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Wave equation and multiplier estimates on $ax + b$ groups

by

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Abstract. Let L be the distinguished Laplacian on certain semidirect products of $\mathbb{R}$ by $\mathbb{R}^n$ which are of $ax+b$ type. We prove pointwise estimates for the convolution kernels of spectrally localized wave operators of the form $e^{it\sqrt{L}}\psi(\sqrt{L}/\lambda)$ for arbitrary time t and arbitrary $\lambda > 0$, where ψ is a smooth bump function supported in $[-2,2]$ if $\lambda \leq 1$ and in $[1,2]$ if $\lambda \geq 1$. As a corollary, we reprove a basic multiplier estimate of Hebisch and Steger [Math. Z. 245 (2003)] for this particular class of groups, and derive Sobolev estimates for solutions to the wave equation associated to L. There appears no dispersive effect with respect to the L^∞-norms for large times in our estimates, so that it seems unlikely that non-trivial Strichartz type estimates hold.

1. Introduction. We denote by G the semidirect product $G = \mathbb{R} \ltimes \mathbb{R}^n$, endowed with the group law

$$(x,y)(x',y') = (x+x', y+e^x y').$$

This subgroup of the affine group of $\mathbb{R}^n$ is a solvable Lie group with exponential volume growth. We call G an *$ax+b$ group*.

A basis for the Lie algebra of left-invariant vector fields is given by

$$X = \partial_x, \quad Y_1 = e^x\partial_{y_1}, \quad \ldots, \quad Y_n = e^x\partial_{y_n}. \tag{1.1}$$

We define the distinguished left-invariant Laplacian to be the second order differential operator

$$L = -X^2 - \sum_{j=1}^n Y_j^2. \tag{1.2}$$

A right-invariant Haar measure on G is given by

$$dg = dx\,dy_1\ldots dy_n.$$

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We will use this right-invariant measure in notions such as $L^p(G)$. An operator on function spaces on G is given by a right convolution kernel k if

$$(1.3)\qquad Tf(g') = f * k(g') = \int f(g^{-1})k(gg')\,dg \quad \text{for } f \in \mathcal{D}(G).$$

The distinguished Laplacian L has a self-adjoint extension in $L^2(G)$ ([13]), and thus we can use spectral calculus to define the operators

$$(1.4)\qquad e^{it\sqrt{L}}m(L),$$

where the multiplier m will lie in a suitable symbol class. The main purpose of this article is to prove Theorem 6.1, which states uniform (with respect to λ and t) pointwise estimates for the convolution kernels of spectrally localized multiplier operators of the form

$$(1.5)\qquad e^{it\sqrt{L}}\psi(\sqrt{L}/\lambda),$$

for arbitrary time $t \in \mathbb{R}$ and arbitrary $\lambda > 0$, where ψ is a bump function supported in $[-2, 2]$ if $\lambda \le 1$, and in $[1, 2]$ if $\lambda \ge 1$. We can use these estimates to give a new proof of the basic multiplier estimate used in [6] (see Theorem 6.1 of [6]), which is based entirely on the wave equation, at least for the class of $ax + b$ groups under consideration.

As a corollary of our main theorem, we shall also deduce Sobolev estimates for solutions to the wave equation associated to L.

Our estimates are mainly controlled by the left-invariant Riemannian distance

$$(1.6)\qquad R = R(x, y) := \operatorname{arcch}\left(\operatorname{ch} x + \frac{1}{2}\,\|y\|^2 e^{-x}\right)$$

of a point (x, y) to the identity element on G, where $y = (y_1, \dots, y_n)$, $\|y\|^2 = \sum_{j=1}^n y_j^2$ and arcch is understood to map $[1, \infty)$ to $[0, \infty)$.

We remark that, for $n = 2$, our main theorem could also be deduced from a transfer principle of Hebisch [5]. In that special situation, the group G is of the form AN where KAN is the Iwasawa decomposition of the complex Lie group $\mathrm{SL}_2(\mathbb{C})$, and Hebisch introduces a mapping from radial functions f on $\mathbb{R}^3$ to functions on the group G, given by

$$Tf(x, y) = Ce^{-x}\,\frac{R}{\operatorname{sh} R}\,f(R),$$

which preserves the L^1-norm and commutes with convolution and with application of the corresponding Laplacians. This allows one to deduce our main Theorem 6.1 from the analogous theorem on $\mathbb{R}^3$. However, this transfer principle is somewhat misleading, since it would suggest for higher dimensions estimates different from the ones which actually hold on G.

We also remark that our results should extend to the distinguished Laplacians which arise from the Laplace–Beltrami operators on rank one symmetric spaces of non-compact type by means of conjugation with the square root of the modular function (see e.g. [1]), by means of refinements of the estimates for spherical functions in [9]. However, we shall not pursue this here, since we prefer to present the completely self-contained proof which we can give for the affine group.

In Section 2, we prove a lemma which describes integration of radial functions over affine groups. In Section 3, we derive an explicit kernel representation for the resolvent of L using the theory of hypergeometric functions. It is known ([8], [4]) that the resolvent kernels can be expressed in terms of special functions, and this has been used in [3] to prove estimates for singular integrals related to L. We chose to derive the integral formula for the resolvent kernel from scratch, even though this could have been done quoting tables of special functions such as [2]. We hope some readers will find benefit of our explicit calculations.

Section 4 presents a subordination argument to obtain convenient integral representations for the convolution kernels of the operators (1.5).

In Section 5, we prove some asymptotic formulas for the hypergeometric functions appearing in Section 3.

Section 6 assembles the results of the previous sections to prove Theorem 6.1, which states pointwise estimates for the convolution kernel of (1.5), and Proposition 6.3, which states $L^1(G)$ estimates for these kernels. We also reprove a multiplier theorem of [6]. In Section 7, we improve on Theorem 6.1 to obtain L^∞ bounds for the convolution kernel of (1.5).

In Section 8, we prove growth estimates for the wave propagator associated with L using spectrally defined Sobolev norms.

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2. Integration of radial functions on the affine group. In this section we discuss integration of "radial" functions. The results will be useful in the estimation of L^1-norms of convolution kernels for functions of the Laplacian L.

Bending the notion of radial function, by *radial function* we mean a function of the type

$$e^{-nx/2}g(R)$$

with R as in (1.6).

We briefly motivate the special form of the radial variable R. For $n = 1$, the affine group G is a subgroup of the group of conformal automorphisms of the upper half plane via the identification of (x, y) with the map $z \mapsto e^x z + y$.

This subgroup acts simply transitively on the upper half plane, thus we can naturally identify G as a set with the upper half plane, identifying the neutral element with the point i. In particular, the hyperbolic metric on the upper half plane turns out to be left-invariant. The pull back of the hyperbolic distance from a point z to the point i (which is $\log|(1+\varrho)/(1-\varrho)|$ with $\varrho = |z-i|/|z+i|$) gives a natural "radial" distance from the origin in the affine group given by

$$\operatorname{arcch}\left(\operatorname{ch} x + \frac{1}{2}\, y^2 e^{-x}\right).$$

Since $R(x,y) = R((x,y)^{-1})$ there is no difference between a left and a right radial variable.

LEMMA 2.1. *Given a function $g : \mathbb{R}^+ \to \mathbb{C}$, we have*

$$\int_G e^{-nx/2} g(R(x,y))\, dx\, dy = \int_0^\infty g(R) J(R)\, dR,$$

where

$$J(R) \sim R^n \quad \text{if } R \le 1, \tag{2.1}$$

$$J(R) \sim R e^{nR/2} \quad \text{if } R \ge 1, \tag{2.2}$$

and $a \sim b$ means that each of the two numbers can be bounded by a constant times the other, the constant depending only on n.

Proof. Define

$$B(r) = \int_{R(x,y)\le r} e^{-nx/2}\, dx\, dy.$$

Then $J(r) = B'(r)$ for all $r \ge 0$. Thus we have to estimate $B'(r)$. Observe that $R(x,y) \le r$ implies $x \le r$ and

$$\|y\| \le (2e^x(\operatorname{ch} r - \operatorname{ch} x))^{1/2}.$$

Doing the y-integration and letting V_n be the Euclidean volume of the unit ball in $\mathbb{R}^n$, we obtain

$$B(r) = V_n \int_{-r}^{r} e^{-nx/2} (2e^x(\operatorname{ch} r - \operatorname{ch} x))^{n/2}\, dx.$$

Simplification and differentiation gives

$$B'(r) = \frac{n}{2}\, 2^{n/2} V_n \int_{-r}^{r} (\operatorname{ch} r - \operatorname{ch} x)^{n/2-1} \operatorname{sh} r\, dx.$$

First assume $r \ge C \gg 1$. We break the integral in the previous display into the sum of

$$I_1 = (\operatorname{sh} r) \int\limits_{|x|<r-1} (\operatorname{ch} r - \operatorname{ch} x)^{n/2-1}\, dx, \tag{2.3}$$

$$I_2 = (\operatorname{sh} r) \int\limits_{r-1\le|x|\le r} (\operatorname{ch} r - \operatorname{ch} x)^{n/2-1}\, dx. \tag{2.4}$$

In the first integral, we have $|x| < r - 1$ and thus

$$\operatorname{ch} R - \operatorname{ch} x \sim e^r, \qquad \operatorname{sh} r \sim e^r.$$

Therefore

$$I_1 \sim re^{nr/2}.$$

Thus I_1 has already the correct order of magnitude which we need to show for I_1+I_2. Since I_1 and I_2 are positive, we only need an upper bound for I_2. Since in the domain of integration of I_2 we still have

$$\operatorname{ch} r - \operatorname{ch} x \le 2e^r, \qquad \operatorname{sh} r \le e^r,$$

we can do the same calculation as before to obtain

$$I_2 \le Ce^{rn/2}.$$

If $1 \le r < C$, we estimate the integrand above by a constant from above and by a positive constant from below in the region $|x| < r/2$ to obtain the desired estimate.

Now assume $r < 1$. We do a similar splitting of the integral as before, now into the regions $|x| < r/2$ and $r/2 \le |x| \le r$. Call the corresponding integrals I_1 and I_2. In the domain $|x| < r/2$ we have

$$\operatorname{ch} r - \operatorname{ch} x \sim r^2, \qquad \operatorname{sh} r \sim r, \qquad e^x \sim 1.$$

Hence

$$I_1 \sim r^{n-1} \int\limits_0^{r/2} dx \sim r^n.$$

As before, in the domain $r/2 \le |x| \le r$ we have the same upper bounds as in the domain $|x| < r/2$, and thus $I_2 \le Cr^n$. This completes the proof of Lemma 2.1. ■

3. An explicit kernel for the resolvent of L. Assume that k is an integrable function on G such that for every compactly supported smooth function φ we have (in the distributional sense, for this purpose we identify G with $\mathbb{R}^{n+1}$)

$$\int \varphi(g)(L-\lambda)k(g)\, dg = \varphi(0). \tag{3.1}$$

Then, by left-invariance of L and right-invariance of dg,

$$\begin{aligned}(L-\lambda)\int \varphi(g^{-1})k(gg')\,dg &= \int \varphi(g^{-1})[(L-\lambda)k](gg')\,dg \\ &= \int \varphi(g'g^{-1})[(L-\lambda)k](g)\,dg = \varphi(g').\end{aligned}$$

Thus the resolvent operator $(L-\lambda)^{-1}$ is given by right convolution with k, which extends to a bounded operator on $L^2(G)$. The following lemma describes such a fundamental solution k.

LEMMA 3.1. *Assume $\lambda \in \mathbb{C} \setminus [0,\infty)$ and choose $\nu := i\sqrt{\lambda}$ with strictly negative real part. Then the resolvent operator $(L-\lambda)^{-1}$ is given by right convolution with the kernel k defined by ($R = R(x,y)$ as in (1.6))*

$$k(x,y) = (-1)^l \frac{2^{-1-n/2}\pi^{-n/2}}{\Gamma(1-n/2+l)}\, e^{-nx/2} \int_R^\infty D^l_{\mathrm{sh},v}[e^{\nu v}](\operatorname{ch} v - \operatorname{ch} R)^{-n/2+l}\,dv,$$

where l is any integer with $-n/2+l > -1$ and we have written $D^l_{\mathrm{sh},v}$ for the lth power of $D_{\mathrm{sh}} : g \to D(g/\mathrm{sh})$ acting on the v variable.

Moreover, the kernel k satisfies the estimate

$$\int_{B_R} |k|\,dx\,dy \le C_n(1+|\nu|)^{n/2}\Big[1 + \int_0^R e^{(\operatorname{Re}\nu)r} r\,dr\Big], \tag{3.2}$$

where B_R is the ball of radius R about the origin in G. In particular, $k \in L^1(G)$.

Proof. Fix λ. For $n > 1$, we will show that k as defined in the lemma is integrable and smooth outside the origin, satisfies $(L-\lambda)k = 0$ outside the origin, and has asymptotics

$$k(x,y) = 2^{-2}\pi^{-(n+1)/2}\Gamma((n-1)/2)(x^2+\|y\|^2)^{-(n-1)/2} + O(x^2+\|y\|^2)^{-(n-2)/2}$$

near the origin. This will give, for any compactly supported φ,

$$\int \varphi(g)(L-\lambda)k(g)\,dg = \int \eta(g)\varphi(g)(L-\lambda)k(g)\,dg,$$

where η is a smooth cutoff function at scale ε, i.e. constantly 1 on an ε-neighbourhood around the origin and zero outside a 2ε-neighbourhood, with the usual control of derivatives. By definition of the distributional derivative, the last display becomes

$$\int (L-\lambda)(\eta\varphi)(g)k(g)\,dg.$$

If we subtract the leading order term from k in this integral, then the remaining integral tends to zero as $\varepsilon \to 0$. Thus we may replace k by the leading order term. Also, we may disregard in the expansion of $(L-\lambda)(\eta\psi)$ by Leibniz' rule all terms other than those taking two derivatives of η, and also we may approximate the coefficient e^x by 1. Thus the last display is equal to ($\Delta = -\sum \partial_j^2$)

$$2^{-2}\pi^{-(n+1)/2}\Gamma((n-1)/2)\int \Delta(\eta\varphi)(x^2+|y|^2)^{-(n-1)/2}\,dx\,dy.$$

Now, standard theory in $\mathbb{R}^m$ ([16, pp. 211, 262]) the last display is equal to $\varphi(0)$, which was to be proved. If $n=1$, we use the same approach; here the asymptotic behaviour of k near the origin is

$$2^{-2}\pi^{-1}\log(x^2+\|y\|^2)+O(1).$$

Now we prove the properties of k claimed above. Define

$$d(x,y) := \operatorname{ch}(R(x,y)) = \operatorname{ch} x + \frac{1}{2}\,\|y\|^2 e^{-x}.$$

Then the kernel k is of the form

$$k(x,y) = e^{-nx/2} f(d(x,y)), \tag{3.3}$$

with a function f which is smooth on $(1,\infty)$. We claim that $(L-\lambda)k=0$ outside the origin is equivalent to f satisfying the ordinary differential equation

$$-\frac{n^2}{4}\,f(d)-(n+1)df'(d)-(d^2-1)f''(d)=\lambda f(d) \tag{3.4}$$

for $d>1$. To verify the claim, we observe

$$\begin{aligned}(Xd)^2+\sum_{j=1}^n (Y_jd)^2 &= \left(\frac{1}{2}\,e^x-\frac{1}{2}\,e^{-x}-\frac{1}{2}\,\|y\|^2e^{-x}\right)^2+\|y\|^2\\ &= \left(\frac{1}{2}\,e^x+\frac{1}{2}\,e^{-x}+\frac{1}{2}\,\|y\|^2e^{-x}\right)^2-1=d^2-1\end{aligned}$$

and

$$\begin{aligned}&X^2d+\sum_{j=1}^n Y_j^2d-nXd\\ &\quad=\left(\frac{1}{2}\,e^x+\frac{1}{2}\,e^{-x}+\frac{1}{2}\,\|y\|^2e^{-x}\right)+ne^x-n\left(\frac{1}{2}\,e^x-\frac{1}{2}\,e^{-x}-\frac{1}{2}\,\|y\|^2e^{-x}\right)\\ &\quad=(n+1)\left(\frac{1}{2}\,e^x+\frac{1}{2}\,e^{-x}+\frac{1}{2}\,\|y\|^2e^{-x}\right)=(n+1)d.\end{aligned}$$

Therefore we have

$$
\begin{aligned}
&\Big(X^2+\sum_{j=1}^{n} Y_j^2\Big)e^{-nx/2}f(d(x,y))\\
&= e^{-nx/2}\\
&\quad\times\left[((Xd)^2+(Yd)^2)f''(d)+(X^2d+Y^2d)f'(d)-n(Xd)f'(d)+\frac{n^2}{4}f(d)\right]\\
&= e^{-nx/2}\\
&\quad\times\left[\frac{n^2}{4}f(d(x,y))+(n+1)d(x,y)f'(d(x,y))+(d(x,y)^2-1)f''(d(x,y)\right]
\end{aligned}
$$

where $(Yd)^2=\sum(Y_jd)^2$ and $Y^2d=\sum Y_j^2d$. Thus if f satisfies the ordinary differential equation (3.4) on $(1,\infty)$, then $(L-\lambda)k=0$ outside the origin, and conversely. Equation (3.4) is a classical hypergeometric differential equation. The associated Riemann symbol [10] is

$$
P\begin{pmatrix} -1 & 1 & \infty & \\ 0 & 0 & n/2+\nu & d \\ -(n-1)/2 & -(n-1)/2 & n/2-\nu & \end{pmatrix}. \tag{3.5}
$$

There is a two-dimensional space of solutions f. However, there is only a one-dimensional space of solutions (those with leading asymptotics $d^{-n/2+\nu}$ as $d\to\infty$) which make k as defined above integrable on G. Of course at most one of these solutions is normalized properly to make k a fundamental solution.

The following lemma provides an explicit solution of the differential equation (3.4) in a certain complex region of parameters n and ν.

LEMMA 3.2. *Assume* $-\operatorname{Re} n/2>-1$ *and* $\operatorname{Re}\nu-\operatorname{Re} n/2<0$. *Then the function*

$$
f_0(d)=\int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v-d)^{-n/2}\,dv, \tag{3.6}
$$

defined for $d>1$, *satisfies the ordinary differential equation* (3.4).

Proof. Under the stated assumptions on n and ν, the integrand defining f_0 is absolutely integrable. We first assume that $-\operatorname{Re} n/2>1$.

Differentiating under the integral sign gives

$$
\begin{aligned}
f_0'(d)&=\frac{n}{2}\int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v-d)^{-n/2-1}\,dv,\\
f_0''(d)&=\frac{n}{2}\Big(\frac{n}{2}+1\Big)\int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v-d)^{-n/2-2}\,dv.
\end{aligned}
$$

Hence,

$$(n+1)df_0' = \left(\frac{n^2}{2}+\frac{n}{2}\right)\left(-f_0 + \int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v - d)^{-n/2-2}((\operatorname{ch} v)^2 - d\operatorname{ch} v)\, dv\right),$$

$$(d^2-1)f_0'' = \left(\frac{n^2}{4}+\frac{n}{2}\right)\left(f_0 - \int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v - d)^{-n/2-2}((\operatorname{ch} v)^2 - 2d\operatorname{ch} v + 1)\, dv\right).$$

It thus remains to prove

$$\int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v - d)^{-n/2-2}\left[-\frac{n^2}{4}(\operatorname{ch} v)^2 - \frac{n}{2} d\operatorname{ch} v + \frac{n^2}{4} + \frac{n}{2}\right] dv = \lambda \int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v - d)^{-n/2}\, dv.$$

However, by partial integrations, the right hand side is equal to

$$\frac{\lambda n}{2\nu}\int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v - d)^{-n/2-1}\operatorname{sh} v\, dv = -\frac{\lambda n}{2\nu^2}\int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v - d)^{-n/2-1}\operatorname{ch} v\, dv + \frac{\lambda}{\nu^2}\left(\frac{n^2}{4}+\frac{n}{2}\right)\int_{\operatorname{arcch} d}^{\infty} e^{\nu v}(\operatorname{ch} v - d)^{-n/2-2}(\operatorname{sh} v)^2\, dv.$$

This proves the lemma for $-\operatorname{Re} n/2 > 1$ since $\lambda/\nu^2 = -1$ and $(\operatorname{sh} v)^2 = (\operatorname{ch} v)^2 - 1$.

The case $-\operatorname{Re} n/2 > -1$ then follows by analytic continuation. ∎

Clearly, any analytic continuation of (3.6) in the parameters n and ν also satisfies the ordinary differential equation (3.4). The following lemma provides explicit expressions for such analytic continuations.

LEMMA 3.3. *Assume* $-\operatorname{Re} n/2 > -1$ *and* $\operatorname{Re}\nu - \operatorname{Re} n/2 < 0$. *Then, for each integer* $l \geq 0$ *and also for* $l = -1$ *in case* $\operatorname{Re} n < 0$, *we have the identity*

$$\Gamma\left(-\frac{n}{2}+1\right)^{-1} f_0(d) = (-1)^l \Gamma\left(l - \frac{n}{2}+1\right)^{-1} \int_{\operatorname{arcch} d}^{\infty} D^l_{\operatorname{sh},v}[e^{\nu v}](\operatorname{ch} v - d)^{-n/2+l}\, dv.$$

The right hand side provides the unique analytic continuation of the left *hand side to the parameter region* $-\operatorname{Re} n/2+l>-1$ *and* $\operatorname{Re}\nu-\operatorname{Re} n/2<0$. *The right hand side satisfies the ordinary differential equation* (3.4) *in this parameter region. Here we have set*, *for* $\operatorname{Re}\nu<0$,

$$D_{\mathrm{sh},v}^{-1}[e^{\nu v}]=\nu^{-1}(\operatorname{sh} v)e^{\nu v}.$$

Proof. By partial integration, we have for $l\ge 1$, and also for $l=0$ if $\operatorname{Re} n<0$,

$$\begin{aligned}\Gamma(l-n/2+1)^{-1}&\int_{\operatorname{arcch} d}^{\infty} D_{\mathrm{sh},v}^{l}[e^{\nu v}](\operatorname{ch} v-d)^{-n/2+l}\,dv\\ &=-\Gamma(l-n/2+1)^{-1}\int_{\operatorname{arcch} d}^{\infty} D_{\mathrm{sh},v}^{l-1}[e^{\nu v}](\operatorname{sh} v)^{-1}D_v[(\operatorname{ch} v-d)^{-n/2+l}]\,dv\\ &=-\Gamma(l-n/2)^{-1}\int_{\operatorname{arcch} d}^{\infty} D_{\mathrm{sh},v}^{l-1}[e^{\nu v}](\operatorname{ch} v-d)^{-n/2+l-1}\,dv.\end{aligned}$$

By induction, this proves the identity of the lemma. ∎

The Riemann symbol (3.5) allows f_0 to have the desired order $-(n-1)/2$ at 1 or the undesired order 0 at 1. Fortunately, the former is the case, as one can infer from the following lemma or deduce from an argument that f_0 cannot be holomorphic at 1. To normalize the resolvent kernel properly, we need to calculate the exact asymptotic behaviour of f_0 near 1.

LEMMA 3.4. *Let* $\operatorname{Re}\nu-\operatorname{Re} n/2<0$ *and assume* n *is not an odd negative integer. For* $R>0$ *near* 0 *and any* $\varepsilon\in(0,1)$ *we have*

$$\Gamma(1-n/2)^{-1}f_0(\operatorname{ch} R)=2^{n/2-1}\pi^{-1/2}\Gamma((n-1)/2)R^{1-n}+O(R^{1-\operatorname{Re} n+\varepsilon})$$

if $n\ne 1$, *and*

$$\Gamma(1-n/2)^{-1}f_0(\operatorname{ch} R)=2^{1/2}\pi^{-1/2}|\log R|+O(1)$$

in case $n=1$. *Here the left hand side is to be understood as an analytic function in the sense of Lemma* 3.3.

Proof. We do the case $1<\operatorname{Re} n<2$. The general case follows by similar calculations or by methods of analytic continuation.

Assume $R\ll 1$. We split the integral

$$f_0(\operatorname{ch} R)=\int_R^{\infty} e^{\nu v}(\operatorname{ch} v-\operatorname{ch} R)^{-n/2}\,dv$$

into the integrals over the intervals $[R,R^{1-\varepsilon}]$ and $[R^{1-\varepsilon},\infty]$. Since $\operatorname{ch} v-1>v^2$, the integral over the second interval is bounded by

$$C\int_{R^{1-\varepsilon}}^{\infty} v^{-\operatorname{Re} n}\,dv\le CR^{(1-\operatorname{Re} n)(1-\varepsilon)}.$$

Thus this integral is negligible. For $v < R^{1-\varepsilon}$ we write

$$e^{\nu v} = 1 + O(R^{1-\varepsilon}), \qquad \operatorname{ch} v - \operatorname{ch} R = \frac{1}{2}\,(v^2 - R^2)(1 + O(R^{2-\varepsilon})).$$

Thus the first integral is

$$2^{n/2} \int\limits_R^{R^{1-\varepsilon}} [v^2 - R^2]^{-n/2}\, dv\, (1 + O(R^{1-\varepsilon})).$$

By an argument as before, we can change the domain of integration back to $[R, \infty]$ producting at most an error of order $R^{(1-n)(1-\varepsilon)}$. Thus we have to get the asymptotics of the integral

$$2^{n/2} \int\limits_R^{\infty} [v^2 - R^2]^{-n/2}\, dw = 2^{n/2-1} R^{1-n} \int\limits_1^{\infty} [w-1]^{-n/2} w^{-1/2}\, dw.$$

The last integral can be expressed in terms of the Gamma function:

$$\begin{aligned}
&\int\limits_1^{\infty} [w-1]^{-n/2} w^{-1/2}\, dw \\
&\quad = \Gamma(1/2)^{-1} \int\limits_1^{\infty} [w-1]^{-n/2} \int\limits_0^{\infty} t^{1/2} e^{-tw}\, \frac{dt}{t}\, dw \\
&\quad = \pi^{-1/2} \int\limits_0^{\infty} t^{1/2} e^{-t} \int\limits_0^{\infty} r^{-n/2} e^{-tr}\, dr\, \frac{dt}{t} \\
&\quad = \pi^{-1/2} \Gamma(1-n/2) \int\limits_0^{\infty} t^{n/2-1/2} e^{-t}\, \frac{dt}{t} = \pi^{-1/2}\Gamma(1-n/2)\Gamma((n-1)/2).
\end{aligned}$$

This proves the asymptotics claimed in the lemma. ∎

With Lemma 3.4 we have completed the proof of the identities for the resolvent kernel k claimed in Lemma 3.1.

It remains to prove the L^1-estimates for k (cf. [8]). We can assume that n is a positive integer. We have to estimate the integral

$$\int\limits_R^{\infty} D^l_{\operatorname{sh},v}[e^{\nu v}](\operatorname{ch} v - \operatorname{ch} R)^{l-n/2}\, dv = H^1_\nu(R) + H^2_\nu(R),$$

where

$$H^1_\nu(R) = \int\limits_R^{R+2} D^l_{\operatorname{sh},v}[e^{\nu v}](\operatorname{ch} v - \operatorname{ch} R)^{l-n/2}\, dv,$$

$$H^2_\nu(R) = \int\limits_{R+2}^{\infty} D^l_{\operatorname{sh},v}[e^{\nu v}](\operatorname{ch} v - \operatorname{ch} R)^{l-n/2}\, dv,$$

and where we choose l such that $l - n/2 \in \{-1/2, 0\}$.

It is easily seen that

$$|H^2_\nu(R)| \lesssim |\nu|^l \int\limits_{R+2}^{\infty} e^{(-l+\operatorname{Re}\nu)v}\, e^{(l-n/2)v}\, dv \lesssim (1+|\nu|)^{n/2} e^{-nR/2+(\operatorname{Re}\nu)R}.$$

To estimate $H^1_\nu(R)$, for $R \geq 1$ we use the fact that in the domain of integration we have

$$D^m((\operatorname{sh} v)^{-1}) \sim e^{-R}, \quad |D^m e^{\nu v}| \sim |\nu|^m e^{(\operatorname{Re}\nu)R}, \quad \operatorname{ch} v - \operatorname{ch} R \sim e^R(v-R).$$

This leads to the same estimate as for $H^2_\nu(R)$.

For $R < 1$ we use the fact that in the domain of integration,

$$D^m((\operatorname{sh} v)^{-1}) \sim v^{-1-m}, \quad \operatorname{ch} v - \operatorname{ch} R \sim v(v-R), \quad |D^m e^{\nu v}| \sim |\nu|^m.$$

Thus

$$\begin{aligned} |H^1_\nu(R)| &\lesssim (1+|\nu|^l) \int\limits_{R}^{R+2} v^{-l-n/2}(v-R)^{-n/2+l}\, dv \\ &\lesssim (1+|\nu|^l) R^{1-n} \int\limits_{1}^{\infty} v^{-l-n/2}(v-1)^{-n/2+l}\, dv \lesssim (1+|\nu|^{n/2}) R^{1-n}. \end{aligned}$$

Altogether, we find that for $\operatorname{Re}\nu < 0$,

$$\begin{aligned} &\Big| \int\limits_{R}^{\infty} D^l_{\operatorname{sh},v}[e^{\nu v}](\operatorname{ch} v - \operatorname{ch} R)^{l-n/2}\, dv \Big| \\ &\qquad\qquad \leq C_n (1+|\nu|)^{n/2} R^{1-n}(1+R^{n-1}) e^{-nR/2+(\operatorname{Re}\nu)R}. \end{aligned}$$

An application of Lemma 2.1 now proves the desired estimate (3.2) and completes the proof of Lemma 3.1.

4. Spectral multipliers. Let $\psi \in C_0(\mathbb{R})$. Since $\psi(L)$ is a bounded linear operator on $L^2(G)$, by Schwartz' kernel theorem and left-invariance of L, there exists a unique convolution kernel $k_\psi \in \mathcal{D}'(G)$ such that $\psi(L)\varphi = \varphi * k_\psi$ for $\varphi \in \mathcal{D}(G)$. In this section we shall derive an integral representation for k_ψ. In what follows, we shall sometimes also use the suggestive notation $k_\psi = \psi(L)\delta_0$.

Since the Gauss kernels $g_\varepsilon(s) = (2\pi\varepsilon)^{1/2} e^{-s^2/2\varepsilon}$, $\varepsilon > 0$, form an approximation to the identity with respect to convolution, we have $\psi_\varepsilon := \psi \star g_\varepsilon \to \psi$, uniformly as $\varepsilon \to 0$. But ψ_ε has an analytic continuation, given by

$$\psi_\varepsilon(\zeta) := \frac{1}{\sqrt{2\pi\varepsilon}} \int\limits_{\mathbb{R}} \psi(t) e^{-(t-\zeta)^2/2\varepsilon}\, dt, \qquad \zeta \in \mathbb{C}.$$

Therefore, $\psi_\varepsilon(L)$ is given by the Cauchy integral

$$\psi_\varepsilon(L) = \frac{1}{2\pi i} \int\limits_{\gamma_\delta} \psi_\varepsilon(\zeta)(L-\zeta)^{-1}\, d\zeta,$$

where, for any $\delta > 0$, we may choose for γ_δ the (clockwise) contour $\gamma_\delta : s \mapsto (s+i\delta)^2$, $s \in \mathbb{R}$. By Lemma 3.1, $\psi_\varepsilon(L)\varphi = \varphi \star k_{\psi_\varepsilon}$ for $\varphi \in \mathcal{S}$, where the kernel k_{ψ_ε} is given, for any $l > n/2 - 1$, by

$$k_{\psi_\varepsilon}(x,y) = c_l e^{-nx/2} \int_{\mathbb{R}} \psi_\varepsilon((s+i\delta)^2) F_R(s+i\delta)(s+i\delta)\, ds, \tag{4.1}$$

where

$$c_l := \frac{(-1)^l}{\pi i} \frac{2^{-1-n/2}\pi^{-n/2}}{\Gamma(-n/2+1+l)}, \tag{4.2}$$

$$F_R(\zeta) := \int_R^\infty D^l_{\mathrm{sh},v}[e^{i\zeta v}](\operatorname{ch} v - \operatorname{ch} R)^{-n/2+l}\, dv \quad \text{for } \operatorname{Im}\zeta > -n/2. \tag{4.3}$$

Estimate (3.2) shows that the mappings $(x,y) \mapsto e^{-nx/2} F_{R(x,y)}(s+i\delta)$ are locally integrable on G, and their integrals over compact subsets of G are of polynomial growth in s, uniformly in $0 < \delta < 1$.

Moreover, since

$$|\mathrm{Re}[(t-(s+i\delta)^2)^2] - s^4| \le C[1+|s|]^3,$$

where C is uniform in $0 \le \delta \le 1$, $t \in \operatorname{supp}\psi$, we have

$$|\psi_\varepsilon((s+i\delta)^2)| \le Ce^{-cs^4},$$

where C and $c > 0$ depend on ε but not on δ.

Therefore, given $\varphi \in \mathcal{D}(G)$, by the dominated convergence theorem the limit of

$$\int \psi_\varepsilon((s+i\delta)^2) \int_G e^{-nx/2} F_{R(x,y)}(s+i\delta)\varphi(x,y)\, dx\, dy\, (s+i\delta)\, ds$$

as δ tends to 0 is equal to the same expression with $\delta = 0$. Therefore, in the sense of distributions,

$$k_{\psi_\varepsilon}(x,y) = c_l e^{-nx/2} \int_{\mathbb{R}} \psi_\varepsilon(s^2) F_{R(x,y)}(s) s\, ds. \tag{4.4}$$

Finally, as $\varepsilon \to 0$, $\psi_\varepsilon(L) \to \psi(L)$ in the operator norm on $L^2(G)$, which implies that $k_{\psi_\varepsilon} \to k_\psi$ in $\mathcal{D}'(G)$. On the other hand, $|\psi_\varepsilon(s^2)| \le Ce^{-s^4/4}$ for $0 < \varepsilon < 1$, so that, again by the dominated convergence theorem and (4.4), $k_{\psi_\varepsilon} \to C_l e^{-nx/2} \int_{\mathbb{R}} \psi(s^2) F_R(s) s\, ds$, in the sense of distributions. We have thus proved

Proposition 4.1. *Let $\psi \in C_0(\mathbb{R})$. Then, for any $l > n/2 - 1$, the convolution kernel k_ψ of $\psi(L)$ is locally integrable on G, and is given by*

$$(4.5)\qquad k_\psi(x,y) = c_l e^{-nx/2}\int_{\mathbb{R}} \psi(s^2)F_{R(x,y)}(s)s\,ds$$
$$= c_l e^{-nx/2}\int_0^\infty \psi(s^2)[F_{R(x,y)}(s) - F_{R(x,y)}(-s)]s\,ds,$$

with c_l given by (4.2).

5. Asymptotics of $F_R(s)$. We denote by S^α the symbol class

$$S^\alpha := \{b \in C^\infty(\mathbb{R}) : \|b\|_{S^\alpha,k} := \sup_s (1+s^2)^{(-\alpha+k)/2}|b^{(k)}(s)| < \infty \text{ for all } k \in \mathbb{N}\}.$$

The spaces S^α are Fréchet spaces, with the topology induced by the sequence of seminorms $\|\cdot\|_{S^\alpha,k}$, $k \in \mathbb{N}$. The product of a function in S^α with a function in S^β is in $S^{\alpha+\beta}$. Moreover, $S^\alpha \subset S^\beta$ if $\alpha < \beta$, and $Db \in S^{\alpha-1}$ if $b \in S^\alpha$. The following general lemma will also be useful:

LEMMA 5.1. *Let $b_\alpha \in S^\alpha$ and $\beta, \gamma > 0$. Then for each $R > 0$ we can write*

$$b_\alpha = R^\beta b_{\alpha+\beta,R} + R^{-\gamma} b_{\alpha-\gamma,R}$$

with $b_{\alpha+\beta} \in S^{\alpha+\beta}$ and $b_{\alpha-\gamma} \in S^{\alpha-\gamma}$ uniformly in R, i.e.,

$$\|b_{\alpha+\beta}\|_{S^{\alpha+\beta},k} \le C_k, \qquad \|b_{\alpha-\gamma}\|_{S^{\alpha-\gamma},k} \le C_k,$$

with constants C_k independent of R.

Note: We have suppressed the R-dependence of the symbols $b_{\alpha+\beta}$, $b_{\alpha-\gamma}$ in the notation, and we will continue to suppress any R-dependence of symbols b throughout the rest of this paper.

Proof. Let χ be a smooth cutoff function which is constantly 1 on $(-\infty, 1]$ and vanishes on $[2, \infty)$. Then we write

$$b_\alpha = R^\beta\,[R^{-\beta}(1-\chi(R(1+s^2)^{1/2}))b_\alpha] + R^{-\gamma}\,[R^\gamma \chi(R(1+s^2)^{1/2})b_\alpha].$$

It is easy to see by Leibniz' rule that this is the desired splitting. ∎

We wish to estimate the function

$$F_R(s) := \int_R^\infty D^l_{\mathrm{sh},v}[e^{isv}](\operatorname{ch} v - \operatorname{ch} R)^{-n/2+l}\,dv \quad s \in \mathbb{R}\ (l > n/2-1).$$

The estimates are stated in Proposition 5.2 for the case $R \ge 1$ and in Proposition 5.7 for the case $0 < R \le 1$.

PROPOSITION 5.2. *If $R \ge 1$, then*

$$(5.1)\qquad F_R(s) = e^{-nR/2}\,e^{iRs}\,b_{n/2-1}(s),$$

where $b_{n/2-1} \in S^{n/2-1}$ uniformly in R.

This proposition will follow from the subsequent lemmas.

LEMMA 5.3. *For $v \geq 1$ we can write*

$$D^l_{\mathrm{sh},v}[e^{isv}] = \sum_{k=0}^{l} s^k q_k(v) e^{-lv}\, e^{isv}$$

with $q_k \in S^0$ for all k.

Proof. This is proved by induction on $l \in \mathbb{N}$, the case $l = 0$ being trivial. Assume the statement is true for some $l \in \mathbb{N}$. Then

$$\begin{aligned} D^{l+1}_{\mathrm{sh},v}[e^{isv}] &= \sum_{k=0}^{l} s^k D_v((\mathrm{sh}\, v)^{-1} q_k(v) e^{-lv} e^{isv}) \\ &= \sum_{k=0}^{l} s^k D_v\biggl(\frac{2}{1-e^{-2v}}\, q_k(v) e^{-(l+1)v} e^{isv}\biggr). \end{aligned}$$

On the interval $[1,\infty)$, the function $1/(1-e^{-2v}) = \sum_{m=0}^{\infty} e^{-2mv}$ coincides with a function in S^0. This easily implies the statement of the lemma for $l+1$. ∎

We can therefore decompose

$$F_R(s) = \sum_{k=0}^{l} s^k \int_R^\infty q_k(v)(\mathrm{ch}\, v - \mathrm{ch}\, R)^{-n/2+l} e^{-lv} e^{isv}\, dv$$

$$= \sum_{k=0}^{l} e^{iRs} e^{-nR/2} s^k \int_0^\infty q_k(R+v)[(\mathrm{ch}(R+v) - \mathrm{ch}\, R) e^{-(R+v)}]^{-n/2+l} e^{-nv/2} e^{isv}\, dv.$$

Fixing $k \in \{0,\dots,l\}$ and writing

$$\gamma_R(v) := q_k(R+v)[(\mathrm{ch}(R+v) - \mathrm{ch}\, R)e^{-(R+v)}]^{-n/2+l} e^{-nv/2},$$

it then suffices to prove that the function

$$f_R(s) := \int_0^\infty \gamma_R(v) e^{isv} dv, \quad s \in \mathbb{R}, \tag{5.2}$$

lies in $S^{n/2-l-1}$ uniformly with respect to $R \geq 1$.

Let χ be a smooth cutoff function which is constantly 1 on $(-\infty, 1]$ and vanishes on $[2,\infty)$. It suffices to show that

$$f_{R,1}(s) := \int_0^\infty \chi(v)\gamma_R(v) e^{isv} dv, \qquad s \in \mathbb{R}, \tag{5.3}$$

$$f_{R,2}(s) := \int_0^\infty (1-\chi(v))\gamma_R(v) e^{isv} dv, \quad s \in \mathbb{R}, \tag{5.4}$$

are in $S^{n/2-l-1}$ uniformly with respect to $R \geq 1$.

The following lemma settles the question for $f_{R,2}$.

Lemma 5.4. *The function* $(1-\chi)\gamma_R$ *is in the Schwartz class uniformly in* $R>0$.

Proof. Since the function q_k is in S^0, all derivatives $D_v^l q_k$ are bounded. It then suffices to show that also all derivatives of

$$[(\operatorname{ch}(R+v)-\operatorname{ch}R)e^{-(R+v)}]^{-n/2+l} \tag{5.5}$$

are bounded on $[1,\infty)$, uniformly in $R>0$. However,

$$(\operatorname{ch}(R+v)-\operatorname{ch}R)e^{-(R+v)} = \frac{\operatorname{ch}v-1}{e^v}\,\frac{\operatorname{ch}R}{e^R}+\frac{\operatorname{sh}v}{e^v}\,\frac{\operatorname{sh}R}{e^R}, \tag{5.6}$$

and this function and all its derivatives are bounded on $[1,\infty)$ uniformly in $R>0$. Moreover, (5.6) is also bounded below by some $\varepsilon>0$ on $[1,\infty)$ uniformly in $R>0$. Therefore, all derivatives of (5.5) are bounded, which completes the proof of the lemma. ∎

It remains to show that $f_{R,1}$ is in $S^{n/2-l-1}$. This will follow from the next two lemmas.

Lemma 5.5. *For* $v>0$ *we can write*

$$\chi\gamma_R = g_R(v)v^{-n/2+l},$$

where g_R *is supported in* $v\le 2$ *and* $D^k g_R$ *is bounded uniformly in* $R\ge 1$ *for all* $k\in\mathbb{N}$.

Proof. The Taylor expansion of $\operatorname{sh}v$ and $\operatorname{ch}v$ in the expression (5.6) gives, for $v\le 2$,

$$(\operatorname{ch}(R+v)-\operatorname{ch}R)e^{-(R+v)} = v\widetilde{g}(v)$$

for some function $\widetilde{g}$ which is bounded below by $\varepsilon>0$ and has all derivatives bounded above uniformly in $R\ge 1$. This proves the lemma. ∎

Lemma 5.6. *Let* $g\in S^0$ *be supported in* $[-2,2]$ *and* $\alpha>-1$. *Then*

$$f(s) := \int_0^\infty g(v)v^\alpha e^{ivs}\,dv, \quad s\in\mathbb{R},$$

is in $S^{-\alpha-1}$ *with seminorms* $\|f\|_{S^{-\alpha-1},k}$ *controlled by* $\|g\|_{S^0,k}$.

Proof. Since

$$D^j f(s) = i^j\int_0^\infty g(v)v^{\alpha+j}e^{ivs}\,dv,$$

it suffices to show that $|f(s)|\le C|s|^{-\alpha-1}$. Assume without loss of generality that $s>0$. By a change of variables, we need to show

$$\int_0^\infty g(v/s)v^\alpha e^{iv}\,dv \le C.$$

Let χ be again a smooth cutoff function which is constantly 1 on $(-\infty, 1]$ and vanishes on $[2, \infty)$. It suffices to estimate separately the terms

$$\int_0^\infty \chi(v) g(v/s) v^\alpha e^{iv}\, dv, \qquad \int_0^\infty (1-\chi(v)) g(v/s) v^\alpha e^{iv}\, dv.$$

The first term is clearly bounded. The second term, after N integrations by part, can be estimated by

$$\int_0^{2s} |D^N[(1-\chi(v)) g(v/s) v^\alpha]|\, dv.$$

By Leibniz' rule, this is dominated by a constant times

$$\int_1^{2s} v^{\alpha - N} dv,$$

which is finite if N is chosen sufficiently large. ■

This completes the proof of Proposition 5.2.

PROPOSITION 5.7. *Assume that* $0 < R \le 1$.

(a) *If* $n = 1$, *then*

$$F_R(s) = e^{iRs} R^{-1/2} b_{-1/2}(s),$$

where $b_{-1/2} \in S^{-1/2}$ *uniformly in* $R \in (0, 1]$.

(b) *If* $n \ge 2$, *then*

$$F_R(s) = e^{iRs}[R^{-n/2} b_{n/2-1}(s) + R^{1-n} b_0(s)],$$

where $b_{n/2-1} \in S^{n/2-1}$ *and* $b_0 \in S^0$ *uniformly in* $R \in (0, 1]$.

For the proof, we decompose $F_R = F_R^1 + F_R^2$, with

$$F_R^1(s) := \int_R^\infty \chi(v) D^l_{\mathrm{sh},v}[e^{isv}](\operatorname{ch} v - \operatorname{ch} R)^{-n/2+l}\, dv,$$

$$F_R^2(s) := \int_2^\infty (1-\chi(v)) D^l_{\mathrm{sh},v}[e^{isv}](\operatorname{ch} v - \operatorname{ch} R)^{-n/2+l}\, dv.$$

Here, χ is a smooth cutoff function such that

$$\chi(v) = 1 \quad \text{if } |v| \le 2, \qquad \chi(v) = 0 \quad \text{if } |v| \ge 4.$$

The function F_R^2 can again be estimated by means of Lemma 5.3 as in Lemma 5.4, which shows that F_R^2 is in $\mathcal{S}(\mathbb{R})$, uniformly in R. Thus it remains to estimate F_R^1.

Lemma 5.8. *For $0 < v \le 4$, we can write*

$$D^l_{\mathrm{sh},v}[e^{isv}] = \sum_{k=0}^{l} s^k q_k(v) v^{k-2l} e^{isv},$$

where $q_k \in S^0$.

Proof. This follows by induction, the case $l = 0$ being trivial. Assume the statement is true for some $l \in \mathbb{N}$. Then

$$D^{l+1}_{\mathrm{sh},v}[e^{isv}] = \sum_{k=0}^{l} s^k D_v((\operatorname{sh} v)^{-1} q_k(v) v^{k-2l} e^{isv}).$$

However, on the interval $[0, 4]$,

$$(\operatorname{sh} v)^{-1} = g(v) v^{-1}$$

for some $g \in S^0$. This easily implies the statement of the lemma for $l + 1$. ∎

Lemma 5.9. *For $0 \le v \le 4$, we have*

$$\operatorname{ch} v - \operatorname{ch} R = \gamma(v)(v + R)(v - R) \tag{5.7}$$

for some $\gamma \in S^0$ with $\gamma(v) > \varepsilon > 0$ for $0 \le v \le 4$, uniformly in $R \in (0, 1]$.

Proof. This follows immediately from

$$\operatorname{ch} v - \operatorname{ch} R = \sum_{n=1}^{\infty} \frac{(v^2)^n - (R^2)^n}{(2n)!}. \quad ∎$$

We can therefore decompose

$$\begin{aligned} F^1_R(s) &= \sum_{k=0}^{l} s^k \int_R^\infty \widetilde{\gamma}_{k,R}(v) v^{k-2l} (v + R)^{-n/2+l} (v - R)^{-n/2+l} e^{isv}\, dv \\ &= \sum_{k=0}^{l} s^k e^{iRs} \int_0^\infty \gamma_{k,R}(v)(R + v)^{k-2l} (2R + v)^{-n/2+l} v^{-n/2+l} e^{isv}\, dv, \end{aligned} \tag{5.8}$$

with $\widetilde{\gamma}_{k,R}, \gamma_{k,R} \in S^0$ uniformly in $R \in (0, 1]$ and $\gamma_{k,R}(v) = 0$ for $v > 4$.

Lemma 5.10. *Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ be such that $\alpha = \alpha_1 + \alpha_2 > 0$ and $\beta > -1$. Let $\gamma \in S^0$ be such that $\gamma(v) = 0$ for $v > 4$. Consider the function*

$$f(s) = \int_0^\infty \gamma(v)(R + v)^{-\alpha_1} (2R + v)^{-\alpha_2} v^\beta e^{isv}\, dv.$$

Then, for every $\delta \ge 0$ with

$$\delta < \alpha \quad \text{and} \quad \delta \le \beta + 1, \tag{5.9}$$

$R^{\alpha-\delta} f$ is in $S^{\delta-\beta-1}$ uniformly in $R \in (0, 1]$.

Proof. As in the proof of Lemma 5.6, it suffices to prove that

(5.10) $$|f(s)| \le CR^{-\alpha+\delta}(1+|s|)^{\delta-\beta-1}, \quad s \in \mathbb{R},$$

with C independent of R. If $\delta = \beta + 1$, then $-\alpha + \beta < -1$ and we have

$$|f(s)| \lesssim R^{-\alpha+\beta+1} \int_0^\infty (1+v)^{-\alpha} v^\beta \, dv,$$

which proves the desired estimate.

Now assume $\delta < \beta + 1$. Consider first the case $|s| \le 1$. We estimate

(5.11) $$\begin{aligned}(R+v)^{-\alpha_1}(2R+v)^{-\alpha_2} &\lesssim (R+v)^{-\alpha} \\ &= (R+v)^{\delta-\alpha}(R+v)^{-\delta} \le R^{\delta-\alpha} v^{-\delta}.\end{aligned}$$

This gives

$$|f(s)| \lesssim \int_0^4 R^{\delta-\alpha} v^{\beta-\delta} \, dv,$$

which proves the desired estimate in view of $\delta < \beta + 1$.

It remains to consider $|s| > 1$. We write $f = f_1 + f_2$ with

$$f_1(s) = s^{-\beta-1} \int_0^\infty \chi(v)\gamma(v/s)(R+v/s)^{-\alpha_1}(2R+v/s)^{-\alpha_2} v^\beta e^{iv} \, dv,$$

$$f_2(s) = s^{-\beta-1} \int_1^\infty (1-\chi(v))\gamma(v/s)(R+v/s)^{-\alpha_1}(2R+v/s)^{-\alpha_2} v^\beta e^{iv} \, dv,$$

where χ is a smooth cutoff function which is constantly 1 on $(-\infty, 1]$ and vanishes on $[2,\infty)$. To estimate f_1, we split $(R+v/s)^{-\alpha}$ analogously to (5.11) and obtain

$$|f_1(s)| \lesssim s^{-\beta-1} \int_0^\infty |\chi(v)| s^\delta R^{\delta-\alpha} v^{\beta-\delta} \, dv.$$

This proves the desired estimate for f_1 in view of $\delta < \beta + 1$.

To estimate f_2, we do N times partial integration. The functions $(1-\chi(v))\gamma(v/s)$ are in S^0 uniformly in $|s| > 1$, and v^β is in S^β. Since

$$\begin{aligned}|D_v^k[(R+v/s)^{-\alpha_1}(2R+v/s)^{-\alpha_2}]| &\lesssim (v/s)^k (R+v/s)^{-\alpha-k} v^{-k} \\ &\lesssim (R+v/s)^{-\alpha} v^{-k} \lesssim s^\delta R^{\delta-\alpha} v^{-\delta-k},\end{aligned}$$

we therefore obtain

$$|f_2(s)| \lesssim s^{\delta-\beta-1} R^{\delta-\alpha} \int_1^\infty v^{\beta-N} \, dv,$$

which proves the desired estimate for f_2. ∎

We are now in a position to estimate F_R^1. Assume first that n is even and choose $l = n/2$.

Applying Lemma 5.10 to (5.8) with $\delta = 0$ in case $k = l$ and $\delta = 1$ in case $k < l$ gives

$$F_R^1 = \Big[\sum_{k=1}^{l-1} s^k e^{iRs} R^{k-l-n/2+1} b_{n/2-l}\Big] + s^l e^{iRs} R^{-n/2} b_{n/2-l-1}$$
$$= \Big[\sum_{k=1}^{l-1} e^{iRs} R^{k-l-n/2+1} b_{k+n/2-l}\Big] + e^{iRs} R^{-n/2} b_{n/2-1},$$

where b_α generally denotes a function in S^α (uniformly in $R \in (0,1]$) which may be different at different places in the argument. Applying Lemma 5.1 gives

$$F_R^1 = e^{iRs}[R^{-l-n/2+1} b_{n/2-l} + R^{-n/2} b_{n/2-1}].$$

As $l = n/2$, we obtain the desired estimate.

Assume next that n is odd and $n \geq 3$. We choose $l = (n-1)/2$. Then we apply Lemma 5.10 with $\delta = 0$ for $k = l$ and with $\delta = 1/2$ for $k < l$ and obtain

$$F_R^1 = \Big[\sum_{k=1}^{l-1} s^k e^{iRs} R^{k-l-(n-1)/2} b_{(n-1)/2-l}\Big] + s^l e^{iRs} R^{-n/2} b_{n/2-l-1}$$
$$= \Big[\sum_{k=1}^{l-1} e^{iRs} R^{k-l-(n-1)/2} b_{k+(n-1)/2-l}\Big] + e^{iRs} R^{-n/2} b_{n/2-1}.$$

Applying Lemma 5.1 again and using $l = (n-1)/2$ gives the desired estimate.

Finally, assume $n = 1$. We choose $l = 0$. Applying Lemma 5.10 with $\delta = 0$ gives

$$F_R^1 = e^{iRs} R^{-n/2} b_{n/2-1},$$

which proves the desired estimate.

6. Spectrally localized estimates for the wave propagator. The following theorem states pointwise estimates for the convolution kernel of spectrally localized wave propagators on the $ax + b$ group.

THEOREM 6.1. *Let $t \in \mathbb{R}$, $\lambda > 0$, and let ψ be an even bump function in $C_0^\infty(\mathbb{R})$ supported in $[-2, 2]$. If $\lambda \geq 1$, assume in addition that ψ vanishes on $[-1, 1]$. Then the convolution kernel of*

$$m_\lambda^t(L) := \psi(\sqrt{L}/\lambda)\cos(t\sqrt{L})$$

is of the form

$$k_\lambda^t(x, y) = e^{-nx/2} e^{-nR/2} [G_\lambda(R, R - t) + G_\lambda(R, R + t)],$$

where the function G_λ satisfies for every $N \in \mathbb{N}$ the following estimates:

(a) *If* $R \geq 1$, *then*

$$|G_\lambda(R,\varrho)| \lesssim \begin{cases} \lambda^{n/2+1}(1+|\lambda\varrho|)^{-N} & \text{if } \lambda \geq 1, \\ \lambda^2(1+|\lambda\varrho|)^{-N} & \text{if } \lambda < 1. \end{cases}$$

(b) *If* $0 \leq R \leq 1$, *then*, *for* $n = 1$,

$$|G_\lambda(R,\varrho)| \lesssim \begin{cases} R^{-1/2}\lambda^{3/2}(1+|\lambda\varrho|)^{-N} & \text{if } \lambda \geq 1, \\ R^{-1/2}\lambda^2(1+|\lambda\varrho|)^{-N} & \text{if } \lambda < 1, \end{cases}$$

and for $n \geq 2$,

$$|G_\lambda(R,\varrho)| \lesssim \begin{cases} (R^{1-n}\lambda^2 + R^{-n/2}\lambda^{n/2+1})(1+|\lambda\varrho|)^{-N} & \text{if } \lambda \geq 1, \\ R^{1-n}\lambda^2(1+|\lambda\varrho|)^{-N} & \text{if } \lambda < 1. \end{cases}$$

The constants in these estimates depend only on the C^N*-norms of* ψ.

Proof. We consider first the case $R \geq 1$. By Propositions 4.1 and 5.2, the kernel k_λ^t can be written as

$$\frac{c_l}{2}\, e^{-nx/2} e^{-nR/2} \int_{\mathbb{R}} \psi(s/\lambda) b_{n/2-1}(s) s [e^{i(R-t)s} + e^{i(R+t)s}]\, ds,$$

where $b_{n/2-1}$ is in $S^{n/2-1}$ uniformly in $R \geq 1$. Then the desired estimate follows by an application of Lemma 6.2 below with $j = 2$.

The case $0 < R \leq 1$ is done similarly using Proposition 5.7 instead of Proposition 5.2. ∎

Lemma 6.2. *Let* $b \in S^\beta$, *let* $j \geq 1$ *be an integer*, *and let* $\lambda > 0$. *Consider*

$$M_\lambda(\varrho) := \int \psi(s/\lambda) b(s) s^{j-1} e^{i\varrho s}\, ds, \quad \varrho \in \mathbb{R}.$$

Then, *for every* $N \in \mathbb{N}$,

$$|M_\lambda(\varrho)| \leq C_N \begin{cases} \lambda^{\beta+j}(1+|\lambda\varrho|)^{-N} & \text{if } \lambda \geq 1 \text{ and } \operatorname{supp}\psi \subset [-2,-1]\cup[1,2], \\ \lambda^j(1+|\lambda\varrho|)^{-N} & \text{if } \lambda < 1 \text{ and } \operatorname{supp}\psi \subset [-2,2]. \end{cases}$$

Here, *the constants* C_N *depend only on* N *and the seminorms* $\|b\|_{S^\beta,k}$ *of* b, *and on the* C^N*-norm of* ψ.

Proof. For ease of notation, we shall write $A \lesssim B$ if $A \leq C_N B$ where C_N is an "admissible" constant in the sense described in the lemma. We may and shall assume that $\varrho \geq 0$. We write

$$M_\lambda(\varrho) = \lambda^j \int \psi(s) b(\lambda s) s^{j-1} e^{i\lambda\varrho s}\, ds.$$

Assume $\lambda \geq 1$ and ψ is supported in $[-2,-1]\cup[1,2]$. By the symbol estimates for b we have, for s in the support of ψ,

$$|D_s^k b(\lambda s)| \lesssim \lambda^k(1+|\lambda s|)^{\beta-k} \lesssim \lambda^\beta(1/\lambda+|s|)^{\beta-k} \lesssim \lambda^\beta.$$

Integrating by parts N times, we thus find that $|M_\lambda(\varrho)| \lesssim \lambda^{\beta+j}(1+|\lambda\varrho|)^{-N}$.

Next, assume $\lambda \le 1$ and ψ is supported in $[-2,2]$. We then have, for s in the support of ψ,

$$|D_s^k b(\lambda s)| \lesssim \lambda^k (1+|\lambda s|)^{\beta-k} \lesssim 1.$$

Integrating again by parts N times, we find that $|M_\lambda(\varrho)| \lesssim \lambda^j (1+|\lambda\varrho|)^{-N}$. ∎

As a consequence of Theorem 6.1, we obtain estimates of the L^1-norms of the convolution kernels of $\psi(\sqrt{L}/\lambda)\cos(t\sqrt{L}/\lambda)$. Notice that the corresponding multipliers result from a re-scaling of the multiplier for the case $\lambda = 1$, but the kernels cannot just be obtained by some scaling argument from the case $\lambda = 1$, since the operator L is not homogeneous.

Proposition 6.3. *Let* $W_\lambda^t := k_\lambda^{t/\lambda}$ *denote the convolution kernel of* $\psi(\sqrt{L}/\lambda)\cos(t\sqrt{L}/\lambda)$, *and let* $\varepsilon \ge 0$.

(a) *If* $\lambda \ge 1$ *and* $\operatorname{supp}\psi \subset [-2,-1]\cup[1,2]$, *then*

$$\int_G |W_\lambda^t(x,y)| R(x,y)^\varepsilon \, dx\, dy \lesssim \begin{cases} \lambda^{-\varepsilon}(1+t)^{n/2+\varepsilon} & \text{if } t \le \lambda, \\ \lambda^{n/2-1-\varepsilon} t^{1+\varepsilon} & \text{if } t \ge \lambda. \end{cases} \tag{6.1}$$

(b) *If* $0 < \lambda \le 1$ *and* $\operatorname{supp}\psi \subset [-2,2]$, *then*

$$\int_G |W_\lambda^t(x,y)| R(x,y)^\varepsilon \, dx\, dy \lesssim \lambda^{-\varepsilon}(1+t)^{1+\varepsilon}. \tag{6.2}$$

In particular, *if* $\lambda \ge 1$, *then*

$$\int_G |W_\lambda^t(x,y)| (1+\lambda R(x,y))^\varepsilon \, dx\, dy \lesssim \begin{cases} (1+t)^{n/2+\varepsilon} & \text{if } n \ge 2, \\ (1+t)^{1+\varepsilon} & \text{if } n = 1, \end{cases} \tag{6.3}$$

and, *if* $0 < \lambda \le 1$, *then*

$$\int_G |W_\lambda^t(x,y)| (1+\lambda R(x,y))^\varepsilon \, dx\, dy \lesssim (1+t)^{1+\varepsilon}, \tag{6.4}$$

in each instance uniformly in λ. *The constants in these estimates depend only on the* C^N*-norms of* ψ.

Proof. Without loss of generality, we shall assume that $t \ge 0$. Then the dominant term in Theorem 6.1 is the one containing $G_\lambda(R, R-t)$, to which we shall therefore restrict ourselves.

We consider first $\lambda \ge 1$. By Theorem 6.1 and Lemma 2.1, we can estimate the left-hand side of (6.1) by

$$\lambda^2 \int_0^1 (1+|\lambda R - t|)^{-N} R^{1+\varepsilon}\, dR + \lambda^{n/2+1} \int_0^1 (1+|\lambda R - t|)^{-N} R^{n/2+\varepsilon}\, dR \\ + \lambda^{n/2+1} \int_1^\infty (1+|\lambda R - t|)^{-N} R^{1+\varepsilon}\, dR. \tag{6.5}$$

If $t \le \lambda/2$, then we can estimate this using Lemma 6.4 below by

$$\lambda^{-\varepsilon}(1+t)^{1+\varepsilon} + \lambda^{-\varepsilon}(1+t)^{n/2+\varepsilon} + \lambda^{n/2+1-N}.$$

Similarly, if $\lambda/2 \le t \le 2\lambda$, we estimate (6.5) by

$$\lambda^{-\varepsilon}(1+t)^{1+\varepsilon} + \lambda^{-\varepsilon}(1+t)^{n/2+\varepsilon} + \lambda^{n/2-1-\varepsilon}(1+t)^{1+\varepsilon},$$

and if $2\lambda \le t$ we estimate (6.5) by

$$(1+t)^{-N} + \lambda^{n/2-1-\varepsilon}(1+t)^{1+\varepsilon}.$$

In each case we easily verify (6.1), taking into account that for $n = 1$ the first of the three summands of (6.5) is not present.

If $0 < \lambda \le 1$, we estimate the left-hand side of (6.2) by

$$\lambda^2 \int_0^\infty (1+|\lambda R - t|)^{-N} R^{1+\varepsilon}\, dR$$

if $n \ge 2$, and by

$$\lambda^2 \int_0^1 (1+|\lambda R - t|)^{-N} R^{1/2+\varepsilon}\, dR + \lambda^2 \int_1^\infty (1+|\lambda R - t|)^{-N} R^{1+\varepsilon}\, dR$$

if $n = 1$. In either case it is easy to verify (6.2) using Lemma 6.4 below. Estimates (6.3) and (6.4) follow immediately from (6.1) and (6.2). ∎

Lemma 6.4. *For $\alpha \ge 0$, $t \ge 0$ and $N > \alpha + 1$, let*

$$I_0 := \int_0^1 (1+|\lambda R - t|)^{-N} R^\alpha\, dR, \qquad I_\infty := \int_1^\infty (1+|\lambda R - t|)^{-N} R^\alpha\, dR.$$

Then

$$I_0 \le C \begin{cases} (1+\lambda)^{-\alpha-1}(1+t)^\alpha & \text{if } t \le 2\lambda, \\ (1+t)^{-N} & \text{if } t \ge 2\lambda, \end{cases} \tag{6.6}$$

$$I_\infty \le C \begin{cases} \lambda^{-\alpha-1}(1+\lambda)^{-N+\alpha+1} & \text{if } t \le \lambda/2, \\ \lambda^{-\alpha-1}(1+t)^\alpha & \text{if } t \ge \lambda/2. \end{cases} \tag{6.7}$$

Proof. We begin with I_0. If $t \ge 2\lambda$, then clearly $I_0 \lesssim (1+t)^{-N}$. If $t \le 2\lambda$, we write

$$I_0 = \lambda^{-\alpha-1} \int_{-t}^{\lambda-t} (1+|v|)^{-N}(v+t)^\alpha\, dv.$$

If $\lambda \ge 1$, one easily deduces from this representation that $I_0 \lesssim \lambda^{-\alpha-1}(1+t)^\alpha$, and if $\lambda \le 1$, the original formula for I_0 immediately implies $I_0 \lesssim 1$, so that we obtain (6.6).

As for I_∞, if $t \le \lambda/2$, then clearly

$$I_\infty \lesssim \int_1^\infty (1+\lambda R)^{-N} R^\alpha \, dR = \lambda^{-\alpha-1} \int_\lambda^\infty (1+R)^{-N} R^{-\alpha} \, dR,$$

hence $I_\infty \lesssim \lambda^{-\alpha-1}(1+\lambda)^{-N+\alpha+1}$. If $t \ge \lambda/2$, we write

$$I_\infty = \lambda^{-\alpha-1} \int_{\lambda-t}^\infty (1+|v|)^{-N}(v+t)^\alpha \, dv.$$

If $t \le 1$, this implies $I_\infty \lesssim \lambda^{-\alpha-1}$, and if $t \ge 1$, one finds that $I_\infty \lesssim \lambda^{-\alpha-1}t^\alpha$, so that also (6.7) is verified. ∎

By means of the subordination principle described e.g. in [12], we immediately obtain:

Corollary 6.5 (cf. [6, Theorem. 6.1]). *If $\varepsilon > 0$, $s_0, s_1 > 3/2 + \varepsilon$ and $s_1 > (n+1)/2 + \varepsilon$, then there exists a constant C such that, for every continuous function F supported in $[1,2]$ and $0 < \lambda \le 1$,*

$$\int_G |F(L/\lambda^2)\delta_0(x,y)|(1+\lambda R(x,y))^\varepsilon \, dx\, dy \le C\|F\|_{H(s_0)},$$

while for $\lambda \ge 1$,

$$\int_G |F(L/\lambda^2)\delta_0(x,y)|(1+\lambda R(x,y))^\varepsilon \, dx\, dy \le C\|F\|_{H(s_1)}.$$

Proof. Choose an even function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi = 1$ on $[1,2]$ and $\operatorname{supp}\psi \subset [-4,-1/2] \cup [1/2,4]$. Proposition 6.3 holds for such ψ as well. Put $f(v) := F(v^2)$. Then $\|f\|_{H(s)} \sim \|F\|_{H(s)}$ for any $s \ge 0$, and $F(L/\lambda^2) = f(\sqrt{L}/\lambda) = \psi(\sqrt{L}/\lambda) f(\sqrt{L}/\lambda)$. Moreover, by the Fourier inversion formula and Fubini's theorem, one easily obtains

$$f(\sqrt{L}/\lambda) = \frac{1}{\pi}\int_0^\infty \widehat{f}(t)\cos(t\sqrt{L}/\lambda)\, dt,$$

since f is an even function. Thus

$$F(L/\lambda^2) = \frac{1}{\pi}\int_0^\infty \widehat{f}(t)\psi(\sqrt{L}/\lambda)\cos(t\sqrt{L}/\lambda)\, dt,$$

which implies

$$\begin{aligned} I_\lambda &:= \int_G |F(L/\lambda^2)\delta_0|(1+\lambda R)^\varepsilon \, dx\, dy \\ &\lesssim \int_0^\infty |\widehat{f}(t)| \Big[\int |W_\lambda^t|(1+\lambda R)^\varepsilon \, dx\, dy\Big]\, dt. \end{aligned}$$

Thus, if $0<\lambda\le 1$, then, by (6.4),

$$I_\lambda \lesssim \int_{\mathbb{R}} |\widehat{f}(t)|(1+|t|)^{1+\varepsilon}\,dt \lesssim \Big(\int_{\mathbb{R}} |\widehat{f}(t)(1+|t|)^{s_0}|^2\,dt\Big)^{1/2} = \|f\|_{H(s_0)} \sim \|F\|_{H(s_0)}.$$

The case $\lambda\ge 1$ can be treated in the same way. ∎

For the class of groups G considered here, we have thus established a completely different approach to the basic Theorem 6.1 in [6], entirely based on the wave equation.

7. Improvements on the estimates in Theorem 6.1 for small R. The estimates in Theorem 6.1 are already good enough for L^1-estimates, but not yet for L^∞-estimates, since they exhibit singularities at $R=0$. One knows that the singular support of the wave propagator for time t is the sphere $R=|t|$, so that the singularities at $R=0$ allowed for in Theorem 6.1 are in fact not present. We shall show in this section how to improve on our estimates when $R\le 1$, which we shall assume throughout this section.

To this end we observe that by formula (4.5), we may replace $F_R(s)$ in the previous discussions by $F_R(s)-F_R(-s)=2i\widetilde{F}_R(s)$, where

$$\widetilde{F}_R(s) := \int_R^\infty D^l_{\mathrm{sh},v}[\sin(sv)](\operatorname{ch} v-\operatorname{ch} R)^{-n/2+l}\,dv \qquad (s\ge 0,\ l>n/2-1).$$

Working with $\widetilde{F}_R(s)$ in place of $F_R(s)$, we can prove the following theorem.

THEOREM 7.1. *If $R\le 1$, then the estimates in Theorem* 6.1 *can be improved by the following additional estimates*, *valid for any $N\in\mathbb{N}$*:

$$|G_\lambda(R,\varrho)| \lesssim \begin{cases} \lambda^{n+1}(1+|\lambda\varrho|)^{-N} & \text{if } \lambda\ge 1,\\ \lambda^2(1+|\lambda\varrho|)^{-N} & \text{if } \lambda<1.\end{cases} \tag{7.1}$$

In order to prove this result, as in Section 5 we split $\widetilde{F}_R(s)=\widetilde{F}^1_R(s)+\widetilde{F}^2_R(s)$, where

$$\widetilde{F}^1_R(s) := \int_R^\infty \chi(v)D^l_{\mathrm{sh},v}[\sin(sv)](\operatorname{ch} v-\operatorname{ch} R)^{-n/2+l}dv.$$

$\widetilde{F}^2_R(s)$ is again a Schwartz function, uniformly for $0\le R\le 1$, and its contribution to k^t_λ can easily be seen to satisfy the estimates in Theorem 7.1.

In order to deal with $\widetilde{F}^1_R(s)$, we need the following substitute for Lemma 5.8.

LEMMA 7.2. *For $0 < sv < \pi/2$ and $0 < v < 4$ we can write*

$$D^l_{\mathrm{sh},v}[\sin(sv)] = \sum_{k=0}^{l} s^{2k} q_k(sv, v), \tag{7.2}$$

where $q_k(y, v)$ has a power series expansion of the form

$$\sum_{m,n=0}^{\infty} a_{mn} y^{2m+1} v^{2n}. \tag{7.3}$$

Moreover,

$$|D^j_s[q_k(sv, v)]| \le C_{j,k}(1 + s^2)^{-j/2}, \quad \textit{uniformly in } v. \tag{7.4}$$

Proof. We proceed by induction on l, the case $l = 0$ being clear. Assume that $q_k(y, v)$ is given by (7.3). Then

$$\frac{q_k(sv, v)}{\mathrm{sh}\, v} = s g_k(sv, v),$$

where $g_k(y, v)$ has an expansion of the form

$$g_k(y, v) = \sum_{m,n=0}^{\infty} b_{mn} y^{2m} v^{2n}.$$

Then

$$\begin{aligned} D_v[s g_k(sv, v)] &= \sum_{m\ge 1,\, n\ge 0} b_{mn} 2m s^2 (sv)^{2m-1} v^{2n} \\ &\quad + \sum_{m\ge 0,\, n\ge 1} s b_{mn} 2n (sv)^{2m} v^{2n-1} \\ &= s^2 h^1_k(sv, v) + h^2_k(sv, v), \end{aligned}$$

where $h^1_k(y, v)$ and $h^2_k(y, v)$ are of the form (7.3). This shows that (7.2) also holds with $l + 1$ in place of l.

Moreover, (7.4) is obvious for $|s| \le 1$, in view of (7.3), and if $|s| \ge 1$, it follows from

$$D^j_s[q_k(sv, v)] = v^j (D^j_y q_k)(sv, v) = s^{-j} (sv)^j (D^j_y q_k)(sv, v). \ \blacksquare$$

Now, if $\lambda R \ge 1/2$, then $\lambda \ge 1/2$, and the estimates (7.1) follow immediately from Theorem 6.1. Let us therefore assume that $\lambda R \le 1/2$.

We write

$$\widetilde{F}^1_R(s) = H^1_R(s) + H^2_R(s),$$

where

$$H^1_R(s) := \int_R^{1/2\lambda} \chi(v) D^l_{\mathrm{sh},v}[\sin(sv)](\mathrm{ch}\, v - \mathrm{ch}\, R)^{-n/2+l}\, dv,$$

$$H_R^2(s) := \int_{1/2\lambda}^{\infty} \chi(v) D_{\mathrm{sh},v}^l[\sin(sv)](\operatorname{ch} v - \operatorname{ch} R)^{-n/2+l} dv.$$

We need information on the asymptotics of these functions for $|s| \leq 2\lambda$.

As for $H_R^1(s)$, notice that in the integral defining it we have $|sv| \leq 1$ if $|s| \leq 2\lambda$. Therefore, from Lemmas 7.2 and 5.9 we find that, for any $l > -n/2+1$,

$$H_R^1(s) = \sum_{k=0}^{l} s^{2k} \int_R^{1/2\lambda} \gamma_k(sv, v)(v+R)^{-n/2+l}(v-R)^{-n/2+l}\, dv,$$

where $\gamma_k(sv, v)$ is supported where $0 \leq v \leq \min(2, 1/2\lambda)$ and satisfies $\gamma_k(-sv, v) = -\gamma_k(sv, v)$ and

$$|D_s^j[\gamma_k(sv, v)]| \leq C_{j,k}(1+s^2)^{-j/2} \quad \text{for every } j \in \mathbb{N}.$$

Choose l large enough so that $l - n/2 \geq 0$, and let

$$J(\lambda) := \int_R^{4/(1+\lambda)} (v+R)^{-n/2+l}(v-R)^{-n/2+l}\, dv.$$

Then clearly $J(\lambda) \leq (1+\lambda)^{-2l+n-1}$, and we find that, for $|s| \leq 2\lambda$,

$$H_R^1(s) = (1+\lambda)^{-2l+n-1} \sum_{k=0}^{l} s^{2k} b_{0,k}(s),$$

where $b_{0,k}$ is an odd function in S^0, uniformly in R and λ. From Lemma 6.2 we therefore obtain

$$\begin{aligned}\Big|\int_0^{\infty} \psi(s/\lambda) H_r^1(s) s \cos(ts)\, ds\Big| &= \frac{1}{2}\Big|\int_{\mathbb{R}} \psi(s/\lambda) H_r^1(s) s \cos(ts)\, ds\Big| \\ &\lesssim C_N \sum_{k=0}^{l} (1+\lambda)^{-2l+n-1} \lambda^{2k+2}(1+|\lambda t|)^{-N},\end{aligned}$$

hence

$$\Big|\int_0^{\infty} \psi(s/\lambda) H_r^1(s) s \cos(ts)\, ds\Big| \lesssim C_N \begin{cases} \lambda^{n+1}(1+|\lambda t|)^{-N} & \text{if } \lambda \geq 1, \\ \lambda^2(1+|\lambda t|)^{-N} & \text{if } \lambda < 1. \end{cases} \tag{7.5}$$

Next, we consider $H_R^2(s)$, again for $|s| \leq 2\lambda$.

Observe first that $H_R^2 \equiv 0$, unless $\lambda \geq 1/4$. In the latter case, one finds that $H_R^2(s)$ behaves like $F_R^1(s)$, only with R replaced by $1/\lambda \leq 4$. Replacing R by $1/\lambda$ and $R \pm t$ by $\pm t$ in Theorem 6.1(b), we therefore find that

$$\Big|\int_0^{\infty} \psi(s/\lambda) H_r^1(s) s \cos(ts)\, ds\Big|$$

also satisfies estimates (7.5).

Noticing finally that $1+|\lambda t| \sim 1+|\lambda\varrho|$ if $\varrho = R\pm t$, since $\lambda R \le 1/2$, we obtain the conclusion of Theorem 7.1.

Corollary 7.1. *Assume that* $\lambda \ge 1$ *and* $t \ge 0$. *Then*

$$\|k_\lambda^t\|_\infty \lesssim (1+t^{-n/2})\lambda^{n/2+1}. \tag{7.6}$$

Remark 7.2. Notice that, for small times, this estimate agrees with the one valid for the Laplacian on Euclidean space $\mathbb{R}^{n+1}$, as is to be expected, since L is elliptic. However, for large times, there appears no dispersive effect (definitely not for $n=2$, by Hebisch's transfer principle), so that it seems unlikely that non-trivial Strichartz-type estimates will hold for large times.

Proof of Corollary 7.1. First we observe that $e^{-nx/2}e^{-nR/2} \le 1$, and equality holds if $y=0$ and $x \le 0$. Therefore,

$$\|k_\lambda^t\|_\infty \lesssim \sup_{R\ge 0} |G_\lambda(R,R-t)|.$$

If $R \ge 1$, then, by Theorem 6.1,

$$|G_\lambda(R,R-t)| \lesssim \lambda^{n/2+1}.$$

So, assume that $R \le 1$. Then, by Theorem 7.1,

$$|G_\lambda(R,R-t)| \lesssim \lambda^{n+1}(1+\lambda|R-t|)^{-N} \tag{7.7}$$

for every $N \in \mathbb{N}$. If $\lambda t \le 1$, this implies

$$|G_\lambda(R,R-t)| \lesssim \lambda^{n+1} \le \lambda^{n/2+1}t^{-n/2}.$$

Assume next that $\lambda t \ge 1$. If $R \le t/2$, then $|R-t| \sim t$, so that (7.7) implies

$$|G_\lambda(R,R-t)| \lesssim \lambda^{n+1}(\lambda t)^{-N}$$

for every $N \in \mathbb{N}$, hence

$$|G_\lambda(R,R-t)| \lesssim \lambda^{n/2+1}t^{-n/2}. \tag{7.8}$$

If $R \ge t/2$, then for $n \ge 2$ Theorem 6.1 implies (7.8), since $t^{1-n}\lambda^2 \le t^{-n/2}\lambda^{n/2+1}$, and (7.8) is also valid for $n=1$. ∎

Remark 7.3. The group G can be considered as an Iwasawa AN-subgroup of the Lorentz group $S = \mathrm{SO}(1,n+1)$, and hence may be identified as a manifold with the symmetric space $K\backslash S$, where K is a maximal compact subgroup of S. The spherical function φ_0 of order zero on $K\backslash S$ is comparable to $(R/\operatorname{sh} R)^{n/2}$ in these coordinates, as one finds from Harish-Chandra's spherical function expansion (see e.g. [7]). In view of the well-known estimates for the wave propagators in Euclidean space, a naive extrapolation of Hebisch's transfer principle to this situation (where S is not a complex semisimple Lie group, unless $n=2$) would lead to the following "conjecture":

$$k_\lambda^t(x,y) = e^{-nx/2}e^{-nR/2}P_\lambda^t(R),$$

where:

(a) if $R \geq 1$, then

$$|P_\lambda^t(R)| \lesssim \begin{cases} t^{-n/2}\lambda^{n/2+1}R^{n/2}(1+\lambda|R-t|)^{-N} & \text{if } \lambda t \geq 1, \\ \lambda^{n+1}R^{n/2}(1+\lambda R)^{-N} & \text{if } \lambda t < 1; \end{cases}$$

(b) if $0 \leq R \leq 1$, then

$$|P_\lambda^t(R)| \lesssim \begin{cases} t^{-n/2}\lambda^{n/2+1}(1+\lambda|R-t|)^{-N} & \text{if } \lambda t \geq 1, \\ \lambda^{n+1}(1+\lambda R)^{-N} & \text{if } \lambda t < 1, \end{cases}$$

From Theorems 6.1 and 7.1, one can indeed easily verify these estimates if $\lambda \geq 2$, say.

However, if $\lambda \leq 1$, and if we choose $R = t \geq 1$ and $\lambda t \geq 1$, the "conjecture" would predict a size of order $\lambda^{n/2+1}$ for $|P_\lambda^t(R)|$, whereas we find the order λ^2.

8. Growth estimates for solutions to the wave equation in terms of spectral Sobolev norms

THEOREM 8.1. *Given a symbol $m \in S^{-\alpha}$, we define operators $T_1^t := m(\sqrt{L})\cos(t\sqrt{L})$ and $T_2^t := m(\sqrt{L})\sin(t\sqrt{L})/\sqrt{L}$, a priori on $L^2(G)$, for $t \in \mathbb{R}$. Let $1 \leq p \leq \infty$, and put $\alpha_n(p) := \max(2,n)|1/p - 1/2|$.*

(a) *If $\alpha > \alpha_n(p)$, then T_1^t extends from $L^p \cap L^2(G)$ to a bounded operator on $L^p(G)$, and*

$$\|T_1^t\|_{L^p\to L^p} \leq C_p(1+|t|)^{2|1/p-1/2|}.$$

(b) *If $\alpha > \alpha_n(p) - 1$, then T_2^t extends from $L^p \cap L^2(G)$ to a bounded operator on $L^p(G)$, and*

$$\|T_2^t\|_{L^p\to L^p} \leq C_p(1+|t|)^{2|1/p-1/2|}.$$

Note that the extension is unique if $1 \leq p < \infty$.

Proof. (a) Let $\chi \in C_0^\infty(\mathbb{R})$ be an even function such that $\chi(s) = 1$ if $|s| \leq 1/2$, and $\chi(s) = 0$ if $|s| \geq 1$. Put $\psi_0(s) := \chi(s/2)$ and $\psi_j(s) := \chi(2^{-j-1}s) - \chi(2^{-j}s) = \psi(2^{-j}s)$, $j = 1, \ldots, \infty$, where $\psi(s) := \chi(s/2) - \chi(s)$ is supported in $\{s : 1/2 \leq |s| \leq 2\}$. Then ψ_0 is supported in $[-2,2]$, ψ_j in $\{s : 2^{j-1} \leq |s| \leq 2^{j+1}\}$ for $j \geq 1$, and

$$\sum_{j=0}^{\infty} \psi_j(s) = 1, \qquad s \in \mathbb{R}. \tag{8.1}$$

We shall restrict ourselves to the case $1 \leq p < 2$, since the case $p = 2$ is trivial and the case $p > 2$ follows from the case $p < 2$ by duality. Using

(8.1), we decompose the symbol m as

$$m(s) = \sum_{j=0}^{\infty} m_j(2^{-j}s),$$

where $m_0 = m\chi$ and $m_j(s) := (m\psi_j)(2^j s) = m(2^j s)\psi$ if $j \geq 1$. Notice that

$$\|m_j\|_{C^N} \leq C2^{-\alpha j}, \tag{8.2}$$

where the constant C depends on the seminorms $\|m\|_{S^{-\alpha},k}$ only.

Then, for every $f \in L^2(G)$,

$$T_1^t f = \sum_{j=0}^{\infty} T_j f \quad \text{in } L^2(G), \tag{8.3}$$

where $T_j := m_j(\sqrt{L}/2^j)\cos((2^j t)\sqrt{L}/2^j)$. Estimating the operator norms $\|T_j\|_{L^1\to L^1}$ of T_j on $L^1(G)$ by means of Proposition 6.3 and (8.2), and interpolating these estimates with the trivial L^2-estimate $\|T_j\|_{L^2\to L^2} \lesssim 2^{-\alpha j}$, we obtain the following inequalities (we assume $t \geq 0$ without loss of generality):

$$\|T_j\|_{L^p\to L^p} \lesssim 2^{-\alpha j}[2^{nj/2}(1+t)]^{2|1/p-1/2|}. \tag{8.4}$$

The estimate in (a) follows immediately from (8.4) by summation over all $j \geq 0$.

As for (b), observe first that if we replace $m_\lambda^t(s) = \psi(\sqrt{s}/\lambda)\cos(t\sqrt{s})$ in Section 6 by $\widetilde{m}_\lambda^t(s) = \psi(\sqrt{s}/\lambda)\sin(t\sqrt{s})/\sqrt{s}$, then the factor $s(e^{i(R-t)s} + e^{i(R+t)s})$ in the corresponding kernel $k_\lambda^t(R)$ has to be replaced by $i(e^{i(R-t)s} - e^{i(R+t)s})$. By Lemma 6.2 with $j = 1$, the estimates for the function $\widetilde{k}_\lambda^t$ associated to $\widetilde{m}_j^t$ are therefore the same as for k_λ^t, except for an additional factor λ^{-1}. Moreover,

$$\sup_s \left| m_j\left(\frac{s}{2^j}\right)\frac{\sin(ts)}{s}\right| \lesssim \begin{cases} 2^{-\alpha j}2^{-j} & \text{if } j \geq 1, \\ 2^{-\alpha j}(1+t) & \text{if } j = 0. \end{cases}$$

Together, this implies that for $j \geq 1$, the operators $\widetilde{T}_j$ arising in the dyadic decomposition of T_2^t satisfy the same estimates as T_j, except for an additional factor 2^{-j}. And, for $j = 0$,

$$\|\widetilde{T}_0\|_{L^2\to L^2} \lesssim 1+t, \qquad \|\widetilde{T}_0\|_{L^1\to L^1} \lesssim 1+t,$$

hence $\|\widetilde{T}_0\|_{L^p\to L^p} \lesssim 1+t$. The estimates in (b) thus follow by summing over all j. ■

Let $u = u(t,x) = u_t(x)$ be the solution of the Cauchy problem

$$\frac{\partial^2}{\partial t^2}u - \Big(X^2 + \sum_{j=1}^{n} Y_j^2\Big)u = 0, \qquad u_0 = f, \qquad \frac{\partial}{\partial t}u\Big|_{t=0} = g. \tag{8.5}$$

Then, a priori for $f, g \in L^2(G)$, u_t is given by $u_t = \cos(t\sqrt{L})f + \frac{\sin(t\sqrt{L})}{\sqrt{L}}g$. If we define adapted Sobolev norms

$$\|\varphi\|_{L^p_\alpha} := \|(1+L)^{\alpha/2}\varphi\|_{L^p}, \qquad \alpha \in \mathbb{R},$$

we therefore immediately obtain from Theorem 8.1 the following

COROLLARY 8.2. *If* $1 \le p < \infty$, *then for* $\alpha_0 > \alpha_n(p)$ *and* $\alpha_1 > \alpha_n(p) - 1$,

$$\|u_t\|_{L^p} \le C_p((1+|t|)^{2|1/p-1/2|}\|f\|_{L^p_{\alpha_0}} + (1+|t|)\|g\|_{L^p_{\alpha_1}}), \tag{8.6}$$

where $\alpha_n(p)$ *is defined as in Theorem* 8.1.

REMARK 8.1. It is likely that the estimate (8.6) even holds for $\alpha_0 = \alpha_n(p)$ and $\alpha_1 = \alpha_n(p) - 1$, if $1 < p < \infty$. This would be the counterpart to the corresponding results by Peral [14] and Miyachi [11] in the Euclidean setting (see also [15] for a local variable coefficient version). The sharp result would require an introduction of a suitable Hardy respectively BMO-space on G. There is strong evidence that such spaces exist on G, in view of the ideas in [9] and [6], but we shall not pursue these issues here.

Notice also that the exponent $2|1/p-1/2|$ of $1+|t|$ in (8.6) is independent of n, in contrast to the corresponding estimate for the Euclidean case, where it agrees with $\alpha_n(p)$ (if $n \ge 2$).

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