of Theorem 1.1 we intend to characterize those parameters p,m and w for which the space $G\Gamma(p,m,w)$ is reflexive. This question was studied in [FRZ], where a number of results were deduced from the assumption that $G\Gamma(p,m,w)$ is reflexive, and also a sufficient condition for reflexivity was given.

To pave the way to a characterization we shall first single out those spaces $G\Gamma(p,m,w)$ which have absolutely continuous norms. We restrict here to the case when $1 < p, m < \infty$. Such a result is of independent interest since it might be handy when compactness of operators and embeddings between function spaces is studied (see e.g. [LZ, FMP, KP, PP, Sl1, Sl2]). A Banach function space X on $(\mathcal{R},\mu)$ is said to have *absolutely continuous norm* if for each sequence $\{E_n\}$ of μ-measurable subsets of $\mathcal{R}$ satisfying $E_n \downarrow \emptyset$ one has $\|\chi_{E_n} f\|_X \to 0$ for every $f \in X$.

THEOREM 1.5. *Let $1 < p, m < \infty$ and let w be a weight on $(0,b)$. Then the space $G\Gamma(p,m,w)$ has absolutely continuous norm if and only if at least one of the following conditions holds:*

$$b < \infty, \tag{1.12}$$

$$\int_0^b t^{m/p} w(t)\,dt = \infty. \tag{1.13}$$

Our next theorem shows that for the associate space of $G\Gamma(p,m,w)$, the absolute continuity of norm is granted unconditionally.

THEOREM 1.6. *Let $1 < p, m < \infty$ and let w be a weight on $(0,b)$. Then the associate space to $G\Gamma(p,m,w)$ has an absolutely continuous norm.*

Now we can state our last result. Again, some particular cases are known [FR2, Theorem 5].

THEOREM 1.7. *Let $1 < p, m < \infty$ and let w be a weight on $(0,b)$. Then the space $G\Gamma(p,m,w)$ is reflexive if and only if at least one of the conditions* (1.12) *and* (1.13) *holds.*

EXAMPLES 1.8. (a) If $b < \infty$, $0 < m < \infty$, $1 \le p < \infty$ and $\int_0^b w(s)\,ds < \infty$, then it is not difficult to verify that the space $G\Gamma(p,m,w)$ degenerates to the Lebesgue space L^p (regardless of m). Indeed, on the one hand, we have

$$\begin{aligned}\|g\|_{G\Gamma(p,m,w)} &= \Big(\int_0^b w(t)\Big(\int_0^t g^*(s)^p\,ds\Big)^{m/p} dt\Big)^{1/m} \\ &\le \Big(\int_0^b g^*(s)^p\,ds\Big)^{1/p}\Big(\int_0^b w(t)\,dt\Big)^{1/m} \\ &= C\|g\|_{L^p}\end{aligned}$$

with $C = \big(\int_0^b w(t)\,dt\big)^{1/m} < \infty$, while, on the other hand, due to the monotonicity of g^* and positivity of w, one has

$$\begin{aligned}\|g\|_{G\Gamma(p,m,w)} &= \Big(\int_0^b w(t)\Big(\int_0^t g^*(s)^p\,ds\Big)^{m/p}dt\Big)^{1/m}\\ &\ge \Big(\int_0^{b/2} g^*(s)^p\,ds\Big)^{1/p}\Big(\int_{b/2}^b w(t)\,dt\Big)^{1/m}\\ &\ge c\|g\|_{L^p}\end{aligned}$$

with $c = 2^{-p}(\int_{b/2}^b w(t)\,dt)^{1/m} > 0$. A simple argument shows that, for this choice of parameters, we have $u(t) \approx t^{m/p}$, and it is easy to check that the appropriate choice of part (i), (ii), (iv) or (vi) of Theorem 1.1 yields $\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{p'}}$ for every measurable function g. For example, if $p = 1$ and $1 < m < \infty$, then by Theorem 1.1(iv) we obtain

$$\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{p'}} + \|g\|_{L^\infty} + \|g\|_{L^1} \approx \|g\|_{L^\infty},$$

since $b < \infty$. We note that cases (iii) and (v) of Theorem 1.1 are inapplicable here since (1.5) is false.

(b) If $1 \le p < \infty$ and $\int_0^b w(s)\,ds < \infty$ but $b = \infty$, then the upper bound for $\|g\|_{G\Gamma(p,m,w)}$ from (a) still applies, but the lower bound does not work, since, in accord with our convention, $b/2 = \infty$, and therefore the integral $\int_{b/2}^b w(t)\,dt$ is zero. Thus, the inclusion $L^p \subset G\Gamma(p,m,w)$ still holds, but the converse need not be satisfied.

(c) We shall now analyze the situation when

$$0 < p < \infty, \qquad m > p, \qquad m \ge 1, \qquad w(t) = t^{-m/p} \qquad \text{for every } t \in (0,b).$$

Then

$$\|g\|_{G\Gamma(p,m,w)} = \Big(\int_0^b\Big(\frac{1}{t}\int_0^t g^*(s)^p\,ds\Big)^{m/p}dt\Big)^{1/m}.$$

Therefore, by the classical Hardy inequality (see e.g. [BS, Chapter 3, Lemma 3.9]) together with (1.1) we get

$$\|g\|_{G\Gamma(p,m,w)} \approx \Big(\int_0^b g^*(t)^m\,dt\Big)^{1/m},$$

whence, for this choice of parameters, the space $G\Gamma(p,m,w)$ always degenerates to the Lebesgue space L^m. We shall now check that the results deduced from Theorem 1.1 are consistent with the classical duality relations between Lebesgue spaces. It will be useful to note that $u(t) \approx t$ for every $t \in (0,b)$.

First, let $m = 1$ and $0 < p < 1$. Then Theorem 1.1(i) implies that

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b/2)} g^{**}(t) = \|g\|_{L^\infty},$$

as required.

Next, assume that $p = 1$, $1 < m < \infty$ and $b = \infty$. Then, obviously, (1.5) is satisfied. Hence Theorem 1.1(iii) applies, and we get

$$\|g\|_{G\Gamma(p,m,w)'} \approx \left(\int_0^\infty g^{**}(t)^{m'} \frac{t^{m'+m-1} \cdot t \cdot t^{1-m}}{t^{m'+1}}\, dt\right)^{1/m'} \approx \|g\|_{L^{m'}},$$

by the Hardy inequality and (1.1), again.

If $p = 1$, $1 < m < \infty$ and $b < \infty$, then

$$\int_0^t w(s)\, ds = \infty \qquad \text{for every } t \in (0,\infty)$$

but

$$\int_t^b w(s) s^{m/p}\, ds < \infty \qquad \text{for every } t \in (0,\infty),$$

hence (1.5) is not satisfied. Consequently, we have to use Theorem 1.1(iv) this time. We get

$$\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{m'}} + b^{-1/m}\|g\|_{L^1}.$$

Because $b < \infty$, we have, by Hölder's inequality, $\|g\|_{L^1} \le b^{1/m}\|g\|_{L^{m'}}$.

Thus, altogether, we again obtain

$$\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{m'}},$$

as desired.

Let $1 < p < m < \infty$ and $b = \infty$. Then (1.5) holds and we can use Theorem 1.1(v). We obtain

$$\begin{aligned}\|g\|_{G\Gamma(p,m,w)'} &\approx \left(\int_0^\infty \left(\int_t^\infty g^{**}(s)^{p'}\, ds\right)^{m'/p'} \frac{t^{m'/p+m/p-1} \cdot t \cdot t^{1-m/p}}{t^{m'+1}}\, dt\right)^{1/m'} \\ &\approx \left(\int_0^\infty \left(\frac{1}{t}\int_t^\infty g^{**}(s)^{p'}\, ds\right)^{m'/p'} dt\right)^{1/m'}.\end{aligned}$$

We claim that

$$\left(\int_0^\infty \left(\frac{1}{t}\int_t^\infty g^{**}(s)^{p'}\, ds\right)^{m'/p'} dt\right)^{1/m'} \approx \|g\|_{L^{m'}}. \tag{1.14}$$

The lower bound is easy, we only have to observe that

$$\begin{aligned}\left(\int_0^\infty \left(\frac{1}{t}\int_t^\infty g^{**}(s)^{p'}\, ds\right)^{m'/p'} dt\right)^{1/m'} &\ge \left(\int_0^\infty \left(\frac{1}{t}\int_t^{2t} g^{**}(s)^{p'}\, ds\right)^{m'/p'} dt\right)^{1/m'} \\ &\ge \left(\int_0^\infty g^{**}(2t)^{m'}\, dt\right)^{1/m'} \approx \|g\|_{L^{m'}},\end{aligned}$$

where the last relation follows by a simple change of variables. As for the upper bound, we first claim that there exists a positive constant C such that, for every $t \in (0,\infty)$ and every $g \in \mathfrak{M}(\mathcal{R})$, one has

$$\Big(\int_t^\infty g^{**}(s)^{p'}\,ds\Big)^{1/p'} \le C\Big(\int_t^\infty g^{**}(s)^{m'}(s-t)^{m'/p'-1}\,ds\Big)^{1/m'}. \tag{1.15}$$

Clearly, (1.15) will follow once we show that

$$\int_t^\infty h^*(s)\,ds \le C\Big(\int_t^\infty h^*(s)^{m'/p'}(s-t)^{m'/p'-1}\,ds\Big)^{p'/m'} \tag{1.16}$$

for some $C>0$, every $t\in(0,\infty)$ and every $h\in\mathfrak{M}(\mathcal{R})$, on applying the last estimate to the particular choice $h^* = (g^{**})^{p'}$. The proof of (1.16) is similar to the classical proof of embeddings between Lorentz spaces (see e.g. [BS, Chapter 4, Proposition 4.2]). Indeed,

$$\begin{aligned}\int_t^\infty h^*(s)\,ds &= \int_t^\infty h^*(s)^{m'/p'}h^*(s)^{1-m'/p'}(s-t)^{1-m'/p'}(s-t)^{m'/p'-1}\,ds\\ &\le \Big(\sup_{y\in(t,\infty)} h^*(y)(y-t)\Big)^{1-m'/p'}\int_t^\infty h^*(s)^{m'/p'}(s-t)^{m'/p'-1}\,ds.\end{aligned}$$

However, for every $y\in(t,\infty)$, we have

$$\begin{aligned}h^*(y)(y-t) &\approx h^*(y)\Big(\int_t^y (s-t)^{m'/p'-1}\,ds\Big)^{p'/m'}\\ &\le \Big(\int_t^y h^*(s)^{m'/p'}(s-t)^{m'/p'-1}\,ds\Big)^{p'/m'}\\ &\le \Big(\int_t^\infty h^*(s)^{m'/p'}(s-t)^{m'/p'-1}\,ds\Big)^{p'/m'}.\end{aligned}$$

So, combining the last two estimates, we get (1.16), hence also (1.15). Now, using (1.15) and the Fubini theorem, we arrive at

$$\begin{aligned}&\Big(\int_0^\infty\Big(\frac{1}{t}\int_t^\infty g^{**}(s)^{p'}\,ds\Big)^{m'/p'}\,dt\Big)^{1/m'} \\ &\qquad\le C\Big(\int_0^\infty t^{-m'/p'}\int_t^\infty g^{**}(s)^{m'}(s-t)^{m'/p'-1}\,ds\,dt\Big)^{1/m'}\\ &\qquad= C\Big(\int_0^\infty g^{**}(s)^{m'}\int_0^s t^{-m'/p'}(s-t)^{m'/p'-1}\,dt\,ds\Big)^{1/m'}.\end{aligned} \tag{1.17}$$

Changing variables, we get, for every fixed $s \in (0,\infty)$,

$$\int_0^s t^{-m'/p'}(s-t)^{m'/p'-1}\,dt = \int_0^1 y^{-m'/p'}(1-y)^{m'/p'-1}\,dy.$$

Thus, denoting

$$K = \int_0^1 y^{-m'/p'}(1-y)^{m'/p'-1}\,dy,$$

we obtain

$$\int_0^s t^{-m'/p'}(s-t)^{m'/p'-1}\,dt \le K \quad \text{for every } s \in (0,\infty).$$

Plugging this into (1.17), we get the upper bound in (1.14). Altogether, also in this case, we conclude that

$$\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{m'}}.$$

Finally, let $1 < p < m < \infty$ and $b < \infty$. Then

$$\int_0^b w(s)s^{m/p}\,ds < \infty,$$

hence the weight is degenerate, and we have to use Theorem 1.1(vi). The first term on the right-hand side is equivalent to $\|g\|_{L^{m'}}$ just as in the preceding case, and the last one is obviously equivalent to $\|g\|_{L^1}$. Furthermore, the middle term disappears.

(d) If $1 < p < \infty$, $m = 1$, $b = 1$ and $w(t) = t^{-1}\big(\log\frac{2}{t}\big)^{-1/p}$, then $G\Gamma(p,m,w)$ coincides with the small Lebesgue space ([F], [FK]). Hence, Theorem 1.1 provides a new characterization of the grand Lebesgue space.

(e) A similar functional to the one in Theorem 1.1(ii) appears in [CP2, Theorem 1.2] in connection with a sharp Sobolev embedding into a Morrey space. Spaces generated by similar functionals are also treated in [Kr].

2. Proofs

Proof of Theorem 1.2. Fix $g \in L^1_{\mathrm{loc}}(\mathcal{R},\mu)$ and $t \in (0, b/2)$. Then

$$\begin{aligned}
\frac{1}{t}\int_t^b g^{**}(s)^{p'}\,ds &\ge \frac{1}{t}\int_t^{2t} g^{**}(s)^{p'}\,ds \\
&= \frac{1}{t}\int_t^{2t} s^{-p'}\Big(\int_0^s g^*(y)\,dy\Big)^{p'}\,ds \\
&\ge \frac{1}{t}\Big(\int_0^t g^*(s)\,ds\Big)^{p'}\int_t^{2t}\frac{ds}{s^{p'}} \\
&= cg^{**}(t)^{p'}
\end{aligned}$$

with $c = (p-1)(1-2^{1-p'})$. Since the estimate

$$\frac{1}{t}\int_t^b g^{**}(s)^{p'-1} g^*(s)\,ds \le \frac{1}{t}\int_t^b g^{**}(s)^{p'}\,ds$$

follows immediately from (1.1), we obtain

$$g^{**}(t) + \Big(\frac{1}{t}\int_t^b g^{**}(s)^{p'-1} g^*(s)\,ds\Big)^{1/p'} \le C\Big(\frac{1}{t}\int_t^b g^{**}(s)^{p'}\,ds\Big)^{1/p'}$$

with C depending only on p. Conversely, integrating by parts, we get

$$\begin{aligned}\frac{1}{t}\int_t^b g^{**}(s)^{p'-1} g^*(s)\,ds &= \frac{1}{t}\int_t^b \frac{1}{s^{p'-1}}\Big(\int_0^s g^*(y)\,dy\Big)^{p'-1} g^*(s)\,ds \\ &= \frac{1}{p't}\Big(\lim_{s\to b_-}\frac{1}{s^{p'-1}}\Big(\int_0^s g^*(y)\,dy\Big)^{p'} - \frac{1}{t^{p'-1}}\Big(\int_0^t g^*(s)\,ds\Big)^{p'}\Big) \\ &\quad + \frac{1}{pt}\int_t^b g^{**}(s)^{p'}\,ds \\ &\ge \frac{1}{pt}\int_t^b g^{**}(s)^{p'}\,ds - \frac{1}{p'} g^{**}(t)^{p'},\end{aligned}$$

hence

$$g^{**}(t) + \Big(\frac{1}{t}\int_t^b g^{**}(s)^{p'-1} g^*(s)\,ds\Big)^{1/p'} \ge c'\Big(\frac{1}{t}\int_t^b g^{**}(s)^{p'}\,ds\Big)^{1/p'}$$

with a suitable $c' > 0$. The assertion now follows from the combination of both estimates. ■

Proof of Theorem 1.1. Assume first that $b = \infty$. Rewriting the norm in (1.2) in a more convenient way and setting $h^* = (f^*)^p$, we get

$$\begin{aligned}\|g\|_{G\Gamma(p,m,w)'} &= \sup_{f\not\equiv 0}\frac{\int_0^b f^*(t)g^*(t)\,dt}{\|f\|_{G\Gamma(p,m,w)}} \\ &= \sup_{f\not\equiv 0}\frac{\int_0^b f^*(t)g^*(t)\,dt}{\big(\int_0^b w(t)\big(\int_0^t f^*(s)^p\,ds\big)^{m/p}\,dt\big)^{1/m}} \\ &= \sup_{h\not\equiv 0}\frac{\int_0^b h^*(t)^{1/p}g^*(t)\,dt}{\big(\int_0^b w(t)\big(\int_0^t h^*(s)\,ds\big)^{m/p}\,dt\big)^{1/m}} \\ &= \sup_{h\not\equiv 0}\frac{\int_0^b h^*(t)^{1/p}g^*(t)\,dt}{\big(\int_0^b h^{**}(t)^{m/p}t^{m/p}w(t)\,dt\big)^{1/m}}.\end{aligned}$$

Raising this to the power p, we arrive at

$$\|g\|^p_{G\Gamma(p,m,w)'} = \sup_{h\not\equiv 0} \frac{\left(\int_0^b h^*(t)^{1/p} g^*(t)\,dt\right)^p}{\left(\int_0^b h^{**}(t)^{m/p} t^{m/p} w(t)\,dt\right)^{p/m}}.$$

Let $0 < m \le 1$. Then, by a slight modification of [GP, Theorem 4.2(i)] and its proof, we obtain

$$\|g\|^p_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b)} \frac{\left(\int_0^t g^*(s)\,ds\right)^p}{u(t)^{p/m}}.$$

Taking the pth root, we get

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b)} g^{**}(t)\frac{t}{u(t)^{1/m}}.$$

Since $b=\infty$, and therefore, by our convention, also $b/2=\infty$, this completes the proof of (i).

If $1<p<\infty$ and $0<m\le 1$, then [GP, Theorem 4.2(iii)] yields

$$\begin{aligned}\|g\|^p_{G\Gamma(p,m,w)'} &\approx \sup_{t\in(0,b)} \frac{\left(\int_0^t g^*(s)\,ds\right)^p + t\left(\int_t^b \left(\int_0^s g^*(y)\,dy\right)^{p'-1} g^*(s) s^{1-p'}\,ds\right)^{p-1}}{u(t)^{p/m}} \\ &= \sup_{t\in(0,b)} \frac{t^p g^{**}(t)^p + t\left(\int_t^b g^{**}(s)^{p'-1} g^*(s)\,ds\right)^{p-1}}{u(t)^{p/m}}.\end{aligned}$$

Thus,

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b)} \left(g^{**}(t) + \left(\frac{1}{t}\int_t^b g^{**}(s)^{p'-1} g^*(s)\,ds\right)^{1/p'}\right)\frac{t}{u(t)^{1/m}}.$$

By Theorem 1.2, this yields

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b)} \left(\int_t^b g^{**}(s)^{p'}\,ds\right)^{1/p'} \frac{t^{1/p}}{u(t)^{1/m}},$$

establishing (ii).

Now assume that $1<m<\infty$, $0<p\le 1$, and (1.5) is satisfied. Then, using [GP, Theorem 4.2(ii)], we get

$$\begin{aligned}&\|g\|^p_{G\Gamma(p,m,w)'} \\ &\approx \left(\int_0^b \sup_{y\in(t,b)} \left(\int_0^y g^*(\tau)\,d\tau\right)^{m'} y^{-m'/p} \frac{t^{m'/p+m/p-1}\int_0^t w(s)s^{m/p}\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt\right)^{p/m'} \\ &\approx \left(\int_0^b \sup_{y\in(t,b)} g^{**}(y)^{m'} y^{m'(p-1)/p} \frac{t^{m'/p+m/p-1}\int_0^t w(s)s^{m/p}\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt\right)^{p/m'}.\end{aligned}$$

Since $p \le 1$, the expression $g^{**}(y)^{m'} y^{m'(p-1)/p}$ is in fact non-increasing on (t, b), hence it takes its largest value at t. With a little algebra, (iii) follows.

Next, let $1 < p, m < \infty$ and let (1.5) hold. Then, by [GP, Theorem 4.2(iv)], we have

$$\begin{aligned}
&\|g\|^p_{G\Gamma(p,m,w)'} \\
&\approx \Bigg(\int_0^b \frac{\big(\big(\int_0^t g^*(s)\,ds\big)^{p'} + t^{p'-1}\int_t^b \big(\int_0^s g^*(y)\,dy\big)^{p'-1} g^*(s) s^{1-p'}\,ds\big)^{m'/p'}}{u(t)^{m'+1}} \\
&\qquad\qquad \times t^{m/p-1}\int_0^t s^{m/p} w(s)\,ds \int_t^b w(s)\,ds\,dt\Bigg)^{p/m'} \\
&\approx \Bigg(\int_0^b \bigg(g^{**}(t) + \bigg(\frac{1}{t}\int_t^b g^{**}(s)^{p'-1} g^*(s)\,ds\bigg)^{1/p'}\bigg)^{m'} \\
&\qquad\qquad \times \frac{t^{m/p-1}\int_0^t s^{m/p} w(s)\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt\Bigg)^{p/m'}.
\end{aligned}$$

By Theorem 1.2, this implies

$$\begin{aligned}
&\|g\|^p_{G\Gamma(p,m,w)'} \\
&\approx \Bigg(\int_0^b \bigg(\frac{1}{t}\int_t^b g^{**}(s)^{p'}\,ds\bigg)^{m'/p'} \frac{t^{m/p+m'-1}\int_0^t s^{m/p} w(s)\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt\Bigg)^{p/m'},
\end{aligned}$$

and (v) follows on taking the pth root.

If $1 < m < \infty$ and (1.5) is violated, then, in order to prove the statements (iv) and (vi), the results of [GP] cannot be used directly, because degenerate weights are not treated there. In this case we have either to use the result of Sinnamon [Si] or modify the argument in [GP]. We omit the technical details.

Now let $b < \infty$. Then, in order to finish the proof of (i), we need to show that

$$\sup_{t\in(0,b)} g^{**}(t)\frac{t}{u(t)^{1/m}} \approx \sup_{t\in(0,b/2)} g^{**}(t)\frac{t}{u(t)^{1/m}}.$$

To this end, denote

$$K = \bigg(\frac{u(b/3)}{\int_0^{b/2} w(s) s^{m/p}\,ds}\bigg)^{1/m}.$$

Then, for every $t \in [b/2, b)$, one has

$$\frac{t}{u(t)^{1/m}} < \frac{b}{\big(\int_0^{b/2} w(s) s^{m/p}\,ds\big)^{1/m}} = 3K\frac{b/3}{u(b/3)^{1/m}}.$$

Thus, using also the fact that g^{**} is non-increasing on $(0,b)$, we get, for every $t\in[b/2,b)$,

$$g^{**}(t)\frac{t}{u(t)^{1/m}}\le 3Kg^{**}(b/3)\frac{b/3}{u(b/3)^{1/m}}\le 3K\sup_{t\in(0,b/2)}g^{**}(t)\frac{t}{u(t)^{1/m}}.$$

Consequently,

$$\sup_{t\in(0,b)}g^{**}(t)\frac{t}{u(t)^{1/m}}\le\max\{1,3K\}\sup_{t\in(0,b/2)}g^{**}(t)\frac{t}{u(t)^{1/m}}.$$

Since the converse inequality is trivial, this completes the proof of (i). The proof of the remaining statements is analogous and therefore omitted. ∎

Proof of Theorem 1.3. First, the 'only if' part of the assertion follows simply on testing the norm in $G\Gamma(p,m,w)$ on characteristic functions of sets of finite measure.

Let us prove the 'if' part. All the assertions in (P1) except the triangle inequality are obvious. Fix $t\in(0,b)$ and let f,g be μ-measurable real functions on $\mathcal{R}$. Denote

$$E_t=\{x\in\mathcal{R}\colon f(x)+g(x)>(f+g)^*(t)\}.$$

Then $\mu(E_t)\le t$ [BS, Chapter 2, Proposition 1.7]. Combining this fact with the Minkowski inequality for the norm in the space $L^p(E_t)$ and the Hardy–Littlewood inequality [BS, Chapter 2, Theorem 2.2], we obtain

$$\begin{aligned}\Big(\int_0^t(f+g)^*(s)^p\,ds\Big)^{1/p}&=\Big(\int_{E_t}(f+g)(s)^p\,d\mu\Big)^{1/p}\\&\le\Big(\int_{E_t}f(s)^p\,d\mu\Big)^{1/p}+\Big(\int_{E_t}g(s)^p\,d\mu\Big)^{1/p}\\&\le\Big(\int_0^{\mu(E_t)}f^*(s)^p\,ds\Big)^{1/p}+\Big(\int_0^{\mu(E_t)}g^*(s)^p\,ds\Big)^{1/p}\\&\le\Big(\int_0^t f^*(s)^p\,ds\Big)^{1/p}+\Big(\int_0^t g^*(s)^p\,ds\Big)^{1/p}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|f+g\|_{G\Gamma(p,m,w)}&=\Big(\int_0^b\Big(\int_0^t(f+g)^*(s)^p\,ds\Big)^{m/p}w(t)\,dt\Big)^{1/m}\\&=\Big(\int_0^b\|(f+g)^*\|^m_{L^p(0,t)}w(t)\,dt\Big)^{1/m}\\&\le\big\|\|g^*\|_{L^p(0,t)}+\|f^*\|_{L^p(0,t)}\big\|_{L^m_w(0,b)}\end{aligned}$$

$$
\begin{aligned}
&\le \big\| \|g^*\|_{L^p(0,t)} \big\|_{L^m_w(0,b)} + \big\| \|f^*\|_{L^p(0,t)} \big\|_{L^m_w(0,b)} \\
&= \|f\|_{G\Gamma(p,m,w)} + \|g\|_{G\Gamma(p,m,w)},
\end{aligned}
$$

as desired.

Next, (P2) follows immediately from the definition and (P3) from the Monotone Convergence Theorem applied first to the inner integral and then on the outer one.

As for (P4) and (P5), let E be a subset of $\mathcal{R}$ of finite measure. Then

$$
\|\chi_E\|_{G\Gamma(p,m,w)} = \Big(\int_0^b \min(t,\mu(E))^{m/p} w(t)\,dt\Big)^{1/m} < \infty,
$$

which establishes (P4).

Finally, if $b=\infty$ and f is a non-negative measurable function on $\mathcal{R}$, then

$$
\begin{aligned}
\Big(\int_0^b \Big(\int_0^t f^*(s)^p\,ds\Big)^{m/p} w(t)\,dt\Big)^{1/m} &\ge \Big(\int_{\mu(E)}^b \Big(\int_0^{\mu(E)} f^*(s)^p\,ds\Big)^{m/p} w(t)\,dt\Big)^{1/m} \\
&\ge \Big(\int_{\mu(E)}^b \Big(\int_E f(s)^p\,ds\Big)^{m/p} w(t)\,dt\Big)^{1/m} \\
&= \|f\|_{L^p(E)} \Big(\int_{\mu(E)}^b w(t)\,dt\Big)^{1/m} \\
&\ge C_E \|f\|_{L^1(E)} \Big(\int_{\mu(E)}^b w(t)\,dt\Big)^{1/m},
\end{aligned}
$$

for an appropriate C_E, while, when $b<\infty$, we have

$$
\begin{aligned}
\Big(\int_0^b \Big(\int_0^t f^*(s)^p\,ds\Big)^{m/p} w(t)\,dt\Big)^{1/m} &\ge \Big(\int_{b/2}^b \Big(\int_0^{b/2} f^*(s)^p\,ds\Big)^{m/p} w(t)\,dt\Big)^{1/m} \\
&= \Big(\int_0^{b/2} f^*(s)^p\,ds\Big)^{1/p} \Big(\int_{b/2}^b w(t)\,dt\Big)^{1/m} \\
&\ge \frac{1}{2}\Big(\int_0^b f^*(s)^p\,ds\Big)^{1/p} \Big(\int_{b/2}^b w(t)\,dt\Big)^{1/m} \\
&\ge C_E \|f\|_{L^1(E)} \Big(\int_{\mu(E)}^b w(t)\,dt\Big)^{1/m},
\end{aligned}
$$

showing (P5) again. ■

Proof of Theorem 1.4. The assumption (1.10) obviously implies (1.9). Therefore, we know from Theorem 1.3 that $G\Gamma(m,p,w)(0,b)$ is a rearrangement-invariant Banach function space. We can thus apply [CPS1, Theorem 6.1] (for the first-order case see also [CP1, Theorem 3.5]), which states that the Sobolev embedding (1.11) is equivalent to the condition

$$t^{-1+k/n} \in G\Gamma(m,p,w)'(0,b). \tag{2.1}$$

So, we only have to analyze when (2.1) is satisfied.

First note that if $k \geq n$, then in fact obviously

$$t^{-1+k/n} \in L^\infty(0,b),$$

which immediately implies (2.1), since, by a classical fact, the space L^∞ is embedded into any rearrangement-invariant space over a finite-measure space (and we have $b < \infty$ here).

Assume now that $k \leq n-1$. We then denote $g(t) = t^{-1+k/n}$ for $t \in (0,b)$ and note that $g^{**} \approx g^* = g$ on $(0,b)$.

Let $m = 1$. Then it follows from Theorem 1.1(i)&(ii) that

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b/2)} \frac{t^{k/n}}{u(t)}$$

if $p = 1$, and

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b/2)} \Big(\int_t^b g^{**}(s)^{p'}\,ds\Big)^{1/p'} \frac{t^{1/p}}{u(t)}$$

if $p \in (1,\infty)$. Now, a calculation shows that, for $t \in (0,b/2)$, we have

$$\begin{aligned}\Big(\int_t^b g^{**}(s)^{p'}\,ds\Big)^{1/p'} &\approx \Big(\int_t^b s^{(-1+k/n)p'}\,ds\Big)^{1/p'}\\ &\approx \begin{cases} t^{k/n-1/p} & \text{if } 1<p<n/k,\\ \big(\log\frac{b}{t}\big)^{1-k/n} & \text{if } p=n/k,\\ 1 & \text{if } p\in(n/k,\infty).\end{cases}\end{aligned}$$

This establishes (i)–(iii). The remaining three statements can be proved in an analogous way. ∎

Proof of Theorem 1.5. Assume first that (1.12) holds. Let $\{E_n\}$ be a sequence of μ-measurable subsets of $\mathcal{R}$ with $E_n \downarrow \emptyset$, and let $f \in G\Gamma(p,m,w)$. Then

$$\begin{aligned}\|f\chi_{E_n}\|^m_{G\Gamma(p,m,w)} &= \int_0^b \Big(\int_0^t (f\chi_{E_n})^*(s)^p\,ds\Big)^{m/p} w(t)\,dt\\ &= \int_0^b \Big(\int_0^{\min(t,\mu(E_n))} f^*(s)^p\,ds\Big)^{m/p} w(t)\,dt.\end{aligned}$$

Since $b<\infty$, $E_n\downarrow\emptyset$ implies $\mu(E_n)\downarrow 0$. Therefore,

$$\lim_{n\to\infty}\int_0^{\min(t,\mu(E_n))} f^*(s)^p\,ds=0$$

for all $t\in(0,b)$. Consequently,

$$\lim_{n\to\infty}\Big(\int_0^t (f\chi_{E_n})^*(s)^p\,ds\Big)^{m/p}w(t)=0.$$

By the Dominated Convergence Theorem with $(\int_0^t f^*(s)^p\,ds)^{m/p}w(t)$ as an integrable majorant, we obtain $\|f\chi_{E_n}\|_{G\Gamma(p,m,w)}\to 0$, as desired.

Assume now that (1.13) is satisfied and $b=\infty$. Then, by the assumption, for every $f\in G\Gamma(p,m,w)$ and $k\in\mathbb{N}$, the set $F_k=\{x\in\mathcal{R}\colon f(x)\geq 1/k\}$ has finite measure. Let $\{E_n\}$ be a sequence of μ-measurable subsets of $\mathcal{R}$ satisfying $E_n\downarrow\emptyset$. Set $f_n=f\chi_{E_n}$, $f_{n,k}=f_n\chi_{F_k}$, and choose $\varepsilon>0$. Then

$$\|f_n\|_{G\Gamma(p,m,w)}\leq\|f_n-f_{n,k}\|_{G\Gamma(p,m,w)}+\|f_{n,k}\|_{G\Gamma(p,m,w)}.$$

Fix $k\in\mathbb{N}$. Then, for every $n\in\mathbb{N}$,

$$\begin{aligned}\|f_n-f_{n,k}\|^m_{G\Gamma(p,m,w)}&=\int_0^\infty\Big(\int_0^t (|f-f\chi_{F_k}|\chi_{E_n})^*(s)^p\,ds\Big)^{m/p}w(t)\,dt\\&\geq\int_0^\infty\Big(\int_0^t (|f-f\chi_{F_k}|\chi_{E_{n+1}})^*(s)^p\,ds\Big)^{m/p}w(t)\,dt\\&=\|f_{n+1}-f_{n+1,k}\|^m_{G\Gamma(p,m,w)}.\end{aligned}$$

Now, for a change, fix $n\in\mathbb{N}$. Then, for every $k\in\mathbb{N}$,

$$\|f_n-f_{n,k}\|^m_{G\Gamma(p,m,w)}\leq\int_0^\infty\Big(\int_0^t (\min(f(y),1/k))^*(s)^p\,ds\Big)^{m/p}w(t)\,dt.$$

For every $t>0$ we clearly have

$$\lim_{k\to\infty}\Big(\int_0^t (\min(f(s),1/k))^p\,ds\Big)^{m/p}w(t)=0.$$

Therefore,

$$\lim_{k\to\infty}\|f_n-f_{n,k}\|_{G\Gamma(p,m,w)}=0,$$

by the Dominated Convergence Theorem. Observe that, for every $k\in\mathbb{N}$,

$$\lim_{n\to\infty}\|f_{n,k}\|_{G\Gamma(p,m,w)}=0,$$

which follows from the first part of the proof since the sets F_k have finite measure.

We first choose $k \in \mathbb{N}$ such that $\|f_1 - f_{1,k}\|_{G\Gamma(p,m,w)} < \varepsilon$. With this k now fixed, we find $n_0 \in \mathbb{N}$ such that $\|f_{n,k}\|_{G\Gamma(p,m,w)} < \varepsilon$ for all $n > n_0$. Then

$$\begin{aligned}\|f_n\|_{G\Gamma(p,m,w)} &\le \|f_n - f_{n,k}\|_{G\Gamma(p,m,w)} + \|f_{n,k}\|_{G\Gamma(p,m,w)}\\ &\le \|f_{1,k} - f_1\|_{G\Gamma(p,m,w)} + \|f_{n,k}\|_{G\Gamma(p,m,w)}\\ &\le 2\varepsilon,\end{aligned}$$

establishing the 'if' part of the theorem.

To prove the 'only if' part, assume that $b = \infty$ and $\int_0^\infty w(t)t^{m/p}\,dt < \infty$. Since $\mathcal{R}$ is σ-finite, there exists a sequence of finite-measure sets $\{D_n\}$ satisfying $D_n \uparrow \mathcal{R}$. For $n \in \mathbb{N}$, define $E_n = \mathcal{R} \setminus D_n$, and set $f \equiv 1$ on $\mathcal{R}$. Then $E_n \downarrow \emptyset$ and, for every $n \in \mathbb{N}$, $(f\chi_{E_n})^* \equiv 1$ on $(0,\infty)$. Therefore, for every $n \in \mathbb{N}$, we have

$$\|f\|_{G\Gamma(p,m,w)} = \|f\chi_{E_n}\|_{G\Gamma(p,m,w)} = \int_0^\infty w(t)t^{m/p}\,dt,$$

which means, due to the assumption, that f belongs to $G\Gamma(p,m,w)$ but does not have absolutely continuous norm. ■

Proof of Theorem 1.6. Let $p, m \in (1,\infty)$ and let w be a weight on $(0,b)$. Assume first that (1.5) holds. Then, by Theorem 1.1(v),

$$\|g\|_{(G\Gamma(p,m,w))'} \approx \bigg(\int_0^{b/2} \bigg(\int_t^b g^{**}(s)^{p'}\,ds\bigg)^{m'/p'} \frac{t^{m'/p+m/p-1}\int_0^t w(s)s^{m/p}\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt\bigg)^{1/m'}.$$

Let $\{E_n\}$ be a sequence of sets such that $E_n \downarrow \emptyset$. Denote $f_n = f\chi_{E_n}$ and $F_n(t) = \int_0^t f_n^*(s)\,ds$, $t \in (0,b)$. For every $f \in G\Gamma(p,m,w)'$, the right side of the last displayed formula is finite. Therefore, by the Dominated Convergence Theorem, it only suffices to verify that

$$\lim_{n\to\infty} F_n(t) = 0 \quad \text{for every } t \in (0,\infty).$$

Fix $t \in (0,b)$. Then the sequence $F_n(t)$ is non-increasing. Therefore the limit $\lim_{n\to\infty} F_n(t)$ exists. Suppose that $\lim_{n\to\infty} F_n(t) > \varepsilon$ for some $\varepsilon > 0$. Then the sets

$$P_n = \left\{ s \in (0,t) \colon f_n^*(s) > \frac{\varepsilon}{2t} \right\}$$

have positive measure. Clearly, $P_n \supset P_{n+1}$ for every $n \in \mathbb{N}$. Moreover,

$$\int_{(0,t)\setminus P_n} f_n^*(s)\,ds \le \frac{\varepsilon}{2},$$

hence

$$\int_{P_n} f_n^*(s)\,ds \geq \frac{\varepsilon}{2}.$$

Furthermore, if $|P_n| \to 0$ then

$$\int_{P_n} f^*(s)\,ds \geq \int_{P_n} f_n^*(s)\,ds \geq \frac{\varepsilon}{2},$$

which is impossible due to the absolute continuity of the Lebesgue integral. So, the only option left is $|\bigcap P_n| > 0$. That, however, leads to a contradiction since

$$\Big|\bigcap P_n\Big| = \mu\{x \in \mathcal{R} \colon f_n(x) > \varepsilon \text{ for every } n \in \mathbb{N}\}.$$

Therefore $\lim_{n\to\infty} F_n(t) = 0$.

If (1.5) is violated, then the above proof works just as well, the only extra observation we have to make is that all functions in $L^{p'}$ and in L^1 have absolutely continuous norms. ■

Proof of Theorem 1.7. A Banach function space X is reflexive if and only if both X and its associate space X' have absolutely continuous norm [BS, Chapter 1, Corollary 4.4]. Thus, the assertion follows from Theorems 1.5 and 1.6. ■

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