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Some examples of cocycles with simple continuous singular spectrum

by

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Abstract. We study spectral properties of Anzai skew products $T_\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$T_\varphi(z,\omega) = (e^{2\pi i\alpha}z, \varphi(z)\,\omega),$$

where α is irrational and $\varphi : \mathbb{T} \to \mathbb{T}$ is a measurable cocycle. Precisely, we deal with the case where φ is piecewise absolutely continuous such that the sum of all jumps of φ equals zero. It is shown that the simple continuous singular spectrum of T_φ on the orthocomplement of the space of functions depending only on the first variable is a "typical" property in the above-mentioned class of cocycles, if α admits a sufficiently fast approximation.

1. Introduction. By $\mathbb{T}$ we denote the circle group $\{z \in \mathbb{C} : |z| = 1\}$ which will most often be treated as the interval $[0,1)$ with addition mod 1; λ will denote Lebesgue measure on $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *piecewise absolutely continuous* (PAC for short) if there exist $\beta_0, \ldots, \beta_k \in \mathbb{T}$ $(0 \le \beta_0 < \ldots < \beta_k < 1)$ such that $f|_{(\beta_j,\beta_{j+1})}$ is absolutely continuous $(\beta_{k+1} = \beta_0)$. Then we set

$$f_+(x) = \lim_{y\to x^+} f(y) \quad \text{and} \quad f_-(x) = \lim_{y\to x^-} f(y).$$

Let $d_j = f_+(\beta_j) - f_-(\beta_j)$ for $j = 0, \ldots, k$ and

$$S(f) = \sum_{j=0}^{k} d_j = -\sum_{j=0}^{k}(f_-(\beta_j) - f_+(\beta_j)) = -\int_{\mathbb{T}} Df(x)\,d\lambda(x).$$

We call a function $\varphi : \mathbb{T} \to \mathbb{T}$ piecewise absolutely continuous if there exists a PAC function $\widetilde{\varphi} : \mathbb{T} \to \mathbb{R}$ such that $\varphi(e^{2\pi ix}) = e^{2\pi i\widetilde{\varphi}(x)}$. Set $S(\varphi) = S(\widetilde{\varphi})$. Since the number $S(\widetilde{\varphi})$ is independent of the choice of the function $\widetilde{\varphi}$, the number $S(\varphi)$ is well defined and will be called the *sum of jumps* of φ.

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Let $\alpha \in \mathbb{T}$ be irrational. Denote by $Tz = e^{2\pi i\alpha}z$ ($Tx = x + \alpha \bmod 1$) the corresponding ergodic rotation on $\mathbb{T}$. We will study spectral properties of measure preserving automorphisms of $\mathbb{T}^2$ (called *Anzai skew products*) defined by

$$T_\varphi(z,\omega) = (Tz, \varphi(z)\,\omega)$$

where $\varphi : \mathbb{T} \to \mathbb{T}$ is a PAC function.

Consider the Koopman unitary operator $U_{T_\varphi} : L^2(\mathbb{T}\times\mathbb{T}, \lambda\otimes\lambda) \to L^2(\mathbb{T}\times\mathbb{T}, \lambda\otimes\lambda)$ associated with the Anzai skew product T_φ and defined by $U_{T_\varphi} = f \circ T_\varphi$. Let us decompose

$$L^2(\mathbb{T}\times\mathbb{T}, \lambda\otimes\lambda) = \bigoplus_{m\in\mathbb{Z}} H_m$$

where

$$H_m = \{g : g(z,\omega) = f(z)\,\omega^m,\ f \in L^2(\mathbb{T},\lambda)\}.$$

Observe that H_m is a closed U_{T_φ}-invariant subspace of $L^2(\mathbb{T}\times\mathbb{T}, \lambda\otimes\lambda)$. Moreover the operator $U_{T_\varphi} : H_m \to H_m$ is unitarily equivalent to the operator $U_\varphi^{(m)} : L^2(\mathbb{T},\lambda) \to L^2(\mathbb{T},\lambda)$ given by

$$(U_\varphi^{(m)} f)(z) = \varphi(z)^m f(Tz).$$

This leads to the problem of spectral classification of unitary operators $V_g : L^2(\mathbb{T},\lambda) \to L^2(\mathbb{T},\lambda)$ given by $V_g f(z) = g(z) f(Tz)$, where $g : \mathbb{T} \to \mathbb{T}$ is a measurable function.

Let U be a unitary operator on a separable Hilbert space $\mathcal{H}$. For any $f \in \mathcal{H}$ we define the *cyclic space* $\mathbb{Z}(f) = \operatorname{span}\{U^n f : n \in \mathbb{Z}\}$. By the *spectral measure* σ_f of f we mean a Borel measure on $\mathbb{T}$ determined by the equalities

$$\widehat{\sigma}_f(n) = \int_{\mathbb{T}} z^n\, d\sigma_f(z) = (U^n f, f)$$

for $n \in \mathbb{Z}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathbb{Z}(f_n) \quad \text{and} \quad \sigma_{f_1} \gg \sigma_{f_2} \gg \ldots$$

The spectral type of σ_{f_1} (the equivalence class of measures) will be called the *maximal spectral type* of U. We say that U has *Lebesgue* (resp. *continuous singular*, *discrete*) *spectrum* if σ_{f_1} is equivalent to Lebesgue (resp. continuous singular, discrete) measure on the circle. A number $m \in \mathbb{N}\cup\{\infty\}$ is called the *maximal spectral multiplicity* of U if $\sigma_{f_n} \not\equiv 0$ for $n \le m$ and $\sigma_{f_n} \equiv 0$ for $n > m$. We say that U has *simple spectrum* if the maximal spectral multiplicity of U equals 1.

The notion of the skew product was introduced in 1951 by Anzai (see [1]) to give some examples of ergodic transformations with some special spectral types. Anzai skew products or more generally operators V_g have a well known property called the purity law. Precisely, each operator V_g has either Lebesgue or continuous singular or discrete spectrum (see [6] and [10]).

In the case where $\varphi : \mathbb{T} \to \mathbb{T}$ is a smooth cocycle, the spectral properties of T_φ depend on the value of the topological degree of φ, which equals $-S(\varphi)$. For example, if φ is of class C^2 and $S(\varphi) \neq 0$, then T_φ has countable Lebesgue spectrum on $H_0^\perp$ (see [2] and [10]). On the other hand, $S(\varphi) = 0$ implies singular spectrum for absolutely continuous φ (see [3]). In this case, numerous dynamical properties of the skew product depend on properties of the continued fraction expansion of α. For example, each smooth cocycle with zero degree is cohomologous to a constant if α admits a sufficiently slow approximation. It follows that the skew product has pure discrete spectrum. On the other hand, if α admits a sufficiently fast approximation, then the skew product associated with a generic C^r-cocycle ($r \in \mathbb{N} \cup \{\infty\}$) with zero degree has simple continuous singular spectrum of T_φ on $H_0^\perp$ (see [8]). Generally, we also have some information about multiplicity of $U_\varphi^{(m)}$. For every absolutely continuous $g : \mathbb{T} \to \mathbb{T}$, the multiplicity of V_g is at most $\max(1, |S(g)|)$ (see [5]).

In the case where $\varphi : \mathbb{T} \to \mathbb{T}$ is PAC, the spectral properties of T_φ also depend on the value $S(\varphi)$. For example, $S(\varphi) \neq 0$ implies continuous spectrum on $H_0^\perp$ (see [9]). Moreover, if φ has a single discontinuity with $S(\varphi) \in \mathbb{R} \setminus \mathbb{Z}$, then T_φ has continuous singular spectrum on $H_0^\perp$.

In the paper we deal with the case where $S(\varphi) = 0$. Generally, it is shown in [5] that the multiplicity of each operator $U_\varphi^{(m)}$ is at most the number of discontinuities of φ. However, every piecewise constant cocycle such that all the discontinuities of φ are multiples of α is cohomologous to a constant cocycle, because each cocycle of the form $\varphi(e^{2\pi ix}) = e^{2\pi ia\mathbf{1}_{[0,k\alpha)}(x)+b}$, $k \in \mathbb{Z}$, is cohomologous to a constant cocycle (see [7], p. 82). Then T_φ has discrete spectrum. If φ has only rational jumps (i.e. $d_0, \ldots, d_k \in \mathbb{Q}$), then φ^m is constant for a nonzero m, hence $U_\varphi^{(m)}$ also has discrete spectrum. On the other hand, we will show that the simple continuous singular spectrum of T_φ on $H_0^\perp$ is a "typical" property for PAC cocycles whose sum of jumps equals zero, if α admits a sufficiently fast approximation.

For every natural k define

$$\mathbb{T}_+^k = \{(x_1, \ldots, x_k) \in \mathbb{T}^k : 0 \leq x_1 < \ldots < x_k < 1\}.$$

We will prove the following assertion.

THEOREM 1.1 [Main Theorem]. *Let $\alpha \in \mathbb{T}$ be an irrational number with unbounded partial quotients in its continued fraction expansion. For every $k \in \mathbb{N}$, there exists a subset $B_{k+1} \subset \mathbb{T}_+^{k+1}$ of full Lebesgue measure such that if $\varphi : \mathbb{T} \to \mathbb{T}$ is a PAC function with*

- $S(\varphi)=0$;
- *at least one of its jumps being irrational*;
- $k+1$ *discontinuities* $\beta_0,\ldots,\beta_k$ *satisfying* $(\beta_0,\ldots,\beta_k)\in B_{k+1}$,

then T_φ *has simple continuous singular spectrum on* $H_0^\perp$.

To prove this theorem we will use the idea of δ-weak mixing. Let δ be a complex number such that $|\delta|\le 1$. We say that a unitary operator $U:\mathcal{H}\to\mathcal{H}$ is *δ-weakly mixing* along a sequence $\{q_n\}_{n\in\mathbb{N}}$ if

$$\lim_{n\to\infty}(U^{q_n}f,f)=\delta(f,f)$$

for any $f\in\mathcal{H}$.

A simple spectral analysis gives the following well known fact.

Proposition 1. *Let $U_i:\mathcal{H}_i\to\mathcal{H}_i$, $i=1,2$, be a unitary operator on a separable Hilbert space. Assume that the U_i are δ_i-weakly mixing along a common sequence $\{q_n\}_{n\in\mathbb{N}}$. If $\delta_1\ne\delta_2$, then the maximal spectral types of U_i are mutually singular.*

We will apply the concept of the δ-weak mixing to the family of unitary operators $(U_\varphi^{(m)})$. We say that an increasing sequence $\{q_n\}_{n\in\mathbb{N}}$ of natural numbers is a *rigid time* for T if

$$\lim_{n\to\infty}\|q_n\alpha\|=0$$

where $\|t\|$ is the distance of t from the set of integers. For given $\varphi:\mathbb{T}\to\mathbb{T}$ and $q\in\mathbb{N}$ let

$$\varphi^{(q)}(z)=\varphi(z)\varphi(Tz)\ldots\varphi(T^{q-1}z).$$

Proposition 2 (see [4]). *Assume that*

$$\lim_{n\to\infty}\int_{\mathbb{T}}(\varphi^{(q_n)}(z))^m dz=\delta_m$$

where $\{q_n\}_{n\in\mathbb{N}}$ is a rigid time for T. Then the operator $U_\varphi^{(m)}$ is δ_m-weakly mixing along $\{q_n\}_{n\in\mathbb{N}}$.

2. The definition of the set B_k. Assume that $\alpha\in[0,1)$ is irrational with continued fraction expansion

$$\alpha=[0;a_1,a_2,\ldots].$$

Let (p_n/q_n) be the convergents of α; then

$$\|\alpha q_n\|=|q_n\alpha-p_n|<1/q_{n+1}$$

and $(T^j[0,\|\alpha q_{n-1}\|))_{0\le j<q_n}$ is a tower (i.e. a family of pairwise disjoint sets). We also have

$$\|\alpha q_{n+1}\|=a_{n+1}\|\alpha q_n\|+\|\alpha q_{n-1}\|\quad\text{and}\quad\|\alpha q_{n-1}\|q_n+\|\alpha q_n\|q_{n-1}=1.$$

We shall consider α with unbounded partial quotients, i.e. we can choose a subsequence, still denoted by n, such that $\lim_{n\to\infty} a_{n+1} = \infty$. Then, with the previous relations, $q_n\|q_n\alpha\| \to 0$ and $q_n\|q_{n-1}\alpha\| \to 1$.

LEMMA 2.1. *Let $0 < \tau < 1$ and let $W = \prod_{i=1}^k [v_i, w_i]$ be a closed cube in $\mathbb{T}^k$ with $\lambda^k(W) > 0$. For almost every $(\beta_1, \ldots, \beta_k) \in \mathbb{T}^k$ there exists a subsequence $\{q_{n_j}\}_{j\in\mathbb{N}}$ such that*

$$\lim_{j\to\infty} q_{n_j}\|q_{n_j}\alpha\| = 0, \qquad \lim_{j\to\infty} (\{q_{n_j}\beta_1\}, \ldots, \{q_{n_j}\beta_k\}) = (\gamma_1, \ldots, \gamma_k) \in W$$

and

$$\beta_1, \ldots, \beta_k \in \bigcup_{\tau q_{n_j} < t < q_{n_j}} T^t[0, \|q_{n_j-1}\alpha\|)$$

for every natural j.

Proof. Assume that $\{\Xi_n\}_{n\in\mathbb{N}}$ is a sequence of towers for the rotation T for which $\liminf_{n\to\infty} \lambda(\Xi_n) > 0$ and $\text{height}(\Xi_n) \to \infty$. Then

$$\lambda(B \cap \Xi_n) - \lambda(B)\lambda(\Xi_n) \to 0 \tag{1}$$

for any measurable $B \subset \mathbb{T}$ (see King [11], Lemma 3.4). It follows that for almost all $\beta \in \mathbb{T}$ there exist infinitely many n such that $\beta \in \Xi_n$.

Applying this fact for subsequences of the towers

$$\{(T^j[v_i\|\alpha q_{n-1}\|, w_i\|\alpha q_{n-1}\|))_{\tau q_n < j < q_n}\}_{n\in\mathbb{N}}$$

successively for $i = 1, \ldots, k$, we conclude that for λ^k-a.e. $(\beta_1, \ldots, \beta_k) \in \mathbb{T}^k$ there exist sequences $\{n_j\}_{j\in\mathbb{N}}$, $\{t_i^{(j)}\}_{j\in\mathbb{N}}$, $i = 1, \ldots, k$, of natural numbers such that $\tau q_{n_j} < t_i^{(j)} < q_{n_j}$ and

$$\begin{aligned}\beta_i \in T^{t_i^{(j)}}&[v_i\|\alpha q_{n_j-1}\|, w_i\|\alpha q_{n_j-1}\|) \\ &= [v_i\|\alpha q_{n_j-1}\| + t_i^{(j)}\alpha, w_i\|\alpha q_{n_j-1}\| + t_i^{(j)}\alpha).\end{aligned}$$

We can assume that $(\{q_{n_j}\beta_1\}, \ldots, \{q_{n_j}\beta_k\}) \to (\gamma_1, \ldots, \gamma_k) \in \mathbb{T}$. Then

$$\{q_{n_j}\beta_i\} \in [v_i q_{n_j}\|q_{n_j-1}\alpha\| + t_i^{(j)}\|q_{n_j}\alpha\|, w_i q_{n_j}\|q_{n_j-1}\alpha\| + t_i^{(j)}\|q_{n_{j_l}}\alpha\|).$$

Since

$$t_i^{(j)}\|q_{n_j}\alpha\| \le q_{n_j}\|q_{n_j}\alpha\| \to 0 \quad \text{and} \quad q_{n_j}\|q_{n_j-1}\alpha\| \to 1,$$

as $j \to \infty$, we have $v_i \le \gamma_i \le w_i$ for $i = 1, \ldots, k$ and finally $(\gamma_1, \ldots, \gamma_k) \in W$. ∎

Let $\Gamma \subset \mathbb{T}^k$ denote the set of all $(\gamma_1, \ldots, \gamma_k) \in \mathbb{T}^k$ such that

$$\forall_{m_1,\ldots,m_k\in\{0,\pm1,\pm2\}} \qquad m_1\gamma_1 + \ldots + m_k\gamma_k \in \mathbb{Z} \;\Rightarrow\; m_1, \ldots, m_k = 0.$$

Since the set Γ is open and dense, we can choose a cube $W = \prod_{i=1}^k [v_i, w_i]$

(with $0 < w_i < v_{i+1} < 1$ for $i = 1,\dots,k-1$) such that $W \subset \Gamma$ and $\lambda^k(W) > 0$. Fix $1/2 < \tau < 1$. Let B' denote the set of all $(\beta_1,\dots,\beta_k) \in \mathbb{T}^k$ such that there exists a subsequence $\{q_{n_j}\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} q_{n_j}\|q_{n_j}\alpha\| = 0, \qquad \lim_{j\to\infty}(\{q_{n_j}\beta_1\},\dots,\{q_{n_j}\beta_k\}) = (\gamma_1,\dots,\gamma_k) \in W$$

and

$$\beta_1,\dots,\beta_k \in \bigcup_{\tau q_{n_j} < t < q_{n_j}} T^t[0, \|q_{n_j-1}\alpha\|)$$

for any natural j. Then $0 = \gamma_0 < \gamma_1 < \dots < \gamma_k < \gamma_{k+1} = 1$. By Lemma 2.1, $\lambda^k(B') = 1$. Define $B_k = B' \cap \mathbb{T}^k_+$.

3. Proof of the Main Theorem. For given $f : \mathbb{T} \to \mathbb{R}$ and $q \in \mathbb{N}$ let

$$f^{(q)}(x) = f(x) + f(x+\alpha) + \dots + f(x+(q-1)\alpha).$$

Proof of Theorem 1.1. Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a PAC cocycle and let $0 = \beta_0 < \beta_1 < \dots < \beta_k < \beta_{k+1} = 1$ be all of the points of discontinuity of φ. Assume that $S(\varphi) = 0$, φ has at least one irrational jump and $(\beta_1,\dots,\beta_k) \in B_k$. Choose a PAC function $\widetilde{\varphi} : \mathbb{T} \to \mathbb{R}$ such that $\varphi(x) = e^{2\pi i\widetilde{\varphi}(x)}$ and $0 = \beta_0 < \beta_1 < \dots < \beta_k < \beta_{k+1} = 1$ are all of the points of discontinuity of $\widetilde{\varphi}$. Let $\{q_n\}_{n\in\mathbb{N}}$ be a subsequence of denominators of α with the properties of Lemma 2.1.

As will be shown in Lemma 3.2 (see §3.2), for all $m \in \mathbb{Z}$ and $r \in \mathbb{N}$ there exists $\delta_r^{(m)} \in \mathbb{C}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{T}} e^{2\pi i m\widetilde{\varphi}^{(rq_n)}(x)}\, dx = \delta_r^{(m)}.$$

This leads to the following statement: each unitary operator $U_\varphi^{(m)}$ is $\delta_r^{(m)}$-weakly mixing along $\{rq_n\}_{n\in\mathbb{N}}$, by Proposition 2. Moreover, it will be proved in Lemma 3.3 (see §3.2) that for every $m \in \mathbb{Z}\setminus\{0\}$ there exists $r \in \mathbb{N}$ such that $0 < |\delta_r^{(m)}| < 1$ and for all distinct $m_1, m_2 \in \mathbb{Z}\setminus\{0\}$, there exists $r \in \mathbb{N}$ such that $\delta_r^{(m_1)} \neq \delta_r^{(m_2)}$. It follows that the maximal spectral types of the operators $U_\varphi^{(m)}$ (for $m \neq 0$) are continuous singular and they are mutually singular, by Proposition 1. The simplicity of the spectrum of $U_\varphi^{(m)}$ will be proved in Lemma 3.1 (see §3.1).

Hence each of the operators $U_{T_\varphi} : H_m \to H_m$ for $m \neq 0$ has simple singular continuous spectrum and their maximal spectral types are pairwise orthogonal. It follows that T_φ has simple singular continuous spectrum on $H_0^\perp$. ■

3.1. *Simplicity of spectrum.* Let $V_g : L^2(\mathbb{T},\lambda) \to L^2(\mathbb{T},\lambda)$ be the unitary operator given by

$$V_g f(e^{2\pi ix}) = e^{2\pi ig(x)} f(Te^{2\pi ix}),$$

where $g : \mathbb{T} \to \mathbb{R}$ is a measurable function. We need the following:

LEMMA 3.1. *Let $g : \mathbb{T} \to \mathbb{R}$ be a PAC function with $S(g) = 0$. Let $0 = \beta_0 < \beta_1 < \ldots < \beta_k < \beta_{k+1} = 1$ be all of the points of discontinuity of g. If $(\beta_1, \ldots, \beta_k) \in B_k$, then V_g has simple spectrum.*

To prove this lemma we apply the following proposition proved in [5].

PROPOSITION 3. *Let $\{\Xi_n\}_{n\in\mathbb{N}}$ be a sequence of towers for the rotation T. Let C_n denote the base of Ξ_n. Suppose that $h_n = \mathrm{height}(\Xi_n) \to \infty$ and $\lambda(\bigcup_{j=0}^{h_n-1} T^j C_n) \to \nu$. If there exists $c < \nu$ such that for any $f \in L^2(\mathbb{T},\lambda)$ with $\|f\|_{L^2} = 1$ we have*

$$\limsup_{n\to\infty} 2\pi \sum_{j=0}^{h_n-1} \int_{T^j C_n} |f|^2\, d\lambda \iint_{C_n^2} |g^{(j)}(x) - g^{(j)}(y)| \frac{dx\, dy}{\lambda(C_n)^2} \le c,$$

then the maximal spectral multiplicity of V_g is at most $1/(\nu - c)$.

Proof of Lemma 3.1. Since $(\beta_1, \ldots, \beta_k) \in B_k$, we can choose a subsequence $\{q_n\}_{n\in\mathbb{N}}$ of denominators of α with the properties of Lemma 2.1, i.e.

$$\lim_{n\to\infty} q_n\|q_n\alpha\| = 0 \quad \text{and} \quad \beta_1, \ldots, \beta_k \in \bigcup_{\tau q_n < t < q_n} T^t[0, \|q_{n-1}\alpha\|). \tag{2}$$

We apply Proposition 3 for the tower $\Xi_n = (T^j[0, \|q_{n-1}\alpha\|))_{0\le j<\tau q_n}$. Then $\lambda(\bigcup_{j=0}^{h_n-1} T^j C_n) \to \tau$. Represent g as the sum of an absolutely continuous function $g_1 : \mathbb{T} \to \mathbb{R}$ and a piecewise constant $g_2 : \mathbb{T} \to \mathbb{R}$. From (2), the function $g_2^{(j)}$ is constant on C_n for $0 \le j < \tau q_n$. Therefore,

$$\sum_{0\le j<\tau q_n} \int_{T^j C_n} |f|^2\, d\lambda \iint_{C_n^2} |g^{(j)}(x) - g^{(j)}(y)| \frac{dx\, dy}{\lambda(C_n)^2}$$
$$= \sum_{0\le j<\tau q_n} \int_{T^j C_n} |f|^2\, d\lambda \iint_{C_n^2} |g_1^{(j)}(x) - g_1^{(j)}(y)| \frac{dx\, dy}{\lambda(C_n)^2}.$$

Applying Lemma 4.1 of [5], we can assert that for any $\varepsilon > 0$ there exists a subsequence $\{\Xi_{n_l}\}_{l\in\mathbb{N}}$ such that

$$\limsup_{j\to\infty} 2\pi \sum_{0\le j<\tau q_{n_l}} \int_{T^j C_{n_l}} |f|^2\, d\lambda \iint_{C_{n_l}^2} |g_1^{(j)}(x) - g_1^{(j)}(y)| \frac{dx\, dy}{\lambda(C_{n_l})^2} \le \varepsilon.$$

Since $\tau > 1/2$, we can take $\varepsilon < \tau - 1/2$. Applying Proposition 3 for the sequence $\{\Xi_{n_l}\}_{l\in\mathbb{N}}$, we conclude that the maximal spectral multiplicity of V_g is at most $1/(\tau-\varepsilon)<2$. ∎

3.2. $\delta_r^{(m)}$*-weak mixing*

Lemma 3.2. *There exists a real number a such that for all natural m and r we have*

$$\lim_{n\to\infty}\int_{\mathbb{T}} e^{2\pi i m\widetilde{\varphi}^{(rq_n)}(x)}\,dx=\delta_r^{(m)}=e^{2\pi i m r a}\sum_{u=0}^{k}(\gamma_{u+1}-\gamma_u)e^{2\pi i m r\sum_{i=1}^{u}d_i}.$$

Proof. Set

$$\phi(x)=\int_0^x\widetilde{\varphi}(y)\,dy-\int_0^1\int_0^z\widetilde{\varphi}(y)\,dy\,dz$$

and $\psi=\widetilde{\varphi}-\phi$. Then $\phi:\mathbb{T}\to\mathbb{R}$ is absolutely continuous with zero integral. Moreover $\psi:\mathbb{T}\to\mathbb{R}$ is constant on each interval (β_i,β_{i+1}) and $\psi_-(\beta_i)-\psi_+(\beta_i)=\widetilde{\varphi}_-(\beta_i)-\widetilde{\varphi}_+(\beta_i)=d_i$ for $i=0,\dots,k$. Of course, we can assume that $\widetilde{\varphi}$ is right continuous. Then

$$\psi=\psi(0)+\sum_{i=1}^{k+1}d_i\mathbf{1}_{[\beta_i,1)},$$

where $d_{k+1}=d_0$. Since $\phi^{(rq_n)}$ converges uniformly to 0 (see for instance [7], p. 189), and $\widetilde{\varphi}^{(rq_n)}=\phi^{(rq_n)}+\psi^{(rq_n)}$, we see that it suffices to find the limit of the sequence

$$\int_{\mathbb{T}} e^{2\pi i m\psi^{(rq_n)}(x)}\,dx.$$

Since for any $a,b,x\in\mathbb{T}$,

$$\mathbf{1}_{[b,1)}(x+a)-\mathbf{1}_{[b,1)}(a)=\mathbf{1}_{[b-a,1)}(x)-\mathbf{1}_{[1-a,1)}(x)$$

we have

$$\begin{aligned}\psi(x+a)-\psi(a)&=\sum_{i=1}^{k+1}d_i(\mathbf{1}_{[\beta_i,1)}(x+a)-\mathbf{1}_{[\beta_i,1)}(x))\\&=\sum_{i=1}^{k+1}d_i(\mathbf{1}_{[\beta_i-a,1)}(x)-\mathbf{1}_{[1-a,1)}(x))\\&=\sum_{i=1}^{k+1}d_i\mathbf{1}_{[\beta_i-a,1)}(x).\end{aligned}$$

Therefore for any $r,q\in\mathbb{N}$ we have

$$\psi^{(rq)}=\psi^{(rq)}(0)+\sum_{h=0}^{q-1}\sum_{s=0}^{r-1}\sum_{i=1}^{k+1}d_i\mathbf{1}_{[\beta_i-(sq+h)\alpha,1)}.\tag{3}$$

Let $\varrho_{r,q}:\mathbb{T}\to\mathbb{R}$ be defined by

$$\varrho_{r,q}=\psi^{(rq)}(0)+r\sum_{j=0}^{q-1}\sum_{i=1}^{k+1}d_i\mathbf{1}_{[(j+\gamma_i)/q,1)}.$$

For given $1\le i\le k+1$ and $0\le j<q_n$ let $h_i^{(j)}$ be the unique integer with $0\le h_i^{(j)}<q_n$ such that

$$h_i^{(j)}p_n+j=[q_n\beta_i]\bmod q_n.$$

Then

$$\begin{aligned}\beta_i-h_i^{(j)}\alpha&=\frac{[q_n\beta_i]}{q_n}+\frac{\{q_n\beta_i\}}{q_n}-h_i^{(j)}\frac{p_n}{q_n}-h_i^{(j)}\frac{\|q_n\alpha\|}{q_n}\\&=\frac{j}{q_n}+\frac{1}{q_n}(\{q_n\beta_i\}-h_i^{(j)}\|q_n\alpha\|).\end{aligned}\tag{4}$$

Therefore

$$\psi^{(rq_n)}-\varrho_{r,q_n}=\sum_{j=0}^{q_n-1}\sum_{s=0}^{r-1}\sum_{i=1}^{k+1}d_i(\mathbf{1}_{[\beta_i-(sq_n+h_i^{(j)})\alpha,1)}-\mathbf{1}_{[(j+\gamma_i)/q_n,1)}),$$

and

$$\|\psi^{(rq_n)}-\varrho_{r,q_n}\|_{L^1}\le D\sum_{j=0}^{q_n-1}\sum_{s=0}^{r-1}\sum_{i=1}^{k+1}|\beta_i-(sq_n+h_i^{(j)})\alpha-(j+\gamma_i)/q_n|,$$

where $D=\max_{i=1,\dots,k+1}|d_i|$. We conclude from (4) that

$$\begin{aligned}\|\psi^{(rq_n)}-\varrho_{r,q_n}\|_{L^1}&\le D\sum_{j=0}^{q_n-1}\sum_{s=0}^{r-1}\sum_{i=1}^{k+1}\left|\frac{\{q_n\beta_i\}-\gamma_i}{q_n}-\left(s+\frac{h_i^{(j)}}{q_n}\right)\|q_n\alpha\|\right|\\&\le Dr\sum_{i=1}^{k}|\{q_n\beta_i\}-\gamma_i|+Dkr^2q_n\|q_n\alpha\|,\end{aligned}$$

and hence that

$$\lim_{n\to\infty}\|\psi^{(rq_n)}-\varrho_{r,q_n}\|_{L^1}=0.\tag{5}$$

On the other hand

$$\begin{aligned}\varrho_{r,q}&=\psi^{(rq)}(0)+r\sum_{j=0}^{q-1}\sum_{i=1}^{k+1}d_i\Big(\sum_{u=i}^{k}\mathbf{1}_{[(j+\gamma_u)/q,(j+\gamma_{u+1})/q)}+\mathbf{1}_{[(j+1)/q,1)}\Big)\\&=\psi^{(rq)}(0)+r\sum_{j=0}^{q-1}\sum_{u=1}^{k}\sum_{i=1}^{u}d_i\mathbf{1}_{[(j+\gamma_u)/q,(j+\gamma_{u+1})/q)}\end{aligned}$$

and consequently

$$(6)\quad \int_{\mathbb{T}} e^{2\pi i m \varrho_{r,q}(x)}\,dx = e^{2\pi i m\psi^{(rq)}(0)} \sum_{j=0}^{q-1}\sum_{u=1}^{k} \frac{1}{q}(\gamma_{u+1}-\gamma_u) e^{2\pi i m r\sum_{i=1}^{u} d_i}$$
$$= e^{2\pi i m\psi^{(rq)}(0)} \sum_{u=0}^{k}(\gamma_{u+1}-\gamma_u) e^{2\pi i m r\sum_{i=1}^{u} d_i}.$$

Without loss of generality we can assume that

$$\lim_{n\to\infty} e^{2\pi i\psi^{(q_n)}(0)} = e^{2\pi i a}.$$

Then

$$(7)\qquad \lim_{n\to\infty} e^{2\pi i\psi^{(rq_n)}(0)} = e^{2\pi i r a}.$$

Indeed, since $\{q_n\beta_i\}\to\gamma_i>\gamma_1>0$ and $q_n\|q_n\alpha\|\to 0$, we have

$$q_n\|q_n\alpha\| < \min_{i=1,\dots,k}\{q_n\beta_i\}/r$$

for sufficiently large n. Then for any $i=1,\dots,k$, $j=0,\dots,q_n$, we have

$$(r-1)\|q_n\alpha\| < \frac{\{q_n\beta_i\}}{q_n} - \|q_n\alpha\| \le \frac{\{q_n\beta_i\}}{q_n} + \frac{j}{q_n} - \frac{h_i^{(j)}\|q_n\alpha\|}{q_n} = \beta_i - h_i^{(j)}\alpha.$$

It follows that $\psi^{(q_n)}(0) = \psi^{(q_n)}(q_n\alpha) = \ldots = \psi^{(q_n)}((r-1)q_n\alpha)$, by (3). Since

$$\psi^{(rq_n)}(0) = \psi^{(q_n)}(0) + \psi^{(q_n)}(q_n\alpha) + \ldots + \psi^{(q_n)}((r-1)q_n\alpha),$$

we have $\psi^{(rq_n)}(0) = r\psi^{(q_n)}(0)$. From (5)–(7), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{T}} e^{2\pi i m\psi^{(rq_n)}(x)}\,dx = e^{2\pi i m r a}\sum_{u=0}^{k}(\gamma_{u+1}-\gamma_u)e^{2\pi i m r\sum_{i=1}^{u} d_i}. \ \blacksquare$$

LEMMA 3.3. *For every $m\in\mathbb{Z}\setminus\{0\}$ there exists $r\in\mathbb{N}$ such that $0<|\delta_r^{(m)}|<1$ and for all distinct $m_1,m_2\in\mathbb{Z}\setminus\{0\}$ there exists $r\in\mathbb{N}$ such that $\delta_r^{(m_1)}\neq\delta_r^{(m_2)}$.*

Proof. Let $G\subset\mathbb{T}$ be the subgroup generated by 1, $e^{2\pi i d_1}$, $e^{2\pi i(d_1+d_2)}$, $\ldots$, $e^{2\pi i(d_1+\ldots+d_k)}$. Let us decompose

$$G = e^{2\pi i\alpha_1\mathbb{Z}}\oplus\ldots\oplus e^{2\pi i\alpha_g\mathbb{Z}}\oplus G_1,$$

where G_1 is a finite group ($c=\operatorname{card} G_1$) and $\alpha_1,\dots,\alpha_g,1$ are independent over $\mathbb{Q}$. As some of d_j is irrational, we have $g=\operatorname{rank}(G)>0$. Let $[a_{ij}]_{i=1,\dots,g;j=1,\dots,k}$ be an integer matrix such that

$$e^{2\pi i c(d_1+\ldots+d_j)} = e^{2\pi i(a_{j1}\alpha_1+\ldots+a_{jg}\alpha_g)}$$

for $j = 1, \ldots, k$. Define $\omega_j = e^{2\pi i\alpha_j}$ for $j = 1, \ldots, g$ and $\omega_0 = e^{2\pi ica}$. Set $\lambda_j = \gamma_{j+1} - \gamma_j$ for $j = 0, \ldots, k$. Then $\lambda_0, \ldots, \lambda_k > 0$ and $\lambda_0 + \ldots + \lambda_k = 1$. Let Q denote the trigonometric polynomial on $\mathbb{T}^g$ given by

$$Q(z_1, \ldots, z_g) = \lambda_0 + \lambda_1 z_1^{a_{11}} \ldots z_g^{a_{1g}} + \ldots + \lambda_k z_1^{a_{k1}} \ldots z_g^{a_{kg}}.$$

Then

$$\delta_{cr}^{(m)} = \omega_0^{mr} Q(\omega_1^{mr}, \omega_2^{mr}, \ldots, \omega_g^{mr}).$$

Since some of $d_1 + \ldots + d_j$ for $j = 1, \ldots, k$ are irrational, it is easy to see that $|\delta_{cr}^{(m)}| < 1$ for all $m, r \neq 0$.

We now show that for any $m \neq 0$ there exists $r \in \mathbb{N}$ such that

$$0 < |Q(\omega_1^{mr}, \ldots, \omega_g^{mr})| < 1.$$

Suppose that for all $r \in \mathbb{N}$, we have $Q(\omega_1^{mr}, \ldots, \omega_g^{mr}) = 0$. Since $\alpha_1, \ldots, \alpha_g, 1$ are independent over $\mathbb{Q}$, $Q(z_1, \ldots, z_g) = 0$ for any $(z_1, \ldots, z_g) \in \mathbb{T}^g$. Hence $0 = Q(1, \ldots, 1) = 1$, a contradiction.

Let us show that if $|m| \neq |m'|$, $m, m' \neq 0$, then there exists $r \in \mathbb{N}$ such that

$$|Q(\omega_1^{mr}, \ldots, \omega_g^{mr})| \neq |Q(\omega_1^{m'r}, \ldots, \omega_g^{m'r})|. \tag{8}$$

Suppose, contrary to our claim, that equality occurs in (8) for any $r \in \mathbb{N}$. Then

$$|Q(z_1^m, \ldots, z_g^m)| = |Q(z_1^{m'}, \ldots, z_g^{m'})| \quad \text{for any } (z_1, \ldots, z_g) \in \mathbb{T}^g.$$

Let P denote the trigonometric polynomial on $\mathbb{T}$ given by

$$P(z) = |Q(z^m, 1, \ldots, 1)|^2 = |Q(z^{m'}, 1, \ldots, 1)|^2.$$

Since

$$\max_{i,j=0,\ldots,k} |m(a_{i1} - a_{j1})| = \max_{i,j=0,\ldots,k} |m'(a_{i1} - a_{j1})| = \deg P > 0,$$

where $a_{01} = 0$, we obtain $|m| = |m'|$, a contradiction.

Let us show that for any $m \neq 0$ there exists $r \in \mathbb{N}$ such that

$$\omega_0^{mr} Q(\omega_1^{mr}, \ldots, \omega_g^{mr}) \neq \omega_0^{-mr} Q(\omega_1^{-mr}, \ldots, \omega_g^{-mr}). \tag{9}$$

Suppose that equality occurs in (9) for all $r \in \mathbb{N}$. Then

$$\omega_0^{mr} Q(\omega_1^{mr}, \ldots, \omega_g^{mr}) \in \mathbb{R} \quad \text{for all } r \in \mathbb{Z}.$$

Set $G_0 = \{(\omega_1^r, \ldots, \omega_g^r) : r \in \mathbb{Z}\}$. Let $F : G_0 \to \mathbb{T}$ be the group homomorphism given by

$$F(\omega_1^r,\dots,\omega_g^r)=\omega_0^{2mr}=\frac{Q(\omega_1^{-mr},\dots,\omega_g^{-mr})}{Q(\omega_1^{mr},\dots,\omega_g^{mr})}.$$

Then $(\omega_1^{r_n},\dots,\omega_g^{r_n})\to(1,\dots,1)$ implies

$$F(\omega_1^{r_n},\dots,\omega_g^{r_n})=\frac{Q(\omega_1^{-mr_n},\dots,\omega_g^{-mr_n})}{Q(\omega_1^{mr_n},\dots,\omega_g^{mr_n})}\to\frac{Q(1,\dots,1)}{Q(1,\dots,1)}=F(1,\dots,1).$$

Since F is a continuous group homomorphism and $\overline{G}_0=\mathbb{T}^g$, there exists a continuous group homomorphism $\overline{F}:\mathbb{T}^g\to\mathbb{T}$ such that $\overline{F}|_{G_0}=F$ and

$$\overline{F}(z_1,\dots,z_g)=z_1^{c_1}\dots z_g^{c_g},$$

where $c_1,\dots,c_g\in\mathbb{Z}$. Therefore

$$\omega_0^{2m}=F(\omega_1,\dots,\omega_g)=\omega_1^{c_1}\dots\omega_g^{c_g}$$

and consequently

$$\omega_1^{c_1r}\dots\omega_g^{c_gr}Q(\omega_1^{2mr},\dots,\omega_g^{2mr})\in\mathbb{R}$$

for all $r\in\mathbb{Z}$. It follows that the trigonometric polynomial

$$z_1^{c_1}\dots z_g^{c_g}Q(z_1^{2m},\dots,z_g^{2m})$$

has only real values. Hence there exist $m_0,\dots,m_k\in\{0,1,-1\}$ such that $\sum_{j=0}^k m_j\lambda_j=0$ and there exist j_1,j_2 such that $m_{j_1}=1$ and $m_{j_2}=-1$, contrary to $(\gamma_1,\dots,\gamma_k)\in\varGamma$. ■

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