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On the geometry of Banach spaces with modulus of convexity of power type 2

by

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Abstract. We use one-dimensional differential inequalities to estimate the squareness and type of Banach spaces with modulus of convexity of power type two. The estimates obtained are sharp and the constants involved moderate.

1. Introduction. The moduli of convexity and smoothness of a Banach space X,

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\right\}, \qquad 0 \le \varepsilon \le 2,$$

and

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\| - 2}{2} : \|x\| = \|y\| = 1\right\}, \qquad \tau \ge 0,$$

respectively, play an important role in Banach space theory. The duality between them reveals itself in the Lindenstrauss formula (e.g. [13, p. 61])

$$\rho_{X^*}(\tau) = \sup\{\tau\varepsilon/2 - \delta_X(\varepsilon) : 0 \le \varepsilon \le 2\}.$$

According to the Nordlander Theorem [17], every Hilbert space H is in a sense the most convex and the most smooth among Banach spaces, that is,

$$\delta_X(\varepsilon) \le \delta_H(\varepsilon) = 1 - \sqrt{1-\varepsilon^2/4} = \varepsilon^2/8 + O(\varepsilon^4)$$

and

$$\rho_X(\varepsilon) \ge \rho_H(\varepsilon) = \sqrt{1+\tau^2} - 1 = \tau^2/2 + O(\tau^4)$$

for arbitrary X. We write the Taylor expansion not only for the sake of greater clarity but also because it is the asymptotic behaviour at 0 of δ_X and ρ_X which matters the most.

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In this connection, it is rather difficult to compute precisely, or even up to equivalence, the modulus of convexity (or smoothness) of a given space. Notable among the special cases for which such calculations were carried through are L_p spaces [8] (see also [15]), more general Orlicz spaces [14] (see also [6]), and Lorentz spaces [1, 2].

By a surprisingly simple argument Figiel [6] proved a sharp estimate for the case when X is a p-convex and q-concave Banach lattice, $1 < p \leq 2 \leq q < \infty$:

$$\delta_X(\varepsilon) \geq \frac{p-1}{8}\,\varepsilon^q. \tag{1}$$

(For the definition of p-convexity and q-concavity we refer to [13, p. 40].)

Recently the class of Banach spaces with modulus of convexity of power type 2 has been studied intensively in connection with the problem of extension of Lipschitz mappings [16, 3, 4].

In [9] we estimate some geometrical quantities for a Banach space with modulus of convexity of power type 2 in terms of the constant $p \in (1, 2]$ (see (4) below). In this paper we elaborate on the method of [9] in order to find new asymptotically sharp estimates.

Before stating the main result we make the convention that $k, k_1, k_2, \ldots$ will denote positive absolute constants. In particular, we set

$$k = 2 + \sqrt{2}, \tag{2}$$

$$k_2 = k/2. \tag{3}$$

We also recall a couple of well-known notions. The quantity

$$d_2(X) = \sup\{d(Y, l_2^{(2)}) : Y \subset X,\ \dim Y = 2\},$$

where d is the Banach–Mazur distance, obviously measures how far from an ellipse the two-dimensional sections of S_X are. By the John Theorem [11], $d_2(X) \leq \sqrt{2}$, and by the Jordan–von Neumann Theorem [12], if $d_2(X) = 1$ then X is Hilbert. So, we are inclined to assume that the smaller $d_2(X)$ the nicer X is. Indeed, by the James Theorem, X is superreflexive if, for some equivalent norm, $d_2(X) < \sqrt{2}$ (see [10]).

We refer to [13, p. 72] for the definition of type.

Theorem 1.1. *Suppose*

$$\liminf_{\varepsilon \to 0} \frac{\delta_X(\varepsilon)}{\varepsilon^2} \geq \frac{p-1}{8} \tag{4}$$

for some $p \in (1, 2]$. *Then*

$$d_2(X) \leq \frac{(p-1) + \sqrt{2}\,k_2(2-p)}{(p-1) + k_2(2-p)}. \tag{5}$$

(In particular, (5) *holds for every* p*-convex and* 2*-concave Banach lattice.) Also,*

$$\text{type}\, X \geq 1 + \frac{p-1}{p-1+\sqrt{2}\,k(2-p)}. \tag{6}$$

We immediately get the following

Corollary 1.2. *If the Banach space* X *has no type greater than* $p \in (1,2]$ *then*

$$\liminf_{\varepsilon\to 0} \frac{\delta_X(\varepsilon)}{\varepsilon^2} \leq \frac{1}{8}\biggl(1 - \frac{\sqrt{2}(2-p)}{k(p-1)+\sqrt{2}(2-p)}\biggr). \tag{7}$$

(In particular, (7) *holds for any Banach space which contains uniformly* $l_p^{(n)}$ *for all* $n \in \mathbb{N}$*.)*

In conclusion we present some clarifying remarks.

Note first that from (5) it follows, as it should, that $d_2(X) < \sqrt{2}$ for all $p \in (1,2]$.

Remark 1.3. The estimate (5) is asymptotically sharp when $p \to 1$ or $p \to 2$.

Indeed, let $\mathcal{Y}_p$ be the class of all Banach spaces that satisfy (4) and let

$$D_p = \sup\{d_2(X) : X \in \mathcal{Y}_p\}.$$

From (1) we know that $l_p^{(2)} \in D_p$. Also, as is well-known, $d_2(l_p^{(2)}) = 2^{1/p-1/2}$. Therefore, (5) implies

$$2^{1/p-1/2} \leq D_p \leq \frac{(p-1)+\sqrt{2}\,k_2(2-p)}{(p-1)+k_2(2-p)}.$$

Considering the difference between the rightmost and leftmost sides of the above as a function of $p \in [1,2]$, we see that it is zero at $p=1$ and $p=2$ and has bounded second derivative on $[1,2]$. So, this difference is smaller than $k_1(p-1)(2-p)$, meaning that the estimate (5) of D_p is asymptotically sharp.

Remark 1.4. The estimates (6) and (7) are also asymptotically sharp for both $p \to 1$ and $p \to 2$.

We reason in similar fashion using l_p as an example: $l_p \in \mathcal{Y}_p$, $\text{type}\, l_p = p$ and l_p has no type strictly greater than p.

Remark 1.5. An asymptotically sharp estimate of the form (6) can be deduced from the renorming result of [9], since an equivalent norm with modulus of smoothness of power type p implies type p (e.g. [13, p. 100]). However, the constant $\sqrt{2}\,k$ in (6) is much smaller than what can be obtained from [9]. Also, the main result of the latter depends upon much deeper results.

Remark 1.6. For $X = l_p$ the constant $(p-1)/8$ in (1) is the best possible, since $\delta_{l_p}(\varepsilon) = (p-1)\varepsilon^2/8 + o(\varepsilon^2)$ (see [8, 15]). It is interesting that for $p = 2$ the constant $1/8$ is best possible even for equivalent norms, due to the Nordlander Theorem. It is open if the same is true for l_p.

2. Differential inequality. The proof of Theorem 1.1 is based on the following

Proposition 2.1. *Let $r = r(\theta)$ be a real 2π-periodic function with absolutely continuous first derivative such that*

(i) $0 \le r(r'' + r) \le 1 + a$ *a.e. for some* $a \ge 0$,
(ii) $0 \le r \le 1$,
(iii) *for every closed interval I of length $\pi/2$ there is $\alpha \in I$ with $r(\alpha) = 1$,*

Then

$$r \ge \frac{1 + k_2 a}{1 + \sqrt{2}\, k_2 a}. \tag{8}$$

We postpone the proof of the proposition to the next section. Now we deduce Theorem 1.1 from it.

In [9] we have introduced the shorthand notation

$$a(X) = 2 \limsup_{\tau \to 0} \frac{\rho_X(\tau)}{\tau^2} - 1, \qquad b(X) = \left(8 \liminf_{\varepsilon \to 0} \frac{\delta_X(\varepsilon)}{\varepsilon^2}\right)^{-1} - 1.$$

That is,

$$\liminf_{\varepsilon \to 0} \frac{\delta_X(\varepsilon)}{\varepsilon^2} = \frac{1}{8(1 + b(X))}, \qquad \limsup_{\tau \to 0} \frac{\rho_X(\tau)}{\tau^2} = \frac{a(X) + 1}{2},$$

$a(X), b(X) \ge 0$ and $a(X^*) = b(X)$.

The main point is the following proposition whose proof is an elaboration on some techniques in [9].

Proposition 2.2. *Each Banach space X satisfies*

$$d_2(X) \le \frac{1 + \sqrt{2}\, k_2 b(X)}{1 + k_2 b(X)} \tag{9}$$

and

$$\log_2 d_2(X) \le \frac{k b(X)}{2(\sqrt{2} + k b(X))}. \tag{10}$$

Proof. Since we are concerned with estimating, for the case $\dim X = 2$, the quantity

$$d_2(X) := \inf\{\|T\|\,\|T^{-1}\|;\ T : X \leftrightarrow l_2^2\}$$

with respect to $b(X)$, and $d_2(X) = d_2(X^*)$, we assume that we are given $a = a(X^*) = b(X) \in (0, \infty)$ and try to estimate $d_2(X)$.

As in [9], we assume that $Y = X^*$ is realised on the plane $\mathbb{R}^2$ in such a way that the Euclidean sphere

$$S = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$$

is the John sphere for B_Y. That is, the Euclidean norm $|\cdot|$ satisfies $|\cdot| \geq \|\cdot\|$ and there is no ellipse of area greater than π included in B_Y. It is well known (see e.g. [5, p. 68] or [7]) that $|\cdot| \leq \sqrt{2}\,\|\cdot\|$. Let e_1, e_2 be the unit vector basis in $\mathbb{R}^2$ and $r(\theta) = \|\cos\theta\, e_1 + \sin\theta\, e_2\|$. Then

$$S_Y = \left\{ \frac{1}{r(\theta)}(\cos\theta, \sin\theta) : \theta \in [-\pi, \pi] \right\}.$$

Since $\|\cdot\| \leq |\cdot| \leq \sqrt{2}\,\|\cdot\|$, we have $1/\sqrt{2} \leq r(\theta) \leq 1$ for all θ. In [19, Lemma 1] it is shown that each arc of the Euclidean sphere S with Euclidean length $\pi/2$ has a point of contact $w \in S \cap S_Y$. Thus, r satisfies (ii) and (iii) of Proposition 2.1.

In [9, Lemma 3.1] it is shown that r has Lipschitz continuous first derivative. Also in [9, pp. 385–386] it is proved that r satisfies the right hand inequality of Proposition 2.1(i) for $a = a(Y)$, while the left hand inequality follows from the convexity of the unit ball.

We see that r satisfies all the hypotheses of Proposition 2.1 and that $d_2(Y) = \max\{1/r(\theta)\}$. So from (8) we get (9):

$$d_2(X) = d_2(Y) \leq \frac{1 + \sqrt{2}\, k_2 a}{1 + k_2 a} = \frac{1 + \sqrt{2}\, k_2 b(X)}{1 + k_2 b(X)}.$$

To complete the proof we need the following elementary inequality that follows from the convexity of the function $t \log t$:

$$2 \log_2 t \leq \frac{\sqrt{2}(t-1)}{(\sqrt{2}-1)t}, \qquad t \in [1, \sqrt{2}]. \tag{11}$$

Also, the function on the right hand side of this inequality is increasing. Thus we can put $t = d_2(X)$ in this inequality and replace t with the estimate of $d_2(X)$ given in (9) to get

$$2 \log_2 d_2(X) \leq \frac{\sqrt{2}\, k_2 b(X)}{1 + \sqrt{2}\, k_2 b(X)},$$

which implies (10). ∎

In the proof Theorem 1.1 we will also use the following

PROPOSITION 2.3 ([18, 19]). *Each Banach space has type*

$$\frac{2}{1 + 2 \log_2 d_2(X)}.$$

Proof of Theorem 1.1. From (4) we have
$$b(X) \le \frac{1}{p-1} - 1 = \frac{2-p}{p-1}.$$
The straightforward substitution in (9) gives (5).

Also, from Proposition 2.2 we get
$$2\log_2 d_2(X) \le 1 - \frac{\sqrt{2}}{\sqrt{2}+kb(X)} = 1 - \frac{1}{1+k_3 b(X)} \le 1 - \frac{p-1}{p-1+k_3(2-p)},$$
where $k_3 = k/\sqrt{2} = 1+\sqrt{2}$. So,
$$1 + 2\log_2 d_2(X) \le \frac{(p-1)+2k_3(2-p)}{(p-1)+k_3(2-p)}$$
and
$$\frac{2}{1+2\log_2 d_2(X)} \ge \frac{2(p-1)+2k_3(2-p)}{(p-1)+2k_3(2-p)} = 1 + \frac{p-1}{(p-1)+2k_3(2-p)},$$
which together with Proposition 2.3 implies (6). ■

Proof of Corollary 1.2. Since X has no type greater than p, Proposition 2.3 implies
$$p \ge \frac{2}{1+2\log_2 d_2(X)}.$$
This and (10) give
$$\frac{2}{p} - 1 \le 2\log_2 d_2(X) \le \frac{kb}{2(\sqrt{2}+kb)},$$
where $b = b(X)$. That is,
$$kb \ge \sqrt{2}\,\frac{2-p}{p-1}$$
and
$$\frac{1}{1+b} \le \frac{k(p-1)}{k(p-1)+\sqrt{2}(2-p)} = 1 - \frac{\sqrt{2}(2-p)}{k(p-1)+\sqrt{2}(2-p)}. \ ■$$

3. The proof of Proposition 2.1. For $\alpha \in (0, \pi/4]$ and $m, t \ge 1$ we denote by $P_{\alpha,m,t}$ the following problem:
$$P_{\alpha,m,t}\begin{cases} 0 \le f''(\theta) + f(\theta) \le tm & \text{for almost all } \theta \in (0,\alpha),\\ f(0) = m^{-1}, \quad f(\alpha) = 1, \\ f'(0) = f'(\alpha) = 0. \end{cases}$$
We say that the function f is a solution to $P_{\alpha,m,t}$ if f' is absolutely continuous on $[0,\alpha]$ and f and its derivatives satisfy the above conditions.

Let $r(\theta)$ be a function that satisfies the hypothesis of Proposition 2.1. Translating the independent variable if necessary, we can assume that $r(0) =$

$\min\{r(\theta) : \theta \in [-\pi,\pi]\}$. By property (iii) in the proposition with $I = [-\pi/4,\pi/4]$ we can also assume that there is $\alpha \in (0,\pi/4]$ such that $r(\alpha) = 1 = \max\{r(\theta) : \theta \in [-\pi,\pi]\}$, using a change of sign when α is negative. Thus our function r is a solution of the problem $P_{\alpha,1/r(0),1+a}$.

We prove our proposition by studying the solutions of $P_{\alpha,m,t}$. One simple fact we will use is given in the following

LEMMA 3.1. *Let $\beta \in (0,\pi)$ and the function $f : [0,\beta] \to \mathbb{R}$ be such that f' is absolutely continuous and*

$$f'' + f \geq 0 \quad \text{a.e. on } (0,\beta) \quad \text{and} \quad f(0) = f'(0) = 0.$$

Then $f \geq 0$ on $[0,\beta]$.

Proof. Assume that there is $\beta_1 \in [0,\beta]$ such that $f(\beta_1) < 0$.

For small enough $\eta > 0$ the function $f_1 = f + \eta \sin t$ satisfies

$$f_1'' + f_1 = f'' + f \geq 0 \quad \text{a.e. on } (0,\beta), \tag{12}$$

$f_1(\beta_1) < 0$, $f_1(0) = 0$ and $f_1'(0) = \eta > 0$. Because of the latter, f_1 is positive for small enough t, and since it is continuous and becomes negative at $\beta_1 \in (0,\pi)$, there is $\beta_2 \in (0,\beta_1)$ such that

$$f_1(0) = f_1(\beta_2) = 0, \quad f_1 > 0 \quad \text{on } (0,\beta_2). \tag{13}$$

Let $k = \pi/\beta_2$ and $g = \sin kt$. Then

$$k > 1 \tag{14}$$

and

$$g(0) = g(\beta_2) = 0, \quad g > 0, \quad g'' = -k^2 g \quad \text{on } (0,\beta_2). \tag{15}$$

Integrating twice by parts and using (13) and (15) we get

$$\int_0^{\beta_2} f_1'' g\,dt = -k^2 \int_0^{\beta_2} f_1 g\,dt.$$

Together with (12)–(15), this implies

$$0 \leq \int_0^{\beta_2} (f_1'' + f_1) g\,dt = (1-k^2) \int_0^{\beta_2} f_1 g\,dt < 0,$$

a contradiction. ■

The key to our proof of Proposition 2.1 is

LEMMA 3.2. *For $t \geq 1$ let*

$$\psi_\alpha(t) = \sup\{m : \text{there exists a solution to } P_{\alpha,m,t}\}. \tag{16}$$

Then

$$\psi_\alpha(t) \leq 2\,\frac{t + \sqrt{2} - 2}{\sqrt{2} + \sqrt{2}\,t - 2}. \tag{17}$$

Proof. We fix $t \geq 1$ and observe that $\psi_\alpha(t) \geq 1$, since $f \equiv 1$ is a solution to $P_{\alpha,1,t}$. We then fix $m \geq 1$ such that there is a solution, say f, to $P_{\alpha,m,t}$. In order to prove (17) it is enough to show that

$$m \leq 2\,\frac{t+\sqrt{2}-2}{\sqrt{2}+\sqrt{2}\,t-2}. \tag{18}$$

Let $f_1(\theta) = f(\alpha-\theta) - \cos\theta$ for $\theta \in [0,\alpha]$. Since f is a solution to $P_{\alpha,m,t}$ we see that a.e. on $(0,\alpha)$,

$$f_1''(\theta) + f_1(\theta) = f''(\alpha-\theta) + f(\alpha-\theta) \geq 0,$$

and $f_1(0) = f(\alpha) - 1 = 0$, and $f_1'(0) = f'(\alpha) = 0$, so Lemma 3.1 implies that $f_1 \geq 0$ on $[0,\alpha]$. In other words,

$$f(\theta) = f(\alpha-(\alpha-\theta)) \geq \cos(\alpha-\theta), \qquad \forall\theta \in [0,\alpha]. \tag{19}$$

Applying the same reasoning to the function $f_2(\theta) = tm + (m^{-1}-tm)\cos\theta - f(\theta)$, satisfying $f_2'' + f_2 = tm - f - f'' \geq 0$, $f_2(0) = m^{-1} - f(0) = 0$, and $f_2'(0) = -f'(0) = 0$, we get $f_2 \geq 0$ on $[0,\alpha]$. That is,

$$f(\theta) \leq tm + (m^{-1}-tm)\cos\theta, \qquad \forall\theta \in [0,\alpha]. \tag{20}$$

Combining (19) with (20), we obtain

$$tm + (m^{-1}-tm)\cos\theta \geq \cos(\alpha-\theta)$$

for all $\theta \in [0,\alpha]$. Equivalently,

$$tm \geq g(\theta), \qquad \forall\theta \in [0,\alpha], \tag{21}$$

where $g(\theta) = (tm - m^{-1})\cos\theta + \cos(\alpha-\theta)$.

Since $m, t \geq 1$ we get $tm - m^{-1} \geq 0$. Hence $g(\theta) > 0$ for $\theta \in [0,\alpha]$. Since $g'(\theta) = \sin(\alpha-\theta) - (tm-m^{-1})\sin\theta$, $g'(0) = \sin\alpha > 0$ and

$$g'(\alpha) = -(tm-m^{-1})\sin\alpha \leq 0,$$

there is $\theta_1 \in [0,\alpha]$ such that $g'(\theta_1) = 0$. Then from (21) it follows that

$$\begin{aligned}(tm)^2 \geq g^2(\theta_1) &= g^2(\theta_1) + g'^2(\theta_1)\\ &= [(tm-m^{-1})\cos\theta_1 + \cos(\alpha-\theta_1)]^2\\ &\quad + [\sin(\alpha-\theta_1) - (tm-m^{-1})\sin\theta_1]^2\\ &= (tm-m^{-1})^2 + 1 + 2(tm-m^{-1})\cos\alpha.\end{aligned}$$

The latter is equivalent to

$$2(tm-m^{-1})\cos\alpha \leq 2t - m^{-2} - 1 \leq 2(t-m^{-2}) = 2m^{-1}(tm-m^{-1}).$$

Thus

$$m < \cos^{-1}\alpha \leq \sqrt{2}, \tag{22}$$

and moreover

$$2t - 2tm\cos\alpha \geq m^{-2} + 1 - 2m^{-1}\cos\alpha$$

so that

$$t \geq \frac{m^2 + (1 - 2m\cos\alpha)}{2m^2(1 - m\cos\alpha)} = \frac{1}{m^2} + \frac{m^2 - 1}{2m^2(1 - m\cos\alpha)}$$
$$\geq \frac{1}{m^2} + \frac{(m^2 - 1)}{m^2(2 - m\sqrt{2})} = \frac{1 - \sqrt{2}\,m + m^2}{m^2(2 - \sqrt{2}\,m)}.$$

Let $h(u) := (u^2 - \sqrt{2}u + 1)/u^2$. Then

$$t \geq \frac{h(m)}{2 - \sqrt{2}\,m}. \tag{23}$$

On the other hand, we have

$$h(u) = 1 - \frac{\sqrt{2}}{u} + \frac{1}{u^2}, \qquad h'(u) = \frac{\sqrt{2}}{u^2} - \frac{2}{u^3},$$
$$h''(u) = -\frac{2\sqrt{2}}{u^3} + \frac{6}{u^4} = \frac{2\sqrt{2}}{u^4}\left(\frac{3}{\sqrt{2}} - u\right).$$

Since obviously $3/\sqrt{2} > \sqrt{2}$, we find that $h'' \geq 0$ and h is convex for $u \in [1, \sqrt{2}]$. In particular, h is greater than its tangent at $(1, h(1))$. Since $h(1) = 2 - \sqrt{2}$ and $h'(1) = -(2 - \sqrt{2})$, we get

$$h(u) \geq (2 - \sqrt{2}) - (2 - \sqrt{2})(u - 1) = (2 - \sqrt{2})(2 - u), \qquad u \in [1, \sqrt{2}].$$

From this and (23) it follows that

$$t \geq (2 - \sqrt{2})\,\frac{2 - m}{2 - \sqrt{2}\,m}.$$

Multiplying by $2 - \sqrt{2}\,m \geq 0$ (see (22)), we obtain

$$2t - (\sqrt{2}\,t)m \geq 2(2 - \sqrt{2}) + (\sqrt{2} - 2)m,$$
$$2(t + \sqrt{2} - 2) \geq (\sqrt{2} + \sqrt{2}\,t - 2)m.$$

Since $t \geq 1$ and thus $\sqrt{2} + \sqrt{2}\,t - 2 > 0$, the latter is equivalent to (18). ∎

Proof of Proposition 2.1. Recall that by the assumptions at the beginning of this section, r is a solution to $P_{\alpha,1/r(0),1+a}$ and $r(0) = \min\{r(\theta) : \theta \in [-\pi, \pi]\}$.

From Lemma 3.2 we get

$$\frac{1}{r(0)} \leq 2\,\frac{1 + a + \sqrt{2} - 2}{\sqrt{2} + \sqrt{2}\,(1 + a) - 2} = \sqrt{2}\,\frac{a + \sqrt{2} - 1}{a + 2 - \sqrt{2}}$$
$$= \frac{\sqrt{2}a + 2 - \sqrt{2}}{a + 2 - \sqrt{2}} = \frac{1 + \sqrt{2}\,k_2 a}{1 + k_2 a},$$

recalling that $k_2 = 1/(2 - \sqrt{2}) = k/2$. Hence we have (8):

$$r \geq r(0) \geq \frac{1 + k_2 a}{1 + \sqrt{2}\,k_2 a}. \ ∎$$

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