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Domination of operators in the non-commutative setting

by

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Abstract. We consider majorization problems in the non-commutative setting. More specifically, suppose E and F are ordered normed spaces (not necessarily lattices), and $0 \leq T \leq S$ in $B(E,F)$. If S belongs to a certain ideal (for instance, the ideal of compact or Dunford–Pettis operators), does it follow that T belongs to that ideal as well? We concentrate on the case when E and F are C^*-algebras, preduals of von Neumann algebras, or non-commutative function spaces. In particular, we show that, for C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent: (1) at least one of the two conditions holds: (i) $\mathcal{A}$ is scattered, (ii) $\mathcal{B}$ is compact; (2) if $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, and S is compact, then T is compact.

1. Preliminaries

1.1. Introduction. Following [45, Definition II.1.2], we say that a real Banach space Z is an *ordered Banach space* (*OBS* for short) if it is equipped with a positive cone Z_+, closed in the norm topology. Throughout, we assume that Z_+ is *proper* (or *pointed*)—that is, $Z_+ \cap (-Z_+) = \{0\}$. The positive cone of an OBS Z is called *generating* if $Z_+ - Z_+ = Z$. Equivalently (see [6], [8]), there exists $\mathbf{G}_Z$ (the *generating constant* of Z) such that, for any $z \in Z$, there exist $a, b \in Z_+$ such that $z = a - b$, and $\max\{\|a\|, \|b\|\} \leq \mathbf{G}_Z \|z\|$. Abusing the terminology slightly, we call such OBSs generating. We say that an OBS Z is *normal* if there exists $\mathbf{N}_Z$ (the *normality constant* of Z) such that $\|z\| \leq \mathbf{N}_Z(\|a\| + \|b\|)$ whenever $a \leq z \leq b$. By [8, Section 1.1] or [6], Z is normal iff its dual $Z^\star$ is generating, and vice versa.

In the current article we consider the following question. Suppose $0 \leq T \leq S$ are operators acting between two ordered Banach spaces, and S belongs to a certain class of operators (say, compact or Dunford–Pettis). Does this imply that T belongs to the same class? This question is usually referred to as the *Domination Problem*. For arbitrary ordered normed spaces, the set-up may be too general to obtain meaningful results. In the (rather restrictive)

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setting of operators between Banach lattices, the Domination Problem has been widely investigated (see e.g. [2], [3], [18], [47], [25], [31], [51]).

We concentrate on the non-commutative version of the Domination Problem. More specifically, we consider the case when the domain and/or range of the operators involved is either a C^*-algebra, its dual or predual, or a non-commutative function space. We refer the reader to e.g. [17], or to the survey article [41], for the definition of the latter. Here, we only briefly outline the basic properties of such spaces.

Suppose a von Neumann algebra $\mathcal{A}$ is equipped with a normal faithful semifinite trace τ. An operator x is called *τ-measurable* if it is (i) closed and densely defined; (ii) affiliated with $\mathcal{A}$, in the sense that $ux = xu$ for any unitary $u \in \mathcal{A}'$; and (iii) for some $c > 0$, the spectral projection $\chi_{(c,\infty)}(|x|)$ has finite trace. On the set $\tilde{\mathcal{A}}$ of τ-measurable operators, we define the *generalized singular value function*: for $x \in \mathcal{A}$ and $t \geq 0$, $\mu_x(t) = \inf\{\|xe\| : e \in \mathbf{P}(\mathcal{A}), \tau(e^\perp) \leq t\}$ (see e.g. [41], [23] for other formulae for $\mu_x(\cdot)$). Here and below, $\mathbf{P}(\mathcal{A})$ stands for the set of all projections in $\mathcal{A}$.

Now suppose $\mathcal{E}$ is a linear subset of $\tilde{\mathcal{A}}$, complete in its own norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a *non-commutative function space* if:

(1) $L_1(\tau) \cap \mathcal{A} \subset \mathcal{E} \subset L_1(\tau) + \mathcal{A}$.
(2) For any $x \in \mathcal{E}$ and $a, b \in \mathcal{A}$, we have $axb \in \mathcal{E}$, and $\|axb\|_{\mathcal{E}} \leq \|a\|\,\|x\|_{\mathcal{E}}\|b\|$.

$\mathcal{E}$ is called *symmetric* if, whenever $x \in \mathcal{E}$, $y \in \tilde{\mathcal{A}}$, and $\mu_y \leq \mu_x$, then $y \in \mathcal{E}$, with $\|y\|_{\mathcal{E}} \leq \|x\|_{\mathcal{E}}$. Following [22], we say that $\mathcal{E}$ is *strongly symmetric* if, in addition, for any $x, y \in \mathcal{E}$, with $y \prec\prec x$, we have $\|y\|_{\mathcal{E}} \leq \|x\|_{\mathcal{E}}$. Here, $\prec\prec$ refers to *Hardy–Littlewood domination*: for any $\alpha > 0$, $\int_0^\alpha \mu_y(t)\,dt \leq \int_0^\alpha \mu_x(t)\,dt$. It is known [16] that, as in the commutative case, $y \prec\prec x$ iff there exists an operator T, contractive both on $\mathcal{A}$ and $\mathcal{A}_\star = L_1(\tau)$, such that $y = Tx$. We say that $\mathcal{E}$ is *fully symmetric* if it is strongly symmetric and, for any $x \in \mathcal{E}$ and $y \in \tilde{A}$, we have $y \in \mathcal{E}$ whenever $y \prec\prec x$.

A non-commutative function space is said to be *order continuous* if, for any sequence $x_n \downarrow 0$, we have $\lim_n \|x_n\| = 0$. Emulating the proof of [37, Proposition 1.a.8], one shows that this is equivalent to requiring that, for any net $x_\alpha \downarrow 0$, $\lim_\alpha \|x_\alpha\| = 0$.

Note that, if $-a \leq b \leq a$ for $a, b \in \tilde{\mathcal{A}}$, then $\mu_b \leq \mu_a$. Indeed, pick $t \in \mathbb{R}$ and $\lambda > \mu_a(t)$. Set $e = \chi_{[0,\lambda]}(a)$. Then $\tau(e^\perp) \leq t$. Furthermore, $eae \geq ebe \geq -eae$, hence $\mu_b(t) \leq \|ebe\| \leq \|eae\| \leq \lambda$. Taking the infimum over λ, we obtain $\mu_b \leq \mu_a$.

Consequently, if $a, b \in \mathcal{E}$ satisfy $-a \leq b \leq a$, then $\|b\| \leq \|a\|$. Therefore, $\mathcal{E}$ is normal with constant 2. It is also easy to see that $\mathcal{E}$ is generating with constant 2. Consequently, the duals of $\mathcal{E}$ of all orders are both generating and normal.

Many symmetric non-commutative function spaces arise from their commutative analogues. Indeed, suppose τ is a normal faithful semifinite trace on a von Neumann algebra $\mathcal{A}$. It is known that if $\mathcal{A}$ has no atomic projections, then the range of τ (denoted by $\Omega = \Omega_\tau$) is $[0, \tau(\mathbf{1})]$ (with $\tau(\mathbf{1}) < \infty$), or $[0, \infty)$. On the other hand, if $\mathcal{A}$ is atomic (that is, any projection has a minimal subprojection), then Ω_τ is either $\{0, 1, \ldots, n\}$ or $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Suppose $\mathcal{E}$ is a symmetric function space (in the sense of e.g. [35]) on Ω. We can define the corresponding non-commutative function space $\mathcal{E}(\tau)$, consisting of those $x \in \tilde{A}$ for which the norm $\|x\|_{\mathcal{E}(\tau)} = \|\mu_x\|_{\mathcal{E}}$ is finite. By [32], this procedure yields a Banach space. It is well known (see e.g. [17], [22], [41]) that many properties of the function space $\mathcal{E}$ (for instance, being reflexive or order continuous) pass to the non-commutative space $\mathcal{E}(\tau)$.

In the discrete case ($\mathcal{E}$ is a symmetric sequence space on $\mathbb{N}$, and τ is the canonical trace on $B(H)$), the construction above produces a *non-commutative symmetric sequence space* (often referred to as a *Schatten space*), denoted by $\mathcal{S}_{\mathcal{E}}(H)$ (instead of $\mathcal{E}(\tau)$). When $H = \ell_2$ ($H = \ell_2^n$), we write $\mathcal{S}_{\mathcal{E}}$ (resp. $\mathcal{S}_{\mathcal{E}}^n$) instead of $\mathcal{S}_{\mathcal{E}}(H)$. For properties of Schatten spaces, the reader is referred to e.g. [27], [46]. We must note that any separable symmetric non-commutative sequence space arises from a sequence space [27, Section III.6].

Observe that a symmetric function (or sequence) space is separable iff it is order continuous. Indeed, symmetric function spaces are order complete, and, for such spaces, separability implies order continuity [37, Proposition 1.a.7]. On the other hand, it is well known that any non-negative function is a limit (a.e.) of an increasing sequence of simple functions. Thus, by [35, Theorem II.4.8], any order continuous symmetric function space is separable. Furthermore, by [35, Theorem II.4.10 and its Corollary], such spaces are fully symmetric (equivalently, they are interpolation spaces between L_1 and L_∞). Some non-commutative generalizations of these results are contained in [21].

Surprisingly, the non-commutative Domination Problem has attracted little attention so far. The connections between domination and irreducibility (for maps between von Neumann algebras) were studied in [24]. In [40], domination of linear functionals on Banach $*$-algebras was used to obtain automatic continuity results. Domination of completely positive compact operators has recently been investigated in [20].

The paper is structured as follows. First (Section 1), we prove some preliminary results about the properties of positive operators, order intervals, and positive solids. In Subsection 1.2, we establish some basic facts about non-commutative function spaces. In Subsection 1.3, we investigate compact C^*-algebras, characterizing them in terms of compactness of order intervals. We also show that a C^*-algebra is compact iff it is hereditary in its enveloping algebra. Subsection 1.4 deals with the positive analogues of the

Schur Property. In Subsection 1.5, we study compactness of order intervals in preduals of von Neumann algebras.

Our main results are contained in Section 2. In Subsection 2.1, we investigate whether an operator to or from a non-commutative function space, dominated by a compact operator, must itself be compact. Subsection 2.2 is devoted to the same question for C^*-algebras. In Subsection 2.3, we consider domination by compact multiplication operators on C^*-algebras. In Subsection 2.4, we tackle domination properties of Dunford–Pettis Schur multipliers. Subsection 2.5 is devoted to the domination properties of weakly compact operators.

Other classes of operators are considered in Section 3. In Subsection 3.1, we show that complete positivity and decomposability are not preserved under domination. Subsection 3.2 demonstrates that operator systems have too little structure to meaningfully consider domination.

Throughout the paper, we use standard Banach space results and notation. If a is a (closed densely defined) operator, a^* refers to the adjoint of a. The same notation is used in preduals of von Neumann algebras. If E is a Banach space, $E^\star$ refers to its dual. Similar notation is used for the predual, and for the conjugate of an operator between Banach spaces. $\mathbf{B}(E)$ stands for the unit ball of E. If S is a subset of an ordered Banach space, we denote by S_+ the intersection of S with the positive cone. We denote by $\mathcal{E}^\times$ the *Köthe dual* of a non-commutative symmetric function space $\mathcal{E}$ (see e.g. [17], [41] for the definition and basic properties of Köthe duals).

1.2. Compactness and positivity in Schatten spaces. To work with Schatten spaces, we need to introduce some notation. Denote the canonical basis in ℓ_2 by (e_k). Let P_n be the orthogonal projection onto $\mathrm{span}[e_1, \ldots, e_n]$, and $P_n^\perp = \mathbf{1} - P_n$. For convenience, set $P_0 = 0$. If $\mathcal{E}$ is a non-commutative symmetric sequence space, let Q_n be the projection on $\mathcal{E}$, defined via $Q_n x = P_n x P_n$. Similarly, let $R_n x = P_n^\perp x P_n^\perp$.

LEMMA 1.2.1. *Suppose $\mathcal{E}$ is a non-commutative symmetric sequence space on $B(\ell_2)$, Z is an ordered normed space, and $T : \mathcal{E} \to Z$ is a positive operator. Then, for any $x \in \mathcal{E}_+$, $\|T(x - R_n x - Q_n x)\|^2 \leq 16\mathbf{N}_Z \|T(Q_n x)\| \, \|T(R_n x)\|$, where $\mathbf{N}_Z$ is the normality constant of Z. If Z is a non-commutative symmetric function space, then $\|T(x - R_n x - Q_n x)\|^2 \leq 4\|T(Q_n x)\| \, \|T(R_n x)\|$.*

Proof. For $t \in \mathbb{R} \setminus \{0\}$, consider $U(t) = tP_n + t^{-1}P_n^\perp$ and $V(t) = tP_n - t^{-1}P_n^\perp$. These operators are self-adjoint and invertible, hence $x(t) = U(t)xU(t)$ and $y(t) = V(t)xV(t)$ are positive elements of $\mathcal{E}$. An elementary calculation shows that $x(t) = t^2 Q_n x + t^{-2} R_n x + (x - Q_n x - R_n x)$, and $y(t) = t^2 Q_n x + t^{-2} R_n x - (x - Q_n x - R_n x)$. Let $a(t) = t^2 Q_n x + t^{-2} R_n x$ and

$b = x - Q_n x - R_n x$. By the above, $-a(t) \le b \le a(t)$. Therefore, for any t,

$$\frac{1}{2\mathbf{N}_Z}\|Tb\| \le \|Ta(t)\| \le t^2\|TQ_n x\| + t^{-2}\|TR_n x\|.$$

Taking $t = \|TR_n x\|^{1/4}/\|TQ_n x\|^{1/4}$, we obtain the desired inequality. If, in addition, Z is a non-commutative symmetric function space, then $\|Tb\| \le \|Ta(t)\|$. ∎

COROLLARY 1.2.2. *Suppose $\mathcal{E}$ is a non-commutative symmetric sequence space on $B(\ell_2)$, Z is a normal OBS, and $T : \mathcal{E} \to Z$ is a positive operator. Then*

$$\|T(I - Q_n)\| \le \|TR_n\| + 16\mathbf{N}_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2}.$$

If Z is a non-commutative symmetric function space, then $\|T(I - Q_n)\| \le \|TR_n\| + 8\mathbf{N}_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2}$.

Proof. We prove the corollary for general Z (the case of Z being a non-commutative function space follows with minor modifications). Lemma 1.2.1 shows that, for $x \ge 0$,

$$\|T(I - R_n - Q_n)x\| \le 4\mathbf{N}_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2}\|x\|.$$

A polarization argument implies

$$\|T(I - R_n - Q_n)\| \le 16\mathbf{N}_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2}.$$

By the triangle inequality, $\|T(I - Q_n)\| \le \|TR_n\| + \|T(I - R_n - Q_n)\|$. ∎

For future use, we need to quote a result from [12, Section 2].

LEMMA 1.2.3. *Suppose τ is a normal faithful semifinite trace on a von Neumann algebra $\mathcal{A}$, and a strongly symmetric non-commutative function space $\mathcal{E}$ is order continuous. Suppose, furthermore, that x is an element of $\mathcal{A}$, and a sequence of projections $p_n \in \mathcal{A}$ decreases to 0 in the strong operator topology. Then* $\lim_n \|xp_n\| = \lim_n \|p_n x\| = \lim_n \|p_n x p_n\| = 0$.

Specializing to Schatten spaces, we obtain:

COROLLARY 1.2.4. *Suppose $\mathcal{E}$ is an order continuous symmetric sequence space. Then, for every $x \in \mathcal{S}_{\mathcal{E}}$,* $\lim_n \|x - Q_n x\| = 0$.

Proof. By [17, Section 3], $\mathcal{S}_{\mathcal{E}}$ is order continuous iff $\mathcal{E}$ is order continuous. It suffices to show that, for $x \in \mathbf{B}(\mathcal{S}_{\mathcal{E}})_+$ and $\varepsilon \in (0,1)$, $\|x - Q_n x\| < \varepsilon$ for n sufficiently large. This follows from the estimate $\|x - Q_n x\| = \|P_n^\perp x P_n + x P_n^\perp\| \le \|P_n^\perp x\| + \|x P_n^\perp\|$ and Lemma 1.2.3. ∎

LEMMA 1.2.5. *Suppose $\mathcal{E}$ is an order continuous symmetric sequence space not containing ℓ_1, and $S : \mathcal{S}_{\mathcal{E}} \to Z$ is compact (Z is a Banach space). Then* $\lim_n \|S|_{R_n(\mathcal{S}_{\mathcal{E}})}\| = 0$.

Proof. Suppose not. By Corollary 1.2.4, we have $\lim_n \|(I - Q_n)x\| = 0$. A standard approximation argument yields a sequence $0 = n_0 < n_2 < \cdots$ with the property that for each k there exists $x_k \in \mathcal{S}_\mathcal{E}$ such that $\|x_k\| = 1$, $(P_{n_k} - P_{n_{k-1}})x_k(P_{n_k} - P_{n_{k-1}}) = x_k$, and $\|Sx_k\| > c > 0$. By compactness, the sequence (Sx_k) must have a convergent subsequence (Sx_{k_i}). Then $\lim_N N^{-1}\|\sum_{i=1}^N Sx_{k_i}\| > 0$, while $\lim_N N^{-1}\|\sum_{i=1}^N x_{k_i}\| = 0$. ∎

Next we describe the Schatten spaces not containing ℓ_1.

PROPOSITION 1.2.6. *Let $\mathcal{E}$ be a separable symmetric sequence space. For any infinite-dimensional Hilbert space H, the following are equivalent:*

(1) *$\mathcal{E}$ contains a copy of ℓ_1.*
(2) *$\mathcal{E}$ contains a lattice copy of ℓ_1 positively complemented.*
(3) *$\mathcal{S}_\mathcal{E}(H)$ contains a positively complemented copy of ℓ_1 spanned by a disjoint positive sequence.*
(4) *$\mathcal{S}_\mathcal{E}(H)$ contains a copy of ℓ_1.*

Proof. The implications (2)⇒(1) and (3)⇒(4) are trivial. To show (2)⇒(3), observe that $\mathcal{S}_\mathcal{E}(H)$ contains $\mathcal{E}$ as a diagonal subspace, which is positively complemented. (4)⇒(1) follows directly from [7, Corollary 3.2]. To prove (1)⇒(2), apply a "gliding hump" argument to show that $\mathcal{E}$ contains disjoint vectors (x_i), equivalent to the canonical basis of ℓ_1. Then $X = \mathrm{span}[|x_i| : i \in \mathbb{N}]$ is a sublattice of $\mathcal{E}$, lattice isomorphic to ℓ_1. By [39, Theorem 2.3.11], X is positively complemented. ∎

For a subset $M \subset X_+$ (X is an OBS), define the *positive solid* of M:

$$\mathbf{PSol}(M) = \{x \in X_+ : 0 \le x \le y \text{ and } y \in M\}.$$

LEMMA 1.2.7. *If $\mathcal{E}$ is an order continuous non-commutative symmetric sequence space, and $M \subset \mathcal{E}$ is relatively compact, then $\mathbf{PSol}(M)$ is relatively compact. In particular, any order interval in an order continuous non-commutative symmetric sequence space is compact.*

For the proof, we need two technical results.

LEMMA 1.2.8. *Suppose $\mathcal{E}$ and M are as in Lemma 1.2.7. Then there exists a projection p with separable range such that $M = pMp$.*

Proof. The set M must contain a countable dense subset S. The elements of M are compact operators, hence, for any $x \in S$, there exists a projection p_x with separable range such that $p_x x p_x = x$. Then $p = \bigvee_{x \in S} p_x$ has the desired properties. ∎

LEMMA 1.2.9. *Suppose $\mathcal{E}$ is an order continuous non-commutative symmetric sequence space on $B(\ell_2)$, and M is a relatively compact subset of $\mathcal{E}$. Then* $\lim_n \|R_n|_M\| = 0$.

Proof. For every $\varepsilon > 0$ there are $x_1, \dots, x_k$ in M such that for every $x \in M$ there is an $1 \le i \le k$ such that $\|x - x_i\| < \varepsilon/2$. Pick $N \in \mathbb{N}$ such that $\|R_n x_i\| < \epsilon/2$ for every $n > N$ and $1 \le i \le k$. Hence, $\|R_n x\| \le \|R_n x_i\| + \|R_n\| \, \|x - x_i\| < \epsilon$ for every $x \in M$ and $n > N$. ∎

Proof of Lemma 1.2.7. By Lemma 1.2.8, we can restrict ourselves to spaces on $B(\ell_2)$. As Q_n is a finite rank projection, it suffices to show that, for any $\varepsilon \in (0,1)$, there exists $n \in \mathbb{N}$ such that $\|(I - Q_n)x\| < \varepsilon$ for any $x \in \mathbf{PSol}(M)$. To this end, write $(I - Q_n)x = (x - Q_n x - R_n x) + R_n x$. Reasoning as in the proof of Lemma 1.2.1, we observe that

$$-(t^2 Q_n x + t^{-2} R_n x) \le x - Q_n x - R_n x \le t^2 Q_n x + t^{-2} R_n x$$

for any $t > 0$, hence $\|x - Q_n x - R_n x\| \le t^2 \|Q_n x\| + t^{-2} \|R_n x\|$. Taking $t = \|R_n x\|^{1/2} / \|Q_n x\|^{1/2}$, we obtain $\|x - Q_n x - R_n x\| \le 2 \|R_n x\|^{1/2} \|Q_n x\|^{1/2}$.

By scaling, we can assume that $\sup_{y \in M} \|y\| = 1$. By Lemma 1.2.9, there exists $n \in \mathbb{N}$ such that $\|R_n y\| < \varepsilon^2/16$ for any $y \in M$. For any $x \in \mathbf{PSol}(M)$, there exists $y \in M$ such that $0 \le x \le y$, hence $0 \le R_n x \le R_n y$. By the above, $\|x - Q_n x - R_n x\| \le 2\|R_n y\|^{1/2} < \varepsilon/2$, hence

$$\|(I - Q_n)x\| = \|x - Q_n x - R_n x\| + \|R_n x\| \le \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16} < \varepsilon. \ ∎$$

Recall that if Z is an OBS and $x \in Z_+$, the *order interval* $[0, x]$ is the set $\{y \in Z_+ : y \le x\}$.

COROLLARY 1.2.10. *Suppose $\mathcal{E}$ is a fully symmetric non-commutative sequence space. Then $\mathcal{E}$ is order continuous if and only if any order interval in $\mathcal{E}$ is compact.*

LEMMA 1.2.11. *Suppose $\mathcal{E}$ is a fully symmetric non-commutative function or sequence space which is not order continuous. Then there exists a positive complete isomorphism $j : \ell_\infty \to \mathcal{E}$.*

Proof. In the notation of [22, Section 6], there exists $x \in \mathcal{E}_+ \setminus \mathcal{E}^{an}$. Moreover, there exists a sequence of mutually orthogonal projections $e_i \in \mathcal{A}$ $(i \in \mathbb{N})$ such that $\inf_i \|e_i x e_i\| > 0$. The map $y \mapsto \sum_i e_i y e_i$ is contractive in $\mathcal{A}$, and in its predual, hence $\sum_i e_i y e_i \prec\!\prec y$ for any $y \in \mathcal{A} + \mathcal{A}_\star$. Due to $\mathcal{A}$ being fully symmetric, $\sum_i e_i x e_i \in \mathcal{E}$, and $\| \sum_i e_i x e_i \| \le \|x\|$. Therefore, the map

$$j : \ell_\infty \to \mathcal{E} : (\alpha_i) \mapsto \Big(\sum_i \alpha_i e_i \Big) \Big(\sum_i e_i x e_i \Big) = \sum_i \alpha_i e_i x e_i$$

has the desired properties. ∎

Proof of Corollary 1.2.10. Note that an order interval $[0, x]$ is closed. If $\mathcal{E}$ is order continuous, an application of Lemma 1.2.7 to $M = \{x\}$ shows the compactness of $[0, x]$. If $\mathcal{E}$ is not order continuous, then, for x as in Lemma 1.2.11, $[0, x]$ is not (relatively) compact. ∎

1.3. Compactness of order intervals in C^*-algebras. In this subsection, we investigate the compactness of order intervals in C^*-algebras, and obtain a new description of compact C^*-algebras.

First we introduce some definitions. We say that an element a of a Banach algebra $\mathcal{A}$ is *multiplication compact* if the map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is compact. Combining [57], [58], we see that, for an element a of a C^*-algebra $\mathcal{A}$, the following are equivalent:

(1) a is multiplication compact.
(2) The map $\mathcal{A} \to \mathcal{A} : b \mapsto ab$ is weakly compact.
(3) The map $\mathcal{A} \to \mathcal{A} : b \mapsto ba$ is weakly compact.
(4) The map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is weakly compact.

By [56], there exists a faithful representation $\pi : \mathcal{A} \to B(H)$ such that a is multiplication compact iff $\pi(a)$ is a compact operator on H. If, in addition, $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, then $a \in \mathcal{A}$ is multiplication compact iff a is a compact operator [55].

Suppose $\mathcal{A}$ is a C^*-subalgebra of $B(H)$, where H is a Hilbert space. For $x \in B(H)$ we define an operator $M_x : \mathcal{A} \to B(H) : a \mapsto x^*ax$.

LEMMA 1.3.1. *For an element a of a C^*-algebra $\mathcal{A}$, the following are equivalent:*

(1) *a is multiplication compact.*
(2) *The operator M_a is compact.*
(3) *The operator M_a is weakly compact.*

Proof. (2)⇒(3) is trivial. To show (1)⇒(2), recall that a is multiplication compact iff the map $\mathcal{A} \to \mathcal{A} : b \mapsto ab$ is weakly compact. Passing to the adjoint, we see that the last statement holds iff the map $\mathcal{A} \to \mathcal{A} : b \mapsto ba^*$ is weakly compact, or equivalently, iff a^* is multiplication compact. By [10], this implies the compactness of M_a.

To prove (3)⇒(1), note that $M_a^{\star\star}$ takes $b \in \mathcal{A}^{\star\star}$ to a^*ba. We identify $M_a^{\star\star}$ with M_a, acting on $\mathcal{A}^{\star\star}$. Write $a = cu$, where $c = (aa^*)^{1/2}$ and u (respectively, u^*) is a partial isometry from $(\ker a)^\perp = (\ker c)^\perp$ to $\overline{\operatorname{ran} a} = \overline{\operatorname{ran} c}$ (from $\overline{\operatorname{ran} a^*} = \overline{\operatorname{ran} c}$ to $(\ker a^*)^\perp = (\ker c)^\perp$). Then $M_a = M_u M_c$, and M_u is an isometry on $\operatorname{ran} M_c \subset \mathcal{A}^{\star\star}$. Writing $M_c = M_u^{-1} M_a$, we conclude that M_c is weakly compact. However, $M_c x = cxc$, hence, by the remarks preceding the lemma, c is multiplication compact. The operator $S : \mathcal{A}^{\star\star} \to \mathcal{A}^{\star\star} : b \mapsto aba$ can be written as $S = UM_cV$, where $Vb = ub$ and $Ub = bu$. Then S is weakly compact, and therefore a is multiplication compact. ∎

Multiplication compactness of elements of a C^*-algebra can be described in terms of compactness of order intervals.

Proposition 1.3.2. *For a positive element a of a C^*-algebra $\mathcal{A}$, the following are equivalent:*

(1) *a is multiplication compact.*
(2) *a^α is multiplication compact for any $\alpha > 0$.*
(3) *The order interval $[0, a]$ is compact.*
(4) *The order interval $[0, a]$ is weakly compact.*

Proof. The implications (2)⇒(1) and (3)⇒(4) are immediate. To establish (1)⇒(2), pick a faithful representation π so that a is multiplication compact if and only if $\pi(a)$ is compact, and note that the compactness of $\pi(a)$ is equivalent to the compactness of $\pi(a)^\alpha = \pi(a^\alpha)$.

For (2)⇒(3), assume $\|a\| = 1$. By [13, Lemma I.5.2], for any $x \in [0, a]$ there exists $u \in \mathbf{B}(\mathcal{A})$ such that $x^{1/2} = ua^{1/4}$, hence $x = a^{1/4}u^*ua^{1/4}$. In particular, $[0, a] \subset M_{a^{1/4}}(\mathbf{B}(\mathcal{A}))$. If a is multiplication compact, then so is $a^{1/4}$. Therefore, $[0, a]$ is compact.

To prove (4)⇒(1), suppose a is not multiplication compact. Then $a^{1/2}$ is not multiplication compact, hence $M_{a^{1/2}}(\mathbf{B}(\mathcal{A}))$ is not relatively compact. Note that any element $x \in \mathbf{B}(\mathcal{A})$ can be written as $x = x_1 - x_2 + i(x_3 - x_4)$ with $x_1, x_2, x_3, x_4 \in \mathbf{B}(\mathcal{A})_+$. Thus, $M_{a^{1/2}}(\mathbf{B}(\mathcal{A})_+)$ is not relatively weakly compact. However, $[0, a] \supset M_{a^{1/2}}(\mathbf{B}(\mathcal{A})_+)$. Indeed, if $0 \le y \le 1$, then $0 \le a^{1/2}ya^{1/2} \le a$. Therefore, $[0, a]$ is not relatively weakly compact. ∎

These results allow us to obtain new characterizations of compact C^*-algebras. Recall that a Banach algebra is called *compact* (or *dual*) if all of its elements are multiplication compact. By [1], the compact C^*-algebras are precisely the algebras of the form $\mathcal{A} = (\sum_{i\in I} K(H_i))_{c_0}$, where each H_i is a complex Hilbert space, and $K(H)$ denotes the space of compact operators on H. Several alternative characterizations of compact C^*-algebras can be found in [14, 4.7.20].

Proposition 1.3.3. *For a C^*-algebra $\mathcal{A}$, the following four statements are equivalent:*

(1) *$\mathcal{A}$ is compact.*
(2) *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is compact.*
(3) *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is weakly compact.*
(4) *For any relatively compact $M \subset \mathcal{A}_+$, $\mathbf{PSol}(M)$ is relatively compact.*

Proof. The implications (4)⇒(2)⇒(3) are immediate.

(3)⇒(1). By Proposition 1.3.2, any positive $a \in \mathcal{A}$ is multiplication compact. By [10, Corollary 10.4], the map $\mathcal{A} \to \mathcal{A} : x \mapsto axb$ is compact for any $a, b \in \mathcal{A}_+$. As any $x \in \mathcal{A}$ is a linear combination of four positive elements, it is multiplication compact.

(1)⇒(4). It suffices to show that, for any $\varepsilon > 0$, $\mathbf{PSol}(M)$ admits a finite ε-net. Assume, without loss of generality, that $M \subset \mathbf{B}(\mathcal{A})_+$. The map $\mathcal{A}_+ \to$

$\mathcal{A}_+ : a \mapsto a^{1/4}$ is continuous, hence $M^{1/4} = \{a^{1/4} : a \in M\}$ is compact. Pick $(a_i)_{i=1}^n \subset M$ so that $(a_i^{1/4})_{i=1}^n$ is an $\varepsilon/4$-net in $M^{1/4}$. By Proposition 1.3.2, $a_i^{1/4}$ is multiplication compact for each i, hence $a_i^{1/4}\mathbf{B}(\mathcal{A})_+ a_i^{1/4}$ contains an $\varepsilon/4$-net $(b_{ij})_{j=1}^m$.

Now consider $x \in [0, a]$ for some $a \in M$. As noted in the proof of Proposition 1.3.2, there exists $u \in \mathbf{B}(\mathcal{A})$ such that $x = a^{1/4}u^*ua^{1/4}$. Pick i and j so that $\|a^{1/4} - a_i^{1/4}\| < \varepsilon/4$ and $\|a_i^{1/4}u^*ua_i^{1/4} - b_{ij}\| < \varepsilon/4$. Then

$$\|a^{1/4}u^*ua^{1/4} - b_{ij}\| \le \|(a_i^{1/4} - a^{1/4})u^*ua^{1/4}\| \\ + \|a_i^{1/4}u^*u(a_i^{1/4} - a^{1/4})\| + \|a_i^{1/4}u^*ua_i^{1/4} - b_{ij}\| < \varepsilon. \blacksquare$$

Recall that a C^*-subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$ is called *hereditary* if, for any $a \in \mathcal{A}_+$, we have $\{b \in \mathcal{B} : 0 \le b \le a\} \subset \mathcal{A}$.

PROPOSITION 1.3.4. *A C^*-algebra $\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{\star\star}$ if and only if $\mathcal{A}$ is a compact C^*-algebra.*

Proof. If $\mathcal{A}$ is compact, then it is an ideal in $\mathcal{A}^{\star\star}$ [57]. It is well known (see e.g. [9, Proposition II.5.3.2]) that any ideal in a C^*-algebra is hereditary.

Now suppose $\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{\star\star}$. By [14, Exercise 4.7.20], it suffices to show that, for any $a \in \mathcal{A}_+$, any non-zero point of the spectrum of a is an isolated point. Suppose, for the sake of contradiction, that there exists $a \in \mathcal{A}_+$ whose spectrum contains a strictly positive non-isolated point α. In other words, for every $\delta > 0$, $((\alpha - \delta, \alpha) \cup (\alpha, \alpha + \delta)) \cap \sigma(a) \neq \emptyset$. Without loss of generality, we can assume $0 \le a \le 1$. Thus, we can find countably many mutually disjoint non-empty subsets S_i of $(\alpha/2, \infty) \cap \sigma(a)$. Denote the corresponding spectral projections by p_i (that is, $p_i = \chi_{S_i}(a)$). These projections belong to $\mathcal{A}^{\star\star}$. Furthermore, $p_i \le (\inf S_i)^{-1}a$, hence, by the hereditary property, these projections belong to $\mathcal{A}$.

Now consider the linear map $T : \mathcal{A} \to \mathcal{A} : x \mapsto axa$. Then $T^{\star\star}$ is also implemented by $x \mapsto axa$. If $0 \le x \le \mathbf{1}$, then $axa \le a^2$, hence $axa \in \mathcal{A}$. Therefore, $T^{\star\star}$ takes $\mathcal{A}^{\star\star}$ to $\mathcal{A}$. By Gantmacher's Theorem (see e.g. [4, Theorem 5.23]), T is weakly compact. However, T is an isomorphism on the copy of c_0 spanned by the projections p_i, leading to a contradiction. $\blacksquare$

REMARK 1.3.5. The above result was independently proved in [5], using a different method.

1.4. Positive Schur Property. Compactness of order intervals in Schatten spaces. An OBS X is said to have the *Positive Schur Property* (*PSP*) if every weakly null positive sequence is norm convergent to 0, and X has the *Super Positive Schur Property* (*SPSP*) if every positive weakly convergent sequence is norm convergent. Clearly, the Schur Property implies the SPSP, which, in turn, implies the PSP. Note that, if X has the SPSP, then, by the Eberlein–Šmulian Theorem, any weakly compact subset of X_+ is compact.

The PSP and SPSP of Banach lattices have been investigated earlier. By [52], the Schur Property and the PSP coincide for atomic Banach lattices. In [33], it is shown that ℓ_1 is the only symmetric sequence space with the Schur Property (by Remark 1.4.7 below, the symmetry assumption is essential). [34] gives a criterion for the PSP of Orlicz spaces.

LEMMA 1.4.1. *Suppose $\mathcal{E}$ is a symmetric sequence space, and (A_n) is a positive bounded sequence in $\mathcal{S}_{\mathcal{E}}$ without a convergent subsequence. Then there exist a subsequence (A_{n_k}) and $c > 0$ such that $\|R_k A_{n_k}\| > c$ for every k.*

Proof. Assume there is no such subsequence, that is,

$$\lim_m \sup_n \|R_m A_n\| = 0.$$

Applying Lemma 1.2.1 when T is the identity operator, we obtain the inequality

$$\begin{aligned}\|A_n - Q_m A_n\| &\leq \|A_n - Q_m A_n - R_m A_n\| + \|R_m A_n\| \\ &\leq 2\|Q_m A_n\|^{1/2}\|R_m A_n\|^{1/2} + \|R_m A_n\|.\end{aligned}$$

Thus, $\lim_m \sup_n \|A_n - Q_m A_n\| = 0$. However, Q_m is a finite rank map, hence the set (A_n) is relatively compact, a contradiction. ∎

PROPOSITION 1.4.2. *Suppose $\mathcal{E}$ is a separable symmetric sequence space. Let (A_n) be a weakly null positive sequence in $\mathcal{S}_{\mathcal{E}}(H)$ which contains no convergent subsequences. Then there exists $c > 0$ with the property that, for any $\varepsilon \in (0, 1)$, there exist sequences $1 = n_1 < n_2 < \cdots$ and $0 = m_0 < m_1 < \cdots$ such that $\inf_k \|A_{n_k}\| > c$ and*

$$\sum_k \|A_{n_k} - (P_{m_k} - P_{m_{k-1}}) A_{n_k} (P_{m_k} - P_{m_{k-1}})\| < \varepsilon.$$

Consequently, the sequence (A_{n_k}) is equivalent to a disjoint sequence of positive finite-dimensional operators.

Proof. By the separability (equivalently, order continuity) of $\mathcal{E}$, there exists a projection $p \in B(H)$ with separable range such that $pA_k p = A_k$ for any k. Thus, it suffices to prove our proposition in $\mathcal{S}_{\mathcal{E}}$.

Furthermore, the order continuity of $\mathcal{E}$ implies that the finite rank operators are dense in $\mathcal{S}_{\mathcal{E}}$. It is easy to see that, for any rank 1 operator u, $\lim_n \|u - Q_n u\| = 0$. Thus, $\lim_n \|x - Q_n x\| = 0$ for any $x \in \mathcal{E}$.

By scaling, we can assume $\sup_n \|A_n\| = 1$. Applying Lemma 1.4.1, and passing to a subsequence if necessary, we may assume that $\|R_n A_n\| > c$ for some positive number c. We construct the sequences (n_k) and (m_k) recursively. Set $n_1 = 1$ and $m_0 = 0$. As noted above, there exists $m_1 > m_0$ such that $\|A_{n_1} - P_{m_1} A_{n_1} P_{m_1}\| < \varepsilon/2$.

Suppose we have already selected $0 = m_0 < m_1 < \cdots < m_j$ and $1 = n_1 < n_2 < \cdots < n_j$ so that, for $1 \le j \le k$,

$$\|A_{n_k} - (P_{m_k} - P_{m_{k-1}})A_{n_k}(P_{m_k} - P_{m_{k-1}})\| < 2^{-j}\varepsilon.$$

As Q_m is a finite rank operator for any m, and the sequence (A_n) is weakly null, we deduce $\lim_n \|Q_m A_n\| = 0$. Consequently, there exists $n_{k+1} > n_k$ such that $\|Q_{m_k} A_{n_{k+1}}\| < 2^{-2(k+1)-4}\varepsilon^2$. Then

$$\begin{aligned}\|A_{n_{k+1}} - R_{m_k} A_{n_{k+1}}\| &\le \|A_{n_{k+1}} - R_{m_k} A_{n_{k+1}} - Q_{m_k} A_{n_{k+1}}\| + \|Q_{m_k} A_{n_{k+1}}\| \\ &\le 2\|Q_{m_k} A_{n_{k+1}}\|^{1/2} \|R_{m_k} A_{n_{k+1}}\|^{1/2} + \|Q_{m_k} A_{n_{k+1}}\| < 2^{-(k+2)}\varepsilon.\end{aligned}$$

Now find m_{k+1} such that $\|R_{m_k} A_{n_{k+1}} - Q_{m_{k+1}} R_{m_k} A_{n_{k+1}}\| < 2^{-(k+2)}\varepsilon$. ∎

PROPOSITION 1.4.3. *For any Hilbert space H, $\mathcal{S}_1(H)$ has the SPSP.*

Proof. It suffices to consider the case of infinite-dimensional H. Suppose $A_0, A_1, A_2, \ldots$ are positive elements of $\mathcal{S}_1(H)$, and $A_n \to A_0$ weakly. Then there exist projections $p_0, p_1, p_2, \ldots$ with separable range such that $p_i A_i p_i = A_i$ for every i. Then $p = \bigvee_{i \ge 0} p_i$ has separable range, and $pA_i p = A_i$ for every i. Thus, we can assume that $H = \ell_2$.

By Lemma 1.4.1 there exist $c > 0$ and a subsequence such that $\|R_k A_{n_k}\| > c$. Since $R_m \ge R_k$ when $m \le k$, we have $\mathrm{tr}(R_m A_{n_k}) > c$ for every k. On the other hand we can always pick m such that $\mathrm{tr}(R_m A) = \|R_m A\| < c$. This contradicts $A_n \to A$ weakly. ∎

PROPOSITION 1.4.4. *Suppose $\mathcal{E}$ is a strongly symmetric sequence space, and H is an infinite-dimensional Banach space. Then the following are equivalent:*

(1) $\mathcal{E} = \ell_1$.
(2) *$\mathcal{E}$ has the Schur Property.*
(3) *$\mathcal{E}$ has the PSP.*
(4) *$\mathcal{E}$ has the SPSP.*
(5) *$\mathcal{S}_{\mathcal{E}}(H)$ has the PSP.*
(6) *$\mathcal{S}_{\mathcal{E}}(H)$ has the SPSP.*

Proof. (1)⇒(2) is well known. The implications (2)⇒(4)⇒(3), (6)⇒(4), and (6)⇒(5)⇒(3) are obvious. (1)⇒(6) follows from Proposition 1.4.3.

(3)⇒(1). Assume that a basis (e_n) of $\mathcal{E}$ is not equivalent to the canonical basis of ℓ_1. By symmetry, (e_n) contains no subsequence equivalent to the canonical basis of ℓ_1. By Rosenthal's dichotomy, the sequence (e_n) is weakly null, which contradicts the PSP. ∎

We complete this section by (partially) describing Banach lattices having various versions of the Schur Property.

PROPOSITION 1.4.5. *Any Banach lattice E with the SPSP is atomic.*

Recall that a Banach lattice is called *atomic* if it is the band generated by its atoms.

Proof. Clearly, a Banach lattice with the SPSP cannot contain a lattice copy of c_0. Theorems 2.4.12 and 2.5.6 of [39] show that E is a KB-space. In particular, E is order continuous. By [37, Proposition 1.a.9], without loss of generality, we may assume E is atomless and has a weak unit. Therefore, by [37, Theorem 1.b.4], there exists an atomless probability measure space (Ω, μ) such that $L_\infty(\mu) \subset E \subset L_1(\mu)$. Suppose, furthermore, that $e \in E_+ \setminus \{0\}$. Find $S \subset \Omega$ of finite measure such that $e\chi_S > \alpha\chi_S$ for some positive number α. By the proof of [11, Proposition 2.1], there exists a weakly null sequence (f_n) such that $|f_n| = 1$ μ-a.e. on S, $f_n = 0$ on $\Omega \setminus S$, and $f_n \to 0$ in $\sigma(L_\infty(\mu), L_1(\mu))$. Letting $e_n = e + f_n$, we conclude that $e_n \geq 0$ for every n, and $e_n \to e$ weakly, but not in norm. ∎

PROPOSITION 1.4.6. *For any order continuous Banach lattice E the SPSP, the PSP, and the Schur Property are equivalent.*

Proof. Proposition 1.4.5 implies E is atomic. Therefore the result follows from the fact that the lattice operations are weakly sequentially continuous (see [39, Proposition 2.5.23]). ∎

REMARK 1.4.7. An order continuous atomic Banach lattice with the Schur Property need not be isomorphic to ℓ_1, even as a Banach space. Indeed, suppose (E_n) is a sequence of finite-dimensional lattices. Then $E = (\sum_{n=1}^\infty E_n)_{\ell_1}$ has the Schur Property. If, for instance, $E_n = \ell_2^n$, then E is not isomorphic to ℓ_1. We do not know of any Banach lattice with the Schur Property which is not isomorphic to an ℓ_1 sum of finite-dimensional spaces.

1.5. Compactness of order intervals in preduals of von Neumann algebras. Following [49, Definition III.5.9], we say that a von Neumann algebra $\mathcal{A}$ is *atomic* if every projection in $\mathcal{A}$ has a minimal subprojection. Note that $\mathcal{A}$ is atomic iff it is isomorphic to $(\sum_{i\in I} B(H_i))_{\ell_\infty(I)}$ for some index set I and a collection $(H_i)_{i\in I}$ of Hilbert spaces. Indeed, any von Neumann algebra of the above form is atomic. To prove the converse, note that an atomic algebra must be of type I. Moreover, it can be written as $\mathcal{A} = (\sum_{j\in J} \mathcal{A}_j)_{\ell_\infty(J)}$, where $\mathcal{A}_j$ is an atomic algebra of type I_j. By [49, Theorem V.1.27] (see also [30, Theorem 6.6.5] and [9, III.1.5.3]), $\mathcal{A}_j$ is isomorphic to $\mathcal{C}_j \overline{\otimes} B(H_j)$, where $\mathcal{C}_j$ is the center of $\mathcal{A}_j$. Denote the set of all minimal projections in $\mathcal{C}_j$ by F_j. Then the elements of F_j are mutually orthogonal, and their join equals the identity of $\mathcal{C}_j$. Thus, $\mathcal{C}_j$ is isomorphic to $\ell_\infty(F_j)$. Alternatively, one could use [9, III.1.5.18] and its proof to show that $\mathcal{C}_j$ is an ℓ_∞ space.

THEOREM 1.5.1. *For a von Neumann algebra $\mathcal{A}$, the following are equivalent:*

(1) *$\mathcal{A}$ is an atomic von Neumann algebra.*
(2) *$\mathcal{A}_\star$ has the SPSP.*
(3) *All order intervals in $\mathcal{A}_\star$ are compact.*

REMARK 1.5.2. Note that the predual of any von Neumann algebra has the PSP. Indeed, suppose (f_n) is a sequence of positive elements of $A_\star$ converging weakly to 0. Then $\|f_n\| = \langle f_n, \mathbf{1}\rangle$, hence $\lim_n \|f_n\| = \lim_n \langle f_n, \mathbf{1}\rangle = 0$.

Also, any order interval in the predual of a von Neumann algebra is weakly compact. Indeed, suppose f is a positive element of $\mathcal{A}_\star$. Then $[0, f]$ is convex and closed. For any $g \in [0, f]$ and $a \in \mathcal{A}$, the Cauchy–Schwarz inequality [49, Proposition I.9.5] yields $|g(a)|^2 \le g(\mathbf{1})g(a^*a) \le f(\mathbf{1})f(a^*a)$. By [49, Theorem III.5.4], $[0, f]$ is relatively weakly compact.

To prove Theorem 1.5.1, we need to determine when $\mathcal{A}_\star$ contains an order copy of $L_1(0,1)$, complemented via a positive projection.

PROPOSITION 1.5.3. *For a von Neumann algebra $\mathcal{A}$, the following statements hold:*

(1) *If $\mathcal{A}$ is atomic, then $\mathcal{A}_\star$ does not contain $L_1(0,1)$ isomorphically.*
(2) *If $\mathcal{A}$ is not atomic, then there exist an isometric order isometry $j : L_1(0,1) \to \mathcal{A}_\star$ and a positive projection $P : \mathcal{A}_\star \to \operatorname{ran} j$.*

Proof. (1) Note that, for any Hilbert space H, $\mathcal{S}_1(H)$ does not contain $L_1(0,1)$ isomorphically. Indeed, otherwise, by the separability argument, we would be able to embed $L_1(0,1)$ into $\mathcal{S}_1$. This, however, is impossible, by e.g. [26]. To finish the proof of (1), recall that, if $\mathcal{A}$ is atomic, then it can be identified with $(\sum_i B(H_i))_\infty$, and $\mathcal{A}_\star$ is isometric to $(\sum_i \mathcal{S}_1(H_i))_1$.

(2) We can write $\mathcal{A} = \mathcal{A}_I \oplus \mathcal{A}_{\neg I}$, where $\mathcal{A}_I$ has type *I*, and $\mathcal{A}_{\neg I}$ has no type *I* components (that is, it is a direct sum of von Neumann algebras of types *II* and *III*). Either $\mathcal{A}_I$ is not atomic, or $\mathcal{A}_{\neg I}$ is non-trivial.

If $\mathcal{A}_I$ is not an atomic von Neumann algebra, write $\mathcal{A}_I = (\sum_{s\in S} \mathcal{A}_s)_{\ell_\infty(S)}$ with $\mathcal{A}_s = \mathcal{C}_s \overline{\otimes} B(H_s)$ ($\mathcal{C}_s$ is the center of $\mathcal{A}_s$). By [49, Theorem III.1.18], $\mathcal{C}_s$ is isomorphic to $L_\infty(\nu_s)$ for some locally finite measure ν_s. Consequently, $\mathcal{A}_\star$ contains $L_1(\nu_s) \otimes \mathcal{S}_1(H_s)$ as a positively and completely contractively complemented subspace. As $\mathcal{A}_I$ is not an atomic von Neumann algebra, ν_s is not a purely atomic measure, for some s. By the above, $\mathcal{A}_\star$ contains $L_1(\nu_s) \otimes \mathcal{S}_1(H_s)$ as a positively and completely contractively complemented subspace. Furthermore, $L_1(\nu_s)$ is complemented in $L_1(\nu_s) \otimes \mathcal{S}_1(H_s)$ via a positive projection Q: just pick a rank one projection $e \in B(H_s)$, and set $Q(x) = (I_{L_1(\nu_s)} \otimes e)x(I_{L_1(\nu_s)} \otimes e)$. Finally, $L_1(\nu_s)$ contains a positively complemented copy of $L_1(0,1)$. Indeed, we can represent $L_1(\nu_s)$ as a direct sum of spaces $L_1(\sigma_i)$, where σ_i is a finite measure. Since ν_s is not purely atomic, the same is true for $L_1(\sigma_i)$, for some i. By [49, Theorem III.1.22] (or [30, Theorem 9.4.1]), $L_1(\nu_s)$ contains a positively complemented copy of $L_1(0,1)$.

Now suppose $\mathcal{A}_{\neg I}$ is non-trivial. By the reasoning of [38, p. 217], $\mathcal{A}_{\neg I}$ contains a von Neumann subalgebra $\mathcal{B}$ isomorphic to the hyperfinite II_1 factor $\mathcal{R}$. Furthermore, there exists a normal contractive projection (conditional expectation) $P : \mathcal{A}_{\neg I} \to \mathcal{B}$. By [49, Theorem III.3.4], P is positive. Consequently, $\mathcal{A}_\star$ contains a copy of $\mathcal{R}_\star$, complemented via a positive contractive projection.

Let μ be the "canonical" measure on the Cantor set Δ, defined as follows: represent $\Delta = \{0,1\}^{\mathbb{N}}$, and write $\mu = \nu^{\mathbb{N}}$, where the measure ν on $\{0,1\}$ satisfies $\nu(0) = \nu(1) = 1/2$. For $\alpha = (i_1, \ldots, i_n) \in I = \{0,1\}^{<\mathbb{N}}$, define the function f_α by setting $f_\alpha(j_1, j_2, \ldots) = \prod_{k=1}^n \delta_{i_k, j_k}$ (here, $\delta_{i,j}$ stands for Kronecker's delta). Note that f_α and f_β have disjoint supports if α and β are different bit strings of the same length. Moreover, $f_\alpha = f_{(\alpha,0)} + f_{(\alpha,1)}$. Clearly, $L_1(\mu)$ is the closed linear span of the functions f_α. Subdividing $(0,1)$ appropriately, one can also construct an isometric order isomorphism between $L_1(\mu)$ and $L_1(0,1)$.

It therefore suffices to show that there exists an order isometry $J : L_1(\mu) \to \mathcal{R}_\star$ such that the range of J is the range of a positive projection. To prove this, let $\Delta_n = \{0,1\}^n$, and denote by μ_n the product of n copies of ν. In this notation, $L_1(\mu_n)$ is isometric to $\ell_1^{2^n}$. We can also identify $L_1(\mu_n)$ with $\mathrm{span}[f_\alpha : |\alpha| = n]$. Let i_n be the formal identity $L_1(\mu_{n-1}) \to L_1(\mu_n)$ (taking f_α to itself when $|\alpha| \leq n$).

For $n \in \mathbb{N}$, consider the map $j_n : M_{2^{n-1}} \to M_{2^n} : x \mapsto x \otimes M_2$. Denote by Tr_n the normalized trace on M_{2^n}, and by $M_{2^n}^\star$ the dual of M_{2^n} defined using Tr_n. Then $j_n : M_{2^{n-1}}^\star \to M_{2^n}^\star$ is an isometry. Furthermore, the diagonal embedding $u_n : L_1(\mu_n) \to M_{2^n}^\star$ is an isometry, and $u_n i_n = j_n u_{n-1}$. We can view both $M_{2^{n-1}}^\star$ and $L_1(\mu_n)$ as subspaces of $M_{2^n}^\star$, Furthermore, for any n there exist positive contractive unital projections $p_n : M_{2^n}^\star \to L_1(\mu_n)$ and $q_n : M_{2^n}^\star \to M_{2^{n-1}}^\star$ (the "diagonal" and "averaging" projections, respectively). We then have $p_n j_n = i_n p_{n-1}$.

It is well known (see e.g. [44, Theorem 3.4]) that $\mathcal{R}_\star$ can be viewed as $\overline{\bigcup_n M_{2^n}^\star}$. Moreover, for any n there exists a positive contractive unital projection $\tilde{q}_n : \mathcal{R}_\star \to M_{2^n}^\star$ (with $\tilde{q}_n|_{M_{2^N}^\star} = q_{n+1} \cdots q_N$). Now identify $L_1(\mu)$ with $\overline{\bigcup_n L_1(\mu_n)}$, and define the projection $P : \mathcal{R}_\star \to J(L_1(\mu))$ by setting $P|_{M_{2^n}^\star} = q_n$. ■

Proof of Theorem 1.5.1. If (1) holds, then $\mathcal{A} = (\sum_i B(H_i))_\infty$, hence $\mathcal{A}_\star = (\sum_i \mathcal{S}_1(H_i))_1$. So (2) and (3) follow from Propositions 1.4.4 and 1.2.7, respectively.

Now suppose $\mathcal{A}$ is not atomic. By Proposition 1.5.3, $\mathcal{A}_\star$ contains a (positively and contractively complemented) lattice copy of $L_1(0,1)$. To finish the proof, note that $L_1(0,1)$ fails the SPSP, and has non-compact order intervals. Indeed, let $f = \mathbf{1}$, and $f_n = \mathbf{1} + r_n$, where $r_1, r_2, \ldots$ are Rademacher functions. Then $f_n \to f$ weakly, but not in norm. This witnesses the failure of the SPSP. Moreover, $f_n/2 \in [0, \mathbf{1}]$, hence the order interval $[0, \mathbf{1}]$ is not compact. ∎

2. Main results on majorization

2.1. Compact operators on non-commutative function spaces. First we consider maps from ordered Banach spaces into Schatten spaces.

PROPOSITION 2.1.1. *Suppose $\mathcal{E}$ is a separable symmetric sequence space, H is a Hilbert space, A is a generating OBS, and $0 \le T \le S : A \to \mathcal{S}_{\mathcal{E}}(H)$ (not necessarily linear). If S is compact, then T is compact.*

Proof. It is enough to show $T(\mathbf{B}(A)_+)$ is relatively compact. This follows from Lemma 1.2.7, since $T(\mathbf{B}(A)_+) \subseteq \mathbf{PSol}(S(\mathbf{B}(A)_+))$. ∎

For operators into Schatten spaces, we have:

PROPOSITION 2.1.2. *Suppose $\mathcal{E}$ is a separable symmetric sequence space, and H is a Hilbert space.*

(1) *If $\mathcal{E}$ does not contain ℓ_1, and operators T and S from $\mathcal{S}_{\mathcal{E}}(H)$ to a normal OBS Z satisfy $0 \le T \le S$, then the compactness of $S^\star$ implies the compactness of $T^\star$.*
(2) *Conversely, suppose $\mathcal{E}$ contains ℓ_1, and a Banach lattice Z is either not atomic, or not order continuous. Then there exist $0 \le T \le S : \mathcal{S}_{\mathcal{E}}(H) \to Z$ such that S is compact, but T is not.*

Proof. (1) By [36, Theorem 1.c.9], $\mathcal{E}^\star$ is separable. Now apply Proposition 2.1.1.

(2) By [51], there exist $0 \le \tilde{T} \le \tilde{S} : \ell_1 \to Z$ such that $\tilde{S}$ is compact, but $\tilde{T}$ is not. By Proposition 1.2.6, there exists a lattice isomorphism $j : \ell_1 \to \mathcal{S}_{\mathcal{E}}$ and a positive projection P from $\mathcal{S}_{\mathcal{E}}$ onto $j(\ell_1)$. Then the operators $T = \tilde{T}j^{-1}P$ and $S = \tilde{S}j^{-1}P$ have the desired properties. ∎

Finally we deal with operators on general non-commutative function spaces.

PROPOSITION 2.1.3. *Suppose $\mathcal{E}$ is a strongly symmetric non-commutative function space such that $\mathcal{E}^\times$ is not order continuous. Suppose, furthermore, that a symmetric non-commutative function space $\mathcal{F}$ contains non-compact order intervals. Then there exist $0 \le T \le S : \mathcal{E} \to \mathcal{F}$ such that S has rank 1 and T is not compact.*

Note that many spaces $\mathcal{F}$ contain non-compact order ideals. Suppose, for instance, that $\mathcal{F}$ arises from a von Neumann algebra $\mathcal{A}$ that is not atomic, and is equipped with a normal faithful semifinite trace τ. Using the type decomposition, we can find a projection $p \in \mathcal{A}$ with a finite trace. Then the interval $[0,p]$ is not compact. Indeed, [49, Proposition V.1.35] allows us to construct a family (p_{ni}) ($n \in \mathbb{N}$, $1 \le i \le 2^n$) of projections such that (i) $p = p_{11} + p_{12}$, and $p_{ni} = p_{n+1,2i-1} + p_{n+1,2i}$ for any n and i, and (ii) all projections p_{ni} are equivalent. Then the family $q_n = \sum_{i=1}^{2^{n-1}} p_{n,2i}$ is a sequence in $[0,p]$ with no convergent subsequences.

Note that, for fully symmetric non-commutative sequence spaces, order continuity is fully described by Corollary 1.2.10.

LEMMA 2.1.4. *Suppose $\mathcal{E}$ is a strongly symmetric non-commutative function space such that $\mathcal{E}^\times$ is not order continuous. Then there exists an isomorphism $j : \ell_1 \to \mathcal{E}$ such that both j and j^{-1} are positive and $j(\ell_1)$ is the range of a positive projection.*

Proof. By [17], $\mathcal{E}^\times$ is fully symmetric. By Lemma 1.2.11, there exist $x \in \mathbf{B}(\mathcal{E}^\times)_+$ and a sequence (e_i) of mutually orthogonal projections such that $(\alpha_i) \mapsto \sum \alpha_i e_i x e_i$ determines a positive embedding of ℓ_∞ into $\mathcal{E}^\times$. For each i, find $y_i \in \mathcal{E}_+$ such that $e_i y_i e_i = y_i$, $\|y_i\| < 2\|e_i x e_i\|^{-1}$, and $\langle e_i x e_i, y_i \rangle = 1$. The map $j : \ell_1 \to \mathcal{E} : (\alpha_i) \mapsto \sum_i \alpha_i y_i$ determines a positive isomorphism. Furthermore, define $U : \mathcal{E} \to \ell_1 : y \mapsto (\langle e_i x e_i, y \rangle)_i$. Clearly, U is a bounded positive map, and $Uj = I_{\ell_1}$. Therefore, jU is a positive projection onto $j(\ell_1)$. ∎

Proof of Proposition 2.1.3. In view of Lemma 2.1.4, it suffices to construct $0 \le T \le S : \ell_1 \to \mathcal{F}$ such that S has rank 1, and T is not compact. Pick $y \in \mathcal{F}$ such that $[0,y]$ is not compact. Then find a sequence $(y_i) \subset [0,y]$ without convergent subsequences. Denote the canonical basis of ℓ_1 by (δ_i). Let $\delta_i^\star$ be the biorthogonal functionals in ℓ_∞. Following [51], define S and T by setting $S\delta_i = y$ and $T\delta_i = y_i$. In other words, for $a = (\alpha_i) \in \ell_1$, $Sa = \langle \mathbf{1}, a \rangle y$ and $Ta = \sum_i \langle \delta_i^\star, a \rangle y_i$. It is easy to see that $\operatorname{rank} S = 1$, and $0 \le T \le S$. Moreover, $T(\mathbf{B}(\ell_1))$ contains the non-compact set $\{y_1, y_2, \ldots\}$, hence T is not compact. ∎

2.2. Compact operators on C^*-algebras and their duals. In this section, we determine the C^*-algebras $\mathcal{A}$ with the property that every operator on $\mathcal{A}$ dominated by a compact operator is itself compact. First we introduce some definitions. Let $\mathcal{A}$ be a C^*-algebra, and consider $f \in \mathcal{A}^\star$. Let $e \in \mathcal{A}^{\star\star}$ be its support projection. Following [29], we call f *atomic* if every non-zero projection $e_1 \le e$ dominates a minimal projection (all projections are assumed to "live" in the enveloping algebra $\mathcal{A}^{\star\star}$). Equivalently, f is a sum of pure positive functionals. We say that $\mathcal{A}$ is *scattered* if every positive functional is atomic. By [28], [29], the following three statements are equivalent: (i) $\mathcal{A}$ is scattered; (ii) $\mathcal{A}^{\star\star} = (\sum_{i \in I} B(H_i))_\infty$; (iii) the spectrum of any self-adjoint element of $\mathcal{A}$ is countable. Consequently (see [14, Exercise 4.7.20]), any compact C^*-algebra is scattered. In [53], it is proved that a separable C^*-algebra has separable dual if and only if it is scattered.

The main result of this section is:

THEOREM 2.2.1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, and E is a generating OBS.*

(1) *Suppose $\mathcal{A}$ is a scattered. Then, for any $0 \le T \le S : E \to \mathcal{A}^\star$, the compactness of S implies the compactness of T.*
(2) *Suppose $\mathcal{B}$ is a compact. Then, for any $0 \le T \le S : E \to \mathcal{B}$, the compactness of S implies the compactness of T.*
(3) *Suppose $\mathcal{A}$ is not scattered, and $\mathcal{B}$ is not compact. Then there exist $0 \le T \le S : \mathcal{A} \to \mathcal{B}$ such that S has rank 1, while T is not compact.*

From this, we immediately obtain:

COROLLARY 2.2.2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. Then the following are equivalent:*

(1) *At least one of the two conditions holds:* (i) *$\mathcal{A}$ is scattered,* (ii) *$\mathcal{B}$ is compact.*
(2) *If $0 \le T \le S : \mathcal{A} \to \mathcal{B}$ and S is compact, then T is compact.*

It is easy to see that a von Neumann algebra is scattered if and only if it is finite-dimensional if and only if it is compact. This leads to:

COROLLARY 2.2.3. *If von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ are infinite-dimensional, then there exist $0 \le T \le S : \mathcal{A} \to \mathcal{B}$ such that S has rank 1, while T is not compact.*

We establish similar results about preduals of von Neumann algebras.

LEMMA 2.2.4.

(1) *Suppose $\mathcal{A}$ is an atomic von Neumann algebra, and E is a generating OBS. Then $0 \le T \le S : E \to \mathcal{A}_\star$, where S is a compact operator, implies T is compact.*

(2) *Suppose $\mathcal{A}$ is a non-atomic von Neumann algebra. Then there exist $0 \le T \le S : \mathcal{A}_\star \to \mathcal{A}_\star$ such that S is compact, but T is not.*

Proof. (1) The weak compactness of S implies, by Theorem 2.5.1 below, the weak compactness of T. By Theorem 1.5.1, $\mathcal{A}_\star$ has the SPSP, hence $T(\mathbf{B}(E)_+)$ is relatively compact. Thus, $T(\mathbf{B}(E))$ is relatively compact as well, hence T is compact.

(2) It suffices to show that there exists an order isomorphism $j : L_1(0,1) \to \mathcal{A}_\star$ such that there exists a positive projection P onto $\operatorname{ran} j$. Indeed, by [51], there exist operators $0 \le T_0 \le S_0 : L_1(0,1) \to L_1(0,1)$ such that S_0 is compact and T_0 is not. Then $T = jT_0j^{-1}P$ and $S = jS_0j^{-1}P$ have the

desired properties. The existence of j and P as above follows from the proof of Proposition 1.5.3. ∎

To establish Theorem 2.2.1, we need some auxiliary results.

LEMMA 2.2.5. *Suppose $\mathcal{A}$ is a C^*-algebra for which $\mathcal{A}^\star$ has non-compact order intervals, and a Banach lattice E is not order continuous. Then there exist $0 \le T \le S : \mathcal{A} \to E$ such that S has rank 1, while T is not compact.*

Proof. By [39, Theorem 2.4.2], there exist $y \in E_+$ and normalized elements $y_1, y_2, \ldots \in [0, y]$ with disjoint supports. By our assumption there exist $\psi \in \mathcal{A}^\star_+$ and a sequence $(\phi_i) \subset [0, \psi]$ which does not have convergent subsequences. By Alaoglu's theorem we may assume $\phi_i \to \phi$ in weak* topology. Define two operators via

$$Sx = \psi(x)y \quad \text{and} \quad Tx = \phi(x)y + \sum_{n=1}^{\infty} (\phi_n - \phi)(x) y_n.$$

Note that T is well defined: $(\phi_n - \phi)(x) \to 0$ for all x, hence

$$\Big\| \sum_{n=m+1}^{k} (\phi_n - \phi)(x) y_n \Big\| \le \sup_{n>m} |(\phi_n - \phi)(x)| \, \|y\| \xrightarrow[m\to\infty]{} 0.$$

Moreover, for any $x > 0$ and $N \in \mathbb{N}$ we have

$$\phi(x)y + \sum_{n=1}^{N} (\phi_n - \phi)(x) y_n = \phi(x)\Big(y - \sum_{n=1}^{N} y_n\Big) + \sum_{n=1}^{N} \phi_n(x) y_n \ge 0,$$

and

$$\begin{aligned}
&\psi(x)y - \phi(x)y - \sum_{n=1}^{N} (\phi_n - \phi)(x) y_n \\
&= \psi(x)y - \sum_{n=1}^{n} \phi_n(x) y_n - \phi(x)\Big(y - \sum_{n=1}^{N} y_n\Big) \ge (\psi(x) - \phi(x))\Big(y - \sum_{n=1}^{N} y_n\Big).
\end{aligned}$$

By letting $N \to \infty$, we obtain $0 \le Tx \le Sx$ for every $x > 0$. Clearly, $\operatorname{rank} S = 1$. It remains to show that $T^\star$ is not compact. Note that there exist norm one $f_1, f_2, \ldots \in E^\star$ such that $f_n(y_m) = \delta_{nm}$. It is easy to see that $T^\star f = f(y)\phi + \sum_{n=1}^{\infty} f(y_n)(\phi_n - \phi)$, hence $T^\star f_m = (f_m(y) - 1)\phi + \phi_m$. The sequence $(T^\star f_m)$ has no convergent subsequences, since if it had, (ϕ_m) would have a convergent subsequence, too. This rules out the compactness of $T^\star$. ∎

COROLLARY 2.2.6. *Suppose a C^*-algebra $\mathcal{B}$ is not compact, and $\mathcal{A}^\star$ has non-compact order intervals. Then there exist $0 \le T \le S : \mathcal{A} \to \mathcal{B}$ such that S has rank 1, while T is not compact.*

Proof. By Lemma 2.2.5, it suffices to show that $\mathcal{B}$ contains a Banach lattice which is not order continuous. By [14, Exercise 4.7.20], $\mathcal{B}$ contains a positive element b whose spectrum contains a positive non-isolated point. Then the abelian C^*-algebra $\mathcal{B}_0$ generated by b is not order continuous. Indeed, suppose $\alpha > 0$ is not an isolated point of $\sigma(a)$. Then there exist disjoint subintervals $I_i = (\beta_i, \gamma_i) \subset (\alpha/2, 3\alpha/2)$ such that $\delta_i = (\beta_i + \gamma_i)/2 \in \sigma(b)$ for every $i \in \mathbb{N}$. For each i, consider the function σ_i such that $\sigma_i(\beta_i) = \sigma_i(\gamma_i) = 0$, $\sigma_i((\beta_i + \gamma_i)/2) = 1$, and σ_i is defined by linearity elsewhere. Then the elements $y_i = \sigma_i(b)$ belong to $\mathcal{B}_0$, are disjoint and normalized, and $y_i \leq y = 2\alpha^{-1}b$. ∎

Proof of Theorem 2.2.1. (1) If $\mathcal{A}$ is scattered, then $\mathcal{A}^{\star\star}$ is atomic. Now invoke Lemma 2.2.4(1).

(2) By assumption, $M = S(\mathbf{B}(E)_+)$ is relatively compact, and $T(\mathbf{B}(E)_+) \subset \mathbf{PSol}(M)$. By Proposition 1.3.3, $T(\mathbf{B}(E)_+)$ is relatively compact.

(3) Combine Theorem 1.5.1 with Corollary 2.2.6. ∎

2.3. Comparisons with multiplication operators. Suppose $\mathcal{A}$ is a C^*-subalgebra of $B(H)$, where H is a Hilbert space. For $x \in B(H)$ we define an operator $M_x : \mathcal{A} \to B(H) : a \mapsto x^*ax$. In this section, we study domination of, and by, multiplication operators, in relation to compactness. First, we record some consequences of the results from Section 1.3.

Proposition 2.3.1. *Suppose x is an element of a C^*-algebra $\mathcal{A}$.*

1. *If M_x is weakly compact, and $0 \leq T \leq M_x : \mathcal{A} \to \mathcal{A}$, then T is compact.*
2. *If $0 \leq M_x \leq S : \mathcal{A} \to \mathcal{A}$, and S is weakly compact, then M_x is compact.*

Proof. By passing to the second adjoint if necessary, we can assume $\mathcal{A}$ is a von Neumann algebra. Note that $[0, x^*x] = M_x(\mathbf{B}(\mathcal{A})_+)$. Indeed, if $a \in \mathbf{B}(\mathcal{A})_+$, then $0 \leq a \leq \mathbf{1}$, so $0 \leq M_x a \leq M_x \mathbf{1} = x^*x$, so $M_x a \in [0, x^*x]$. Next we show that any $b \in [0, x^*x]$ belongs to $M_x a \in [0, x^*x]$. By [15, p. 11], there exists $v \in \mathbf{B}(\mathcal{A})$ such that $b^{1/2} = vc$, where $c = (x^*x)^{1/2}$. Write $x = uc$, where u is a partial isometry from $(\ker x)^\perp$ onto $\overline{\operatorname{ran} x}$. Then $c = u^*x = x^*u$, and therefore $b = M_x(uv^*vu^*)$.

Consequently, M_x is (weakly) compact if and only if the interval $[0, x^*x]$ is (weakly) compact. By Proposition 1.3.2, the compactness and weak compactness of $[0, x^*x]$ are equivalent. To establish (1), suppose $0 \leq T \leq M_x$ and M_x is weakly compact. Then $T(\mathbf{B}(\mathcal{A})_+)$ is relatively compact, as a subset of $[0, x^*x]$. Thus, T is compact. (2) is established similarly. ∎

If the "symbol" x of the operator M_x comes from the ambient $B(H)$, we obtain:

PROPOSITION 2.3.2. *Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, $x \in B(H)$, $M_x : \mathcal{A} \to B(H)$ is compact, and $0 \le T \le M_x$. Then T is compact.*

PROPOSITION 2.3.3. *Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, $S : \mathcal{A} \to B(H)$ is compact, $x \in B(H)$, and $0 \le M_x \le S$. Then M_x is compact.*

REMARK 2.3.4. The irreducibility of $\mathcal{A}$ is essential here. Below we construct an abelian C^*-subalgebra $\mathcal{A} \subset B(H)$ and operators $x_1, x_2 \in B(H)$ such that $0 \le M_{x_1} \le M_{x_2}$ and M_{x_2} is compact, while M_{x_1} is not (here, M_{x_1} and M_{x_2} are viewed as taking $\mathcal{A}$ to $B(H)$). By [51], there exist operators $0 \le R_1 \le R_2 : C[0,1] \to C[0,1]$ such that R_2 is compact and R_1 is not. Let λ be the usual Lebesgue measure on $[0,1]$, and let $j : C[0,1] \to B(L_2(\lambda))$ be the diagonal embedding (taking a function f to the multiplication operator $\phi \mapsto \phi f$). By [42, Theorem 3.11], R_1 and R_2 are completely positive. Thus, by the Stinespring Theorem, these operators can be represented as $R_i(f) = V_i^* \pi_i(f) V_i$ ($i = 1, 2$), where $\pi_i : C[0,1] \to B(H_i)$ are representations, and $V_i \in B(L_2(\lambda), H_i)$. Let $H = L_2(\lambda) \oplus_2 H_1 \oplus_2 H_2$. Then $\pi = j \oplus \pi_1 \oplus \pi_2 : C[0,1] \to B(H)$ is an isometric representation. Let $\mathcal{A} = \pi(C[0,1])$. Furthermore, consider the operators x_1 and x_2 on H defined via

$$x_1 = \begin{pmatrix} 0 & 0 & 0 \\ V_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_2 & 0 & 0 \end{pmatrix}.$$

Then, for any $f \in C[0,1]$, $jR_i(f) = x_i^* \pi(f) x_i$. Considering M_{x_1} and M_{x_2} as operators on $\mathcal{A}$, we see that $0 \le M_{x_1} \le M_{x_2}$, and M_{x_2} is compact, while M_{x_1} is not.

The following lemma establishes a criterion for compactness of M_x. This result may be known to experts, but we could not find any references in the literature.

LEMMA 2.3.5. *Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, and $c \in B(H)$. Then $c^* \mathbf{B}(\mathcal{A})_+ c$ is a relatively compact set if and only if c is a compact operator.*

Proof. By polar decomposition, it suffices to consider the case of $c \ge 0$. Indeed, write $c = du$, where $d = (cc^*)^{1/2}$ and u is a partial isometry from $(\ker c)^\perp = \overline{\operatorname{ran} c^*}$ to $(\ker c^*)^\perp = \overline{\operatorname{ran} c}$. Then $M_c = M_u M_d$ and $M_d = M_{u^*} M_c$ (here, we abuse the notation slightly, and allow M_u and M_{u^*} to act on $B(H)$). Therefore, the sets $c^* \mathbf{B}(\mathcal{A})_+ c = M_c(\mathbf{B}(\mathcal{A})_+)$ and $d \mathbf{B}(\mathcal{A})_+ d = M_d(\mathbf{B}(\mathcal{A})_+)$ are compact simultaneously.

If c is compact, then, by [56], $c\mathbf{B}(B(H))c$ is relatively compact. The set $c\mathbf{B}(\mathcal{A})_+c$ is also relatively compact, since it is contained in $c\mathbf{B}(B(H))c$.

Now suppose c is not compact. By scaling, we can assume that the spectral projection $p = \chi_{(1,\infty)}(c)$ has infinite rank. We shall show that, for every $n \in \mathbb{N}$, there exist $a_1, \ldots, a_n \in \mathbf{B}(\mathcal{A})_+$ such that $\|c(a_i - a_j)c\| > 1/3$ for $i \neq j$. Note first that there exist mutually orthogonal unit vectors $\xi_1, \ldots, \xi_n$ in $\operatorname{ran} p$ such that $\langle \xi_i, \xi_j \rangle = \langle c\xi_i, c\xi_j \rangle = 0$ whenever $i \neq j$. Indeed, if $\sigma(c) \cap (1, \infty)$ is infinite, then there exist disjoint Borel sets $E_i \subset (1, \infty)$ $(1 \leq i \leq n)$ such that $\sigma(c) \cap E_i \neq \infty$. Then we can take $\xi_i \in \chi_{E_i}(c)$. On the other hand, if $\sigma(c) \cap (1, \infty)$ is finite, then for some $s \in \sigma(c) \cap (1, \infty)$, the projection $q = \chi_{\{s\}}(c)$ has infinite rank. Then we can take $\xi_1, \ldots, \xi_n \in \operatorname{ran} q$.

Let $\eta_i = c\xi_i/\|c\xi_i\|$ (by construction, these vectors are mutually orthogonal). As $\mathcal{A}$ is irreducible, its second commutant is $B(H)$. By the Kaplansky Density Theorem (see e.g. [13, Theorem I.7.3]), $\mathbf{B}(\mathcal{A})_+$ is strongly dense in $\mathbf{B}(B(H))_+$. Thus, for every $1 \leq i \leq n$ there exist $a_i \in \mathbf{B}(\mathcal{A})_+$ such that $\|a_i \eta_k\| < 1/3$ for $i \neq k$, and $\|a_i \eta_i - \eta_i\| < 1/3$. Consider $b_i = c a_i c \in c(\mathbf{B}(\mathcal{A})_+)c$. For $i \neq j$,

$$\|b_i - b_j\| \geq \langle c(a_i - a_j)c\xi_i, \xi_i \rangle = \|c\xi_i\|^2 \langle (a_i - a_j)\eta_i, \eta_i \rangle > 2/3 - 1/3 = 1/3.$$

As n is arbitrary, we conclude that $c(\mathbf{B}(\mathcal{A})_+)c$ is not relatively compact. ∎

Proof of Proposition 2.3.2. Suppose $x \in B(H)$ is such that $M_x : \mathcal{A} \to B(H)$ is compact. By polar decomposition, we can assume that $x \geq 0$. Then $x\mathbf{B}(A)_+x$ is relatively compact, and therefore, by Lemma 2.3.5, x is a compact operator. By Proposition 1.3.2, $[0, x^2]$ is compact. But $T(\mathbf{B}(\mathcal{A})_+) \subset [0, x^2]$, hence $T(\mathbf{B}(\mathcal{A})_+)$ is relatively compact. By polarization, $T(\mathbf{B}(\mathcal{A}))$ is compact. ∎

To prove Proposition 2.3.3, we need a technical result.

Lemma 2.3.6. *Suppose $z \in B(H)$, and $x, y \in [0, \mathbf{1}_H]$. Then $zxz^* \geq zxyxz^*$.*

Proof. Note that $zxz^* - zxyxz^* = z(x - x^2)z^* + zx(\mathbf{1} - y)xz^*$, and both terms on the right are positive. ∎

Proof of Proposition 2.3.3. As in the proof of Proposition 2.3.2, we can assume that $x \geq 0$, and that $p = \chi_{(1,\infty)}(x)$ is a projection of infinite rank. It suffices to show that there exist $a_0 \geq a_1 \geq \cdots \geq a_n$ in $\mathbf{B}(\mathcal{A})_+$ such that $\|x(a_{k-1} - a_k)x\| > 2/3$ for $1 \leq k \leq n$. Indeed, if S is compact, then there exist $u_1, \ldots, u_m \in B(H)$ such that for every $a \in \mathbf{B}(\mathcal{A})_+$ there exists $j \in \{1, \ldots, m\}$ with $\|Sa - u_j\| < 1/3$. By the pigeon-hole principle, if $n > m$, then there exist $i < j$ in $\{1, \ldots, n\}$ and k in $\{1, \ldots, m\}$ such that $\max\{\|Sa_i - u_k\|, \|Sa_j - u_k\|\} < 1/3$. However, $\|Sa_i - Sa_j\| \geq \|x(a_i - a_j)x\| > 2/3$, leading to a contradiction.

Imitating the proof of Proposition 2.3.2, we use the spectral decomposition of x to find mutually orthogonal unit vectors $\xi_1, \ldots, \xi_n$ in $\operatorname{ran} p$ such that (i) $x^k \xi_i$ is orthogonal to $x^\ell \xi_j$ for any $i \neq j$ and $k, \ell \in \{0, 1, \ldots\}$, and (ii) for any i, $1 = \|\xi_i\| \leq \|x\xi_i\| \leq \|x^2\xi_i\| \leq \cdots$. To construct $a_0, \ldots, a_n$, let $c = (2/3)^{1/(2n+1)}$ and $\eta_i = x\xi_i/\|x\xi_i\|$. By the Kaplansky Density Theorem, for $0 \leq k \leq n$ there exist $b_k \in \mathbf{B}(\mathcal{A})_+$ such that

$$b_k \eta_i = \begin{cases} c\eta_i, & 1 \leq i \leq n-k, \\ 0, & i > n-k \end{cases}$$

(we can take $b_n = 0$). Let $a_0 = b_0$, $a_1 = b_0 b_1 b_0$, $a_2 = b_0 b_1 b_2 b_1 b_0$, etc. By Lemma 2.3.6, $a_0 \geq a_1 \geq \cdots \geq a_n$. Furthermore,

$$a_k \eta_i = \begin{cases} c^{2k-1}\eta_i, & 1 \leq i \leq n-k, \\ 0, & i > n-k, \end{cases}$$

and therefore

$$\begin{aligned} \|x(a_{k-1} - a_k)x\| &\geq \langle x(a_{k-1} - a_k)x\xi_{n-k+1}, \xi_{n-k+1}\rangle \\ &= \langle (a_{k-1} - a_k)\eta_{n-k+1}, \eta_{n-k+1}\rangle = c^{2k-1} > 2/3. \end{aligned}$$

Hence, the sequence $(a_k)_{k=0}^n$ has the desired properties. ■

2.4. Dunford–Pettis Schur multipliers. Recall that a map $T : \mathcal{S}_\mathcal{F} \to \mathcal{S}_\mathcal{E}$ is called a *Schur* (or *Hadamard*) *multiplier* if it can be written in the coordinate form as $(Tx)_{ij} = \phi_{ij} x_{ij}$. The infinite matrix ϕ is called the *symbol* of T, which we denote by $\mathbf{S}_\phi$. The main goal of this section is to prove:

THEOREM 2.4.1. *Suppose* $0 \leq \mathbf{S}_\phi \leq \mathbf{S}_\psi$ *are Schur multipliers from* $\mathcal{S}_1$ *to* $\mathcal{S}_\mathcal{E}$ (*$\mathcal{E}$ is a symmetric sequence space*). *If* $\mathbf{S}_\psi$ *is Dunford–Pettis, then the same is true for* $\mathbf{S}_\phi$.

Recall that an operator is called *Dunford–Pettis* if it maps weakly null sequences to norm null ones. Equivalently, it carries relatively weakly compact sets to relatively norm compact sets. The reader is referred to e.g. [4, Section 5.4] for more information.

The proof relies on several technical lemmas, which may be known to experts.

LEMMA 2.4.2. *A bounded sequence* (x_n) *in* $\mathcal{S}_1$ *is weakly null if and only if the following two conditions are satisfied:*

(1) $\lim_m \sup_n \|R_m x_n\| = 0$, *and*
(2) *for every* m, $\lim_n \|Q_m x_n\| = 0$.

Proof. Suppose first (x_n) is weakly null. As Q_m has finite rank, (2) must be satisfied. If (1) fails, then one can assume, by passing to a subsequence, that there exist $c > 0$ and a sequence $n_1 < n_2 < \cdots$ such that, for every k, $\|Q_{n_{k+1}} R_{n_k} x_k\| > c$, while $\|R_{n_{k+1}} x_k\| + \|Q_{n_k} x_k\| < 10^{-k} c$. Consider the block-diagonal truncation $P : \mathcal{S}_1 \to \mathcal{S}_1 : x \mapsto \sum_k Q_{n_{k+1}} R_{n_k} x$. Clearly, P is contractive. Letting, for every k, $y_k = Q_{n_{k+1}} R_{n_k} x_k$, we see that $\|Px_k - y_k\| < 10^{-k} c$. Thus, for every sequence (α_k),

$$\Big\| \sum_k \alpha_k x_k \Big\| \geq \Big\| \sum_k \alpha_k y_k \Big\| - \sum_k |\alpha_k| \cdot 10^{-k} c > \frac{c}{2} \sum_k |\alpha_k|.$$

Therefore, the sequence (x_k) is equivalent to the canonical basis of ℓ_1, hence is not weakly null.

Now suppose (1) and (2) are satisfied for a bounded sequence (x_n); we show that, for any $f \in B(\ell_2)$, $\lim_n f(x_n) = 0$. Indeed, otherwise, by passing to a subsequence, and by scaling, we can assume that $\sup_n \|x_n\| \leq 1$, and there exists $f \in \mathbf{B}(B(\ell_2))$ such that $\inf_n |f(x_n)| > c$. Pick m with $\sup_n \|R_m x_n\| < c/5$. We now observe that there exists $M > m$ such that $\|(I - Q_M)(I - R_m) f\| < c/5$. Indeed,

$$(I - Q_M)(I - R_m) f = P_M^{\perp} f P_m + P_m f P_M^{\perp}.$$

For a fixed m, $B(\ell_2) P_m$ is isomorphic to a Hilbert space. For every $y \in B(\ell_2) P_m$, $P_M^{\perp} y \to 0$, hence $\lim_M P_M^{\perp} f P_m = 0$. Similarly, $\lim_M P_m f P_M^{\perp} = 0$.

Finally, pick N so that, for $n > N$, $\|Q_M x_n\| < c/5$. As

$$\begin{aligned} \langle f, x_n \rangle &= \langle f, (R_m + (I - R_m) Q_M + (I - Q_M)(I - R_m)) x_n \rangle \\ &= \langle f, R_m x_n \rangle + \langle (I - R_m) f, Q_M x_n \rangle + \langle (I - Q_M)(I - R_m) f, x_n \rangle, \end{aligned}$$

we have, for $n > N$,

$$c < |\langle f, x_n \rangle| \leq \|R_m x_n\| + 2\|Q_M x_n\| + \|(I - Q_M)(I - R_m) f\| < 4c/5,$$

a contradiction. ∎

Corollary 2.4.3. *An operator $T : \mathcal{S}_1 \to X$ is Dunford–Pettis if and only if, for every i, the restrictions of T to* $\mathrm{span}[E_{ij} : j \in \mathbb{N}]$ *and* $\mathrm{span}[E_{ji} : j \in \mathbb{N}]$ *are compact.*

Proof. Suppose the restrictions of T to $\mathrm{span}[E_{ij} : j \in \mathbb{N}]$ and $\mathrm{span}[E_{ji} : j \in \mathbb{N}]$ are compact, and (x_n) is a weakly null sequence in $\mathcal{S}_1$. We have to show that, for every $c > 0$, $\|Tx_n\| < c$ for n large enough. Without loss of generality, assume T is a contraction and $\sup_n \|x_n\| \leq 1$. Find $M > m$ such that $\sup_n \|R_m x_n\| < c/4$, and

$$\|T|_{\mathrm{span}[E_{ij} : j > M]}\| + \|T|_{\mathrm{span}[E_{ji} : j > M]}\| < \frac{c}{4M}.$$

Find $N \in \mathbb{N}$ with $\sup_{n > N} \|Q_M x_n\| < c/4$. Thus, for $n > N$, $\|Tx_n\| < 3c/4$.

Conversely, suppose T is Dunford–Pettis, but its restriction to $\mathrm{span}[E_{ij} : j \in \mathbb{N}]$ is not compact. Then there exist $n_1 < n_2 < \cdots$ and $\alpha_j \in \mathbb{C}$ such that the vectors satisfy $x_k = \sum_{j=n_k+1}^{n_{k+1}} \alpha_j E_{ij}$, hence $\|x_k\| = 1$, and $\limsup_k \|Tx_k\| > 0$. However, the sequence (x_k) is weakly null, while the sequence (Tx_k) is not norm null, yielding a contradiction. The restrictions to $\mathrm{span}[E_{ji} : j \in \mathbb{N}]$ are handled similarly. ∎

Specializing the previous result to Schur multipliers, we immediately obtain:

COROLLARY 2.4.4. *A Schur multiplier with the symbol ϕ, acting from $\mathcal{S}_1$ to $\mathcal{S}_{\mathcal{E}}$, is Dunford–Pettis if and only if, for any i, $\lim_j \phi_{ij} = \lim_j \phi_{ji} = 0$.*

Proof. By Corollary 2.4.3, $\mathbf{S}_\phi : \mathcal{S}_1 \to \mathcal{S}_{\mathcal{E}}$ is Dunford–Pettis iff, for every i, the restrictions of $\mathbf{S}_\phi$ to $\mathrm{span}[E_{ij} : j \in \mathbb{N}]$ and $\mathrm{span}[E_{ji} : j \in \mathbb{N}]$ are compact. By the definition, $\mathbf{S}_\phi$ maps E_{ij} to $\phi_{ij} E_{ij}$. It is well known that, for any $\mathcal{E}$, $\mathrm{span}[E_{ij} : j \in \mathbb{N}] \subset \mathcal{S}_{\mathcal{E}}$ is isometric to ℓ_2, via an isometry sending the matrix units E_{ij} to the elements of the orthonormal basis. Thus, $\mathbf{S}_\phi|_{\mathrm{span}[E_{ij} : j \in \mathbb{N}]}$ is compact iff $\lim_j \phi_{ij} = 0$. Similarly, $\mathbf{S}_\phi|_{\mathrm{span}[E_{ji} : j \in \mathbb{N}]}$ is compact iff $\lim_j \phi_{ji} = 0$. ∎

LEMMA 2.4.5. *Suppose $c > 0$ and $m \in \mathbb{N}$ satisfy $(mc)^2 > m+1$. Suppose, furthermore, that C and D are positive matrices, with entries C_{ij} and D_{ij} $(0 \le i, j \le m)$, respectively, so that $\max_{i,j}\{\max\{|C_{ij}|, |D_{ij}|\}\} \le 1$, $|C_{0j}| > c$ for $1 \le j \le m$, and $|D_{ij}| < 10^{-2(i+j)}$ for $i \ne j$. Then the inequality $C \le D$ cannot hold.*

Proof. Suppose, for the sake of contradiction, that $D \ge C$. Then, for any vector $\xi \in \ell_2^{m+1}$,

$$\|D^{1/2}\xi\|^2 = \langle D^{1/2}\xi, D^{1/2}\xi\rangle = \langle D\xi, \xi\rangle \ge \langle C\xi, \xi\rangle = \|C^{1/2}\xi\|^2,$$

hence there exists a contraction U such that $UD^{1/2}\xi = C^{1/2}\xi$. Thus, $C = D^{1/2}U^*UD^{1/2}$. By [59, Lemma 1.21], the block matrix $\left(\begin{smallmatrix} D & C \\ C & D \end{smallmatrix}\right)$ is positive.

Denote the canonical basis in ℓ_2^{m+1} by $(e_i)_{i=0}^m$. Consider the vector $\xi = \left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right) \in \ell_2^{2(m+1)}$, where $\xi_1 = \alpha e_0$ and $\xi_2 = -\sum_{i=1}^m \omega_i e_i$. Here, $\omega_i = C_{i0}/|C_{i0}|$ and $\alpha = mc$. By the above,

$$0 \le \left\langle \begin{pmatrix} D & C \\ C & D \end{pmatrix} \xi, \xi \right\rangle = \langle D\xi_1, \xi_1\rangle + \langle D\xi_2, \xi_2\rangle + 2\,\mathrm{Re}\langle C\xi_1, \xi_2\rangle. \tag{1}$$

Note that $\langle D\xi_1, \xi_1\rangle = \alpha^2 D_{00} \le \alpha^2$ and

$$\langle D\xi_2, \xi_2\rangle \le \sum_{i=1}^m D_{ii} + 2 \sum_{1 \le i < j \le m} |D_{ij}| \le m + 2 \sum_{1 \le i < j \le m} 10^{-2(i+j)} < m + 1.$$

On the other hand,

$$\langle C\xi_1, \xi_2\rangle = -\alpha \sum_{i=1}^m C_{i0} \cdot \overline{\frac{C_{i0}}{|C_{i0}|}} < -\alpha mc.$$

Returning to (1), we see that

$$\left\langle \begin{pmatrix} D & C \\ C & D \end{pmatrix} \xi, \xi \right\rangle \leq \alpha^2 + m + 1 - 2\alpha mc < 0,$$

a contradiction. ∎

Proof of Theorem 2.4.1. We say that an infinite matrix ϕ is *formally positive* if each of its finite submatrices is positive. By [42, Theorem 3.7], $\mathbf{S}_\sigma \geq 0$ iff σ is formally positive.

Suppose, for the sake of contradiction, that $0 \leq \mathbf{S}_\phi \leq \mathbf{S}_\psi$, where $\mathbf{S}_\psi$ is Dunford–Pettis, while $\mathbf{S}_\phi$ is not. We can assume that $\mathbf{S}_\psi$ is contractive, hence, for any (i,j), $\max\{|\phi_{ij}|, |\psi_{ij}|\} \leq 1$. Corollary 2.4.4 shows that, for any i, $\lim_{j\to\infty} \psi_{ij} = 0$. By rearranging rows and columns if necessary, we can assume the existence of $n_0 < n_1 < n_2 < \cdots$ such that $|\phi_{n_0 n_k}| > c > 0$. Passing to a further subsequence, we obtain $|\psi_{n_i n_j}| < 10^{-2(i+j)}$ for $i \neq j$.

Now select m so that $mc > 4(m+1)$, and define matrices C and D with entries $C_{ij} = \phi_{n_i n_j}$ and $D_{ij} = \psi_{n_i n_j}$ $(0 \leq i, j \leq m)$, respectively. As noted above, the matrices C and D are positive. By Lemma 2.4.5, we cannot have $C \leq D$. Thus, a contradiction. ∎

2.5. Weakly compact operators. In this section, we show that, under certain conditions, weak compactness is inherited under domination. First consider operators on C^*-algebras and their duals.

THEOREM 2.5.1. *Suppose E is an OBS, $\mathcal{A}$ is a C^*-algebra, S is a weakly compact operator, and one of the following holds:*

(1) *E is generating, and $0 \leq T \leq S : E \to \mathcal{A}^\star$.*
(2) *E is normal, and $0 \leq T \leq S : \mathcal{A} \to E$.*

Then T is weakly compact.

Note that, for $\mathcal{A}$ commutative, this theorem follows from [50] and the order continuity of $\mathcal{A}^\star$.

Proof. (1) Suppose, for the sake of contradiction, that $T(\mathbf{B}(E)_+)$ is not weakly compact. By Pfitzner's Theorem [43], there exist a bounded sequence $(a_n) \subset \mathcal{A}$ of positive pairwise orthogonal elements, a sequence $(\phi_n) \subset \mathbf{B}(E)_+$, and $c > 0$ such that $T\phi_n(a_n) > c$. Therefore, $S\phi_n(a_n) > c$, which contradicts the weak compactness of $S(\mathbf{B}(E))$ (once again, by Pfitzner's Theorem).

(2) Apply part (1) to $0 \leq T^\star \leq S^\star$. ∎

REMARK 2.5.2. Theorem 2.5.1 fails for operators from duals of C^*-algebras to C^*-algebras, even in the commutative setting. Indeed, by [4, Theorem 5.31], there exist $0 \leq T \leq S : \ell_1 \to \ell_\infty$ such that S is weakly compact, whereas T is not.

For operators to or from general Banach lattices, we have:

THEOREM 2.5.3. *Suppose either*

(i) *A is a generating OBS, and B is order continuous Banach lattice, or*
(ii) *A is a Banach lattice with order continuous dual, and B is a normal OBS.*

If $0 \leq T \leq S : A \to B$, and S is weakly compact, then T is weakly compact as well.

Proof. The proof of case (i) is contained in the first few lines of the proof of [4, Theorem 5.31]. Case (ii) follows by duality. ∎

Next we obtain a partial generalization of the above results for non-commutative function spaces. In the discrete case, we obtain a characterization of order continuous Banach lattices.

PROPOSITION 2.5.4. *Suppose $\mathcal{E}$ is a symmetric sequence space containing a copy of ℓ_1, H is an infinite-dimensional Hilbert space, and X is a Banach lattice. Then the following are equivalent:*

(1) *If $0 \leq T \leq S : \mathcal{S}_{\mathcal{E}}(H) \to X$, and S is weakly compact, then T is weakly compact.*
(2) *X is order continuous.*

Proof. (2)⇒(1) follows from Theorem 2.5.3.

(1)⇒(2). By Proposition 1.2.6, $\mathcal{S}_{\mathcal{E}}(H)$ contains a positive disjoint sequence that spans a positively complemented copy of ℓ_1. Hence, the result follows from [4, Theorem 5.31]. ∎

Now consider domination by a weakly compact operator for non-commutative function spaces.

Recall that a non-commutative symmetric function space $\mathcal{E}$ is said to have the *Fatou Property* (sometimes referred to as the *Beppo Levi Property*) if for any norm bounded increasing net $(x_i) \subset \mathcal{E}_+$, there exists $x \in \mathcal{E}$ such that $x_i \uparrow x$ and $\|x\| = \sup_i \|x_i\|$. In the commutative setting, any symmetric space with the Fatou Property is order complete.

We say that a non-commutative function space $\mathcal{E}$ is a *KB space* if any increasing norm bounded sequence in $\mathcal{E}$ is norm convergent. Equivalently, $\mathcal{E}$ is order continuous and has the Fatou Property (see [21]). Furthermore, the following are equivalent: (i) $\mathcal{E}$ is a KB space, (ii) $\mathcal{E}$ is weakly sequentially complete, and (iii) $\mathcal{E}$ contains no copy of c_0. It is clear from [17] that, if $\mathcal{E}$ is a symmetric KB function space, then the same is true of $\mathcal{E}(\tau)$.

The following result is contained in [17, Section 5].

PROPOSITION 2.5.5. *Suppose $\mathcal{E}$ is a non-commutative strongly symmetric function space. Then:*

(1) *$\mathcal{E}^\times$ is strongly symmetric.*
(2) *$\mathcal{E}^\times$ coincides with $\mathcal{E}^\star$ if and only if $\mathcal{E}$ is order continuous. In this case, for every $f \in \mathcal{E}^\star$ there exists a unique $y \in \mathcal{E}^\times$ such that $f(x) = \tau(xy)$ for any $x \in \mathcal{E}$.*
(3) *$\mathcal{E}$ coincides with $\mathcal{E}^{\times\times}$ if and only if $\mathcal{E}$ has the Fatou Property.*

PROPOSITION 2.5.6. *Suppose $\mathcal{E} = \mathcal{E}(\tau)$ is a non-commutative strongly symmetric KB function space, X a generating OBS, and $0 \le T \le S : X \to \mathcal{E}$ with S weakly compact. Then T is weakly compact as well.*

Proof. By [17, Section 5], any positive element $\phi \in \mathcal{E}^{\star\star} = (\mathcal{E}^\times)^\star$ can be written as $\phi(f) = \tau(af) + \psi(f)$, where $a \in \mathcal{E}$ is positive and ψ is a positive singular functional. The canonical embedding of $\mathcal{E}$ into its double dual takes a to the linear functional $f \mapsto \tau(fa)$.

S is weakly compact, hence $S^{\star\star}(X) \subset \mathcal{E}$. A normal functional cannot dominate a singular one, hence $T^{\star\star}(\mathbf{B}(X^{\star\star})_+) \subset \mathcal{E}$. As noted in Section 1.1, $X^{\star\star}$ is a generating OBS, hence $T^{\star\star}(\mathbf{B}(X^{\star\star})) \subset \mathcal{E}$. Therefore, T is weakly compact. ∎

Alternatively, one can prove the above result using the characterization of $\sigma(\mathcal{F}^\times, \mathcal{F})$-compact sets given in [19, Proposition 6.2].

REMARK 2.5.7. Note that the assumptions of Proposition 2.5.6 are stronger than those of its commutative counterpart, Theorem 2.5.3. For instance, the statement of Theorem 2.5.3(i) holds when the range space is order continuous. Proposition 2.5.6 is proved under the additional assumption of the Fatou Property. One reason for this is that much more is known about order continuous Banach lattices (see e.g. [39, Section 2.4]). One useful characterization states that a Banach lattice $\mathcal{E}$ is order continuous iff it is an ideal in its second dual. No such description seems to be known in the non-commutative setting.

3. Miscellaneous results

3.1. 2-positivity and decomposability: negative results. In this section we consider stronger versions of positivity, such as 2-positivity and indecomposability, as well as the appropriate notions of domination. We show that these properties are not, in general, inherited by the dominated operator.

PROPOSITION 3.1.1.

(a) *There are $0 \le T \le_c S$ acting on M_2 such that S is completely positive, but T is not 2-positive.*

(b) *There are $0 \le T \le_c S$ acting on M_3 such that S is completely positive, but T is not decomposable.*

For the definition and basic properties of decomposable maps, see e.g. [48]. Note that part (b) is optimal in the sense that any positive map from M_2 to M_3 is decomposable [54].

In the proof below, we use the notation E_{ij} for the matrix with 1 in the (i, j) position, and 0's elsewhere.

Proof. (a) Define $T(a) = a^t$ and $S(a) = \mathrm{tr}(a)\mathbf{1}$ ($\mathrm{tr}(\cdot)$ stands for the canonical trace on M_2). Clearly, $T \geq 0$ and S is completely positive. Indeed, consider $a = \sum_{i,j=1}^{n} E_{ij} \otimes a^{(ij)} \in M_n(M_2) \geq 0$ (here, $a^{(ij)} = (a_{k\ell}^{ij})_{k,\ell=1}^{2} \in M_2$). Passing to submatrices, we see that for $k = 1, 2$, the $n \times n$ matrix $a'_k = (a_{kk}^{(ij)})$ is positive. Thus, $(I_{M_n} \otimes S)a = (a'_1 + a'_2) \otimes (E_{11} + E_{22}) \geq 0$.

The fact that T is not 2-positive is folklore: just apply $I_{M_2} \otimes T$ to $\sum_{i,j=1}^{2} E_{ij} \otimes E_{ij}$. To establish that $S - T \geq_c 0$, note that $(S - T)(a) = uau^*$, where $u = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

(b) Define

$$U\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix},$$

$$V\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{33} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{pmatrix},$$

$$W\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

Let $T = U + V$ and $S = V + 2W$. By [48], T is positive, but not decomposable. On the other hand, the maps V and W are completely positive, hence so is S. Furthermore, $S - T = I$ (the identity map on M_3), hence it is completely positive as well. ■

For powers of operators, we get:

Proposition 3.1.2. *There are $0 \leq T \leq_c S$ acting on M_2 such that S is completely positive, while T is not 2-positive and $T = T^2$.*

Proof. Define $T(a) = (a + a^t)/2$ and $S(a) = (\mathrm{tr}(a)\mathbf{1} + a)/2$. As in the proof of Proposition 3.1.1, we can establish the inequalities $0 \leq_c S$ and $0 \leq T \leq_c S$. Clearly, $T = T^2$. To show that T is not 2-positive, consider $x = \sum_{i,j=1}^{2} E_{ij} \otimes E_{ij} \in M_2 \otimes M_2$. Then x can be identified with the 4×4

matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$(I_{M_2} \otimes T)(x) = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix},$$

which is not positive. ∎

Remark 3.1.3. It is not clear whether we can strengthen Proposition 3.1.1(b) to make the powers of T (not just T itself) non-decomposable. The operator T presented in the proof of Proposition 3.1.1(b) will not work, since T^2 is completely positive. Indeed, [48] shows that $T = U + \mu V$ is not decomposable for $\mu \geq 1$. However, $U^2 = I$ and $UV = VU = V$. Thus, $T^2 = I + 2\mu V + \mu^2 V^2$, which is completely positive.

3.2. A remark on operator systems. In the previous section, we were working with non-commutative function spaces, or with C^*-algebras. This brief section shows that general operator systems have too few positive elements for any results about domination and inheritance of properties.

Recall that an *operator system* is a subspace of $B(H)$ closed under conjugation. It is *unital* if it contains $\mathbf{1}$. If A and B are operator systems, and $T : A \to B$, we say that T is *positive* ($T \geq 0$) if $Ta \geq 0$ for any $a \geq 0$. Moreover, T is *completely positive* ($T \geq_c 0$) if $T \otimes I_{M_n} \geq 0$ for every n. Write $T \geq S$ (resp. $T \geq_c S$) if $T - S \geq 0$ (resp. $T - S \geq_c 0$).

It turns out that little can be said about domination in operator systems. More precisely, there exist a unital operator system A and a rank one $S \in B(A)$ such that $I_A \leq_c S$. We may choose A to be infinite-dimensional, and even non-separable. We describe the construction of A and S below.

Suppose $X \subset B(H)$ is an operator system (not necessarily unital). Using "Paulsen's trick", define A as the set of all block matrices on $H \oplus_2 H$ of the form

$$\begin{pmatrix} \lambda \mathbf{1}_H & x \\ y & \lambda \mathbf{1}_H \end{pmatrix},$$

where $\lambda \in \mathbb{C}$ and $x, y \in X$. It is easy to see that

$$\begin{pmatrix} \lambda \mathbf{1}_K & x \\ y & \lambda \mathbf{1}_H \end{pmatrix} \geq 0 \quad \text{iff} \quad x = y^* \text{ and } \lambda \geq \|x\|.$$

Set

$$S\begin{pmatrix}\lambda\mathbf{1}_H & x\\ y & \lambda\mathbf{1}_H\end{pmatrix} = 2\begin{pmatrix}\lambda\mathbf{1}_H & 0\\ 0 & \lambda\mathbf{1}_H\end{pmatrix} = 2\lambda\mathbf{1}_{H\oplus_2 H}.$$

PROPOSITION 3.2.1. *In the above notation, $S \geq_c I_A$.*

Proof. It suffices to observe that

$$(S-I_A)\begin{pmatrix}\lambda\mathbf{1}_H & x\\ y & \lambda\mathbf{1}_H\end{pmatrix} = \begin{pmatrix}\lambda\mathbf{1}_H & -x\\ -y & \lambda\mathbf{1}_H\end{pmatrix} = u\begin{pmatrix}\lambda\mathbf{1}_H & x\\ y & \lambda\mathbf{1}_H\end{pmatrix}u$$

with

$$u = \begin{pmatrix}\mathbf{1}_H & 0\\ 0 & -\mathbf{1}_H\end{pmatrix}. \blacksquare$$

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